MULTIPLIER IDEAL SHEAVES AND LOCAL GEODESICS FOR PLURISUBHARMONIC FUNCTIONS

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Abstract. Motivated by a study of weak geodesics on the space of Kähler metrics, we study local analytic analogies of weak geodesics on a space of certain class of plurisubharmonic functions. We shall propose a condition using multiplier ideal sheaves to connect with a starting point by a weak geodesic.

1. Introduction

For two plurisubharmonic functions (for short *psh* functions) u_0 and u_1, we can define the weak geodesic $\{u_t\}_{t\in(0,1)}$ joining u_0 and u_1. A weak geodesic measures the difference between singularities of u_0 and u_1, In general, the weak geodesic $\{u_t\}$ may be discontinuous at the start point, that is $\lim_{t\to 0} u_t$ does not equal u_0. The property $\lim_{t\to 0} u_t = u_0$ means that u_0 has worse singularities than u_1 in some sense. Our interest is to analyze the behavior of weak geodesics and psh functions. In this paper, we shall propose some equivalent conditions to $\lim_{t\to 0} u_t = u_0$ in some class of psh functions.

Now we state our results. Let B be the unit ball in $\mathbb{C}^n$. We will denote by $\mathcal{P}(B)$ the set of non-positive psh functions φ on B such that $\lim_{z\to\partial B}\varphi(z) = 0$. In these notations, we investigate relationships between a multiplier ideal sheaf and a Kiselman-Lelong number $\nu^K(\phi, 0, y)$ which is like a Lelong number at the origin in the direction toward $y \in \mathbb{R}^n_{>0}$ (see Definiton 4.1).

Theorem 1.1. *Let $\phi, \psi \in \mathcal{P}(B)$. If $\mathcal{J}(m\phi) \subset \mathcal{J}(m\psi)$ for any $m \in \mathbb{Z}_{>0}$, then $\nu^K(\phi, 0, y) \geq \nu^K(\psi, 0, y)$ for any $y \in \mathbb{R}^n_{>0}$.*

Combining Theorem 1.1, [Gue] and [Hos], we have

Corollary 1.2. *Let u_0 and u_1 be toric psh functions on the unit ball B in $\mathbb{C}^n$ and let u_t be the weak geodesic joining u_0 and u_1. Assume that $u_i^{-1}(-\infty) \subset \{0\}$ and $\lim_{z\to\partial B} u_i(z) = 0$ for $i = 0, 1$. Then the following are equivalent.*

(1) $\lim_{t\to 0} u_t = u_0$ *in capacity.*
(2) $\nu^K(u_0, 0, y) \geq \nu^K(u_1, 0, y)$ *for any $y \in \mathbb{R}^n_{>0}$.*
(3) $\mathcal{J}(mu_0) \subset \mathcal{J}(mu_1)$ *for any $m \in \mathbb{Z}_{>0}$.*

The equivalence between (1) and (2) was known by Hosono [Hos]. Our contribution is the equivalence between (2) and (3). It is easy to see that a toric psh function in Corollary 1.2 is a member of $\mathcal{P}(B)$.

Next, we investigate weak geodesics in the case psh functions have tame singularities introduced in [BFJ] (see Definition 5.1).

THEOREM 1.3. *Let u_0 , $u_1 \in \mathcal{P}(B)$ and let u_t be the weak geodesic joining u_0 and u_1. We assume u_1 has tame singularities. Then the following are equivalent.*

(1) $\lim_{t\to 0} u_t = u_0$ *in capacity.*
(2) $\mathcal{J}(mu_0) \subset \mathcal{J}(mu_1)$ *for any* $m \in \mathbb{Z}_{>0}$.

