MULTIPLIER IDEAL SHEAVES AND LOCAL GEODESICS FOR PLURISUBHARMONIC FUNCTIONS

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ABSTRACT. Motivated by a study of weak geodesics on the space of Kähler metrics, we study local analytic analogies of weak geodesics on a space of certain class of plurisubharmonic functions. We shall propose a condition using multiplier ideal sheaves to connect with a starting point by a weak geodesic.

1. Introduction

For two plurisubharmonic functions (for short psh functions) $u_0$ and $u_1$, we can define the weak geodesic $\{u_t\}_{t \in (0,1)}$ joining $u_0$ and $u_1$. A weak geodesic measures the difference between singularities of $u_0$ and $u_1$. In general, the weak geodesic $\{u_t\}$ may be discontinuous at the start point, that is $\lim_{t \to 0} u_t$ does not equal $u_0$. The property $\lim_{t \to 0} u_t = u_0$ means that $u_0$ has worse singularities than $u_1$ in some sense. Our interest is to analyze the behavior of weak geodesics and psh functions. In this paper, we shall propose some equivalent conditions to $\lim_{t \to 0} u_t = u_0$ in some class of psh functions.

Now we state our results. Let $B$ be the unit ball in $\mathbb{C}^n$. We will denote by $\mathcal{P}(B)$ the set of non-positive psh functions $\varphi$ on $B$ such that $\lim_{z \to \partial B} \varphi(z) = 0$. In these notations, we investigate relationships between a multiplier ideal sheaf and a Kiselman-Lelong number $\nu^K(\varphi,0,y)$ which is like a Lelong number at the origin in the direction toward $y \in \mathbb{R}^n_>$ (see Definition 4.1).

**Theorem 1.1.** Let $\phi, \psi \in \mathcal{P}(B)$. If $\mathcal{J}(m\phi) \subset \mathcal{J}(m\psi)$ for any $m \in \mathbb{Z}_{>0}$, then $\nu^K(\phi,0,y) \geq \nu^K(\psi,0,y)$ for any $y \in \mathbb{R}^n_>$. 

Combining Theorem 1.1, [Gue] and [Hos], we have

**Corollary 1.2.** Let $u_0$ and $u_1$ be toric psh functions on the unit ball $B$ in $\mathbb{C}^n$ and let $u_i$ be the weak geodesic joining $u_0$ and $u_1$. Assume that $u_i^{-1}(-\infty) \subset \{0\}$ and $\lim_{z \to \partial B} u_i(z) = 0$ for $i = 0, 1$. Then the following are equivalent.

1. $\lim_{t \to 0} u_t = u_0$ in capacity.
2. $\nu^K(u_0,0,y) \geq \nu^K(u_1,0,y)$ for any $y \in \mathbb{R}^n_>$. 
3. $\mathcal{J}(mu_0) \subset \mathcal{J}(mu_1)$ for any $m \in \mathbb{Z}_{>0}$. 

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The equivalence between (1) and (2) was known by Hosono [Hos]. Our contribution is the equivalence between (2) and (3). It is easy to see that a toric psh function in Corollary 1.2 is a member of $\mathcal{P}(B)$.

Next, we investigate weak geodesics in the case psh functions have tame singularities introduced in [BFJ] (see Definition 5.1).

**Theorem 1.3.** Let $u_0, u_1 \in \mathcal{P}(B)$ and let $u_t$ be the weak geodesic joining $u_0$ and $u_1$. We assume $u_1$ has tame singularities. Then the following are equivalent.

1. $\lim_{t \to 0} u_t = u_0$ in capacity.
2. $\mathcal{J}(mu_0) \subset \mathcal{J}(mu_1)$ for any $m \in \mathbb{Z}_{>0}$.

This work was intended as an attempt to motivate [Ras] and [Hos]. In these papers, weak geodesics joining two psh functions on a pseudoconvex domain were studied. Rashkovskii [Ras] proved that if $u_0$ and $u_1$ have finite energy then $\lim_{t \to 0} u_t = u_0$ and $\lim_{t \to 1} u_t = u_1$ holds. Hosono [Hos] proposed an example that $\lim_{t \to 0} u_t$ does not equal $u_0$ and an equivalent condition to $\lim_{t \to 0} u_t = u_0$ for toric psh functions. Our motivation is to clarify when $\lim_{t \to 0} u_t = u_0$ holds.

