博士論文

論文題目　Studies on singular Hermitian metrics and their applications in algebraic geometry

(特異エルミート計量の研究と代数幾何学への応用)

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Studies on singular Hermitian metrics and their applications in algebraic geometry

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Contents

Preface

We study proper surjective morphisms by using singular Hermitian metrics and applications of singular Hermitian metrics of vector bundles.

In Chapter 2, we study the following Fujita-type conjecture proposed by Popa and Schnell.

Conjecture 0.0.1 ([**PS14**] Conjecture 1.3). Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties, with Y of dimension n, and L be an ample line bundle on Y. For any $a \geq 1$, the sheaf

$$f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$$

is globally generated for all $b \geq a(n+1)$.

We give a partial answer of this conjecture and we obtain an effective bound on the global generation of a direct image of a pluri-adjoint line bundle on the regular locus.

Theorem 0.0.2. [**Iwa17**] Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties, with Y of dimension n, and L be an ample line bundle on Y. If y is a regular value of f, then for any $a \geq 1$ the sheaf

$$f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$$

is generated by the global sections at y for all $b \geq \frac{n(n-1)}{2} + a(n+1)$.

We also obtain an effective bound on the generic global generation for a Kawamata log terminal $\mathbb{Q}$-pair. We use analytic methods such as m-Bergman type metric on $mK_{X/Y}$, relative Ohsawa-Takegoshi type L^2 extension theorem, L^2 estimates, and injective theorems of cohomology groups.

In Chapter 3, we study a Nadel-Nakano type vanishing theorem of a vector bundle with a singular hermitian metric.

Theorem 0.0.3. [**Iwa18a**] Let (X, ω) be a compact Kähler manifold and (E, h) be a holomorphic vector bundle on X with a singular hermitian metric. We assume the following conditions.

(1) There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.
(2) $he^{-\zeta}$ is a positively curved singular hermitian metric on E for some continuous function ζ on X.

(3) There exists a positive number $\epsilon > 0$ such that $\sqrt{-1}\Theta_{E,h} - \epsilon\omega \otimes Id_E \geq 0$ on $X \setminus Z$ in the sense of Nakano.

Then $H^q(X, K_X \otimes E(h)) = 0$ holds for any $q \geq 1$.

$E(h)$ is a higher rank version of multiplier ideal sheaf. We also obtain a generalization of Griffiths' vanishing theorem and a generalization of Ohsawa's vanishing theorem.

In Chapter 4 we give complex geometric descriptions of the notions of algebraic geometric positivity of vector bundles and torsion-free coherent sheaves, such as nef, big, pseudo-effective and weakly positive, by using singular hermitian metrics.

Theorem 0.0.4. [**Iwa18b**] *Let X be a smooth projective variety and E be a holomorphic vector bundle on X.*

(1) *E is nef iff there exists an ample line bundle A on X such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive smooth hermitian metric for any $k \in \mathbb{N}_{>0}$.*
(2) *E is big iff there exist an ample line bundle A and $k \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^k(E)\otimes A^{-1}$ has a Griffiths semipositive singular hermitian metric.*
(3) *E is pseudo-effective iff there exists an ample line bundle A such that $\mathrm{Sym}^k(E)\otimes A$ has a Griffiths semipositive singular hermitian metric for any $k \in \mathbb{N}_{>0}$.*
(4) *E is weakly positive iff there exist an ample line bundle A and a proper Zariski closed set Z such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive singular hermitian metric h_k for any $k \in \mathbb{N}_{>0}$ and the Lelong number of h_k at x is less than 2 for any $x \in X \setminus Z$.*

As an applications, we obtain a generalization of Mori's result by using the result of [**CMSB02**].

Corollary 0.0.5. [**Iwa18b**] *Let X be a smooth projective n-dimensional variety. If the tangent bundle T_X is big then X is biholommorphic to $\mathbb{CP}^n$.*

In Chapter 5, we develop the theory of singular hermitian metrics on vector bundles. As an application, we give a structure theorem of a projective manifold X with pseudo-effective tangent bundle. This is a joint work with Genki Hosono and Shin-ichi Matsumura.

Theorem 0.0.6. [**HIM19**] *Let X be a projective manifold with pseudo-effective tangent bundle. Then X admits a (surjective) morphism $\phi : X \to Y$ with connected fiber to a smooth manifold Y with the following properties:*

(1) *The morphism $\phi : X \to Y$ is smooth.*
(2) *The image Y admits a finite étale cover $A \to Y$ by an abelian variety A.*
(3) *A general fiber F of ϕ is rationally connected.*
(4) *A general fiber F of ϕ also has the pseudo-effective tangent bundle.*

Moreover, if we further assume that T_X admits a positively curved singular hermitian metric, then we have:

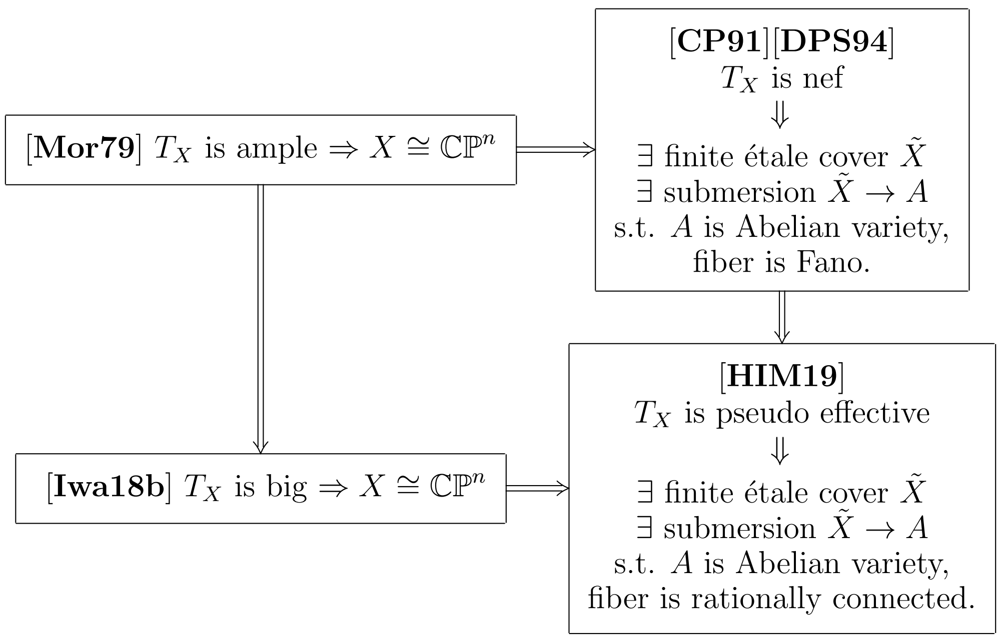
(5) The standard exact sequence of tangent bundles

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow \phi^* T_Y \longrightarrow 0$$

splits.

(6) The morphism $\phi : X \to Y$ is locally trivial (that is, all the fibers are smooth and isomorphic).

To summarize, it is as shown in this table.



In Chapter 6, we prove a few result. In 6.1, we study a Lelong number of singular hermitian metric on vector bundle and apply to augmented base locus. In 6.2 we give an example of a rationally connected manifold with a hermitian metric with negative scalar curvature. This is a counter-example in [**NZ18**, Conjecture 1.6]. In 6.3 we show an existence of a higher Fujita's decomposition of a direct image sheaf of relative pluri-canonical line bundle.

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CHAPTER 1

Preliminary

1.1. Notations

- $\mathbb{N}_{>0}$ is a set of positive integers.
- For any compact Kähler manifold X, $K_X := \det(T_X)^\vee$ is a canonical line bundle, where T_X is a holomorphic tangent bundle of Y.
- We regard Cartier (Weil) divisors as line bundles when the base space is a smooth projective manifold. In particular, a canonical divisor is regarded as a canonical line bundle. We regard locally free coherent sheaves as vector bundles.
- We denote $\mathrm{Hom}(\mathcal{E}, \mathcal{O}_X)$ by $\mathcal{E}^\vee$ for any torsion-free coherent sheaf $\mathcal{E}$.
- For any line bundle L, we also denote $L^\vee$ by $L^{\otimes -1}$. For any integer m, we denote $L^{\otimes m}$ by L^m or $L^{\otimes m}$.

1.2. Singular hermitian metrics on line bundles.

Let X be a connected complex manifold. A function $\varphi : X \to [-\infty, +\infty)$ on X is said to be *quasi-plurisubharmonic* if φ is locally the sum of a plurisubharmonic function and of a smooth function. Let L be a line bundle on X. Fix some smooth metric h_0 on L. h is a singular hermitian metric if $h = h_0 e^{-\varphi}$ for some quasi-plurisubharmonic function φ.

For any quasi-plurisubharmonic function φ on X, the *multiplier ideal sheaf* $\mathcal{J}(e^{-\varphi})$ is a coherent subsheaf of $\mathcal{O}_X$ defined by

$$\mathcal{J}(e^{-\varphi})_x := \{f \in \mathcal{O}_{X,x}; \exists U \ni x, \int_U |f|^2 e^{-\varphi} d\lambda < \infty\},$$

where U is an open coordinate neighborhood of x, and $d\lambda$ is the standard Lesbegue measure in the corresponding open chart of $\mathbb{C}^n$, and the *Lelong number* $\nu(\varphi, x)$ at $x \in X$ is defined by

$$\nu(\varphi, x) := \liminf_{z \to x} \frac{\varphi(z)}{\log|z - x|}.$$

We define the *multiplier ideal sheaf* $\mathcal{J}(h)$ of a singular hermitian metric h on L by $\mathcal{J}(h) := \mathcal{J}(e^{\log(hh_0^{-1})})$ and the *Lelong number* $\nu(h, x)$ of h at $x \in X$ is defined by $\nu(h, x) := \nu(-\log(hh_0^{-1}), x)$. We point out $\mathcal{J}(h)$ and $\nu(h, x)$ do not depend on the choice of h_0. We define the *curvature current* of h by $\sqrt{-1}\Theta_{L,h} := \Theta_{L,h_0} + \sqrt{-1}\partial\overline{\partial}\varphi$.

1.3. Algebraic positivity of line bundles

We define notions of algebraic positivity of line bundles.

Definition 1.3.1. Let X be a smooth projective manifold and L be a line bundle.

(1) L is *ample* if there exist an $m \in \mathbb{N}_{>0}$ and a basis $s_0 \cdots s_N \in H^0(X, L^{\otimes m})$ such that
$$\begin{array}{cccc} \Phi_{|L^{\otimes m}|}: & X & \to & \mathbb{CP}^N \\ & x & \mapsto & (s_0(x) : \cdots : s_N(x)). \end{array}$$
is closed embedding.

(2) L is *nef* if $L.C \geq 0$ for any curve $C \subset X$.

(3) L is *big* if $\limsup_{m\to\infty} \dim H^0(X, L^{\otimes m})/m^{\dim X} > 0$.

(4) L is pseudo-effective if there exists an ample line bundle A such that $L^{\otimes m} \otimes A$ is big for any $m \in \mathbb{N}_{>0}$.

We have the following theorem by Kodaira and Demailly.

Theorem 1.3.2. [**Kod54**] [**Dem92**] Let ω be a Kähler form on X.

(1) L is ample iff L has a smooth metric with positive curvature.

(2) L is nef iff for any $\epsilon > 0$ there exists a smooth metric h_ϵ such that $\sqrt{}\Theta_{L,h_\epsilon} \geq -\epsilon\omega$.

(3) L is big iff there exist an $\epsilon > 0$ and a singular hermitian metric h such that $\sqrt{}\Theta_{L,h} \geq \epsilon\omega$ in the sense of current.

(4) L is pseudo-effective iff L has a singular hermitian metric with semipositive curvature current.

We have the following implications.

$$\begin{array}{ccc} \text{ample} & \Longrightarrow & \text{big} \\ \Downarrow & & \Downarrow \\ \text{nef} & \Longrightarrow & \text{pseudo-effective} \end{array}$$

1.4. Singular hermitian metrics on vector bundles.

Next, we review the definitions of singular hermitian metrics. We adopt the definitions of singular hermitian metrics of vector bundles in [**HPS18**].

Definition 1.4.1. [**HPS18**] A *singular hermitian inner product* on a finite dimensional complex vector space V is a function $|-|_h : V \to [0, +\infty]$ with the following properties:

(1) $|\alpha \cdot v|_h = |\alpha||v|_h$ for any $\alpha \in \mathbb{C} \setminus 0$ and any $v \in V$.

(2) $|0|_h = 0$.

(3) $|v+w|_h \leq |v|_h + |w|_h$ for any $v, w \in V$.
(4) $|v+w|_h^2 + |v-w|_h^2 = 2|v|_h^2 + 2|w|_h^2$ for any $v, w \in V$.

Definition 1.4.2. [**BP08**][**HPS18**] Let X be a connected complex manifold and E be a vector bundle on X. A *singular hermitian metric* on E is a function h that associates to every point $x \in X$ a singular hermitian inner product $|-|_{h,x} \colon E_x \to [0, +\infty]$ on the complex vector space E_x, subject to the following two condtions:

(1) $|v|_{h,x} = 0 \Leftrightarrow v = 0$ for almost everywhere $x \in X$.
(2) $|v|_{h,x} < +\infty$ for any $v \in E_x$ and almost everywhere $x \in X$.
(3) For any open U and any $s \in H^0(U, E)$,

$$|s|_h \colon U \to [0, +\infty] \;\; ; \;\; x \to |s(x)|_{h,x}$$

is measurable.

Definition 1.4.3. [**BP08**] [**PT18**][**HPS18**] Let h be a singular hermitian metric on a vector bundle E.

(1) h is *Griffiths seminegative* or (*semi*)*negatively curved* if $\log |u|_h^2$ is plurisubharmonic for any local holomorphic section u.
(2) h is *Griffiths semipositive* or (*semi*)*positively curved* if a metric $h^\vee := {}^t h^{-1}$ on $E^\vee$ is Griffiths seminegative.

If h is smooth, h is Griffiths semipositive in above definition is same as usual one. These definitions are well-defined even if E is a line bundle. In particular, for any singular hermitian metric h on a line bundle L, h is Griffiths semipositive iff h has semipositive curvature current.

We recall the definition of a singular hermitian metric on a torsion-free coherent sheaf. Let $\mathcal{E} \neq 0$ be a torsion-free coherent sheaf on X. We will denote by $X_{\mathcal{E}}$ the maximal Zariski open set where $\mathcal{E}$ is locally free. We point out $\mathcal{E}|_{X_{\mathcal{E}}}$ is a vector bundle on $X_{\mathcal{E}}$ and $\operatorname{codim}(X \setminus X_{\mathcal{E}}) \geq 2$.

Definition 1.4.4. [**PT18**, Definition 2.4.1] [**HPS18**]

(1) The *singular hermitian metric* h on $\mathcal{E}$ is a singular hermitian metric on the vector bundle $\mathcal{E}|_{X_{\mathcal{E}}}$.
(2) A singular hermitian metric h on $\mathcal{E}$ is *Griffiths seminegative* or (*semi*)*negatively curved* if $h|_{X_{\mathcal{E}}}$ is Griffiths seminegative.
(3) A singular hermitian metric h on $\mathcal{E}$ is *Griffiths semipositive* or (*semi*)*positively curved* if there exists a Griffiths seminegative metric g on $\mathcal{E}^\vee|_{X_{\mathcal{E}}}$ such that $h|_{X_{\mathcal{E}}} = (g|_{X_{\mathcal{E}}})^\vee$.

These are well-defined definitions (see [**PT18**, Remark 2.4.2]). About a Griffiths semipositive singular hermitian metric, Păun and Takayama proved the following Theorem.

THEOREM 1.4.5. [**PT18**, Theorem 1.1] [**HPS18**, Theorem 21.1 and Corollary 21.2] Let $f\colon X \to Y$ be a projective surjective morphism between connected complex manifolds and (L, h) be a holomorphic line bundle with a singular hermitian metric with semipositive curvature current on X. Then $f_*(K_{X/Y} \otimes L \otimes \mathcal{J}(h))$ has a Griffiths semipositive singular hermitian metric.

Moreover if the inclusion morphism

$$f_*(K_{X/Y} \otimes L \otimes \mathcal{J}(h)) \to f_*(K_{X/Y} \otimes L)$$

is generically isomophism, then $f_*(K_{X/Y} \otimes L)$ also has a Griffiths semipositive singular hermitian metric.

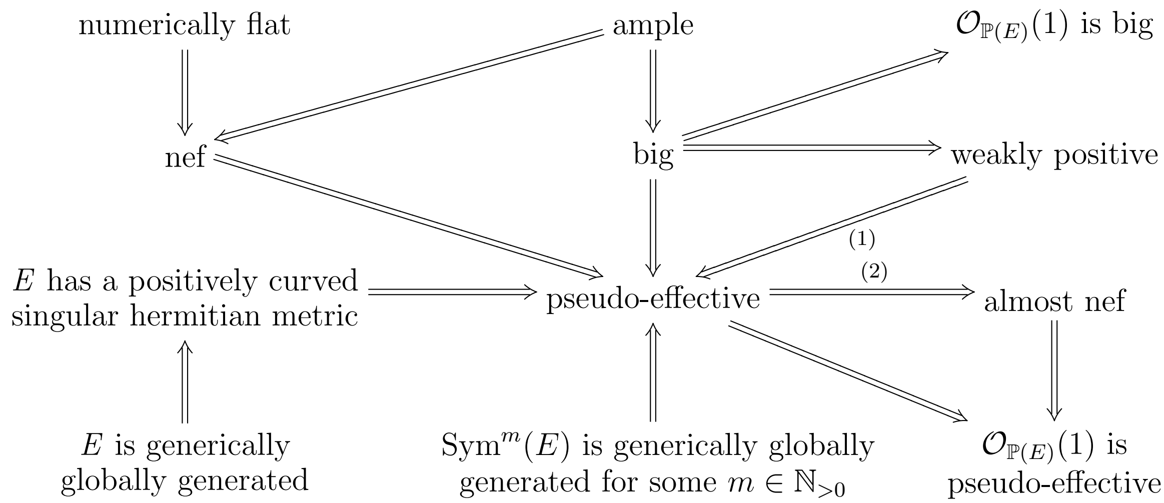
1.5. Algebraic positivity on vector bundles

We summarize the notions of positivity of vector bundles and torsion free coherent sheaves. In this thesis, we will denote by $\pi\colon \mathbb{P}(E) \to X$ the projective bundle of rank one quotients of E and by $\mathcal{O}_{\mathbb{P}(E)}(1)$ the universal quotient of π^*E on $\mathbb{P}(E)$.