This work was intended as an attempt to motivate [Ras] and [Hos]. In these papers, weak geodesics joining two psh functions on a pseudoconvex domain were studied. Rashkovskii [Ras] proved that if u_0 and u_1 have finite energy then $\lim_{t\to 0} u_t = u_0$ and $\lim_{t\to 1} u_t = u_1$ holds. Hosono [Hos] proposed an example that $\lim_{t\to 0} u_t$ does not equal u_0 and an equivalent condition to $\lim_{t\to 0} u_t = u_0$ for toric psh functions. Our motivation is to clearify when $\lim_{t\to 0} u_t = u_0$ holds.

Here is a brief history of weak geodesics. The origin is due to Mabuchi [Mab], in which he defined a metric on the space $\mathcal{H}$ of Kähler metrics. After that, Donaldoson [Don] and Semmes [Sem] proved that the geodesic equation on $\mathcal{H}$ can be written as a homogeneous complex Monge Ampére equation. In general, a smooth geodesic may not necessarily exists. But as shown by Chen [Chen], there exists a certain weak geodesic connecting points of $\mathcal{H}$. Weak geodesics on $\mathcal{H}$ are related to a test configulation, a Kähler-Einstein metric and so on. For more details of geodesics on $\mathcal{H}$, we refer the reader to [PSS]. In [Dar], Darvas generalized the notion of weak geodesics on the space of quasi-psh functions in a Kähler comology class. These weak geodesics are introduced as the upper envelop of the family of some quasi-psh functions. Recently Darvas, DiNezza and Lu [DDNL] introduced a notion of weak geodesics in a big cohomology class, and solved the question in [DGZ].

The organization of the paper is as follows. In section §2, we introduce the notion of a weak geodesic same as [Dar]. In section §3, we show that the condition on multiplier ideal sheaves in Theorem 1.1 leads to some inequality of m-th Bergman approximations, which are approximations of a psh function by using Hilbert space of L^2 integrable functions with the psh weight. This inequality is the key to prove Corollary 1.2 and Theorem 1.3. In section §4, we give a generalization of Demailly's m-th Bergman approximation of psh functions. We investigate relationships between Kiselman-Lelong number and Bergman approximation by the same method as [Dem12, Chapter14]. In section §5, we introduce tame singularities and show that Theorem 1.3.

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2. Weak geodesics joining psh function

Let Ω be a bounded pseudoconvex domain containing the origin in $\mathbb{C}^n$. We will denote by $PSH(\Omega)$ the set of psh functions in Ω. In this section we introduce the notion of weak geodesics joining psh functions, following [Dar], [Ras]. We follow the notation of [Ras, Chapter 3].

Let S be the annulus $\{\zeta \in \mathbb{C} \mid 0 < \log|\zeta| < 1\}$. Given two functions $u_0, u_1 \in PSH(\Omega)$, consider a set of psh functions $W(u_0, u_1)$ defined as follows;

$$W(u_0, u_1) := \{r \in PSH(\Omega \times S) \mid r \leq 0, \limsup_{\log|\zeta| \to 0} r(z, \zeta) \leq u_0, \limsup_{\log|\zeta| \to 1} r(z, \zeta) \leq u_1\}.$$

The class is not empty since $u_0 + u_1$ is a member of $W(u_0, u_1)$. We let $\widetilde{u}$ be the pointwise supremum of all the functions in $W(u_0, u_1)$. Since $\widetilde{u}^*$ is a member of $W(u_0, u_1)$ ($*$ means upper-semi-continuous regularization), $\widetilde{u}$ is a psh function on $\Omega \times S$.

Definition 2.1. The *weak geodesic* $\{u_t\}$ joining u_0 and u_1 is a family of functions $u_t(z) := \tilde{u}(z, e^t)$ $\in W(u_0, u_1)$ for each $t \in (0, 1)$

On the other hand, given two functions $u_0, u_1 \in PSH(\Omega)$, we define the *envelope* by

$$P_{[u_1]}(u_0) := \Big(\sup\{r \in PSH(\Omega) \mid r \leq u_0, r \leq u_1 + O(1)\}\Big)^*.$$

The next result gives a relationship between the weak geodesic and the envelope.