Here is a brief history of weak geodesics. The origin is due to Mabuchi [Mab], in which he defined a metric on the space $\mathcal{H}$ of Kähler metrics. After that, Donaldson [Don] and Semmes [Sem] proved that the geodesic equation on $\mathcal{H}$ can be written as a homogeneous complex Monge Ampère equation. In general, a smooth geodesic may not necessarily exists. But as shown by Chen [Chen], there exists a certain weak geodesic connecting points of $\mathcal{H}$. Weak geodesics on $\mathcal{H}$ are related to a test configuration, a Kähler-Einstein metric and so on. For more details of geodesics on $\mathcal{H}$, we refer the reader to [PSS]. In [Dar], Darvas generalized the notion of weak geodesics on the space of quasi-psh functions in a Kähler comology class. These weak geodesics are introduced as the upper envelop of the family of some quasi-psh functions. Recently Darvas, DiNezza and Lu [DDNL] introduced a notion of weak geodesics in a big cohomology class, and solved the question in [DGZ].

The organization of the paper is as follows. In section §2, we introduce the notion of a weak geodesic same as [Dar]. In section §3, we show that the condition on multiplier ideal sheaves in Theorem 1.1 leads to some inequality of $m$-th Bergman approximations, which are approximations of a psh function by using Hilbert space of $L^2$ integrable functions with the psh weight. This inequality is the key to prove Corollary 1.2 and Theorem 1.3. In section §4, we give a generalization of Demailly’s $m$-th Bergman approximation of psh functions. We investigate relationships between Kiselman-Lelong number and Bergman approximation by the same method as [Dem12, Chapter14]. In section §5, we introduce tame singularities and show that Theorem 1.3.

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2. Weak geodesics joining psh function

Let $\Omega$ be a bounded pseudoconvex domain containing the origin in $\mathbb{C}^n$. We will denote by $\text{PSH}(\Omega)$ the set of psh functions in $\Omega$. In this section we introduce the notion of weak geodesics joining psh functions, following [Dar], [Ras]. We follow the notation of [Ras, Chapter 3].

Let $S$ be the annulus $\{ \zeta \in \mathbb{C} | 0 < \log |\zeta| < 1 \}$. Given two functions $u_0, u_1 \in \text{PSH}(\Omega)$, consider a set of psh functions $W(u_0, u_1)$ defined as follows;

$$W(u_0, u_1) := \{ r \in \text{PSH}(\Omega \times S) | r \leq 0, \limsup \log |\zeta| \to 0 r(z, \zeta) \leq u_0, \limsup \log |\zeta| \to 1 r(z, \zeta) \leq u_1 \}.$$ 

The class is not empty since $u_0 + u_1$ is a member of $W(u_0, u_1)$. We let $\tilde{u}$ be the pointwise supremum of all the functions in $W(u_0, u_1)$. Since $\tilde{u}^*$ is a member of $W(u_0, u_1)$ (∗ means upper-semi-continuous regularization), $\tilde{u}$ is a psh function on $\Omega \times S$.

Definition 2.1. The weak geodesic $\{ u_t \}$ joining $u_0$ and $u_1$ is a family of functions $u_t(z) := \tilde{u}(z, t^j)$ $\in W(u_0, u_1)$ for each $t \in (0, 1)$.

On the other hand, given two functions $u_0, u_1 \in \text{PSH}(\Omega)$, we define the envelope by 

$$P_{[u_1]}(u_0) := \left( \sup\{ r \in \text{PSH}(\Omega) | r \leq u_0, r \leq u_1 + O(1) \} \right)^*.$$ 

The next result gives a relationship between the weak geodesic and the envelope.

Theorem 2.2 ([Dar], [Hos]). Let $u_t$ be the weak geodesic joining $u_0$ and $u_1$. Then $\lim_{t \to 0} u_t = u_0$ in capacity if and only if $P_{[u_1]}(u_0) = u_0$.

3. Key lemma

Definition 3.1. Let $u$ be a psh function on $\Omega$. For any positive integer $m$, let $u_m$ be given 

$$u_m(z) := \frac{1}{2m} \log \sup_{\|f\|_{L^0} \leq 1} |f|^2(z).$$
Here, \( \|f\|_{mu} := \int_{\Omega} |f|^2 e^{-2\mu} dV \) and \( dV := \frac{\sqrt{-1}}{2\pi n!} dz_1 \overline{dz_1} \cdots dz_n \overline{dz_n} \). We call \( u_m \) a \( m \)-th Bergman approximation of \( u \).