DEFINITION 1.5.1 ([**BDPP13**, Definition 7.1],[**DPS94**, Definition 1.17], [**DPS01**, Definition 6.4], [**Nak04**, Definition 3.20]). Let X be a smooth projective manifold.

(1) A vector bundle E is *ample* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a ample line bundle on $\mathbb{P}(E)$.
(2) A vector bundle E is *nef* if $\mathcal{O}_{\mathbb{P}(E)}(1)$ is a nef line bundle on $\mathbb{P}(E)$.
(3) A vector bundle E is *numerically flat* if E is nef and $c_1(E) = 0$.
(4) A vector bundle E is *almost nef* if there exists a countable family of proper subvarieties Z_i of X such that $E|_C$ is nef for any curve $C \not\subset \cup_i Z_i$.
(5) A torsion free coherent sheaf $\mathcal{E}$ is *weakly positive at* $x \in X$ if, for any $a \in \mathbb{N}_{>0}$ and for any ample line bundle A on X, there exists $b \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^{ab}(\mathcal{E})^{\vee\vee} \otimes A^b$ is globally generated at x.
(6) A torsion free coherent sheaf $\mathcal{E}$ is *pseudo-effective* (*weakly positive in the sense of Nakayama*) if $\mathcal{E}$ is weakly positive at some $x \in X$.
(7) A torsion free coherent sheaf $\mathcal{E}$ is *weakly positive* (*weakly positive in the sense of Viehweg*) if there exist a non empty Zariski open set U such that $\mathcal{E}$ is weakly positive at any $x \in U$.
(8) A torsion free coherent sheaf $\mathcal{E}$ is *big* (*V-big, dd-ample, ample modulo double duals*) if there exist $a \in \mathbb{N}_{>0}$ and an ample line bundle A on X such that $\mathrm{Sym}^{a}(\mathcal{E})^{\vee\vee} \otimes A^{-1}$ is pseudo-effective.
(9) A torsion free coherent sheaf $\mathcal{E}$ is *generically globally generated* if $\mathcal{E}$ is globally generated at a general point in X.

The definition of ample (resp. nef, big, or pseudo-effective) vector bundles coincides with the usual one in the case E being a line bundle. Relationships among them can be summarized by the following table:



Even if E is a line bundle, the converse of (1) is unknown.[1] When E is a line bundle, the converses of (2) hold by [**BDPP13**, Theorem 0.2]. However, in a higher rank case, the converse of (2) is unknown.

Example 1.5.2.

(1) $\mathcal{O}_{\mathbb{CP}^1}$ is nef (pseudo-effective) but not ample (big).
(2) We put $E = \mathcal{O}_{\mathbb{CP}^1}(2) \oplus \mathcal{O}_{\mathbb{CP}^1}(-1)$. By Cutkosky criterion ([**Laz04a**]), $\mathcal{O}_{\mathbb{P}(E)}(1)$ is big (pseudo-effective) line bundle on $\mathbb{P}(E)$. However E is not nef (almost nef) vector bundle on $\mathbb{CP}^1$ since the quotient sheaf $\mathcal{O}_{\mathbb{CP}^1}(-1)$ of E does not have semipositive degree.
(3) For any $n \in \mathbb{N}_{>0}$, we put $E_n = \mathcal{O}_{\mathbb{CP}^1} \oplus \mathcal{O}_{\mathbb{CP}^1}(n)$. The *Hirzebruch surface* is defined by $\mathbb{F}_n := \mathbb{P}(E_n)$. We put ruling $\pi : \mathbb{F}_n \to \mathbb{CP}^1$, a general fiber F of π, and $L := \mathcal{O}_{\mathbb{F}_n}(1)$. By [**Bea96**] and [**Laz04a**], we have the followings for any integers a, b.
   - $L^a \otimes F^b$ is pseudo-effective iff $a \geq 0$ and $na + b \geq 0$.
   - $L^a \otimes F^b$ is nef iff $a \geq 0$ and $b \geq 0$.

   Therefore $L^2 \otimes F^{-n}$ is big but not nef. F is nef but not big.
(4) Let C be an elliptic surface. For any $n \in \mathbb{N}_{>0}$ we put $S_n := \mathbb{P}(\mathcal{O}_C \oplus \mathcal{O}_C(np))$, where p is a prime divisor of C. The tangent bundle of S_n is pseudo-effective by Proposition 5.3.2. But by computations, $\mathrm{Sym}^m(T_{S_n})$ is not generically globally generated for any $m \in \mathbb{N}_{>0}$.

[1] John-Lesiutre [**Les14**] proved that there exists a pseudo-effective $\mathbb{R}$-divisor D such that D is not weakly positive. However D is not $\mathbb{Q}$-divisor.

(5) Let C be an elliptic curve. E is defined by the nontrivial exact sequence of vector bundles:
$$0 \to \mathcal{O}_C \to E \to \mathcal{O}_C \to 0.$$
E is nef (pseudo-effective) vector bundle by [**DPS94**, Example 1.7]. By [**Hos17**, Example 5.4], E does not have a Griffiths semipositive singular hermitian metric.

We review some of the standard facts of augmented base locus and restiricted base locus of vector bundles.

Definition 1.5.3. [**BKK+15**, Section 2] Let X be a smooth projective variety and E be a holomorphic vector bundle. The *base locus* of E is defined by
$$\mathrm{Bs}(E) := \{x \in X : H^0(X, E) \to E_x \text{ is not surjective}\},$$
and the *stable base locus* of E is defined by
$$\mathbb{B}(E) := \bigcap_{m>0} \mathrm{Bs}(\mathrm{Sym}^m(E)).$$
Let A be an ample line bundle. We define the *augmented base locus* of E by
$$\mathbb{B}_+(E) = \bigcap_{p,q \in \mathbb{N}_{>0}} \mathbb{B}(\mathrm{Sym}^q(E) \otimes A^{\otimes -p})$$
and the *restricted base locus* of E by
$$\mathbb{B}_-(E) = \bigcup_{p,q \in \mathbb{N}_{>0}} \mathbb{B}(\mathrm{Sym}^q(E) \otimes A^{\otimes p}).$$
We point out $\mathbb{B}_+(E)$ and $\mathbb{B}_-(E)$ do not depend on the choice of the ample line bundle A by [**BKK+15**, Remark 2.7]. By [**BKK+15**], we have $\pi(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))) = \mathbb{B}_-(E)$ and $\pi(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1))) = \mathbb{B}_+(E)$

We point out the relationship between algebraic positivity and base loci.

Theorem 1.5.4. [**BDPP13**, Proposition 7.2.] [**BKK+15**, Definition 5.1] The following are equivalent.

(1) E is pseudo-effective.
(2) $\mathbb{B}_-(E) \neq X$.
(3) $\mathcal{O}_{\mathbb{P}(E)}(1)$ is pseudo-effective on $\mathbb{P}(E)$ and $\pi\Big(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))\Big) \neq X$

Theorem 1.5.5. [**BKK+15**, Definition 5.1] The following are equivalent.

(1) E is weakly positive.
(2) $\overline{\mathbb{B}_-(E)} \neq X$.

Theorem 1.5.6. [**BKK+15**, Theorem 6.4] The following are equivalent.

(1) E is big.

(2) there exist $b \in \mathbb{N}_{>0}$ and an ample line bundle A on X such that $\mathrm{Sym}^b(E) \otimes A^{-1}$ is globally generated at a general point.
(3) $\mathcal{O}_{\mathbb{P}(E)}(1)$ is big on $\mathbb{P}(E)$ and $\pi\Big(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1))\Big) \neq X$
(4) $\mathbb{B}_+(E) \neq X$.

1.6. MRC fibrations

THEOREM 1.6.1. [**Cam92**][**KoMM92**] Let X be a smooth projective manifold. Then there exists a dominant rational map $\varphi : X \dashrightarrow Y$ onto a smooth projective manifold Y with the following properties:

(1) There exists a Zariski open set $Y_0 \subset Y$ such that $\varphi|_{X_0} : X_0 \to Y_0$ is proper holomorphic map, where $X_0 := \varphi^{-1}(Y_0)$.
(2) A general fiber F of φ is an irreducible rationally connected manifold.
(3) If a rational curve R meets a general fiber F, then we have $R \subset F$.

This rational map φ is called *MRC* (*Maximally rationally connected*) *fibration*. An MRC fibration is unique up to birational map.

By Greb-Harris-Starr's result [**GHS03**] and Boucksom-Demailly-Păun-Peternell's result [**BDPP13**], we have the following theorem.

THEOREM 1.6.2. [**GHS03**][**BDPP13**] For any MRC fibration $\varphi : X \dashrightarrow Y$, the canonical bundle K_Y of Y is pseudo-effective.

1.7. Singular Foliations

DEFINITION 1.7.1. [**Lazi**, Chapter 4] Let X be a smooth projective manifold and $\mathcal{E} \subset T_X$ be a coherent sheaf on X.

(1) $\mathcal{E}$ is a *singular foliation* if $\mathcal{E}$ is saturated (i.e. $\mathcal{E}$ and $T_X/\mathcal{E}$ are torsionfree) and $\mathcal{E}$ is closed by Lie bracket.
(2) A subset F is a *leaf* if F is a maximally connected locally closed set such that $T_F = \mathcal{E}|_F$

For any morphism $f : X \to Y$ between smooth projective manifolds, the kernel $\ker df \subset T_X$ of the differential $df : T_X \to f^*(T_Y)$ is a singular foliation. A general fiber F or f is a leaf of $\ker df$.

The following theorems is used in Chapter 5.

THEOREM 1.7.2. [**Hör07**, Corollary 2.11] Let $\mathcal{E} \subset T_X$ be a singular foliation. If $\mathcal{E}$ is locally free and a leaf of $\mathcal{E}$ is compact and rationally connected, then there exists a smooth morphism $f : X \to Y$ onto a smooth projective manifold Y such that $\mathcal{E} = \ker df$. Moreover all fiber F of f is compact and rationally connected.

Theorem 1.7.3. [**Hör07**, Lemma 3.19] Let $\varphi : X \to Y$ be a smooth morphism between smooth projective manifolds. If there exists a vector bundle $V \subset T_X$ such that $T_X = V \oplus T_{X/Y}$, then φ is locally trivial (analytic fiber bundle), i.e. for any $y \in Y$, there exist Euclid open set $y \in V \subset Y$ such that $\varphi^{-1}(V) \cong V \times F$, where F is a fiber of φ.

CHAPTER 2

On the global generation of direct images of pluri-adjoint line bundles

ABSTRACT. We study the Fujita-type conjecture proposed by Popa and Schnell. We obtain an effective bound on the global generation of direct images of pluri-adjoint line bundles on the regular locus. We also obtain an effective bound on the generic global generation for a Kawamata log terminal $\mathbb{Q}$-pair. We use analytic methods such as L^2 estimates, L^2 extensions and injective theorems of cohomology groups.

2.1. Introduction

The aim of this paper is to give a partial answer to the following conjecture by Popa and Schnell. This conjecture is a version of Fujita's conjecture.

CONJECTURE 2.1.1 ([**PS14**] Conjecture 1.3). Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties, with Y of dimension n, and L be an ample line bundle on Y. For any $a \geq 1$, the sheaf

$$f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$$

is globally generated for all $b \geq a(n+1)$.

In [**PS14**], Popa and Schnell proved this conjecture in the case when L is ample and globally generated. After that, Dutta removed the global generation assumption on L making a statement about generic global generation.

THEOREM 2.1.2 ([**Dut17**] Theorem A). Let (X, Δ) be a Kawamata log terminal $\mathbb{Q}$-pair of a normal projective variety and an effective divisor, and Y be a smooth projective n-dimensional variety. Let $f\colon X \to Y$ be a surjective morphism, and L be an ample line bundle on Y. For any $a \geq 1$ such that $a(K_X + \Delta)$ is an integral Cartier divisor, the sheaf

$$f_*\Big(\mathcal{O}_X\big(a(K_X+\Delta)\big)\Big) \otimes L^{\otimes b}$$

is generated by the global sections at a general point $y \in Y$ either

(1) for all $b \geq a\left(\frac{n(n+1)}{2} + 1\right)$, or
(2) for all $b \geq a(n+1)$ when $n \leq 4$.

On the other hand, Deng obtained a linear bound for b when a is large by using analytic methods.

THEOREM 2.1.3 ([**Deng17**] Theorem C). With the above notation and in the setting of Theorem 2.1.2, for any $a \geq 1$ such that $a(K_X + \Delta)$ is an integral Cartier divisor, the sheaf

$$f_*\Big(\mathcal{O}_X\big(a(K_X + \Delta)\big)\Big) \otimes L^{\otimes b}$$

is generated by the global sections at a general point $y \in Y$ either

(1) for all $b \geq n^2 - n + a(n+1)$, or
(2) for all $b \geq n^2 + 2$ when K_Y is pseudo-effective.

Now we state our results. First, we treat the case when X is smooth and $\Delta = 0$. In [**Dut17**], Dutta proved that if $K_X^{\otimes a}$ is relatively free on the regular locus of f, $f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$ is generated by the global sections at any regular value of f for all $b \geq a\big(\frac{n(n+1)}{2} + 1\big)$. In this paper, we can remove this assumption and obtain a better bound for b.

THEOREM 2.1.4. Let $f \colon X \to Y$ be a surjective morphism of smooth projective varieties, with Y of dimension n, and L be an ample line bundle on Y. If y is a regular value of f, then for any $a \geq 1$ the sheaf

$$f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$$

is generated by the global sections at y for all $b \geq \frac{n(n-1)}{2} + a(n+1)$.

In this paper, X_y is a smooth connected variety for any regular value $y \in Y$. In particular, if f is smooth, $f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$ is globally generated for all $b \geq \frac{n(n-1)}{2} + a(n+1)$. We give a partial answer to Conjecture 2.1.1.

Second, we treat a log case. In this case, we obtain the same bound as Theorem 2.1.4 about generic global generation even when X is a complex analytic variety.

THEOREM 2.1.5. Let (X, Δ) be a Kawamata log terminal $\mathbb{Q}$-pair of a normal complex analytic variety in Fujiki's class $\mathcal{C}$ and an effective divisor, and Y be a smooth projective n-dimensional variety. Let $f \colon X \to Y$ be a surjective morphism, and L be an ample line bundle on Y. For any $a \geq 1$ such that $a(K_X + \Delta)$ is an integral Cartier divisor, the sheaf

$$f_*\Big(\mathcal{O}_X\big(a(K_X + \Delta)\big)\Big) \otimes L^{\otimes b}$$

is generated by the global sections at a general point $y \in Y$ either

(1) for all $b \geq \frac{n(n-1)}{2} + a(n+1)$, or
(2) for all $b \geq \frac{n(n-1)}{2} + 2$ when K_Y is pseudo-effective.

REMARK 2.1.6. After the author submitted this paper to arXiv, Dutta told the author that she and Murayama obtained the same bounds as in Theorem 2.1.5 (1) in [**DM17**, Theorem B] by using the algebraic geometric methods when X is a normal projective variety. Also, in [**DM17**, Theorem B], they obtained the linear bound when (X, Δ) is a log canonical $\mathbb{Q}$-pair. For more details, we refer the reader to [**DM17**].

2.2. Preliminary

In this paper we will denote $N := \frac{n(n+1)}{2}$. Angehrn and Siu proved the existence of a quasi-psh function whose multiplier ideal sheaf has isolated zero set at y when we pick one point $y \in Y$.

THEOREM 2.2.1. [**AS95**] Let Y be a smooth projective n-dimensional variety, and We fix $m \in \mathbb{N}$ such that $m(N+1)L$ is very ample. We choose a Kähler form ω_Y on Y and a smooth positive metric h_L on L such that $\sqrt{-1}\Theta_{L,h_L} = \frac{1}{m(N+1)}\omega_Y$, where $N = \frac{n(n+1)}{2}$. Then for any point $y \in Y$, there exist a quasi-psh function φ with neat analytic singularities on Y and a positive number $0 < \varepsilon_0 < 1$, such that

(1) $\sqrt{-1}\Theta_{L^{\otimes N+1}h_L^{N+1}} + \sqrt{-1}\partial\overline{\partial}\varphi \geq \frac{1-\varepsilon_0}{m(N+1)}\omega_Y$

(2) y is an isolated point in the zero variety $V(\mathcal{J}(e^{-\varphi}))$.

By the following theorem, a relative pluricanonical line bundle $K_{X/Y}^{\otimes a}$ has a semipositive singular hermitian metric which is equal to the fiberwise Bergman kernel metric.

THEOREM 2.2.2 ([**BP08**] Theorem 4.2 , [**PT18**] Collary 4.3.2). Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties. Assume that there exists a regular value $y \in Y$ such that $H^0\big(X_y, K_{X_y}^{\otimes a}\big) \neq 0$. Then the bundle $K_{X/Y}^{\otimes a}$ admits a singular hermitian metric h_a with semipositive curvature current such that for any regular value $w \in Y$ and any section $s \in H^0\big(X_w, K_{X_w}^{\otimes a}\big)$ we have

$$|s|_{h_a}^{\frac{2}{a}}(z) \leq \int_{X_w} |s|^{\frac{2}{a}}$$

for any $z \in X_w$ up to the identification of $K_{X/Y}|_{X_w}$ with K_{X_w}. We regard $|s|^{\frac{2}{a}}$ as a semipositive continuous (m,m) form where $m = \dim X_w$.

2.3. Proof of main theorem

In this section, we prove Theorem 2.1.4.