Theorem 2.2 ([Dar], [Hos]). *Let u_t be the weak geodesic joining u_0 and u_1. Then* $\lim_{t\to 0} u_t = u_0$ *in capacity if and only if* $P_{[u_1]}(u_0) = u_0$.

Note that the Monge-Ampére capacity of a Borel set $E \subset \Omega$ is defined by the formula

$$\mathrm{Cap}(E) = \sup\left\{ \int_E (dd^c r)^n \mid r \in PSH(\Omega), -1 \leq r \leq 0\right\}.$$

For a sequence $\{v_i\} \subset PSH(\Omega)$, we say that $\lim_{i\to\infty} v_i = v \in PSH(\Omega)$ *in capacity* if for any $\epsilon > 0$ we have

$$\lim_{i\to\infty} \mathrm{Cap}(\{x \in \Omega \mid |v_i - v| > \epsilon\}) = 0.$$

3. Key lemma

Definition 3.1. Let u be a psh function on Ω. For any positive integer m, let u_m be given

$$u_m(z) := \frac{1}{2m} \log \sup_{\|f\|_{mu} \leq 1} |f|^2(z).$$

Here, $\|f\|_{mu} := \int_\Omega |f|^2 e^{-2mu} dV$ and $dV := \frac{\sqrt{-1}^n}{2^n n!} dz_1 d\overline{z_1} \cdots dz_n d\overline{z_n}$. we call u_m a *m-th Bergman approximation of u.*

For any $u \in \mathcal{P}(B)$, we will denote by $\mathcal{G}_u$ the space of holomorphic functions f on B such that $\|f\|_u < \infty$. We consider the evaluation map $\pi\colon \mathcal{G}_u \otimes \mathcal{O}_B \to \mathcal{O}_B$ and define $\mathcal{H}_u := \mathrm{Im}(\pi)$. We obtain the following proposition by [Dem12, Proposition 5.7].

Proposition 3.2. *$\mathcal{H}_u = \mathcal{J}(u)$ holds on B for any $u \in \mathcal{P}(B)$.*

Now we prove the following key lemma.

Lemma 3.3. *Fix $m \in \mathbb{Z}_{>0}$ and $u, v \in \mathcal{P}(B)$. Let u_m* (resp. v_m) be m-th Bergman approximations of u (resp. v). If $\mathcal{J}(mu) \subset \mathcal{J}(mv)$, then there exists a positive number M_m such that $u_m \le v_m + M_m$

Proof. The proof will be divided into three steps.

§1 We prove that $\|f\|_{mv} < \infty$ for all holomorphic function f on B such that $\|f\|_{mu} \le 1$. By considering a germ of f at an origin o, we obtain $f_o \in \mathcal{J}(mu)_o \subset \mathcal{J}(mv)_o$. By Propositon 3.2, there exists a holomorphic function F on B such that $F_o = f_o$ and $\|F\|_{mv} < \infty$. Since f is holomorphic on B and $F = f$ near o, we obtain $F = f$ on B by the identity theorem. Therefore we obtain $\|f\|_{mv} < \infty$.

§2 We show that there exists a real number M such that $\|f\|_{mv} < M$ for all holomorphic function f on B such that $\|f\|_{mu} \le 1$. Suppose it were false. Then we could find f_i for all positive natural number i such that $\|f_i\|_{mu} \le 1$ and $\|f_i\|_{mv} > i$. We set $\mathcal{F} := \{f \in \mathcal{O}(B) \mid \|f\|_{mu} \le 1\}$. Since $\mathcal{F}$ is uniform bounded on compact subsets, $\mathcal{F}$ is normal family by Montel's theorem. Therefore, there exist $F \in \mathcal{F}$ and a subsequence $\{f_{i_k}\}$ such that $f_{i_k} \to F$ by uniform converge on compact subsets. That $\|F\|_{mv} < \infty$ follows from $F \in \mathcal{F}$. This contradicts our assumption the fact that $\|F\|_{mv} \ge \liminf_k \|f_{i_k}\|_{mv} \ge \infty$.