For any \( u \in \mathcal{P}(B) \), we will denote by \( \mathcal{G}_u \) the space of holomorphic functions \( f \) on \( B \) such that \( \|f\|_u < \infty \). We consider the evaluation map \( \pi: \mathcal{G}_u \otimes \mathcal{O}_B \to \mathcal{O}_B \) and define \( \mathcal{H}_u := \text{Im}(\pi) \). We obtain the following proposition by [Dem12, Proposition 5.7].

**Proposition 3.2.** \( \mathcal{H}_u = \mathcal{J}(u) \) holds on \( B \) for any \( u \in \mathcal{P}(B) \).

Now we prove the following key lemma.

**Lemma 3.3.** Fix \( m \in \mathbb{Z}_{>0} \) and \( u, v \in \mathcal{P}(B) \). Let \( u_m \) (resp. \( v_m \)) be \( m \)-th Bergman approximations of \( u \) (resp. \( v \)). If \( \mathcal{J}(mu) \subset \mathcal{J}(mv) \), then there exists a positive number \( M_m \) such that \( u_m \leq v_m + M_m \).

**Proof.** The proof will be divided into three steps.

§1 We prove that \( \|f\|_{mv} < \infty \) for all holomorphic function \( f \) on \( B \) such that \( \|f\|_{mu} \leq 1 \). By considering a germ of \( f \) at an origin \( o \), we obtain \( f_o \in \mathcal{J}(mu)_o \subset \mathcal{J}(mv)_o \). By Proposition 3.2, there exists a holomorphic function \( F \) on \( B \) such that \( F_o = f_o \) and \( \|F\|_{mv} < \infty \). Since \( f \) is holomorphic on \( B \) and \( F = f \) near \( o \), we obtain \( F = f \) on \( B \) by the identity theorem. Therefore we obtain \( \|f\|_{mv} < \infty \).

§2 We show that there exists a real number \( M \) such that \( \|f\|_{mv} < M \) for all holomorphic function \( f \) on \( B \) such that \( \|f\|_{mu} \leq 1 \). Suppose it were false. Then we could find \( f_i \) for all positive natural number \( i \) such that \( \|f_i\|_{mu} \leq 1 \) and \( \|f_i\|_{mv} > i \). We set \( \mathcal{F} := \{ f \in \mathcal{O}(B) \mid \|f\|_{mu} \leq 1 \} \). Since \( \mathcal{F} \) is uniform bounded on compact subsets, \( \mathcal{F} \) is normal family by Montel’s theorem. Therefore, there exist \( F \in \mathcal{F} \) and a subsequence \( \{f_{i_k}\} \) such that \( f_{i_k} \to F \) by uniform converge on compact subsets. Then \( \|F\|_{mv} < \infty \) follows from \( F \in \mathcal{F} \). This contradicts our assumption the fact that \( \|F\|_{mv} \geq \lim inf_k \|f_{i_k}\|_{mv} \geq \infty \).

§3 We finish proof. For \( f \in \mathcal{F} \),

\[
\frac{1}{2m} \log(|f|^2/\|f\|_{mv}^2) \leq \frac{1}{2m} \log \sup_{\|g\|_{mv} \leq 1} |g|^2 = v_m.
\]

Then we put \( M_m := \frac{1}{m} \log M \),

\[
\frac{1}{2m} \log(|f|^2) \leq v_m + \frac{1}{2m} \log \|f\|_{mv}^2 \leq v_m + \frac{1}{m} \log M.
\]

Therefore, we take supremum for \( f \), which proves the lemma. \( \square \)
4. Kiselman Lelong number and Bergman approximation

**Definition 4.1** ([Kis] Definition 5.1 and [Dem] Chapter 3). Let \( \varphi \in \text{PSH}(\Omega) \) and \( y \in \mathbb{R}^n_{>0} \). Then we define the positive constant to be

\[
y^K(\varphi, 0, y) := \lim_{t \to -\infty} \frac{1}{t} \sup_{|z| \leq 1} \varphi(z_1 e^{by_1}, \ldots, z_n e^{by_n}).
\]

We call \( n^K(\varphi, 0, y) \) Kiselman-Lelong number with coefficients \( y \) at the origin.

**Lemma 4.2.** Let \( \varphi \in \text{PSH}(\Omega) \), and \( r_1, r_2, \ldots, r_n \) be positive real numbers with \( |z| \leq 2r_1 \subset \Omega \). Then there exists a constant \( C \) independent of \( m \) satisfying:

\[
\varphi(w) - \frac{C}{m} \leq \varphi_m(w) \leq \sup_{|z| \leq 1} \varphi(z) - \frac{1}{2m} \log(\pi^n r_1^2 \cdots r_n^2 / n!)
\]

for all \( w \in \{|z| \leq r_1\} \). Here, \( \varphi_m \) is the \( m \)-th Bergman approximation of \( \varphi \).