THEOREM 2.3.1 (= Theorem 2.1.4). Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties, with Y of dimension n, and L be an ample line bundle on Y. If y is a regular value of f, then for any $a \geq 1$ the sheaf

$$f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$$

is generated by the global sections at y for all $b \geq \frac{n(n-1)}{2} + a(n+1)$.

Let us first outline the proof. It is enough to show that for any regular value $y \in Y$, any section $s \in H^0\big(X_y, K_{X_y}^{\otimes a} \otimes f^*(L)^{\otimes b}|_{X_y}\big)$ can be extended to X. By taking an appropriate singular hermitian metric on $K_X^{\otimes a-1} \otimes f^*(L^{\otimes b})$, we can prove there exists a section S_U near X_y such that $S_U|_{X_y} = s$ by an L^2 extension theorem. To extend S_U to X, we solve a $\overline{\partial}$-equation with some weight.

2.3.1. Set up. We fix a regular value $y \in Y$ and a section $s \in H^0\big(X_y, K_{X_y}^{\otimes a} \otimes f^*(L)^{\otimes b}|_{X_y}\big)$. We may assume $s \neq 0$.

Let ω_X be a Kähler form on X. We will denote by h_L the smooth positive metric on L and denote by ω_Y the Kähler form on Y as in Theorem 2.2.1. Since $K_Y \otimes L^{\otimes n+1}$ is semiample by Mori theory and Kawamata's basepoint free theorem (see [**KM98**, Theorem 1.13 and Theorem 3.3]), there exists a smooth semipositive metric h_s on $K_Y \otimes L^{\otimes n+1}$. We take the singular hermitian metric h_a on $K_{X/Y}^{\otimes a}$ as in Theorem 2.2.2.

We will denote by $\overline{L} := K_{X/Y}^{\otimes(a-1)} \otimes f^*(K_Y \otimes L^{\otimes n+1})^{\otimes a-1} \otimes f^*(L^{\otimes N+1+\bar{b}})$ and $\bar{b} := b - \frac{n(n-1)}{2} - a(n+1) \geq 0$. Define $h_{\overline{L}} := h_a^{\frac{a-1}{a}} f^*(h_s^{a-1} h_L^{N+1+\bar{b}})$, which is a singular hermitian metric on $\overline{L}$ with semipositive curvature current. Note that $K_X \otimes \overline{L} = K_X^{\otimes a} \otimes f^*(L^{\otimes b})$.

2.3.2. Local Extension. We choose a coodinate neighborhood V near y and we set $U := f^{-1}(V)$. We may regard V as an open ball in $\mathbb{C}^n$ and y as an origin in $\mathbb{C}^n$. Since $|s|^2_{h_a}$ is bounded above on X_y by Theorem 2.2.2, we obtain

$$\begin{aligned} \|s\|^2_{h_{\overline{L}},\omega_X} &= \int_{X_y} |s|^2_{h_{\overline{L}},\omega_X} dV_{X_y,\omega_X} \\ &= C \int_{X_y} |s|_{h_a}^{\frac{2(a-1)}{a}} |s|_{\omega_X}^{\frac{2}{a}} dV_{X_y,\omega_X} \\ &\leq C' \int_{X_y} |s|_{\omega_X}^{\frac{2}{a}} dV_{X_y,\omega_X} \\ &< +\infty, \end{aligned} \tag{2.3.1}$$

where C and C' are some positive constants. Therefore by the L^2 extension theorem in [**HPS18**, Theorem 14.4], there exists $S_U \in H^0\big(U, K_X \otimes \overline{L} \otimes \mathcal{J}(h_{\overline{L}})\big)$ such that $S_U|_{X_y} = s$.

2.3.3. Global Extension. We denote by φ the quasi-psh function on Y as in Theorem 2.2.1 and denote by $\psi := \varphi \circ f$. By Theorem 2.2.1, we can take a cut-off function ρ near y such that

(1) $\mathrm{supp}(\rho) \subset\subset V$,
(2) $\mathrm{supp}(\overline{\partial}\rho) \not\ni y$,
(3) $\int_{\mathrm{supp}(\overline{\partial}\rho)} e^{-\varphi} dV_{Y,\omega_Y} < +\infty$,

and put $\widetilde{\rho} := \rho \circ f$. We solve the global $\overline{\partial}$-equation $\overline{\partial} F = \overline{\partial}(\widetilde{\rho} S_U)$ on X with the weight of $h_{\overline{L}} e^{-\psi}$.

It is easy to check $\|\widetilde{\rho} S_U\|^2_{h_{\overline{L}},\omega_X} < +\infty$ and $\|\overline{\partial}(\widetilde{\rho} S_U)\|^2_{h_{\overline{L}},\omega_X} < +\infty$. Therefore $\overline{\partial}(\widetilde{\rho} S_U)$ gives rise to a cohomology class $[\overline{\partial}(\widetilde{\rho} S_U)]$ which is $[\overline{\partial}(\widetilde{\rho} S_U)] = 0$ in $H^1\big(X, K_X \otimes \overline{L} \otimes \mathcal{J}(h_{\overline{L}})\big)$. Since $|S_U|^2_{h_{\overline{L}}}$ is bounded above on U by Theorem 2.2.2 (if necessary we take

U small enough), we obtain

$$\begin{aligned}\|\overline{\partial}(\widetilde{\rho}S_U)\|^2_{h_{\overline{L}}e^{-\psi},\omega_X} &= \int_U |\overline{\partial}(\widetilde{\rho}S_U)|^2_{h_{\overline{L}}e^{-\psi}} dV_{X,\omega_X} \\ &\leq C\int_{f^{-1}(\mathrm{supp}(\overline{\partial}\rho))} |S_U|^2_{h_{\overline{L}}} e^{-\psi} dV_{X,\omega_X} \\ &\leq C'\int_{f^{-1}(\mathrm{supp}(\overline{\partial}\rho))} e^{-\psi} dV_{X,\omega_X} \\ &< +\infty,\end{aligned} \tag{2.3.2}$$

where C and C' are some positive constants. Therefore $\overline{\partial}(\widetilde{\rho}S_U)$ is a $\overline{\partial}$-closed $(d,1)$ form with $\overline{L}$ value which is square integrable with the weight of $h_{\overline{L}}e^{-\psi}$, where $d = \dim X$.

We put $\delta := \frac{1-\epsilon_0}{2(N+\epsilon_0)}$. Then we obtain

$$\begin{aligned}&\sqrt{-1}\Theta_{h_{\overline{L}},h_{\overline{L}}} + (1+\alpha\delta)\sqrt{-1}\partial\overline{\partial}\psi \\ &= \frac{a-1}{a}\sqrt{-1}\Theta_{K_{X/Y},h_a} + (a-1)f^*(\sqrt{-1}\Theta_{K_Y\otimes L^{\otimes n+1},h_s}) \\ &\quad + (N+1+\overline{b})f^*(\sqrt{-1}\Theta_{L,h_L}) + (1+\alpha\delta)\sqrt{-1}\partial\overline{\partial}\psi \\ &\geq f^*\Big((N+1)\sqrt{-1}\Theta_{L,h_L} + (1+\alpha\delta)\sqrt{-1}\partial\overline{\partial}\varphi\Big) \\ &= f^*\Big((1+\alpha\delta)(\sqrt{-1}\Theta_{L^{\otimes N+1},h_L^{N+1}} + \sqrt{-1}\partial\overline{\partial}\varphi) - \alpha\delta\sqrt{-1}\Theta_{L^{\otimes N+1},h_L^{N+1}}\Big) \\ &\geq \frac{(2-\alpha)(1-\epsilon_0)}{2m(N+1)} f^*(\omega_Y) \\ &\geq 0\end{aligned} \tag{2.3.3}$$

in the sense of current for any $\alpha \in [0,1]$. Therefore by the injectivity theorem in [**CDM17**, Theorem 1.1], the natural morphism

$$H^1\big(X, K_X\otimes\overline{L}\otimes\mathcal{J}(h_{\overline{L}}e^{-\psi})\big) \to H^1\big(X, K_X\otimes\overline{L}\otimes\mathcal{J}(h_{\overline{L}})\big)$$

is injective. Since $[\overline{\partial}(\widetilde{\rho}S_U)] = 0$ in $H^1\big(X, K_X\otimes\overline{L}\otimes\mathcal{J}(h_{\overline{L}})\big)$, we obtain $[\overline{\partial}(\widetilde{\rho}S_U)] = 0$ in $H^1\big(X, K_X\otimes\overline{L}\otimes\mathcal{J}(h_{\overline{L}}e^{-\psi})\big)$. Hence we obtain a $(d,0)$ form F with $\overline{L}$ value which is square integrable with the weight of $h_{\overline{L}}e^{-\psi}$ such that $\overline{\partial}F = \overline{\partial}(\widetilde{\rho}S_U)$, that is we can solve $\overline{\partial}$ equation.

Now we show that $F|_{X_y} \equiv 0$. To obtain a contradiction, suppose that $F(x) \neq 0$ for some $x \in X_y$. We may assume there exists an open set W near x such that $F(z) \neq 0$ for any $z \in W$ and $\int_W e^{-\psi} dV_{X,\omega_X} = +\infty$ since y is an isolated point in the zero variety $V(\mathcal{J}(e^{-\varphi}))$ by Theorem 2.2.1. Since there exists a positive constant C such that $|F|^2_{h_{\overline{L}}} \geq C$ on W, we have

$$\begin{aligned} +\infty > \|F\|^2_{h_{\overline{L}}e^{-\psi}} = \int_X |F|^2_{h_{\overline{L}}} e^{-\psi} dV_{X,\omega_X} &\geq \int_W |F|^2_{h_{\overline{L}}} e^{-\psi} dV_{X,\omega_X} \\ &\geq C \int_W e^{-\psi} dV_{X,\omega_X} \\ &= +\infty, \end{aligned} \tag{2.3.4}$$

which is impossible.

Hence we put $S := \widetilde{\rho} S_U - F \in H^0\big(X, K_X \otimes \overline{L}\big)$, then $S|_{X_y} = (\widetilde{\rho} S_U - F)|_{X_y} = s$, which completes the proof.

REMARK 2.3.2. In [**Fuj19**], Fujino proved the following theorem.

THEOREM 2.3.3. [**Fuj19**, Theorem 1.5] Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties, with Y of dimension n, and L be an ample line bundle on Y. For any $a, s \geq 1$ and any $b \geq n^2 + min(2, a)$ the sheaf

$$(\otimes^s f_*(K_{X/Y}^{\otimes a}))^{\vee\vee} \otimes K_Y \otimes L^{\otimes b}$$

is generic globally generated.

As stated in [**Fuj19**, Remark 1.6], we proved above theorem by same method in the case of $s = 1$. More precisely we have the follwing theorem.

THEOREM 2.3.4. Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties, with Y of dimension n, and L be an ample line bundle on Y. If y is a regular value of f, then for any $a \geq 1$ and any $b \geq \frac{n(n+1)}{2} + 1$ the sheaf

$$f_*(K_{X/Y}^{\otimes a}) \otimes K_Y \otimes L^{\otimes b}$$

is globally generated at y.

PROOF. We will denote by $\overline{L} := K_{X/Y}^{\otimes(a-1)} \otimes f^*(L^{\otimes b})$. Note that $K_X \otimes \overline{L} = K_{X/Y}^{\otimes a} \otimes f^*(K_Y \otimes L^{\otimes b})$. Define $h_{\overline{L}} := h_a^{\frac{a-1}{a}} f^*(h_L^b)$, which is a singular hermitian metric on $\overline{L}$ with semipositive curvature current. The rest proof of Theorem 2.3.4 is similar to the proof of Theorem 2.1.4. □

2.4. On a log case

In this section, we prove Theorem 2.1.5.

THEOREM 2.4.1 (= Theorem 2.1.5). Let (X, Δ) be a Kawamata log terminal $\mathbb{Q}$-pair of a normal complex analytic variety in Fujiki's class $\mathcal{C}$ and an effective divisor, and Y be a smooth projective n-dimensional variety. Let $f\colon X \to Y$ be a surjective morphism, and L be an ample line bundle on Y. For any $a \geq 1$ such that $a(K_X + \Delta)$ is an integral Cartier divisor, the sheaf

$$f_*\Big(\mathcal{O}_X\big(a(K_X+\Delta)\big)\Big) \otimes L^{\otimes b}$$

is generated by the global sections at a general point $y \in Y$ either

(1) for all $b \geq \frac{n(n-1)}{2} + a(n+1)$, or
(2) for all $b \geq \frac{n(n-1)}{2} + 2$ when K_Y is pseudo-effective.

Proof. The proof is similar to Theorem 2.1.4 and [**Deng17**, Theorem C]. Take a log resolution $\mu\colon X' \to X$ of (X, Δ) we have a compact Kähler manifold X' such that

$$aK_{X'} = \mu^*(a(K_X+\Delta)) + \sum a\alpha_i E_i - \sum a\beta_j F_j,$$

where $a\alpha_i, a\beta_j \in \mathbb{N}_+$ and $\sum_i E_i + \sum_j F_j$ has simple normal crossing supports. Since (X, Δ) is a Kawamata log terminal $\mathbb{Q}$-pair and Δ is effective, E_i is an exceptional divisor and $0 < \beta_j < 1$. We denote by $f' := f \circ \mu$, which is a surjective morphism between compact Kähler manifolds. Since E_i is an exceptional divisor, the natural morphism

$$\begin{aligned}
&H^0\big(X', \mu^*\big(\mathcal{O}_X\big(a(K_X+\Delta)\big) \otimes f^*(L^{\otimes b})\big)\big)\\
&\to H^0\big(X', \mu^*\big(\mathcal{O}_X\big(a(K_X+\Delta)\big) \otimes f^*(L^{\otimes b})\big) \otimes \mathcal{O}_{X'}(\textstyle\sum a\alpha_i E_i)\big)\\
&= H^0\big(X', K_{X'}^{\otimes a} \otimes f'^*(L)^{\otimes b} \otimes \mathcal{O}_{X'}(\textstyle\sum a\beta_j F_j)\big)
\end{aligned} \tag{2.4.1}$$

is isomorphism. Thus it is enough to show that for any general point $y \in Y$, the restriction map

$$\pi_y\colon H^0\big(X', K_{X'}^{\otimes a} \otimes f^*(L^{\otimes b}) \otimes \mathcal{O}_{X'}(\textstyle\sum a\beta_j F_j)\big) \to H^0\big(X'_y, K_{X'_y}^{\otimes a} \otimes f^*(L)^{\otimes b}|_{X'_y} \otimes \mathcal{O}_{X'}(\textstyle\sum a\beta_j F_j)|_{X'_y}\big)$$

is surjective.

In case (1), we choose the canonical singular hermitian metric h_F on $\mathcal{O}_{X'}(\sum a\beta_j F_j)$ as in [**Dem12**, Example 3.13]. We obtain $\mathcal{J}(h_F^{\frac{1}{a}}) = \mathcal{O}_{X'}$ since $\sum_i E_i + \sum_j F_j$ has simple normal crossing supports and $0 < \beta_j < 1$. By [**Cao17**, Theorem 3.5], there exists an a-th Bergman type metric $h_{a,B}$ on $K_{X'/Y}^{\otimes a} \otimes \mathcal{O}_{X'}(\sum a\beta_j F_j)$. We note that for any general point y of f such that $\mathcal{J}(h_F^{\frac{1}{a}}|_{X'_y}) = \mathcal{O}_{X'_y}$, and for any section $s' \in H^0(X'_y, K_{X'_y}^{\otimes a} \otimes \mathcal{O}_{X'}(\sum a\beta_j F_j)|_{X'_y})$, we have $|s'|^{\frac{2}{a}}_{h_{a,B}} \leq \int_{X'_y} |s'|^{\frac{2}{a}}_{h_F} < +\infty$ on X'_y by [**Cao17**, Theorem 3.5].

We will denote by $\overline{L} := K_{X'/Y}^{\otimes a-1} \otimes \mathcal{O}_{X'}(\sum a\beta_j F_j) \otimes f^*(K_Y \otimes L^{\otimes n+1})^{\otimes a-1} \otimes f^*(L^{\otimes \overline{b}})$ and $\overline{b} := b - \frac{n(n-1)}{2} - a(n+1) \geq 0$. Define a singlar hermitian metric $h_{\overline{L}} := h_{a,B}^{\frac{a-1}{a}} h_F^{\frac{1}{a}} f^*(h_s^{a-1} h_L^{\overline{b}})$ on $\overline{L}$. If y is a general point in Y such that $\mathcal{J}(h_F^{\frac{1}{a}}|_{X'_y}) = \mathcal{O}_{X'_y}$, the restriction map π_y

is surjective since the same proof works as in Section 3. By [**Laz04b**, section 9.5.D], $\mathcal{J}(h_F^{\frac{1}{a}}|_{X'_y}) = \mathcal{J}(h_F^{\frac{1}{a}})|_{X'_y} = \mathcal{O}_{X'_y}$ for any general point $y \in Y$. Therefore π_y is surjective for any general point $y \in Y$, which completes the proof.

In case (2), since K_Y is pseudo-effective, $K_Y^{\otimes a-1} \otimes L$ is a big line bundle. Therefore, there exists a singular hermitian metric h_Y on $K_Y^{\otimes a-1} \otimes L$ with neat analytic singularities such that $\sqrt{-1}\Theta_{K_Y^{\otimes a-1} \otimes L, h_Y} > 0$ in the sense of current.

We will denote by $\overline{L} := K_{X'/Y}^{\otimes a-1} \otimes \mathcal{O}_{X'}(\sum a\beta_j F_j) \otimes f^*(K_Y^{\otimes a-1} \otimes L) \otimes f^*(L^{\otimes \overline{b}})$ and $\overline{b} := b - N - 2 \geq 0$. Define a singular hermitian metric $h_{\overline{L}} := h_{a,B}^{\frac{a-1}{a}} h_F^{\frac{1}{a}} f^*(h_Y h_L^{\overline{b}})$ on $\overline{L}$. If y is a general point in Y such that $y \notin \{z \in Y : h_Y(z) = +\infty\}$ and $\mathcal{J}(h_F^{\frac{1}{a}}|_{X'_y}) = \mathcal{O}_{X'_y}$, the restriction map π_y is surjective since the same proof works as in Section 3. Since the set $\{z \in Y : h_Y(z) = +\infty\}$ is Zariski closed, then π_y is surjective for any general point $y \in Y$, which completes the proof.