§3 We finish proof. For $f \in \mathcal{F}$,

$$\frac{1}{2m} \log(|f|^2/\|f\|^2_{mv}) \le \frac{1}{2m} \log \sup_{\|g\|_{mv} \le 1} |g|^2 = v_m.$$

Then we put $M_m := \frac{1}{m} \log M$,

$$\frac{1}{2m} \log(|f|^2) \le v_m + \frac{1}{2m} \log \|f\|^2_{mv} \le v_m + \frac{1}{m} \log M.$$

Therefore, we take supremum for f, which proves the lemma. □

4. Kiselman Lelong number and Bergman approximation

Definition 4.1 ([Kis] Definition 5.1 and [Dem] Chapter 3). Let $\varphi \in PSH(\Omega)$ and $y \in \mathbb{R}^n_{>0}$. Then we define the positive constant to be

$$\nu^K(\varphi, 0, y) := \lim_{t \to -\infty} \frac{1}{t} \sup_{|z_i| \le 1} \varphi(z_1 e^{ty_1}, \ldots, z_n e^{ty_n}).$$

We call $\nu^K(\varphi, 0, y)$ *Kiselman-Lelong number* with coefficients y at the origin.

Lemma 4.2. *Let $\varphi \in PSH(\Omega)$, and $r_1, r_2 \ldots r_n$ be positive real numbers with $\{|z_i| \le 2r_i\} \subset \Omega$. Then there exists a constant C independent of m satisfying:*

$$\varphi(w) - \frac{C}{m} \le \varphi_m(w) \le \sup_{|\zeta_i - w_i| \le r_i} \varphi(\zeta) - \frac{1}{2m} \log(\pi^n r_1^2 \cdots r_n^2 / n!) \tag{4.1}$$

for all $w \in \{|z_i| \le r_i\}$. Here, φ_m is the m-th Bergman approximation of φ.

Proof. In [Dem12, Chapter 14], $\varphi(w) - \frac{C}{m} \le \varphi_m(w)$ is already proved. Thus we need only to show the right hand inequality.

We fix m. By definition of φ_m, we have

$$\varphi_m = \frac{1}{2m} \log \sup_{\|f\|_{m\varphi} \le 1} |f|^2 = \sup_{\|f\|_{m\varphi} \le 1} \frac{1}{2m} \log |f|^2.$$

Here $\|f\|_{m\varphi} = \int_\Omega |f|^2 e^{-2m\varphi} dV$.

We fix a holomorphic function f on Ω satisfying $\|f\|_{m\varphi} \le 1$. Since $\log |f|^2$ is psh, we apply submean value inequality to each variable,

$$\begin{aligned} \log |f|^2(w_1, \ldots, w_n) &\le \frac{\sqrt{-1}}{2\pi r_1^2} \int_{|\zeta_1 - w_1| \le r_1} \log |f|^2(\zeta_1, 0, \ldots, 0) d\zeta_1 d\overline{\zeta_1} \\ &\le \cdots \\ &\le \frac{\sqrt{-1}^n}{2^n \pi^n r_1^2 \cdots r_n^2} \int_{|\zeta_i - w_i| \le r_i} \log |f|^2(\zeta_1, \zeta_2, \ldots, \zeta_n) d\zeta_1 d\overline{\zeta_1} \cdots d\zeta_n d\overline{\zeta_n} \\ &= \frac{n!}{\pi^n r_1^2 \cdots r_n^2} \int_{|\zeta_i - w_i| \le r_i} \log |f|^2(\zeta) dV. \end{aligned} \tag{4.2}$$