**Proof.** In [Dem12, Chapter 14], \( \varphi(w) - \frac{C}{m} \leq \varphi_m(w) \) is already proved. Thus we need only to show the right hand inequality.

We fix \( m \). By definition of \( \varphi_m \), we have

\[
\varphi_m = \frac{1}{2m} \log \sup_{\|f\|_{lev} \leq 1} |f|^2 = \sup_{\|f\|_{lev} \leq 1} \frac{1}{2m} \log |f|^2.
\]

Here \( \|f\|_{lev} = \int_\Omega |f|^2 e^{-2m\varphi} dV \).

We fix a holomorphic function \( f \) on \( \Omega \) satisfying \( \|f\|_{lev} \leq 1 \). Since \( \log |f|^2 \) is psh, we apply submean value inequality to each variable,

\[
\log |f|^2(w_1, \ldots, w_n) \leq \frac{\sqrt{-1}}{2\pi r_1^2} \int_{|z_1| \leq r_1} \log |f|^2(z_1, 0, \ldots, 0) dz_1 d\bar{z}_1
\]

\[
\leq \cdots
\]

\[
\leq \frac{\sqrt{-1}}{2\pi^n r_1^2 \cdots r_n^2} \int_{|z| \leq r} \log |f|^2(z_1, \ldots, z_n) dz_1 d\bar{z}_1 \cdots dz_n d\bar{z}_n
\]

\[
= \frac{n!}{\pi^n r_1^2 \cdots r_n^2} \int_{|z| \leq r} \log |f|^2(\zeta) dV, \quad (4.2)
\]

Since log is a concave function, by Jensen inequality,

\[
\frac{n!}{\pi^n r_1^2 \cdots r_n^2} \int_{|z| \leq r} \log |f|^2(\zeta) dV \leq \log \left( \frac{n!}{\pi^n r_1^2 \cdots r_n^2} \int_{|z| \leq r} |f|^2(\zeta) dV \right)
\]

\[
= \log \left( \int_{|z| \leq r} |f|^2(\zeta) dV \right) + \log \frac{n!}{\pi^n r_1^2 \cdots r_n^2}, \quad (4.3)
\]
Therefore from \( \|f\|_{m,p} = \int_{\Omega} |f|^2 e^{-2m\varphi} dV \leq 1 \), we have

\[
\log\left( \int_{|z_i - w_i| \leq r_i} |f|^2(\zeta) dV \right) \leq \log\left( \int_{|z_i - w_i| \leq r_i} |f|^2 e^{-2m\varphi(\zeta)} dV \right) + \sup_{|z_i - w_i| \leq r_i} 2m\varphi(\zeta)
\]

\[
(4.4)
\]

\[
\leq \log\left( \int_{\Omega} |f|^2 e^{-2m\varphi} dV \right) + \sup_{|z_i - w_i| \leq r_i} 2m\varphi(\zeta)
\]

\[
\leq 2m \sup_{|z_i - w_i| \leq r_i} \varphi(\zeta).
\]

We thus get

\[
\varphi_m(w) = \sup_{\|f\|_{m,p} \leq 1} \frac{1}{2m} \log |f|^2(w)
\]

\[
(4.5)
\]

\[
\leq \sup_{\|f\|_{m,p} \leq 1} \frac{1}{2m} \log\left( \int_{|z_i - w_i| \leq r_i} |f|^2(\zeta) dV \right) + \frac{1}{2m} \log\frac{n!}{\pi^m r_1^2 \cdots r_n^2}
\]

\[
\leq \sup_{|z_i - w_i| \leq r_i} \varphi(\zeta) - \frac{1}{2m} \log(\pi^n r_1^2 \cdots r_n^2/n!).
\]

\[\square\]

**Corollary 4.3.** Under the same assumption in Lemma 4.2, for all \( y \in \mathbb{R}^n \),

\[
(4.6)
\]

\[
y^K(\varphi, 0, y) \geq y^K(\varphi_m, 0, y) \geq y^K(\varphi, 0, y) - \frac{y_1 + \cdots + y_n}{m}
\]

holds. In particular, \( \lim_{m \to \infty} y^K(\varphi_m, 0, y) = y^K(\varphi, 0, y) \).