$\square$

CHAPTER 3

Nadel-Nakano vanishing theorems of vector bundles with singular hermitian metrics

Abstract. We study a singular hermitian metric of a vector bundle. First, we prove that the sheaf of locally square integrable holomorphic sections of a vector bundle with a singular hermitian metric, which is a higher rank analog of a multiplier ideal sheaf, is coherent under some assumptions. Second, we prove a Nadel-Nakano type vanishing theorem of a vector bundle with a singular hermitian metric. We do not use an approximation technique of a singular hermitian metric. We apply these theorems to a singular hermitian metric induced by holomorphic sections and a big vector bundle, and we obtain a generalization of Griffiths' vanishing theorem. Finally, we show a generalization of Ohsawa's vanishing theorem.

3.1. Introduction

The aim of this paper is to study the vanishing theorem of a vector bundle with a singular hermitian metric. Here is a brief history of a singular hermitian metric of a vector bundle. A singular hermitian metric of a vector bundle is a higher rank analog of a singular hermitian metric of a line bundle. The singular hermitian metric was originated by de Cataldo [**deC98**], and was later defined in a different way by Berndtsson and Păun [**BP08**]. We adopt the definition of a singular hermitian metric of a vector bundle in [**BP08**]. They also defined the notion of a singular hermitian metric with positive curvature, called *positively curved*. In [**PT18**], Păun and Takayama proved that a direct image sheaf of an m-th relative canonical line bundle $f_*(mK_{X/Y})$ can be endowed with a positively curved singular hermitian metric for any fibration $f\colon X \to Y$. Recently Cao and Păun [**CP17**] used this result to prove Iitaka's conjecture when the base space is an Abelian variety. For more details, we refer the reader to [**Pau16**].

Although a singular hermitian metric of a vector bundle has been investigated in many papers (for example [**BP08**], [**PT18**], [**Hos17**], [**HPS18**], [**Rau15**], etc.), there exist few results on vanishing theorems for vector bundles with singular hermitian metrics. We explain the details of the investigations of a singular hermitian metric of a vector bundle below. Let (X,ω) be a compact Kähler manifold and (E,h) be a vector bundle with a singular hermitian metric. In [**deC98**], *the sheaf of locally square integrable holomorphic sections of E with respect to h*, denoted by $E(h)$, is defined as

$$E(h)_x = \{f_x \in \mathcal{O}(E)_x \colon |f_x|_h^2 \in L^1_{loc}\} \ \ x \in X,$$

which is a higher rank analog of a multiplier ideal sheaf. In this paper, we will denote by $\mathcal{O}(E)_x$ the stalk of E at x, defined by $\varinjlim_{x\in U} H^0(U, E)$. We consider the following problems.

PROBLEM 3.1.1. (1) Is $E(h)$ a coherent sheaf?
(2) Does there exist a Nadel-Nakano type vanishing theorem, that is, the vanishing of the cohomology group $H^q(X, K_X \otimes E(h))$ for any $q \geq 1$ if h has some positivity?

We do not know if $E(h)$, unlike a multiplier ideal sheaf, is coherent, In [**deC98**], de Cataldo proved that $E(h)$ is coherent and a Nadel-Nakano type vanishing theorem if h has an approximate sequence of smooth hermitian metrics $\{h_\mu\}$ satisfying $h_\mu \uparrow h$ pointwise and $\sqrt{-1}\Theta_{E,h_\mu} - \eta\omega \otimes Id_E \geq 0$ in the sense of Nakano for some positive and continuous function η. However, h does not always have such an approximate sequence (see [**Hos17**, Example 4.4]). Therefore these problems are open.

Nonetheless, we can provide a partial answer to Problem 3.1.1. First we prove the coherence of $E(h)$ under some assumptions.

THEOREM 3.1.2. Let (X, ω) be a Kähler manifold and (E, h) be a holomorphic vector bundle on X with a singular hermitian metric. We assume the following conditions.

(1) There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.
(2) $he^{-\zeta}$ is a positively curved singular hermitian metric on E for some continuous function ζ on X.
(3) There exists a real number C such that $\sqrt{-1}\Theta_{E,h} - C\omega \otimes Id_E \geq 0$ on $X \setminus Z$ in the sense of Nakano.

Then the sheaf $E(h)$ is coherent.

Next we study the cohomology group $H^q(X, K_X \otimes E(h))$ for any $q \geq 1$. We prove a vanishing theorem and an injectivity theorem for vector bundles with singular hermitian metrics under some assumptions.

THEOREM 3.1.3. Let (X, ω) be a compact Kähler manifold and (E, h) be a holomorphic vector bundle on X with a singular hermitian metric. We assume the following conditions.

(1) There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.
(2) $he^{-\zeta}$ is a positively curved singular hermitian metric on E for some continuous function ζ on X.
(3) There exists a positive number $\epsilon > 0$ such that $\sqrt{-1}\Theta_{E,h} - \epsilon\omega \otimes Id_E \geq 0$ on $X \setminus Z$ in the sense of Nakano.

Then $H^q(X, K_X \otimes E(h)) = 0$ holds for any $q \geq 1$.

THEOREM 3.1.4. Let (X,ω) be a compact Kähler manifold, (E,h) be a holomorphic vector bundle on X with a singular hermitian metric and (L,h_L) be a holomorphic line bundle with a smooth metric. We assume the following conditions.

(1) There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.
(2) $he^{-\zeta}$ is a positively curved singular hermitian metric on E for some continuous function ζ on X.
(3) $\sqrt{-1}\Theta_{E,h} \geq 0$ on $X \setminus Z$ in the sense of Nakano.
(4) There exists a positive number $\epsilon > 0$ such that $\sqrt{-1}\Theta_{E,h} - \epsilon\sqrt{-1}\Theta_{L,h_L} \otimes Id_E \geq 0$ on $X \setminus Z$ in the sense of Nakano.

Let s be a non zero section of L. Then for any $q \geq 0$, the multiplication homomorphism

$$\times s : H^q(X, K_X \otimes E(h)) \to H^q(X, K_X \otimes L \otimes E(h))$$

is injective.

Therefore we proved a Nadel-Nakano type vanishing theorem with some assumptions. If E is a holomorphic line bundle, these theorems were proved in [**Fuj12**]. We point out we do not use an approximation sequence of a singular hermitian metric to show these theorems.

Some applications are indicated as follows. First, we treat a singular hermitian metric induced by holomorphic sections, as proposed by Hosono [**Hos17**, Chapter 4]. By calculating the curvature of this metric, we prove that we can apply Theorem 3.1.3 to Hosono's example. Therefore we can apply a Nadel-Nakano type vanishing theorem even if h does not have an approximate sequence such as [**deC98**]. Second, we generalize Griffiths' vanishing theorem. That is, $H^q(X, K_X \otimes \mathrm{Sym}^m(E) \otimes \det E) = 0$ holds for any $m \geq 0$ and $q \geq 1$ if E is an ample vector bundle. We treat the case when E is a big vector bundle. If E is a big vector bundle with some assumptions, $\mathrm{Sym}^m(E) \otimes \det E$ can be endowed with a singular hermitian metric h_m satisfying assumptions such as those in Theorem 3.1.3 (see Section 5.2). Therefore $H^q(X, K_X \otimes (\mathrm{Sym}^m(E) \otimes \det E)(h_m)) = 0$ holds for any $m \geq 0$ and $q \geq 1$.

Finally, we generalize Ohsawa's vanishing theorem.

THEOREM 3.1.5. Let (X,ω) be a compact Kähler manifold and (E,h) be a holomorphic vector bundle on X with a singular hermitian metric. Let $\pi\colon X \to W$ be a proper surjective holomorphic map to an analytic space with a Kähler form σ. We assume the following conditions.

(1) There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.
(2) $he^{-\zeta}$ is a positively curved singular hermitian metric on E for some continuous function ζ on X.
(3) $\sqrt{-1}\Theta_{E,h} - \pi^*\sigma \otimes Id_E \geq 0$ on $X \setminus Z$ in the sense of Nakano.

Then $H^q\big(W, \pi_*(K_X \otimes E(h))\big) = 0$ holds for any $q \geq 1$.

If h is smooth, this theorem was proved by Ohsawa [**Ohs84**].

3.2. Preliminaries

3.2.1. hermitian metrics on vector bundles. We briefly explain definitions and notations of smooth hermitian metrics of vector bundles.

We will denote by (X, ω) a compact Kähler manifold and denote by E a holomorphic vector bundle of rank r on X. For any point $x \in X$, we take a system of local coordinates $(V; z_1, \ldots, z_n)$ near x. Let h be a smooth metric on E and let $e_1, \ldots, e_r$ be a local orthogonal frame of E near x. We denote by

$$\sqrt{-1}\Theta_{E,h} = \sqrt{-1} \sum_{1\leq j,k\leq n,\, 1\leq \lambda,\mu\leq r} c_{jk\lambda\mu}\, dz_j \wedge d\bar{z}_k \otimes e_\lambda^\vee \otimes e_\mu$$

the Chern curvature tensor. For any $u = \sum_{1\leq j\leq n,\, 1\leq\lambda\leq r} u_{j\lambda} \frac{\partial}{\partial z_j} \otimes e_\lambda \in T_x X \otimes E_x$, we denote by

$$\theta_{E,h}(u) = \sum_{1\leq j,k\leq n,\, 1\leq\lambda,\mu\leq r} c_{jk\lambda\mu} u_{j\lambda} \bar{u}_{k\mu}$$

and

$$\theta_{\omega\otimes id_E}(u) = \sum_{1\leq j,k\leq n,\, 1\leq\lambda\leq r} \omega_{jk} u_{j\lambda} \bar{u}_{k\lambda},$$

where $\omega = \sqrt{-1} \sum_{1\leq j,k\leq n} \omega_{jk} dz_j \wedge d\bar{z}_k$.

Definition 3.2.1. [**Dembook**, Chapter 7 §6] For any real number C, we write $\sqrt{-1}\Theta_{E,h} \geq C\omega \otimes id_E$ *in the sense of Nakano* if $\theta_{E,h}(u) - C\theta_{\omega\otimes id_E}(u) \geq 0$ for any $u \in TX \otimes E$.

We prove the following lemma of a positively curved singular hermitian metric.

Lemma 3.2.2. For any point $x \in X$, we take a system of local coordinate $(V; z_1, \ldots, z_n)$ near x and take a local holomorphic frame $e_1, \ldots, e_r$ of E on V. Let $U \Subset V$ be an open set near x. We assume there exists a continuous function ζ on X such that $he^{-\zeta}$ is a positively curved singular hermitian metric on E. Then there exists a positive number M_U such that for any $u \in H^0(V, E)$

$$|u|_h^2 \geq M_U \sum_{1\leq i\leq r} |u_i|^2$$

holds on U, where $u = \sum_{1\leq i\leq r} u_i e_i$.

Proof. We may assume $u = u_1 e_1$. By [**HPS18**, Chapter 16], we obtain

$$|u|_{he^{-\zeta}}(z) = \sup_{f\in E_z^\vee} \frac{|f(u)|(z)}{|f|_{(he^{-\zeta})^\vee}} \geq \frac{|e_1^\vee(u)|(z)}{|e_1^\vee|_{(he^{-\zeta})^\vee}} = \frac{|u_1|(z)}{|e_1^\vee|_{(he^{-\zeta})^\vee}}$$

for any $z \in V$. Since $he^{-\zeta}$ is positively curved, $|e_1^\vee|_{(he^{-\zeta})^\vee}$ is a plurisubharmonic function on V. Therefore $|e_1^\vee|_{(he^{-\zeta})^\vee}$ is bounded above on U. We take a positive number M_1 such that $|e_1^\vee|_{(he^{-\zeta})^\vee} \leq M_1$, then we have $|u|_{he^{-\zeta}} \geq \frac{|u_1|}{M_1}$. Since e^ζ is a positive continuous function, we can take a positive number M such that $e^\zeta \geq M$ on X. We set $M_U := \frac{M^2}{M_1^2}$ and we obtain

$$|u|_h^2 = |u|_{he^{-\zeta}}^2 e^{2\zeta} \geq M_U |u_1|^2,$$

which completes the proof. □

3.2.2. L^2 estimates and harmonic integrals on complete Kähler manifolds. We need an L^2 estimate on a complete Kähler manifold. Let Y be a complete Kähler manifold, ω' be a (not necessarily complete) Kähler form and (E, h) be a vector bundle with a smooth hermitian metric. The L^2 space $L^2_{n,q}(Y, E)_{\omega',h}$ is defined by the set of E-valued (n, q) forms with measurable coefficients on Y such that $\int_Y |f|^2_{\omega',h} dV_{\omega'} < +\infty$, where $dV_{\omega'} := \omega'^n/n!$ is a volume form on Y.

THEOREM 3.2.3. [**Dembook**, Chapter 7 §7 and Chapter 8 §6] [**Dem82**, Lemme 3.2 and Théorème 4.1] *Under the conditions stated above, we also assume that there exists a positive number $\epsilon > 0$ such that $\sqrt{-1}\Theta_{E,h} \geq \epsilon\omega' \otimes Id_E$ in the sense of Nakano. Then for any $q \geq 1$ and any $g \in L^2_{n,q}(Y, E)_{\omega',h}$ such that $\bar\partial g = 0$, there exists $f \in L^2_{n,q-1}(Y, E)_{\omega',h}$ such that $\bar\partial f = g$ and*

$$\int_Y |f|^2_{\omega',h} dV_{\omega'} \leq \frac{1}{q\epsilon} \int_Y |g|^2_{\omega',h} dV_{\omega'}.$$

We use a fact of harmonic integrals to prove Theorem 3.1.4. For more details, we refer the reader to [**Fuj12**, Section 2] or [**Dembook**, Chapter 8]. The maximal closed extension of the $\bar\partial$ operator determines a densely defined closed operator $\bar\partial\colon L^2_{n,q}(Y, E)_{\omega',h} \to L^2_{n,q+1}(Y, E)_{\omega',h}$. Then we obtain the following orthogonal decomposition.

THEOREM 3.2.4. [**Fuj12**, Section 3], [**Dembook**, Chapter 8].

$$L^2_{n,q}(Y, E)_{\omega',h} = \overline{\mathrm{Im}\bar\partial} \oplus \mathcal{H}^{n,q}(Y, E) \oplus \overline{\mathrm{Im}\bar\partial^*_{\omega',h}}$$

holds, where $\bar\partial^*_{\omega',h}$ is the Hilbert space adjoint of $\bar\partial$ and $\mathcal{H}^{n,q}(Y, E)$ is the set of harmonic forms defined by

$$\mathcal{H}^{n,q}(Y, E) := \{f \in L^2_{n,q}(Y, E)_{\omega',h} \colon \bar\partial f = \bar\partial^*_{\omega',h} f = 0\}.$$

3.3. Coherence of $E(h)$

We prove Theorem 3.1.2.

PROOF. We may assume that X is a unit ball in $\mathbb{C}^n$, $E = X \times \mathbb{C}^r$, and ω is a standard Euclidean metric. Let $e_1, \ldots, e_r$ be a local holomorphic frame of E on X. We take an open ball $U \subset\subset X$. It is enough to show that there exists a coherent sheaf $\mathcal{F}$ on U such that $E(h)_x = \mathcal{F}_x$ for any $x \in U$.

We will denote by $\mathcal{G}$ the space of holomorphic sections $g \in H^0(U, E)$ such that $\int_U |g|_h^2 dV_\omega < \infty$. We consider the evaluation map $\pi\colon \mathcal{G} \otimes_{\mathbf{C}} \mathcal{O}_U \to E|_U$. We define $\mathcal{F} := \mathrm{Im}(\pi)$. By Noether's Lemma (see [**GR84**, Chapter 5 §6]), $\mathcal{F}$ is a coherent sheaf on U.

CLAIM 3.3.1. For any $x \in U$ and any positive integer k,

$$\mathcal{F}_x + E(h)_x \cap m_x^k \cdot E_{(x)} = E(h)_x$$

holds, where m_x is a maximal ideal of $\mathcal{O}_x$.

We postpone the proof of Claim 3.3.1 and conclude the proof of Theorem 3.1.2. We fix $x \in U$. By the Artin-Rees lemma, there exists a positive integer l such that

$$E(h)_x \cap m_x^k \cdot E_{(x)} = m_x^{k-l}(E(h)_x \cap m_x^l \cdot E_{(x)})$$

holds for any $k > l$. Therefore by Claim 3.3.1, we have

$$E(h)_x = \mathcal{F}_x + E(h)_x \cap m_x^k \cdot E_{(x)} \subset \mathcal{F}_x + m_x \cdot E(h)_x \subset E(h)_x.$$

By Nakayama's lemma, we obtain $E(h)_x = \mathcal{F}_x$, which completes the proof. $\square$

We now prove Claim 3.3.1.

PROOF. It is easy to check that $\mathcal{F}_x + E(h)_x \cap m_x^k \cdot E_{(x)} \subset E(h)_x$; therefore, we show that $E(h)_x \subset \mathcal{F}_x + E(h)_x \cap m_x^k \cdot E_{(x)}$.

We take $f = \sum_i f_i e_i \in E(h)_x$. Then there exists an open neighborhood $W \subset\subset U$ near x such that f_i is a holomorphic function on W and $\int_W |f|_h^2 dV_\omega < +\infty$. Let ρ be a cut-off function on W. We note that $\bar{\partial}(\rho f)$ is an E-valued $(0,1)$ smooth form such that $\int_X |\rho f|_{\omega,h}^2 dV_\omega < +\infty$. We define the plurisubharmonic function φ_k to be $\varphi_k(z) = (n+k)\log|z-x|^2 + C|z|^2$ such that

$$\sqrt{-1}\Theta_{E,h} + \sqrt{-1}\partial\bar{\partial}\varphi_k \otimes Id_E \geq \omega \otimes Id_E \text{ on } X \setminus Z \text{ in the sense of Nakano,}$$

where C is some positive constant. Since ρ is a cut-off function, we obtain

$$\int_X |\bar{\partial}(\rho f)|_{\omega,h}^2 e^{-\varphi_k} dV_\omega < +\infty.$$

Since $X \setminus Z$ is complete by [**Dem82**, Théorème 0.2], there exists an E-valued $(0,0)$ form $F = \sum_i F_i e_i$ on $X \setminus Z$ such that

$$\int_{X\setminus Z} |F|_h^2 e^{-\varphi_k} dV_\omega \leq \int_X |\bar{\partial}(\rho f)|_{\omega,h}^2 e^{-\varphi_k} dV_\omega < +\infty \quad \text{and} \quad \bar{\partial}F = \bar{\partial}(\rho f)$$

by Theorem 3.2.3. Here we may regard $\bar{\partial}(\rho f)$ as an $(n,1)$ form $\bar{\partial}(\rho f)dz^1 \wedge \cdots \wedge dz^n$ on X with values in $-K_X$.