Since log is a concave function, by Jensen inequality,

$$\begin{aligned} \frac{n!}{\pi^n r_1^2 \cdots r_n^2} \int_{|\zeta_i - w_i| \le r_i} \log |f|^2(\zeta) dV &\le \log \left(\frac{n!}{\pi^n r_1^2 \cdots r_n^2} \int_{|\zeta_i - w_i| \le r_i} |f|^2(\zeta) dV \right) \\ &= \log \left(\int_{|\zeta_i - w_i| \le r_i} |f|^2(\zeta) dV \right) + \log \frac{n!}{\pi^n r_1^2 \cdots r_n^2}. \end{aligned} \tag{4.3}$$

Therefore from $\|f\|_{m\varphi} = \int_\Omega |f|^2 e^{-2m\varphi} dV \leq 1$, we have

$$
\begin{aligned}
\log\left(\int_{|\zeta_i - w_i| \leq r_i} |f|^2(\zeta) dV\right) &\leq \log\left(\int_{|\zeta_i - w_i| \leq r_i} |f|^2 e^{-2m\varphi}(\zeta) dV\right) + \sup_{|\zeta_i - w_i| \leq r_i} 2m\varphi(\zeta) \\
&\leq \log\left(\int_\Omega |f|^2 e^{-2m\varphi} dV\right) + \sup_{|\zeta_i - w_i| \leq r_i} 2m\varphi(\zeta) \\
&\leq 2m \sup_{|\zeta_i - w_i| \leq r_i} \varphi(\zeta).
\end{aligned} \tag{4.4}
$$

We thus get

$$
\begin{aligned}
\varphi_m(w) &= \sup_{\|f\|_{m\varphi} \leq 1} \frac{1}{2m} \log |f|^2(w) \\
&\leq \sup_{\|f\|_{m\varphi} \leq 1} \frac{1}{2m} \log\left(\int_{|\zeta_i - w_i| \leq r_i} |f|^2(\zeta) dV\right) + \frac{1}{2m} \log \frac{n!}{\pi^n r_1^2 \cdots r_n^2} \\
&\leq \sup_{|\zeta_i - w_i| \leq r_i} \varphi(\zeta) - \frac{1}{2m} \log(\pi^n r_1^2 \cdots r_n^2 / n!).
\end{aligned} \tag{4.5}
$$

□

Corollary 4.3. *Under the same asuumption in Lemma 4.2, for all* $y \in \mathbb{R}^n_{>0}$,

$$
\nu^K(\varphi, 0, y) \geq \nu^K(\varphi_m, 0, y) \geq \nu^K(\varphi, 0, y) - \frac{y_1 + \cdots + y_n}{m} \tag{4.6}
$$

holds. In particular, $\lim_{m\to\infty} \nu^K(\varphi_m, 0, y) = \nu^K(\varphi, 0, y)$.

Proof. By Lemma 4.2, if we take supremum on $w \in \{|z_i| \leq r_i\}$, we have

$$
\sup_{|w_i| \leq r_i} \varphi(w) - \frac{C}{m} \leq \sup_{|w_i| \leq r_i} \varphi_m(w) \leq \sup_{|w_i| \leq 2r_i} \varphi(\zeta) - \frac{\log r_1 \cdots r_n}{m} - \frac{1}{2m} \log(\pi^n / n!). \tag{4.7}
$$

Therefore if we take $t < 0$, put $r_i = e^{ty_i}$ and multiply $1/t$, we have

$$
\frac{1}{t} \sup_{|w_i| \leq e^{ty_i}} \varphi(w) - \frac{C}{mt} \geq \frac{1}{t} \sup_{|w_i| \leq e^{ty_i}} \varphi_m(w) \geq \frac{1}{t} \sup_{|w_i| \leq 2e^{ty_i}} \varphi(\zeta) - \frac{ty_1 + \cdots + ty_n}{mt} - \frac{1}{2mt} \log(\pi^n / n!). \tag{4.8}
$$

Therefore we take the limit for $t \to -\infty$,

$$
\nu^K(\varphi, 0, y) \geq \nu^K(\varphi_m, 0, y) \geq \nu^K(\varphi, 0, y) - \frac{y_1 + \cdots + y_n}{m}.
$$

□

Theorem 4.4 (= Theorem 1.1). *Let* $\phi, \psi \in \mathcal{P}(B)$. *If* $\mathcal{J}(m\phi) \subset \mathcal{J}(m\psi)$ *for any* $m \in \mathbb{Z}_{>0}$, *then* $\nu^K(\phi, 0, y) \geq \nu^K(\psi, 0, y)$ *for any* $y \in \mathbb{R}^n_{>0}$.