**Proof.** By Lemma 4.2, if we take supremum on \( w \in \{|z_i| \leq r_i\} \), we have

\[
(4.7)
\]

\[
\sup_{|w| \leq r_i} \varphi(w) - C m \leq \sup_{|w| \leq r_i} \varphi_m(w) \leq \sup_{|w| \leq 2r_i} \varphi(\zeta) - \frac{\log r_1 \cdots r_n}{m} - \frac{1}{2m} \log(\pi^n n!).
\]

Therefore if we take \( t < 0 \), put \( r_i = e^{ht} \) and multiply \( 1/t \), we have

\[
(4.8)
\]

\[
\frac{1}{t} \sup_{|w| \leq e^{ht}} \varphi(w) - C \frac{m t}{t} \geq \frac{1}{t} \sup_{|w| \leq e^{ht}} \varphi_m(w) \geq \frac{1}{t} \sup_{|w| \leq 2e^{ht}} \varphi(\zeta) - \frac{ty_1 + \cdots + ty_n}{mt} - \frac{1}{2mt} \log(\pi^n n!).
\]

Therefore we take the limit for \( t \to -\infty \),

\[
y^K(\varphi, 0, y) \geq y^K(\varphi_m, 0, y) \geq y^K(\varphi, 0, y) - \frac{y_1 + \cdots + y_n}{m}.
\]

\[\square\]

**Theorem 4.4 (= Theorem 1.1).** Let \( \phi, \psi \in \mathcal{P}(B) \). If \( \mathcal{J}(m\phi) \subset \mathcal{J}(m\psi) \) for any \( m \in \mathbb{Z}_{>0} \), then \( y^K(\phi, 0, y) \geq y^K(\psi, 0, y) \) for any \( y \in \mathbb{R}^n_{>0} \).
Proof. Fix \( y \in \mathbb{R}_+^n \). Since \( \mathcal{I}(m\varphi) \subset \mathcal{I}(m\psi) \) for any \( m \in \mathbb{Z}_{>0} \), by Lemma 3.3,
\[
\nu^K(\phi_m, 0, y) \geq \nu^K(\psi_m, 0, y).
\]
Therefore by Corollary 4.3, by taking the limit \( m \to \infty \), we have
\[
\nu^K(\phi, 0, y) \geq \nu^K(\psi, 0, y).
\]
\( \square \)

Corollary 4.5 (=Corollary 1.2). Let \( u_0 \) and \( u_1 \) be toric psh functions on the unit ball \( B \) in \( \mathbb{C}^n \) and let \( u_t \) be the weak geodesic joining \( u_0 \) and \( u_1 \). Then the following are equivalent.

1. \( \lim_{t \to 0} u_t = u_0 \) in capacity.
2. \( \nu^K(u_0, 0, y) \geq \nu^K(u_1, 0, y) \) for any \( y \in \mathbb{R}_+^n \).
3. \( \mathcal{I}(mu_0) \subset \mathcal{I}(mu_1) \) for any \( m \in \mathbb{Z}_{>0} \).

Proof. The equivalence between (1) and (2) was known by Hosono [Hos]. From Theorem 1.1, (3) implies (2), thus we only need to show that (2) implies (3). It is easy to check that according to [Gue, Theorem 1.20], \( \mathcal{I}(u_0) \) is monomial and
\[
z_1^{a_1}z_2^{a_2} \cdots z_n^{a_n} \in \mathcal{I}(u_0) \iff \sup_{y \in \mathbb{R}_+^n} \frac{\nu^K(u_0, 0, y)}{\nu^K(z_1^{a_1}z_2^{a_2} \cdots z_n^{a_n}, 0, y) + \sum_{i=1}^n a_iy_i} < 1.
\]

5. Tame singularity and weak geodesics

Definition 5.1 ([BFJ] section 5.3). We say that \( u \in PSH(B) \) has tame singularities with coefficient \( c > 0 \), if
\[
u u + O(1) \leq u_m \leq (1 - c/m)u + O(1)
\]
holds, where the \( O(1) \) term is independent of \( m \).

Remark 5.2. According to [BFJ, Lemma 5.10], if \( u \) is exponential \( \alpha \)-holder for some \( \alpha > 0 \) (i.e. \( e^u \) is \( \alpha \)-holder continuous), then \( u \) has tame singularities. In particular, if \( u \) has algebraic singularities (i.e. \( u \) is written as sum of \( C^\infty \)function and \( \log |f| \) for some holomorphic function \( f \)), then \( u \) has tame singularities.