Let $G := \rho f - F = \sum_i G_i e_i$, which is an E-valued $(0,0)$ form on $X \setminus Z$. We obtain

$$\int_{X\setminus Z} |G|_h^2 dV_\omega < +\infty \quad \text{and} \quad \bar{\partial}G = 0.$$

By Lemma 3.2.2 we have $\sum_i \int_{U\setminus Z} |G_i|^2 dV_\omega < +\infty$, and therefore G_i extends to the whole of U and G_i is holomorphic on U by the Riemann extension theorem. Hence we obtain $G \in \mathcal{G}$ and $G_x \in \mathcal{F}_x$.

Let W' be the set of interior points in $\{z \in U : \rho(z) = 1\}$; then we have $F = f - G$ on $W' \setminus Z$. Then F extends on W' and F is holomorphic on W'. It is obvious that $F_x \in E(h)_x$ from $f_x \in E(h)$ and $G_x \in \mathcal{F}_x \subset E(h)_x$. By $\int_{X\setminus Z} |F|_h^2 e^{-\varphi_k} dV_\omega < +\infty$ and Lemma 3.2.2, we have

$$\sum_i \int_{W'} |F_i|^2 e^{-(n+k)\log|z-x|^2} dV_\omega < +\infty.$$

Therefore we obtain $(F_i)_x \in m_x^k$ and $F_x \in m_x^k \cdot E_{(x)}$.

Thus we have $f_x = G_x + F_x \in \mathcal{F}_x + E(h)_x \cap m_x^k \cdot E_{(x)}$, which completes the proof of Claim 3.3.1. □

3.4. Vanishing theorems and injectivity theorems

Let (X,ω) be a compact Kähler manifold and (E,h) be a holomorphic vector bundle with a singular hermitian metric on X. We assume the conditions (1) – (3) in Theorem 3.1.2. We will denote $Y := X\setminus Z$. By [**Fuj12**, Section 3], there exists a complete Kähler form ω' on Y such that $\omega' \geq \omega$ on Y. We study the cohomology group $H^q(X, K_X \otimes E(h))$.

Theorem 3.4.1. Under the conditions stated above, we obtain the following isomorphism:

$$H^q(X, K_X \otimes E(h)) \cong \frac{L^2_{n,q}(Y,E)_{\omega',h} \cap \mathrm{Ker}\bar{\partial}}{\mathrm{Im}\bar{\partial}}$$

for any $q \geq 0$.

Proof. The proof will be divided into three steps.

Step 1 Setup.

Let $\mathcal{U} = \{U_j\}_{j\in\Lambda}$ be a finite Stein cover of X. By Theorem 3.1.2, the sheaf cohomology $H^q(X, K_X \otimes E(h))$ is isomorphic to the Čech cohomology $H^q(\mathcal{U}, K_X \otimes E(h))$. If necessary we take U_j small enough, we may assume that there exists a Stein open set V_j, a smooth plurisubharmonic function φ_j on V_j and a positive number $C_j > 0$ such that

(1) $U_j \subset\subset V_j$,

(2) $C_j^{-1} < e^{-\varphi_j} < C_j$ on V_j, and
(3) $\sqrt{-1}\Theta_{E,h} + \sqrt{-1}\partial\bar{\partial}\varphi_j \geq \omega' \otimes Id_E$ on $V_j \setminus Z$

for any $j \in \Lambda$. We set $U_{i_0 i_1 \dots i_q} := U_{i_0} \cap U_{i_1} \cap \dots \cap U_{i_q}$, which is a Stein open set.

With the conditions above, it is easy to check the following two claims.

Claim 3.4.2. [**Fuj12**, Remark 2.19] For any E-valued (n, q) form u on Y with measurable coefficients, $|u|^2_{\omega',h} dV_{\omega'} \leq |u|^2_{\omega,h} dV_\omega$ holds. If $q = 0$, $|u|^2_{\omega',h} dV_{\omega'} = |u|^2_{\omega,h} dV_\omega$ holds.

Claim 3.4.3. For any $q \geq 1$ and any $g \in L^2_{n,q}(U_{i_0 i_1 \dots i_q} \setminus Z, E)_{\omega',h}$ such that $\bar{\partial} g = 0$, there exists $f \in L^2_{n,q-1}(U_{i_0 i_1 \dots i_q} \setminus Z, E)_{\omega',h}$ such that $\bar{\partial} f = g$ and

$$\int_{U_{i_0 i_1 \dots i_q} \setminus Z} |f|^2_{\omega',h} dV_{\omega'} \leq C'^2 \int_{U_{i_0 i_1 \dots i_q} \setminus Z} |g|^2_{\omega',h} dV_{\omega'},$$

where $C' := max_{i \in \Lambda} C_i$.

Since $U_{i_0 i_1 \dots i_q} \setminus Z$ is a complete Kähler manifold and $\sqrt{-1}\Theta_{E,h} + \sqrt{-1}\partial\bar{\partial}\varphi_{i_0} \geq \omega' \otimes Id_E$ holds on $U_{i_0 i_1 \dots i_q} \setminus Z$, we can prove Claim 3.4.3 by Theorem 3.2.3.

Step 2 Construction of a homomorphism from Čech cohomology to Dolbeault cohomology.

We fix $c = \{c_{i_0 i_1 \dots i_q}\} \in H^q(\mathcal{U}, K_X \otimes E(h))$. By the definition of Čech cohomology, we have

(1) $c_{i_0 i_1 \dots i_q} \in H^0(U_{i_0 i_1 \dots i_q}, K_X \otimes E(h))$ and
(2) $\delta c := \sum_{k=0}^{q+1} (-1)^k c_{i_0 i_1 \dots \check{i_k} \dots i_{q+1}}|_{U_{i_0 i_1 \dots i_{q+1}}} = 0$.

Let $\{\rho_i\}_{i \in \Lambda}$ be a partition of unity subordinate to $\mathcal{U}$. For each $k \in \{0, 1, \dots, q-1\}$, we define an E-valued form $b_{i_0 i_1 \dots i_k}$ by

$$b_{i_0 i_1 \dots i_k} := \begin{cases} \sum_{j \in \Lambda} \rho_j c_{j i_0 i_1 \dots i_{q-1}} & \text{if } k = q-1 \\ \sum_{j \in \Lambda} \rho_j \bar{\partial} b_{j i_0 i_1 \dots i_k} & otherwise. \end{cases}$$

Then, we have

$$\delta\{b_{i_0 i_1 \dots i_{q-1}}\}_{i_0 i_1 \dots i_q} = \sum_{k=0}^{q} (-1)^k b_{i_0 i_1 \dots \check{i_k} \dots i_q} = \sum_{k=0}^{q} \sum_{j \in \Lambda} (-1)^k \rho_j c_{j i_0 i_1 \dots \check{i_k} \dots i_q} = \sum_{j \in \Lambda} \rho_j \sum_{k=0}^{q} (-1)^k c_{j i_0 i_1 \dots \check{i_k} \dots i_q}$$

From $\delta c = 0$, we have

$$\sum_{j \in \Lambda} \rho_j \sum_{k=0}^{q} (-1)^k c_{j i_0 i_1 \dots \check{i_k} \dots i_q} = \sum_{j \in \Lambda} \rho_j c_{i_0 i_1 \dots i_q} = c_{i_0 i_1 \dots i_q}.$$

Therefore, we obtain $\delta\{b_{i_0 i_1 \dots i_{q-1}}\} = c$. Similarly we obtain $\delta\{b_{i_0 i_1 \dots i_k}\} = \{\bar{\partial} b_{i_0 i_1 \dots i_{k+1}}\}$ for each $k \in \{0, 1, \dots, q-2\}$.

Therefore we obtain $\bar{\partial} b_{i_0}|_{U_{i_0}\setminus Z}$, which is an E-valued (n,q) $\bar{\partial}$-closed form on $U_{i_0}\setminus Z$. Since we have

$$\delta\{\bar{\partial} b_{i_0}\}=0 \quad\text{and}\quad \int_{U_{i_0}\setminus Z}|\bar{\partial} b_{i_0}|^2_{\omega',h}dV_{\omega'}\le\int_{U_{i_0}}|\bar{\partial} b_{i_0}|^2_{\omega,h}dV_{\omega}<+\infty$$

by Claim 3.4.2, we can define $\alpha(c):=\{\bar{\partial} b_{i_0}\}\in L^2_{n,q}(Y,E)_{\omega',h}\cap \mathrm{Ker}\bar{\partial}$. By the above construction, we obtain the homomorphism

$$\alpha\colon H^q(\mathcal{U},K_X\otimes E(h))\to\frac{L^2_{n,q}(Y,E)_{\omega',h}\cap\mathrm{Ker}\bar{\partial}}{\mathrm{Im}\bar{\partial}}.$$

Step 3 Construction of a homomorphism from Dolbeault cohomology to Čech cohomology.

We fix $u\in L^2_{n,q}(Y,E)_{\omega',h}\cap\mathrm{Ker}\bar{\partial}$ and define $D:=\int_Y|u|^2_{\omega',h}dV_{\omega'}<+\infty$. By Claim 3.4.3, there exists $v_{i_0}\in L^2_{n,q-1}(U_{i_0}\setminus Z,E)_{\omega',h}$ such that

$$\bar{\partial} v_{i_0}=u|_{U_{i_0}\setminus Z}\quad\text{and}\quad\int_{U_{i_0}\setminus Z}|v_{i_0}|^2_{\omega',h}dV_{\omega'}\le C'^2D.$$

We set $u^1:=\delta\{v_{i_0}\}$. From $\bar{\partial} u^1=0$, there exists $v_{i_0i_1}\in L^2_{n,q-2}(U_{i_0i_1}\setminus Z,E)_{\omega',h}$ such that

$$\bar{\partial} v_{i_0i_1}=u^1_{i_0i_1}\quad\text{and}\quad\int_{U_{i_0i_1}\setminus Z}|v_{i_0i_1}|^2_{\omega',h}dV_{\omega'}\le 2C'^2D$$

by Claim 3.4.3. We set $u^2:=\delta\{v_{i_0i_1}\}$ and we have $\bar{\partial} u^2=0$.

By repeating this procedure, we obtain $v_{i_0i_1\dots i_{q-1}}\in L^2_{n,0}(U_{i_0i_1\dots i_{q-1}}\setminus Z,E)_{\omega',h}$ and $u^q=\delta\{v_{i_0i_1\dots i_{q-1}}\}$. By $\bar{\partial} u^q_{i_0i_1\dots i_q}=0$, $u^q_{i_0i_1\dots i_q}$ is a holomorphic E-valued $(n,0)$ form on $U_{i_0i_1\dots i_q}\setminus Z$. Since we obtain

$$\int_{U_{i_0i_1\dots i_q}\setminus Z}|u^q_{i_0i_1\dots i_q}|^2_{\omega,h}dV_{\omega}=\int_{U_{i_0i_1\dots i_q}\setminus Z}|u^q_{i_0i_1\dots i_q}|^2_{\omega',h}dV_{\omega'}\le q!C'^2D<+\infty$$

by Claim 3.4.2, $u^q_{i_0i_1\dots i_q}|_{U_{i_0i_1\dots i_q}\setminus Z}$ extends on $U_{i_0i_1\dots i_q}$ and $u^q_{i_0i_1\dots i_q}|_{U_{i_0i_1\dots i_q}\setminus Z}$ is a holomorphic E-valued $(n,0)$ form on $U_{i_0i_1\dots i_q}$ by the Riemann extension theorem and Lemma 3.2.2. Therefore we can define $\beta(u):=\{u^q_{i_0i_1\dots i_q}|_{U_{i_0i_1\dots i_q}\setminus Z}\}\in H^q(\mathcal{U},K_X\otimes E(h))$. By the above construction, we obtain the homomorphism

$$\beta\colon\frac{L^2_{n,q}(Y,E)_{\omega',h}\cap\mathrm{Ker}\bar{\partial}}{\mathrm{Im}\bar{\partial}}\to H^q(\mathcal{U},K_X\otimes E(h)).$$

It is easy to check whether α and β induce the isomorphism in Theorem 3.4.1. □

We finish this section by proving Theorem 3.1.3 and 3.1.4.

Proof of Theorem 3.1.3. By Theorem 3.4.1, we have $H^q(X,K_X\otimes E(h))\cong\frac{L^2_{n,q}(Y,E)_{\omega',h}\cap\mathrm{Ker}\bar{\partial}}{\mathrm{Im}\bar{\partial}}$. By Theorem 3.2.3, we have $\frac{L^2_{n,q}(Y,E)_{\omega',h}\cap\mathrm{Ker}\bar{\partial}}{\mathrm{Im}\bar{\partial}}=0$, which completes the proof.

Proof of Theorem 3.1.4. By Theorem 3.1.2, $K_X \otimes E(h)$ is a coherent sheaf on X. Therefore, by the argument of [**Fuj12**, Claim 1], Theorem 3.2.4 and Theorem 3.4.1, we obtain $\overline{\mathrm{Im}\bar\partial} = \mathrm{Im}\bar\partial$, $\overline{\mathrm{Im}\bar\partial^*_{\omega',h}} = \mathrm{Im}\bar\partial^*_{\omega',h}$ and $H^q(X, K_X \otimes E(h)) \cong \mathcal{H}^{n,q}(Y, E)$. Similarly, we obtain $H^q(X, K_X \otimes L \otimes E(h)) \cong \mathcal{H}^{n,q}(Y, L \otimes E)$. By [**Fuj12**, Claim 2], the multiplication homomorphism $\times s \colon \mathcal{H}^{n,q}(Y, E) \to \mathcal{H}^{n,q}(Y, L \otimes E)$ is well-defined and injective, which completes the proof.

3.5. Applications

3.5.1. Hosono's example. In this subsection, we study a singular hermitian metric induced by holomorphic sections, proposed by Hosono [**Hos17**, Chapter 4].

In this section, we assume that E has holomorphic sections $s_1, \dots, s_N \in H^0(X, E)$ such that E_y is generated by $s_1(y), \dots, s_N(y)$ for a general point y. For any point $x \in X$, we take a local coordinate $(U; z_1, \dots, z_n)$ near x and take a local holomorphic frame $e_1, \dots, e_r$ of E on U. Write $s_i = \sum_{1 \le j \le r} f_{ij} e_j$, where f_{ij} are holomorphic functions on U. A singular hermitian metric h induced by $s_1, \dots, s_N$ is given by

$$h^{-1}_{jk} := \sum_{1 \le i \le N} \bar f_{ij} f_{ik}.$$

By [**Hos17**, Example 3.6 and Proposition 4.1], h is positively curved and $E(h)$ is a coherent sheaf. Hosono pointed out that we can easily calculate the curvature of h in the case $N = r$.

LEMMA 3.5.1. In the case $N = r$, there exists a proper analytic subset Z such that $\sqrt{-1}\Theta_{E,h} = 0$ on $X \setminus Z$. In particular we obtain $\sqrt{-1}\Theta_{E,h} \ge 0$ on $X \setminus Z$ in the sense of Nakano.

PROOF. We take a finite Stein open covering $\{U_i\}_{i \in \Lambda}$. Under the conditions stated above, an $r \times r$ matrix $A^{(i)}$ on U_i is defined by

$$A^{(i)}_{jk} = f_{jk}.$$

Set $Z_i := \{z \in U_i \colon \operatorname{rank} A^{(i)}(z) < r\}$ and $W = \{z \in X \colon h \text{ is not smooth at } z\}$. We have $h = (h^{-1})^{-1} = \frac{\widetilde{h^{-1}}}{\det h^{-1}}$, where $\widetilde{h^{-1}}$ is a cofactor matrix of h^{-1}. Since the (i,j) element of $\widetilde{h^{-1}}$ is a smooth function on X for any $1 \le i, j \le r$, we have $W = \{z \in X \colon \det h^{-1}(z) = 0\}$. By [**Hos17**, Lemma 4.3], we have

$$\det h^{-1} = \sum_{1 \le i_1 < i_2 < \dots < i_r \le N} |\det(s_{i_1}, s_{i_2}, \dots, s_{i_r})|,$$

and therefore W is a proper analytic subset. We write $Z := \cup_{i \in \Lambda} Z_i \cup W$, which is a proper analytic subset.

By an easy computation, we have

$$\sqrt{-1}\Theta_{E,h} = \sqrt{-1}\bar\partial(\bar h^{-1}\partial \bar h) = \sqrt{-1}(\partial\bar\partial \bar h^{-1} - \partial \bar h^{-1} \bar h \bar\partial \bar h^{-1})\bar h.$$

For any $z \in X \setminus Z$, we may assume $f_{ij}(z) = \delta_{ij}$. From $\bar{h}^{-1}_{jk} = \sum_{1\le i\le r} f_{ij}\bar{f_{ik}}$, we have

$$\partial \bar{h}^{-1}_{jk}(z) = \partial f_{kj}(z) \text{ and } \bar{\partial}\bar{h}^{-1}_{jk}(z) = \bar{\partial}\bar{f_{jk}}(z).$$

Thus, we obtain

$$(\partial \bar{h}^{-1}\bar{h}\bar{\partial}\bar{h}^{-1})_{jk}(z) = \sum_{1\le i\le r} \partial f_{ij}\bar{\partial}\bar{f_{ik}}(z) = \partial\bar{\partial}\bar{h}^{-1}_{jk}(z),$$

which completes the proof. □

By Lemma 3.5.1 and Theorem 3.1.3, we obtain the following corollary.