Proof. Fix $y \in \mathbb{R}^n_{>0}$. Since $\mathcal{J}(m\phi) \subset \mathcal{J}(m\psi)$ for any $m \in \mathbb{Z}_{>0}$, by Lemma 3.3,

$$\nu^K(\phi_m, 0, y) \geq \nu^K(\psi_m, 0, y).$$

Therefore by Corollary 4.3, by taking the limit $m \to \infty$, we have

$$\nu^K(\phi, 0, y) \geq \nu^K(\psi, 0, y).$$

□

Corollary 4.5 (=Corollary 1.2). *Let u_0 and u_1 be toric psh functions on the unit ball B in $\mathbb{C}^n$ and let u_t be the weak geodesic joining u_0 and u_1. Then the following are equivalent.*

(1) $\lim_{t\to 0} u_t = u_0$ *in capacity.*
(2) $\nu^K(u_0, 0, y) \geq \nu^K(u_1, 0, y)$ *for any* $y \in \mathbb{R}^n_{>0}$.
(3) $\mathcal{J}(mu_0) \subset \mathcal{J}(mu_1)$ *for any* $m \in \mathbb{Z}_{>0}$.

Proof. The equivalence between (1) and (2) was known by Hosono [Hos]. From Theorem 1.1, (3) implies (2), thus we only need to show that (2) implies (3). It is easy to check that according to [Gue, Theorem 1.20], $\mathcal{J}(u_0)$ is monomial and

$$z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n} \in \mathcal{J}(u_0) \Longleftrightarrow \sup_{y \in \mathbb{R}^n_{>0}} \frac{\nu^K(u_0, 0, y)}{\nu^K(z_1^{a_1} z_2^{a_2} \cdots z_n^{a_n}, 0, y) + \sum_{i=1}^n a_i y_i} < 1.$$

□

5. Tame sinularity and weak geodesics

Definition 5.1 ([BFJ] section 5.3). We say that $u \in PSH(B)$ has *tame singularities* with coefficient $c > 0$, if

$$u + O(1) \leq u_m \leq (1 - c/m)u + O(1)$$

holds, where the $O(1)$ term is independent of m.

Remark 5.2. According to [BFJ, Lemma 5.10], if u is exponential α-holder for some $\alpha > 0$ (i.e. e^u is α-hölder continuous), then u has tame singularities. In paticular, if u has algebraic singularities (i.e. u is written as sum of C^∞function and $\log|f|$ for some holomorphic function f), then u has tame singularities.

Theorem 5.3 (=Theorem 1.3). *Let $u_0, u_1 \in \mathcal{P}(B)$ and let u_t be a weak geodesic joining u_0 and u_1. We assume u_1 has tame singularities. Then the following are equivalent.*

(1) $\lim_{t\to 0} u_t = u_0$ *in capacity.*
(2) $\mathcal{J}(mu_0) \subset \mathcal{J}(mu_1)$ *for any* $m \in \mathbb{Z}_{>0}$.

Proof. By Theorem 2.2, it suffices to prove that the equivalence between $P_{[u_1]}(u_0) = u_0$ and $\mathcal{J}(mu_0) \subset \mathcal{J}(mu_1)$ for any $m \in \mathbb{Z}_{>0}$.