Theorem 5.3 (=Theorem 1.3). Let \( u_0, u_1 \in \mathcal{P}(B) \) and let \( u_t \) be a weak geodesic joining \( u_0 \) and \( u_1 \). We assume \( u_1 \) has tame singularities. Then the following are equivalent.

1. \( \lim_{t \to 0} u_t = u_0 \) in capacity.
2. \( \mathcal{I}(mu_0) \subset \mathcal{I}(mu_1) \) for any \( m \in \mathbb{Z}_{>0} \).
Chapter 14, there exists a positive number $C$ such that $\mathcal{J}(u_0, u_1 + C) \leq u_1 + C$, therefore $\mathcal{J}(u_0) = \mathcal{J}(P(u_0, u_1 + C)) \subset \mathcal{J}(u_1)$.

Conversely, we assume $\mathcal{J}(mu_0) \subset \mathcal{J}(mu_1)$ for any $m \in \mathbb{Z}_{>0}$. We will denote $u_{i,m}$ by the $m$-th Bergman approximation of $u_i$ for $i = 0, 1$. By Demailly’s approximation theorem [Dem12, Chapter 14], there exists a positive number $C_1$ independent of $m$ such that $u_0 - \frac{C_1}{m} \leq u_{0,m}$. From Lemma 3.3, we obtain $u_0 \leq u_{1,m} + O(1)$. It follows that $P_{[u_{1,m}]}(u_0) = u_0$.

Since $u_1$ has tame singularities, $u_{1,m} \leq (1 - \frac{C}{m})u_1 + O(1)$ for all $m \in \mathbb{Z}_{>0}$. For all positive number $C$ and all $r \in PS H(B)$ satisfying $r \leq \min(u_0, u_{1,m} + C)$,

$$r + \frac{c}{m}u_1 \leq u_0, r + \frac{c}{m}u_1 \leq u_1 + C + O(1).$$

By the definition of $P_{[u_{1,m}]}(u_0)$, we have $r + \frac{c}{m}u_1 \leq P_{[u_{1,m}]}(u_0)$. Thus we take supremum for $r$ and $C$, $P_{[u_{1,m}]}(u_0) + \frac{c}{m}u_1 \leq P_{[u_{1,m}]}(u_0)$ holds. Since $P_{[u_{1,m}]}(u_0) = u_0$, it follows that $u_0 + \frac{c}{m}u_1 \leq P_{[u_{1,m}]}(u_0)$. Consequently we can take the limit for $m$ outside the origin, we have $u_0 \leq P_{[u_{1,m}]}(u_0)$ on $B \setminus \{0\}$. On the other hand, $P_{[u_{1,m}]}(u_0) \leq u_0$ by definition. Hence we have $u_0 = P_{[u_{1,m}]}(u_0)$ on $B \setminus \{0\}$. From Lemma 5.4 as below, we get $u_0 = P_{[u_{1,m}]}(u_0)$.

**Lemma 5.4 ([Gun] Chapter A).** Let $u, v \in PS H(\Omega)$. If $u = v$ almost everywhere with respect to Lebesgue measure, then $u = v$ on $\Omega$.

**Proof.** First, we show that

$$(5.1) \quad u(a) = \lim_{\epsilon \to 0} \frac{1}{\text{vol}(B(a, \epsilon))} \int_{B(a, \epsilon)} u(z) dV$$

Since $u$ is an upper-semi-continuous function, we have

$$u(a) = \limsup_{z \to a} u(z)$$

$$= \lim_{\epsilon \to 0} \sup_{z \in B(a, \epsilon)} u(z)$$

$$\geq \lim_{\epsilon \to 0} \frac{1}{\text{vol}(B(a, \epsilon))} \int_{B(a, \epsilon)} u(z) dV.$$  

On the other hand, according to the submean value inequality,

$$(5.3) \quad u(a) \leq \lim_{\epsilon \to 0} \frac{1}{\text{vol}(B(a, \epsilon))} \int_{B(a, \epsilon)} u(z) dV.$$  

We thus get (5.1).
Applying (5.1),

\[
\begin{align*}
    u(a) &= \lim_{\epsilon \to 0} \frac{1}{\text{vol}(B(a, \epsilon))} \int_{B(a, \epsilon)} u(z) dV \\
    &= \lim_{\epsilon \to 0} \frac{1}{\text{vol}(B(a, \epsilon))} \int_{B(a, \epsilon)} v(z) dV \\
    &= v(a) .
\end{align*}
\]

(5.4)

\[\Box\]

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