Corollary 3.5.2. Let (L, h_L) be a holomorphic line bundle with a singular hermitian metric. We assume there exist a proper analytic subset Z and a positive number $\epsilon > 0$ such that h_L is smooth on $X \setminus Z$ and $\sqrt{-1}\Theta_{L,h_L} \ge \epsilon\omega$ on X.

Then, $H^q(X, K_X \otimes L \otimes E(hh_L)) = 0$ holds for all $q \ge 1$ for any holomorphic vector bundle E and a singular hermitian metric h induced by $s_1 \cdots s_r \in H^0(X, E)$.

In particular $H^q(X, K_X \otimes L \otimes E(h)) = 0$ holds for all $q \ge 1$ if L is ample.

We point out that such a metric h_L on L as in Corollary 3.5.2 always exists if L is big.

Now, we introduce Hosono's example [**Hos17**, Example 4.4]. Set $X = \mathbb{C}^2$ and let $E = X \times \mathbb{C}^2$ be the trivial rank-two bundle. We choose sections $s_1 = e_1$ and $s_2 = ze_1 + we_2$. Then the singular hermitian metric h_E induced by s_1, s_2 can be written by

$$h_E = \frac{1}{|w|^2}\begin{pmatrix} |w|^2 & -w\bar{z} \\ -z\bar{w} & |z|^2+1 \end{pmatrix}.$$

Hosono proved the following theorem by calculating the standard approximation by convolution of h_E.

Theorem 3.5.3. [**Hos17**, Theorem 1.2] The standard approximation defined by convolution of h_E does not have a uniformly bounded curvature from below in the sense of Nakano.

Therefore, we can not apply the vanishing theorem of [**deC98**] to this example. However, we can apply Corollary 3.5.2 to this example. Thus our results are new results.

Remark 3.5.4. We ask whether there exists a proper analytic subset Z such that $\sqrt{-1}\Theta_{E,h} \ge 0$ on $X \setminus Z$ in the sense of Nakano for any singular hermitian metric h induced by $s_1 \cdots s_N \in H^0(X, E)$ in the case $N > r$. This calculation is very complicated and this question is open, but it is likely that the answer is "No".

3.5.2. Big vector bundles. Let $\tilde{\omega}$ be a Kähler form on $\mathbb{P}(E)$. Inayama communicated to the author the following lemma.

LEMMA 3.5.5. Let E be a vector bundle and $\tilde{h}$ be a singular hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$. We assume that there exist a positive number $\epsilon > 0$ and a proper analytic subset $\tilde{Z} \subset \mathbb{P}(E)$ such that $\tilde{h}$ is smooth on $\mathbb{P}(E) \setminus \tilde{Z}$, $\pi(\tilde{Z}) \neq X$, and $\sqrt{-1}\Theta_{\mathcal{O}_{\mathbb{P}(E)}(1),\tilde{h}} \geq \epsilon\tilde{\omega} \otimes id_{\mathcal{O}_{\mathbb{P}(E)}(1)}$.

Then $\tilde{h}$ induces a singular hermitian metric h_m on $\mathrm{Sym}^m(E) \otimes \det E$ such that

(1) h_m is smooth on $X \setminus \pi(\tilde{Z})$,
(2) h_m is a positively curved singular hermitian metric, and
(3) $\sqrt{-1}\Theta_{\mathrm{Sym}^m(E)\otimes\det E,h_m} \geq \epsilon\omega \otimes Id_{\mathrm{Sym}^m(E)\otimes\det E}$ on $X \setminus \pi(\tilde{Z})$ in the sense of Nakano.

PROOF. From $\mathrm{Sym}^m(E)\otimes\det E = \pi_*(K_{\mathbb{P}(E)/X}\otimes\mathcal{O}_{\mathbb{P}(E)}(m+r))$, $\mathrm{Sym}^m(E)\otimes\det E$ can be endowed with the L^2 metric h_m with respect to $\tilde{h}$. Therefore by the argument of [**Ber09**, Theorem 1.2, Theorem 1.3, and Section 4], (1) and (3) are proved. By [**HPS18**] and [**PT18**], (2) is proved. □

REMARK 3.5.6. If E is big, such a metric $\tilde{h}$ on $\mathcal{O}_{\mathbb{P}(E)}(1)$ as in the assumption of Lemma 3.5.5 always exists.

Thus, we can apply Theorem 3.1.3 to $(\mathrm{Sym}^m(E) \otimes \det E, h_m)$ and we have the following corollary.

COROLLARY 3.5.7. Under the conditions stated in Lemma 3.5.5, $H^q(X, K_X \otimes (\mathrm{Sym}^m(E) \otimes \det E)(h_m)) = 0$ holds for any $m \geq 0$ and $q \geq 1$.

This corollary is a generalization of Griffiths' vanishing theorem in [**Gri69**].

3.6. On Ohsawa's vanishing theorem

We use the results of [**Ohs84**]. Let Y be a complete Kähler manifold, ω' be a Kähler form and (E, h) be a vector bundle with a smooth hermitian metric. Let τ be a smooth semipositive $(1, 1)$ form on Y. Write

$$L^2_{n,q}(Y,E)_{\tau,h} := \{f \in L^2_{n,q}(Y,E)_{\omega'+\tau,h}\,;\ \lim_{\epsilon\downarrow 0}\int_Y |f|^2{}_{\epsilon\omega'+\tau,h}dV_{\epsilon\omega'+\tau} < +\infty\}.$$

By [**Ohs84**, Proposition 2.4], $\lim_{\epsilon\downarrow 0}\int_Y |f|^2{}_{\epsilon\omega'+\tau,h}dV_{\epsilon\omega'+\tau}$ and $L^2_{n,q}(Y,E)_{\tau,h}$ do not depend on the choice of the metric ω'. We use Ohsawa's L^2 estimate.

THEOREM 3.6.1. [**Ohs84**, Theorem 2.8] Under the conditions stated above, we also assume that $\sqrt{-1}\Theta_{E,h} - \tau \otimes Id_E \geq 0$ on Y. For any $q \geq 1$ and $f \in L^2_{n,q}(Y,E)_{\tau,h}$ such that $\bar{\partial}f = 0$, there exists $g \in L^2_{n,q-1}(Y,E)_{\tau,h}$ such that $\bar{\partial}g = f$ and

$$\lim_{\epsilon\downarrow 0}\int_Y |g|^2{}_{\epsilon\omega'+\tau,h}dV_{\epsilon\omega'+\tau} \leq q\lim_{\epsilon\downarrow 0}\int_Y |f|^2{}_{\epsilon\omega'+\tau,h}dV_{\epsilon\omega'+\tau}.$$

Now we prove Theorem 3.1.5.

PROOF. We take a complete Kähler form ω' on $Y := X \setminus Z$ as in Section 4. The proof of Theorem 3.1.5 is similar to those of [**Ohs84**, Theorem 3.1] and Theorem 3.4.1 with a slight modification.

Let $\mathcal{U} = \{U_j\}_{j\in\Lambda}$ be a finite Stein cover of W. By Theorem 3.1.2 and the Grauert direct image theorem, $\pi_*(K_X \otimes E(h))$ is coherent. Therefore the sheaf cohomology $H^q(W, \pi_*(K_X \otimes E(h)))$ is isomorphic to the Čech cohomology $H^q(\mathcal{U}, \pi_*(K_X \otimes E(h)))$. We point out the following claim.

CLAIM 3.6.2. [**Ohs84**, Lemma 3.2] For any form g on W, $|\pi^* g(x)|_{\omega+\pi^*\sigma} \leq |g(\pi(x))|_\sigma$ holds at any $x \in X$.

We fix $c = \{c_{i_0 i_1 \dots i_q}\} \in H^q(\mathcal{U}, \pi_*(K_X \otimes E(h)))$. By the definition of Čech cohomology, we have

(1) $c_{i_0 i_1 \dots i_q} \in H^0(U_{i_0 i_1 \dots i_q}, \pi_*(K_X \otimes E(h))) = H^0(\pi^{-1}(U_{i_0 i_1 \dots i_q}), K_X \otimes E(h))$ and

(2) $\delta c := \sum_{k=0}^{q+1} (-1)^k c_{i_0 i_1 \dots \check{i_k} \dots i_{q+1}}|_{\pi^{-1}(U_{i_0 i_1 \dots i_{q+1}})} = 0$.

Let $\{\rho_j\}_{j\in\Lambda}$ be a partition of unity of $\mathcal{U}$. Based on Section 4, for each $k \in \{0, 1, \dots, q-1\}$, we define an E-valued form $b_{i_0 i_1 \dots i_k}$ by

$$b_{i_0 i_1 \dots i_k} := \begin{cases} \sum_{j\in\Lambda} \pi^*(\rho_j) c_{j i_0 i_1 \dots i_{q-1}} & \text{if } k = q-1 \\ \sum_{j\in\Lambda} \pi^*(\rho_j) \bar{\partial} b_{j i_0 i_1 \dots i_k} & \textit{otherwise.} \end{cases}$$

As in Step 2 in the proof of Theorem 3.4.1, we obtain

$$\delta\{b_{i_0 i_1 \dots i_{q-1}}\} = c, \quad \text{and} \quad \delta\{b_{i_0 i_1 \dots i_k}\} = \{\bar{\partial} b_{i_0 i_1 \dots i_{k+1}}\}$$

for each $k \in \{0, 1, \dots, q-2\}$.

Therefore we obtain $\bar{\partial} b_{i_0}|_{\pi^{-1}(U_{i_0})\setminus Z}$, which is an E-valued (n, q) $\bar{\partial}$-closed form on $\pi^{-1}(U_{i_0}) \setminus Z$. By Claim 3.6.2, $|\bar{\partial}(\pi^*\rho_j)|_{\epsilon\omega+\pi^*\sigma}$ are bounded above by $|\bar{\partial}(\rho_j)|_\sigma$ for any $\epsilon > 0$ and $|c_{i_0 i_1 \dots i_q}|^2_{\epsilon\omega+\pi^*\sigma} dV_{\epsilon\omega+\pi^*\sigma}$ are independent of ϵ by Claim 3.4.2. Therefore we have $\delta\{\bar{\partial} b_{i_0}\} = 0$ and

$$\begin{aligned} \int_{\pi^{-1}(U_{i_0})\setminus Z} |\bar{\partial} b_{i_0}|^2_{\epsilon\omega'+\pi^*\sigma,h} dV_{\epsilon\omega'+\pi^*\sigma} &\leq \int_{\pi^{-1}(U_{i_0})} |\bar{\partial} b_{i_0}|^2_{\epsilon\omega+\pi^*\sigma,h} dV_{\epsilon\omega+\pi^*\sigma} \\ &\leq \lim_{\epsilon\downarrow 0} \int_{\pi^{-1}(U_{i_0})} |\bar{\partial} b_{i_0}|^2_{\epsilon\omega+\pi^*\sigma,h} dV_{\epsilon\omega+\pi^*\sigma} \\ &< +\infty \end{aligned}$$

for any $\epsilon > 0$ by Claim 3.4.2. Thus, we may regard $\{\bar{\partial} b_{i_0}\}$ as an element of $L^2_{n,q}(Y, E)_{\sigma,h}$ and denote by $b := \bar{\partial} b_{i_0}$. By Theorem 3.6.1, there exists $a \in L^2_{n,q-1}(Y, E)_{\sigma,h}$ such that

$$\bar{\partial} a = b \quad \text{and} \quad \lim_{\epsilon\downarrow 0} \int_{Y\setminus Z} |a|^2_{\epsilon\omega'+\pi^*\sigma,h} dV_{\epsilon\omega'+\pi^*\sigma} < +\infty.$$

Write $d^1_{i_0} := b_{i_0} - a \in L^2_{n,q-1}(\pi^{-1}(U_{i_0}) \setminus Z, E)_{\sigma,h}$ and $d^1 := \{d^1_{i_0}\}$. We point out

$$\delta d^1 = \delta\{b_{i_0}\} = \{\bar\partial b_{i_0 i_1}\} \quad \text{and} \quad \bar\partial d^1 = 0.$$

By Theorem 3.6.1, there exists $a_{i_0} \in L^2_{n,q-2}(\pi^{-1}(U_{i_0}) \setminus Z, E)_{\sigma,h}$ such that

$$\bar\partial a_{i_0} = d^1_{i_0} \quad \text{and} \quad \lim_{\epsilon \downarrow 0} \int_{U_{i_0} \setminus Z} |a_i|^2_{\epsilon\omega' + \pi^*\sigma, h} dV_{\epsilon\omega' + \pi^*\sigma} < +\infty.$$

We write $d^2_{i_0 i_1} := b_{i_0 i_1} - a_{i_0} + a_{i_1} \in L^2_{n,q-2}(\pi^{-1}(U_{i_0 i_1}) \setminus Z, E)_{\sigma,h}$ and $d^2 := \{d^2_{i_0 i_1}\}$. We point out that

$$\delta d^2 = \delta\{b_{i_0 i_1}\} = \{\bar\partial b_{i_0 i_1 i_2}\} \quad \text{and} \quad \bar\partial d^2 = 0.$$

By repeating this procedure, we obtain $d^{q-1}_{i_0 i_1 \ldots i_{q-1}} \in L^2_{n,0}(\pi^{-1}(U_{i_0 i_1 \ldots i_{q-1}}) \setminus Z, E)_{\sigma,h}$ and $d^{q-1} := \{d^{q-1}_{i_0 i_1 \ldots i_{q-1}}\}$ such that

$$\delta d^{q-1} = \delta\{b_{i_0 i_1 \ldots i_{q-1}}\} = c \quad \text{and} \quad \bar\partial d^{q-1} = 0.$$

We have

$$\begin{aligned} \int_{\pi^{-1}(U_{i_0 i_1 \ldots i_{q-1}}) \setminus Z} |d^{q-1}_{i_0 i_1 \ldots i_{q-1}}|^2_{\omega,h} dV_\omega &= \int_{\pi^{-1}(U_{i_0 i_1 \ldots i_{q-1}}) \setminus Z} |d^{q-1}_{i_0 i_1 \ldots i_{q-1}}|^2_{\omega' + \pi^*\sigma, h} dV_{\omega' + \pi^*\sigma} \\ &= \lim_{\epsilon \downarrow 0} \int_{\pi^{-1}(U_{i_0 i_1 \ldots i_{q-1}}) \setminus Z} |d^{q-1}_{i_0 i_1 \ldots i_{q-1}}|^2_{\epsilon\omega' + \pi^*\sigma, h} dV_{\epsilon\omega' + \pi^*\sigma} \\ &< +\infty. \end{aligned}$$

By Lemma 3.2.2 and the Riemann extension theorem, $d^{q-1}_{i_0 i_1 \ldots i_{q-1}}$ extends on $\pi^{-1}(U_{i_0 i_1 \ldots i_{q-1}})$ and $d^{q-1}_{i_0 i_1 \ldots i_{q-1}}$ is holomorphic on $\pi^{-1}(U_{i_0 i_1 \ldots i_{q-1}})$. Therefore we obtain

$$d^{q-1}_{i_0 i_1 \ldots i_{q-1}} \in H^0(\pi^{-1}(U_{i_0 i_1 \ldots i_{q-1}}), K_X \otimes E(h)) \ \textit{and} \ \delta d^{q-1} = c,$$

which completes the proof. □

REMARK 3.6.3. We ask whether, under the assumptions of singular hermitian metrics as in Theorems 3.1.3 - 3.1.5, we can show higher rank analogies of a generalization of the Kollár-Ohsawa type vanishing theorem by Matsumura [**Mat16**], an injectivity theorem of higher direct images by Fujino [**Fuj13**], an injectivity theorem of pseudoeffective line bundles by Fujino and Matsumura [**FujM16**] and so on. It is likely the answer is "Yes" and the proof may be similar to the original proof with a slight modification.

CHAPTER 4

Characterization of pseudo-effective vector bundles by singular hermitian metrics

ABSTRACT. In this paper, we give complex geometric descriptions of the notions of algebraic geometric positivity of vector bundles and torsion-free coherent sheaves, such as nef, big, pseudo-effective and weakly positive, by using singular hermitian metrics. As an applications, we obtain a generalization of Mori's result.

4.1. Introduction

In [**Kod54**], Kodaira proved that a line bundle L is ample if and only if L has a smooth hermitian metric with positive curvature. After that, Demailly [**Dem92**] gave complex geometric descriptions of nef, big and pseudo-effective line bundles. For example, he proved that a line bundle L is pseudo-effective if and only if L has a singular hermitian metric with semipositive curvature current. Ample, nef, big and pseudo-effective are notions of algebraic geometric positivity. Thus, their works related algebraic geometry to complex geometry.

The aim of this paper is to give complex geometric descriptions of notions of algebraic geometric positivity of vector bundles and torsion-free coherent sheaves. Griffiths [**Gri69**] proved that if a vector bundle E has a Griffiths positive metric, then E is ample (i.e. $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample). The inverse implication is unknown. We do not know whether an ample vector bundle has a Griffiths positive metric. This is so-called Griffiths' conjecture, which is one of longstanding open problems. In recent years, Liu, Sun and Yang [**LSY13**] gave a partial answer to this conjecture.

THEOREM 4.1.1. [**LSY13**, Theorem 1.2 and Corollary 4.6] Let X be a smooth projective variety and E be a holomorphic vector bundle on X. If E is ample, then there exists $k \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^k(E)$ has a Griffiths (Nakano) positive smooth hermitian metric.

Throughout this paper, we will denote by $\mathrm{Sym}^k(E)$ the k-th symmetric power of E and denote by $\mathbb{N}_{>0}$ the set of positive integers. Inspired by the works of Liu, Sun and Yang, we study notions of algebraic geometric positivity of vector bundles by using smooth and singular hermitian metrics.

THEOREM 4.1.2. Let X be a smooth projective variety and E be a holomorphic vector bundle on X.