We assume $P_{[u_1]}(u_0) = u_0$. We need only consider $\mathcal{J}(u_0) \subset \mathcal{J}(u_1)$ since $P_{[mu_1]}(mu_0) = mP_{[u_1]}(u_0)$. By the strong openness property of a multiplier ideal sheaf by Guan and Zhou [GZ], we can take a large positive number C such that $\mathcal{J}(u_0) = \mathcal{J}(P_{[u_1]}(u_0)) = \mathcal{J}(P(u_0, u_1 + C))$. By definition $P(u_0, u_1 + C) \le u_1 + C$, therefore $\mathcal{J}(u_0) = \mathcal{J}(P(u_0, u_1 + C)) \subset \mathcal{J}(u_1)$.

Conversely, we assume $\mathcal{J}(mu_0) \subset \mathcal{J}(mu_1)$ for any $m \in \mathbb{Z}_{>0}$. We will denote $u_{i,m}$ by the m-th Bergman approximation of u_i for $i = 0, 1$. By Demailly's approximation theorem [Dem12, Chapter 14], there exists a positive number C_1 independent of m such that $u_0 - \frac{C_1}{m} \le u_{0,m}$. From Lemma 3.3, we obtain $u_0 \le u_{1,m} + O(1)$. It follows that $P_{[u_{1,m}]}(u_0) = u_0$.

Since u_1 has tame singularities, $u_{1,m} \le (1 - \frac{c}{m})u_1 + O(1)$ for all $m \in \mathbb{Z}_{>0}$. For all positive number C and all $r \in PSH(B)$ satisfying $r \le \min(u_0, u_{1,m} + C)$,

$$r + \frac{c}{m}u_1 \le u_0, r + \frac{c}{m}u_1 \le u_1 + C + O(1).$$

By the definition of $P_{[u_1]}(u_0)$, we have $r + \frac{c}{m}u_1 \le P_{[u_1]}(u_0)$. Thus we take supremum for r and C, $P_{[u_{1,m}]}(u_0) + \frac{c}{m}u_1 \le P_{[u_1]}(u_0)$ holds. Since $P_{[u_{1,m}]}(u_0) = u_0$, it follows that $u_0 + \frac{c}{m}u_1 \le P_{[u_1]}(u_0)$. Consequencely we can take the limit for m outside the origin, we have $u_0 \le P_{[u_1]}(u_0)$ on $B \setminus \{0\}$. On the other hand, $P_{[u_1]}(u_0) \le u_0$ by definition. Hence we have $u_0 = P_{[u_1]}(u_0)$ on $B \setminus \{0\}$. From Lemma 5.4 as below, we get $u_0 = P_{[u_1]}(u_0)$. □

Lemma 5.4 ([Gun] Chapter A). *Let $u, v \in PSH(\Omega)$. If $u = v$ almost everywhere with respect to Lesbegue measure, then $u = v$ on Ω.*

Proof. First, we show that

$$u(a) = \lim_{\epsilon \to 0} \frac{1}{vol(B(a, \epsilon))} \int_{B(a,\epsilon)} u(z) dV \tag{5.1}$$

Since u is an upper-semi-continuous function, we have

$$\begin{aligned} u(a) &= \limsup_{z \to a} u(z) \\ &= \lim_{\epsilon \to 0} \sup_{z \in B(a,\epsilon)} u(z) \\ &\ge \lim_{\epsilon \to 0} \frac{1}{vol(B(a, \epsilon))} \int_{B(a,\epsilon)} u(z) dV. \end{aligned} \tag{5.2}$$

On the other hand, according to the submean value inequality,

$$u(a) \le \lim_{\epsilon \to 0} \frac{1}{vol(B(a, \epsilon))} \int_{B(a,\epsilon)} u(z) dV. \tag{5.3}$$

We thus get (5.1).

Applying (5.1),

$$\begin{aligned} u(a) &= \lim_{\epsilon\to 0} \frac{1}{vol(B(a,\epsilon))} \int_{B(a,\epsilon)} u(z)dV \\ &= \lim_{\epsilon\to 0} \frac{1}{vol(B(a,\epsilon))} \int_{B(a,\epsilon)} v(z)dV \\ &= v(a). \end{aligned} \tag{5.4}$$

□

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