(1) E is nef iff there exists an ample line bundle A on X such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive smooth hermitian metric for any $k \in \mathbb{N}_{>0}$.
(2) E is big iff there exist an ample line bundle A and $k \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^k(E) \otimes A^{-1}$ has a Griffiths semipositive singular hermitian metric.
(3) E is pseudo-effective iff there exists an ample line bundle A such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive singular hermitian metric for any $k \in \mathbb{N}_{>0}$.
(4) E is weakly positive iff there exist an ample line bundle A and a proper Zariski closed set Z such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive singular hermitian metric h_k for any $k \in \mathbb{N}_{>0}$ and the Lelong number of h_k at x is less than 2 for any $x \in X \setminus Z$.

We will explain the definitions of big, pseudo-effective and weakly positive in Section 2. Further, we obtain similar results in case of torsion-free coherent sheaves. We will discuss about torsion-free coherent sheaves in Section 5.

Nef, big, pseudo-effective and weakly positive are notions of algebraic geometric positivity of vector bundles and torsion-free coherent sheaves. In particular, Viehweg [**Vie83a**] proved that a direct image sheaf of an m-th relative canonical line bundle $f_*(mK_{X/Y})$ is weakly positive for any fibration $f\colon X \to Y$. By using this result, he studied Iitaka's conjecture. A Griffiths semipositive singular hermitian metric, which is a analogy of a singular hermitian metric of a line bundle and a Griffiths semipositive metric, was investigated in many papers. By using Griffiths semipositive singular hermitian metrics, Cao and Păun [**CP17**] proved Iitaka's conjecture when the base space is an Abelian variety. Therefore, our results also relate algebraic geometry to complex geometry.

We have some applications about our results.

Corollary 4.1.3. Let X be a smooth projective n-dimensional variety. If the tangent bundle T_X is big then X is biholommorphic to $\mathbb{CP}^n$.

This corollary is a generalization of Mori's result: "If the tangent bundle T_X is ample then X is biholomorphic to $\mathbb{CP}^n$" since an ample vector bundle is big. This Corollary was proved by Fulger and Murayama [**FulM19**, Corollary 7.8] by using Seshadri constants of vector bundles . We give an another proof by using singular hermitian metrics.

4.2. A singular hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$

We study a singular hermitian metric on $\mathcal{O}_{\mathbb{P}(E)}(1)$ induced by a singular hermitian metric on E.

Lemma 4.2.1. Let X be a smooth projective n-dimensional variety, E be a holomorphic vector bundle of rank r on X, and A be a line bundle on X. Assume that there exists $m \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^m(E) \otimes A$ has a Griffiths semipositive singular hermitian metric h_m. Then $\mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^*A$ has a singular hermitian metric g_m with semipositive curvature current.

Moreover for any $x \in X$, there exist an open set $x \in V$ and a positive constant C_V such that $g_m \le C_V \pi^*(\det h_m)$ on $\pi^{-1}(V)$.

PROOF. We will denote by $\pi_m \colon \mathbb{P}(\mathrm{Sym}^m(E) \otimes A) \to X$. We have $\mathbb{P}(\mathrm{Sym}^m(E)) = \mathbb{P}(\mathrm{Sym}^m(E) \otimes A)$ and $\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^m(E))}(1) \otimes \pi_m^*(A) = \mathcal{O}_{\mathbb{P}(\mathrm{Sym}^m(E) \otimes A)}(1)$. Let $\mu_m \colon \mathbb{P}(E) \to \mathbb{P}(\mathrm{Sym}^m(E))$ be a standard m-th Veronese embedding. Then we have $\pi = \pi_m \circ \mu_m$ and

$$\mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^* A = \mu_m^*(\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^m(E))}(1)) \otimes \pi^*(A) = \mu_m^*(\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^m(E) \otimes A)}(1)).$$

By [**PT18**, Proposition 2.3.5], $\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^m(E) \otimes A)}(1)$ can be endowed a singular hermitian metric $\widetilde{g_m}$ with semipositive curvature current. Therefore, we put $g_m := \pi_m^* \widetilde{g_m}$, which is a singular hermitian metric with semipositive curvature current on $\mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^* A$.

We fix a point $x \in X$. Since $\pi_m^{-1}(x)$ is compact, there exist an open set $x \in V$ and a positive constant C_V such that $\widetilde{g_m} \le C_V \pi_m^*(\det h_m)$ on $\pi_m^{-1}(V)$ by [**PT18**, Proposition 2.3.5]. Therefore, we have $g_m \le C_V \pi^*(\det h_m)$ on $\pi^{-1}(V)$. □

LEMMA 4.2.2. Let X be a smooth projective n-dimensional variety, E be a holomorphic vector bundle of rank r on X and A be a line bundle on X. Assume there exist $m \in \mathbb{N}_{>0}$ and a point $x \in X$ such that $\mathrm{Sym}^m(E) \otimes A$ is globally generated at x. Then there exist a singular hermitian metric g with semipositive curvature current on $\mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^* A$ and a proper Zariski closed set $Z \subset X$ such that g is smooth outside $\pi^{-1}(Z)$.

Moreover if there exists a Zariski open set $U \subset X$ such that $\mathrm{Sym}^m(E) \otimes A$ is globally generated at x for any $x \in U$, we can take Z such that $Z \cap U = \varnothing$.

PROOF. Let $\{U_i\}$ be a finite open cover of X such that U_i is a coordinate neighborhood and $\pi^{-1}(U_i)$ is biholomorphic to $U_i \times \mathbb{P}^{r-1}$. We take a local holomorphic frame $e_1, \ldots, e_r$ of E on U_i and a local holomorphic frame e_A of A on U_i. Let $s_1, \ldots, s_N$ be a basis on $H^0(X, \mathrm{Sym}^m(E) \otimes A)$. We put $M := \binom{m+r-1}{r}$. Write

$$s_j = \sum_{\alpha} f_{j\alpha} e_1^{\alpha_1} \cdots e_r^{\alpha_r} e_A,$$

where $f_{j\alpha}$ are holomorphic function on U_i and the sum is taken over $\alpha = (\alpha_1, \cdots, \alpha_r) \in \mathbb{N}_{>0}^r$ such that $\alpha_1 + \cdots + \alpha_r = m$. The $N \times M$ matrix $B^{(i)}$ is defined by $B^{(i)} = f_{j\alpha}$. Set $Z_i := \{z \in U_i \colon \operatorname{rank} B^{(i)}(z) < M\}$ and $Z := \cup Z_i$. Since $\mathrm{Sym}^m(E) \otimes A$ is globally generated at x, we have $N \ge M$ and Z is a proper Zariski closed set of X.

We define the singular hermitian metric g with semipositive curvature current on $\mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^* A$, induced by the global sections $\pi^*(s_1), \cdots, \pi^*(s_N) \in H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^* A)$ (see [**Dem12**, Example 3.14]). We will show that g is smooth outside $\pi^{-1}(Z)$.

We will denote by $e_1^\vee, \ldots, e_r^\vee$ the dual frame on $E^\vee$. The corresponding holomorphic coordinate on $E^\vee$ are denoted by $(W_1, \cdots, W_r)$. We may regard $\pi^{-1}(U_i)$ as $U_i \times \mathbb{P}^{r-1}$. We take the chart $\{[W_1 : \cdots : W_r] \in \mathbb{P}^{r-1} \colon W_r \neq 0\}$. We will define the isomorphism by

$$\begin{array}{ccc} U_i \times \{W_r \neq 0\} & \to & U_i \times \mathbb{C}^{r-1} \\ (z, [W_1 : \cdots : W_r]) & \to & (z, \frac{W_1}{W_r}, \cdots, \frac{W_{r-1}}{W_r}) \end{array}$$

and we may regard $U_i \times \{W_r \neq 0\}$ as $U_i \times \mathbb{C}^{r-1}$. Put $\eta_l := \frac{W_l}{W_r}$ for $1 \leq l \leq r-1$ and $\eta_r := 1$. In this setting, we have

$$\mathcal{O}_{\mathbb{P}(E)}(-1)|_{U_i \times \mathbb{C}^{r-1}} = \{(z, \eta, \xi) \in U_i \times \mathbb{C}^{r-1} \times \mathbb{C}^r : \eta_i \xi_j = \eta_j \xi_i\}$$

and the local section

$$e_{\mathcal{O}_{\mathbb{P}(E)}(-1)}(z, (\eta_1, \cdots, \eta_{r-1})) := (z, (\eta_1, \cdots, \eta_{r-1}), (\eta_1, \cdots, \eta_{r-1}, 1)).$$

The local section $e_{\mathcal{O}_{\mathbb{P}(E)}(1)}$ of $\mathbb{P}(E)(1)$ is defined by the dual of $e_{\mathcal{O}_{\mathbb{P}(E)}(-1)}$. Then we have

$$\pi^*(s_j)|_{U_i \times \mathbb{C}^{r-1}} = \sum_{\alpha} f_{j\alpha}(z) \eta_1^{\alpha_1} \cdots \eta_{r-1}^{\alpha_{r-1}} 1^{\alpha_r} e^m_{\mathcal{O}_{\mathbb{P}(E)}(1)} \pi^*(e_A)$$

by using the isomorphism $H^0(X, \mathrm{Sym}^m(E) \otimes A) \simeq H^0(\mathbb{P}(E), \mathcal{O}_{\mathbb{P}(E)}(m) \otimes \pi^*(A))$.

Since g is defined by $1/(\sum_{1 \leq j \leq N} |\pi^*(s_j)|^2)$, g is described on $U_i \times \mathbb{C}^{r-1}$ by

$$H := \Big(\sum_{1 \leq j \leq N} | \sum_{\alpha} f_{j\alpha}(z) \eta_1^{\alpha_1} \cdots \eta_{r-1}^{\alpha_{r-1}} 1^{\alpha_r} |^2 \Big)^{-1}.$$

Therefore it is enough to show that $H^{-1}(z, \eta_1, \cdots, \eta_{r-1}) \neq 0$ for any $(z, \eta_1, \cdots, \eta_{r-1}) \in (U_i \setminus W) \times \mathbb{C}^{r-1}$. It is easily to check by the definition of Z and the standard linear algebra.

The second statement is also easily proved by the definition of Z. □

Corollary 4.2.3. Let X be a smooth projective n-dimensional variety, E be a holomorphic vector bundle of rank r on X and A be a line bundle on X. Assume there exist $m, b \in \mathbb{N}_{>0}$ and a point $x \in X$ such that $\mathrm{Sym}^{(m+r)b}(E) \otimes (A \otimes \det E^\vee)^b$ is globally generated at x. Then there exist a Griffiths semipositive singular hermitian metric h on $\mathrm{Sym}^m(E) \otimes A$ and a proper Zariski closed set $Z \subset X$ such that h is smooth outside Z.

Moreover if there exists a Zariski open set $U \subset X$ such that $\mathrm{Sym}^{(m+r)b}(E) \otimes (A \otimes \det E^\vee)^b$ is globally generated at x for any $x \in U$, we can take Z such that $Z \cap U = \varnothing$.

Proof. By Lemma 4.2.2 and dividing by b, there exist a singular hermitian metric g with semipositive curvature current on $\mathcal{O}_{\mathbb{P}(E)}(m+r) \otimes \pi^*(A \otimes \det E^\vee)$ and a proper Zariski closed set $Z \subset X$ such that g is smooth outside $\pi^{-1}(Z)$. From $\det E \simeq \pi_*(K_{\mathbb{P}(E)/X} \otimes \mathcal{O}_{\mathbb{P}(E)}(r))$, we have

$$\mathrm{Sym}^m(E) \otimes A \simeq \pi_*(K_{\mathbb{P}(E)/X} \otimes \mathcal{O}_{\mathbb{P}(E)}(m+r) \otimes \pi^*(A \otimes \det E^\vee))$$

and the inclusion morphism

$$\pi_*(K_{\mathbb{P}(E)/X} \otimes \mathcal{O}_{\mathbb{P}(E)}(m+r) \otimes \pi^*(A \otimes \det E^\vee) \otimes \mathcal{J}(g)) \to \pi_*(K_{\mathbb{P}(E)/X} \otimes \mathcal{O}_{\mathbb{P}(E)}(m+r) \otimes \pi^*(A \otimes \det E^\vee))$$

is generically isomorphism. By Theorem 1.4.5, $\mathrm{Sym}^m(E)\otimes A$ has a Griffiths semipositive singular hermitian metric h such that h is smooth outside Z (see [**HPS18**, Chapter 22]).

The proof of the second statement is same as above. $\square$

4.3. Proof of main theorems

In this section, we prove Theorem 4.1.2. First, we study a pseudo-effective vector bundle .

THEOREM 4.3.1. Let X be a smooth projective n-dimensional variety and E be a holomorphic vector bundle of rank r on X. The following are equivalent.

(A) E is pseudo-effective.
(B) There exists an ample line bundle A such that $\mathrm{Sym}^k(E)\otimes A$ has a Griffiths semipositive singular hermitian metric h_k for any $k\in\mathbb{N}_{>0}$. Moreover, for any $k\in\mathbb{N}_{>0}$, there exists a proper Zariski closed set $Z_k\subset X$ such that h_k is smooth outside Z_k.
(C) There exists an ample line bundle A such that $\mathrm{Sym}^k(E)\otimes A$ has a Griffiths semipositive singular hermitian metric h_k for any $k\in\mathbb{N}_{>0}$.

Moreover if E satisfies the condition (C), then E is weakly positive at any $x\in X\setminus \cup_{k\in\mathbb{N}_{>0}}\{z\in X\colon \nu(\det h_k,z)\geq 2)\}$.

PROOF. (A) $\Rightarrow$ (B). We take a point $x\in X$ such that E is weakly positive at x and take an ample line bundle A such that $A\otimes\det E^\vee$ is ample. For any $a\in\mathbb{N}_{>0}$, there exists $b\in\mathbb{N}_{>0}$ such that $\mathrm{Sym}^{(a+r)b}(E)\otimes(A\otimes\det E^\vee)^b$ is globally generated at x. By Corollary 4.2.3, the proof is complete.

(B) $\Rightarrow$ (C). Clear.

(C) $\Rightarrow$ (A). The proof will be divided into 3 steps.

Step 1. **Preliminary** We fix an ample line bundle H. By Siu's Theorem [**Dem12**, Corollary 13.3], the set $Z_k:=\{z\in X\colon \nu(\det h_k,z)\geq 2)\}$ is a proper Zariski closed set. We fix a point $x\in X\setminus\cup_k Z_k$. We take a local coordinate $(U;z_1,\cdots,z_n)$ near x. Let $\varphi=\eta(n+1)\log|z-x|^2$, where η is a cut-off function such that $\eta\equiv 1$ near x. Let h_H be a positive smooth hermitian metric on H. We take $b\in\mathbb{N}_{>0}$ such that

(1) $A^{-1}\otimes K_X^{-1}\otimes\det E^\vee\otimes H^b$ is ample, and
(2) $b\sqrt{-1}\Theta_{H,g_H}+\sqrt{-1}\partial\overline{\partial}\varphi\geq 0$ in the sense of current.

From $\mathrm{Sym}^{2ab}(E)\otimes H^{2b}\simeq\pi_*(\mathcal{O}_{\mathbb{P}(E)}(2ab)\otimes\pi^*H^{2b})$, it is enough to show that the restriction map

$$H^0(\mathbb{P}(E),\mathcal{O}_{\mathbb{P}(E)}(2ab)\otimes\pi^*H^{2b})\to H^0(\pi^{-1}(x),\mathcal{O}_{\mathbb{P}(E)}(2ab)\otimes\pi^*H^{2b}|_{\pi^{-1}(x)})$$

is surjective for any $a\in\mathbb{N}_{>0}$.

Step 2. **Taking a singular hermitian metric** From $\pi^*(\det E) \simeq K_{\mathbb{P}(E)/X} \otimes \mathcal{O}_{\mathbb{P}(E)}(r)$, we have

$$\mathcal{O}_{\mathbb{P}(E)}(2ab) \otimes \pi^* H^{2b} \simeq K_{\mathbb{P}(E)} \otimes (\mathcal{O}_{\mathbb{P}(E)}(2ab+r) \otimes \pi^* A) \otimes \pi^*(A^{-1} \otimes K_X^{-1} \otimes \det E^\vee \otimes H^b) \otimes \pi^*(H^b).$$

Since $\mathrm{Sym}^{2ab+r}(E) \otimes A$ has a Griffiths semipositive singular hermitian metric h_{2ab+r}, by Lemma 4.2.1, $\mathcal{O}_{\mathbb{P}(E)}(2ab+r) \otimes \pi^* A$ has a singular hermitian metric g_{2ab+r} with semipositive curvature current. By Skoda's theorem [**Dem12**, Lemma 5.6] and Lemma 4.2.1, there exist an open set $x \in V \subset X$ and a positive constant C such that

(1) $g_{2ab+r} \leq C \pi^*(\det h_{2ab+r})$ holds on $\pi^{-1}(V)$,
(2) $\det h_{2ab+r} \in L^1(V)$, and
(3) $\varphi = (n+1) \log |z - x|^2$ holds on V.

Since $A^{-1} \otimes K_X^{-1} \otimes \det E^\vee \otimes H^b$ is ample, there exists a smooth positive metric g_1 on $A^{-1} \otimes K_X^{-1} \otimes \det E^\vee \otimes H^b$.

We put $\widetilde{L} := (\mathcal{O}_{\mathbb{P}(E)}(2ab+r) \otimes \pi^* A) \otimes \pi^*(A^{-1} \otimes K_X^{-1} \otimes \det E^\vee \otimes H^b) \otimes \pi^*(H^b)$, $\widetilde{g} := g_{2ab+r} \pi^*(g_1 g_H^b)$, and $\psi := \frac{n}{n+1} \pi^* \varphi$. Then the following conditions hold.

(1) $K_{\mathbb{P}(E)} \otimes \widetilde{L} \simeq \mathcal{O}_{\mathbb{P}(E)}(2ab) \otimes \pi^* H^{2b}$.
(2) $\widetilde{g}$ is a singular hermitian metric with semipositive curvature current on $\widetilde{L}$.
(3) For any $\alpha \in [0,1]$, we have $\sqrt{-1}\Theta_{\widetilde{L},\widetilde{g}} + (1 + \frac{\alpha}{n})\sqrt{-1}\partial\overline{\partial}\psi \geq 0$ in the sense of current.

Step 3. **Global extension by an L^2 estimate** Fix a Kähler form $\omega_{\mathbb{P}(E)}$ on $\mathbb{P}(E)$. If necessarily we take V small enough, we may assume $\pi^{-1}(V)$ is biholomorphic on $V \times \mathbb{P}^{r-1}$. Therefore, there exists $s_V \in H^0(\pi^{-1}(V), K_{\mathbb{P}(E)} \otimes \widetilde{L})$ such that $s_V|_{\pi^{-1}(x)} = s$. We take a cut-off function ρ on V such that

(1) $\rho = 1$ near x, and
(2) $\inf_{supp(\overline{\partial}\rho)} \varphi > -\infty$.

We put $\widetilde{\rho} := \pi^*\rho$. We solve the global $\overline{\partial}$-equation $\overline{\partial}F = \overline{\partial}(\widetilde{\rho} s_V)$ with the weight $\widetilde{g}e^{-\psi}$.

First, we have

$$\|\widetilde{\rho} s_V\|^2_{\widetilde{g},\omega_{\mathbb{P}(E)}} = \int_{\pi^{-1}(V)} |\widetilde{\rho} s_V|^2_{\widetilde{g},\omega_{\mathbb{P}(E)}} dV_{\omega_{\mathbb{P}(E)},\mathbb{P}(E)} \leq C_1 \int_{\pi^{-1}(V)} |\pi^* \det h| dV_{\omega_{\mathbb{P}(E)},\mathbb{P}(E)} < +\infty,$$

where C_1 is some positive constant. Similarly, it is easy to check $\|\overline{\partial}(\widetilde{\rho} s_V)\|^2_{\widetilde{g},\omega_{\mathbb{P}(E)}} < +\infty$. Therefore $\overline{\partial}(\widetilde{\rho} s_V)$ gives rise to a cohomology class $[\overline{\partial}(\widetilde{\rho} s_V)]$ which is $[\overline{\partial}(\widetilde{\rho} s_V)] = 0$ in $H^1(\mathbb{P}(E), K_{\mathbb{P}(E)} \otimes \widetilde{L} \otimes \mathcal{J}(\widetilde{g}))$.

Second, we have

$$\begin{aligned}\|\overline{\partial}(\widetilde{\rho}s_V)\|^2_{\widetilde{g}e^{-\psi},\omega_{\mathbb{P}(E)}} &= \int_{\pi^{-1}(V)} |\overline{\partial}(\widetilde{\rho}s_V)|^2_{\widetilde{g}e^{-\psi},\omega_{\mathbb{P}(E)}} dV_{\omega_{\mathbb{P}(E)},\mathbb{P}(E)} \\ &\leq C_2 \int_{\pi^{-1}(supp(\overline{\partial}\rho))} |\pi^*(\det h)| e^{-\psi} dV_{\omega_{\mathbb{P}(E)},\mathbb{P}(E)} \\ &< +\infty,\end{aligned}$$

where C_2 is some positive constant. Therefore $\overline{\partial}(\widetilde{\rho}s_V)$ is a $\overline{\partial}$-closed $(n+r-1,1)$ form with $\widetilde{L}$ value which is square integrable the weight of $\widetilde{g}e^{-\psi}$.

By the injectivity theorem in [**CDM17**, Theorem 1.5], the natural morphism

$$H^1(\mathbb{P}(E), K_{\mathbb{P}(E)} \otimes \widetilde{L} \otimes \mathcal{J}(\widetilde{g}e^{-\psi})) \to H^1(\mathbb{P}(E), K_{\mathbb{P}(E)} \otimes \widetilde{L} \otimes \mathcal{J}(\widetilde{g}))$$

is injective. Since $[\overline{\partial}(\widetilde{\rho}s_V)] = 0$ in $H^1(\mathbb{P}(E), K_{\mathbb{P}(E)} \otimes \widetilde{L} \otimes \mathcal{J}(\widetilde{g}))$, we have $[\overline{\partial}(\widetilde{\rho}s_V)] = 0$ in $H^1(\mathbb{P}(E), K_{\mathbb{P}(E)} \otimes \widetilde{L} \otimes \mathcal{J}(\widetilde{g}e^{-\psi}))$. Hence we obtain a $(n+r-1,1)$ form F with $\widetilde{L}$ value which is square integrable with the weight $\widetilde{g}e^{-\psi}$ such that $\overline{\partial}F = \overline{\partial}(\widetilde{\rho}s_V)$.

We will show that $F|_{\pi^{-1}(x)} \equiv 0$. To obtain a contradiction, we assume $F(z) \neq 0$ for some point $z \in \pi^{-1}(x)$. We take an open set $x \in W \subset\subset V$, an open set $W' \subset \mathbb{P}^{r-1}$ and a positive constant C_3 such that $W \times W' \subset\subset \pi^{-1}(V)$ and $|F|^2_{\widetilde{g}} \geq C_3$ on W. Thus we have

$$\begin{aligned}\|F\|^2_{\widetilde{g}e^{-\psi},\omega_{\mathbb{P}(E)}} = \int_{\mathbb{P}(E)} |F|^2_{\widetilde{g}e^{-\psi},\omega_{\mathbb{P}(E)}} dV_{\omega_{\mathbb{P}(E)},\mathbb{P}(E)} &\geq \int_{W\times W'} |F|^2_{\widetilde{g}e^{-\psi},\omega_{\mathbb{P}(E)}} dV_{\omega_{\mathbb{P}(E)},\mathbb{P}(E)} \\ &\geq C_4 \int_{W\times W'} e^{-\psi} dV_{\omega_{\mathbb{P}(E)},\mathbb{P}(E)} \\ &\geq C_5 \int_{W\times W'} e^{-n\log|z-x|^2} dV_{\omega_{\mathbb{P}(E)},\mathbb{P}(E)} \\ &= +\infty,\end{aligned}$$

where C_4 and C_5 are some positive constant. This is a contradiction from $\|F\|^2_{\widetilde{g}e^{-\psi},\omega_{\mathbb{P}(E)}} < +\infty$.

Therefore we put $S := \widetilde{\rho}s_V - F \in H^0(\mathbb{P}(E), K_{\mathbb{P}(E)} \otimes \widetilde{L})$, then $S|_{\pi^{-1}(x)} = (\widetilde{\rho}s_V - F)|_{\pi^{-1}(x)} = s$, which completes the proof. Therefore, E is weakly positive at any $x \in X \setminus \cup_k Z_k$. □

By the same argument, we have the following Corollary.

Corollary 4.3.2. Let X be a smooth projective n-dimensional variety and E be a holomorphic vector bundle of rank r on X. The following are equivalent.

(A) E is weakly positive.

(B) There exist an ample line bundle A and a proper Zariski closed set $Z \subset X$ such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive singular hermitian metric h_k for any $k \in \mathbb{N}_{>0}$ and h_k is smooth outside Z.

(C) There exist an ample line bundle A and a proper Zariski closed set $Z \subset X$ such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive singular hermitian metric h_k for any $k \in \mathbb{N}_{>0}$ and $\cup_k \{z \in X \colon \nu(\det h_k, z) \geq 2\} \subset Z$.

Proof. (B) $\Rightarrow$ (C) is clear. By Theorem 4.3.1, we obtain (C) $\Rightarrow$ (A). We give a proof of (A) $\Rightarrow$ (B). By the definition, there exists a Zariski open set U such that E is a weakly positive at any $x \in U$. We take an ample line bundle A such that $A \otimes \det E^\vee$ is ample. Fix $a \in \mathbb{N}_{>0}$. For any $m \in \mathbb{N}_{>0}$, we define Z_m by the Zariski closed set of points $x \in X$ such that $\mathrm{Sym}^{(a+r)m} E \otimes (A \otimes \det E^\vee)^m$ is not globally generated at x. Then we obtain $b \in \mathbb{N}_{>0}$ such that $Z_b = \cap_{m \in \mathbb{N}_{>0}} Z_m$. Thus, $\mathrm{Sym}^{(a+r)b} E \otimes (A \otimes \det E^\vee)^b$ is globally generated at any $x \in X \setminus U$ by $Z_b \subset X \setminus U$. By Corollary 4.2.3, $\mathrm{Sym}^a E \otimes A$ has a Griffiths semipositive singular hermitian metric h and h is smooth on U, the proof is complete. □

The following corollary was already proved in [**PT18**]. We give an another proof.

Corollary 4.3.3. [**PT18**, Proposition 2.3.5]
Let X be a smooth projective variety and E be a holomorphic vector bundle on X. If E has a Griffiths semipositive singular hermitian metric h, then E is weakly positive at any $x \in \{z \in X \colon \nu(\det h, z) = 0\}$. In particular, E is pseudo-effective.

Proof. Since E has a Griffiths semipositive singular hermitian metric, $\mathrm{Sym}^k(E)$ also has a Griffiths semipositive singular hermitian metric $\mathrm{Sym}^k(h)$ for any $k \in \mathbb{N}_{>0}$ induced by h. Therefore, for any ample line bundle, $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive singular hermitian metric $\mathrm{Sym}^k(h)h_A$, where h_A is a smooth metric with positive curvature on A. Since we have

$$\cup_{k \in \mathbb{N}_{>0}} \{z \in X \colon \nu(\det \mathrm{Sym}^k(h)h_A, z) \geq 2)\} = \{z \in X \colon \nu(\det h, z) > 0\},$$

E is weakly positive at any $x \in \{z \in X \colon \nu(\det h, z) = 0\}$ by Theorem 4.3.1. □

Remark 4.3.4. By Hosono [**Hos17**, Example 5.4], there exists a nef vector bundle E_H such that E_H does not have a Griffiths semipositive singular hermitian metric. Therefore, pseudo-effective (weakly positive) does not always imply the existence of a Griffiths semipositive singular hermitian metric.

Next, we treat big vector bundles.

Corollary 4.3.5. Let X be a smooth projective n-dimensional variety and E be a holomorphic vector bundle of rank r on X. The following are equivalent.

(A) E is big.

(B) There exist $k \in \mathbb{N}_{>0}$, an ample line bundle A and a proper Zariski closed set $Z \subset X$ such that $\mathrm{Sym}^k(E) \otimes A^{-1}$ has a Griffiths semipositive singular hermitian metric h and h is smooth outside Z.

(C) There exist an ample line bundle A and $k \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^k(E) \otimes A^{-1}$ has a Griffiths semipositive singular hermitian metric h.

Proof. (A) $\Rightarrow$ (B). There exist an ample line bundle A and $b \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^b(E) \otimes A^{-1}$ is pseudo-effective. By Theorem 4.3.1, there exists an ample line bundle H such that $\mathrm{Sym}^{kb}(E) \otimes A^{-k} \otimes H$ has Griffiths semipositive singular hermitian metric h_k for any $k \in \mathbb{N}_{>0}$. Moreover, there exists a proper Zariski closed set Z_k such that h_k is smooth outside Z_k. Therefore we take $k \in \mathbb{N}_{>0}$ such that $A^k \otimes H^{-1}$ is ample, which completes the proof.

(B) $\Rightarrow$ (C). Clear.

(C) $\Rightarrow$ (A). For any $a \in \mathbb{N}_{>0}$, we have $\mathrm{Sym}^a(\mathrm{Sym}^k(E) \otimes A^{-1}) \otimes A$ has a Griffiths semipositive singular hermitian metric. By Theorem 4.3.1, $\mathrm{Sym}^k(E) \otimes A^{-1}$ is pseudo-effective, which completes the proof. □

Proof of Corollary 4.1.3 . By Corollary 4.3.5, there exist $k \in \mathbb{N}_{>0}$, an ample line bundle A and a proper Zariski closed set $Z \subset X$ such that $\mathrm{Sym}^k(T_X) \otimes A^{-1}$ has a Griffiths semipositive singular hermitian metric h and h is smooth on $X \setminus Z$. it is enough to show that $K_X^{-1}.C \geq n+1$ for any $x \in X \setminus Z$ and for any rational curve C through x by [**CMSB02**, Cor 0.4] since X is uniruled.

Fix $x \in X \setminus Z$ and a rational curve C through x . First we will show that $T_X|_C$ is ample. By [**Laz04b**, Theorem 6.4.15], it is enough to show that any quotient bundle of $T_X|_C$ has positive degree. Fix a quotient bundle G of $T_X|_C$ and a smooth positive metric h_A on A. $\mathrm{Sym}^k G$ has a quotient metric $h_{\mathrm{Sym}^k G}$ induced by $(hh_A)|_C$ on $\mathrm{Sym}^k(T_X|_C)$. Therefore $\det G$ has a singular hermitian metric $h_{\det G}$ with positive curvature current by some root of $\det h_{\mathrm{Sym}^k G}$. We have

$$\deg G = \int_C c_1(G) = \int_C c_1(\det G, h_{\det G}) = \int_C \frac{\sqrt{-1}}{2\pi}\Theta_{\det G, h_{\det G}} > 0,$$

thus $T_X|_C$ is ample. Since C is a rational curve, we obtain

$$T_X|_C \cong \mathcal{O}_C(a_1) \oplus \cdots \oplus \mathcal{O}_C(a_n),$$

where a_i is integer for any $1 \leq i \leq n$, $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_1 \geq 2$. Since $T_X|_C$ is ample, we have $a_n \geq 1$, therefore $K_X^{-1}.C = a_1 + \cdots + a_n \geq n+1$, which completes the proof.

Finally, we study a nef vector bundle.

Proposition 4.3.6 (cf. [**DPS94**] Theorem 1.12). Let X be a smooth projective variety and E be a holomorphic vector bundle of rank r on X. E is nef (i.e. $\mathcal{O}_{\mathbb{P}(E)}(1)$

is nef) iff there exists an ample line bundle A on X such that $\mathrm{Sym}^k(E)\otimes A$ has a Griffiths positive smooth hermitian metric for any $k\in\mathbb{N}_{>0}$.

Proof. ($\Rightarrow$) We assume E is nef. We take an ample line bundle H on X such that $H\otimes\det E^\vee$ is ample. There exists $N\in\mathbb{N}_{>0}$ such that $E\otimes(H\otimes\det E^\vee)^N$ is ample, that is, $\mathcal{O}_{\mathbb{P}(E)}(1)\otimes\pi^*(H\otimes\det E^\vee)^N$ is ample. For any $k\in\mathbb{N}_{>0}$, we have

$$\mathrm{Sym}^k(E)\otimes H\otimes(H\otimes\det E^\vee)^{N-1}\simeq\pi_*(K_{\mathbb{P}(E)/X}\otimes\mathcal{O}_{\mathbb{P}(E)}(k+r)\otimes\pi^*(H\otimes\det E^\vee)^N).$$

Since $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef, $\mathcal{O}_{\mathbb{P}(E)}(k+r)\otimes\pi^*(H\otimes\det E^\vee)^N$ is ample. Therefore, $\mathrm{Sym}^k(E)\otimes H\otimes(H\otimes\det E^\vee)^{N-1}$ has a Griffiths semipositive smooth hermitian metric for any $k\in\mathbb{N}_{>0}$. We put $A:=H^2\otimes(H\otimes\det E^\vee)^{N-1}$, the proof is complete.

($\Leftarrow$) Let $\mu_k\colon\mathbb{P}(E)\to\mathbb{P}(\mathrm{Sym}^k(E))=\mathbb{P}(\mathrm{Sym}^k(E)\otimes A)$ be a standard k-th Veronese embedding. Since $\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^k(E)\otimes A)}(1)$ is ample and $\mathcal{O}_{\mathbb{P}(E)}(k)\otimes\pi^*A=\mu_k^*(\mathcal{O}_{\mathbb{P}(\mathrm{Sym}^k(E)\otimes A)}(1))$, $\mathcal{O}_{\mathbb{P}(E)}(k)\otimes\pi^*A$ is ample for any $k\in\mathbb{N}_{>0}$. Therefore $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef. □

Example 4.3.7 (Cutkosky's criterion). Let X be a smooth projective variety and $L_1,\dots,L_r$ be holomorphic line bundles. The vector bundle E is defined by $E:=\oplus_{i=1}^r L_i$. By [**Laz04a**, Chapter 2.3.B], we have the following criterions.

(1) E is ample if and only if any L_i is ample.
(2) E is nef if and only if any L_i is nef.

We give a generalization of Cutkosky's criterion of big and pseudo-effective.

Lemma 4.3.8. (1) E is big if and only if any L_i is big.
(2) E is pseudo-effective if and only if any L_i is pseudo-effective. Moreover E is pseudo-effective if and only if E has a Griffiths semipositive singular hermitian metric.

Proof. (1) ($\Rightarrow$) If E is big, then there exist an ample line bundle A, $c\in\mathbb{N}_{>0}$ and a Zariski open set U such that $\mathrm{Sym}^c(E)\otimes A^{-1}$ is globally generated at any $x\in U$. For any $1\le i\le r$, $L_i^{\otimes c}\otimes A^{-1}$ is globally generated at any $x\in U$. Therefore L_i is big.

(1) ($\Leftarrow$) Let A be an ample line bundle and h_A be a smooth metric with positive curvature on A such that $\omega=\sqrt{-1}\Theta_{A,h_A}$ is a Kähler form on X. Since L_i is big, there exist a singular hermitian metric h_i and positive number ϵ_i such that $\sqrt{-1}\Theta_{L_i,h_i}\ge\epsilon_i\omega$. We define a singular hermitian metric h on E by $h=\oplus_{i=1}^r h_i$. We take $c\in\mathbb{N}_{>0}$ such that $\min_{1\le i\le r}\epsilon_i>2/c$. Then $\mathrm{Sym}^c(E)\otimes A^{-1}$ has a Griffiths semipositive singular hermitian metric $\mathrm{Sym}^c(h)h_A^{-1}$, which completes the proof.

(2) ($\Rightarrow$) We fix $x\in X$ such that E is weakly positive at x. For any ample line bundle A and $a\in\mathbb{N}_{>0}$ there exists $b\in\mathbb{N}_{>0}$ such that $\mathrm{Sym}^{ab}(E)\otimes A^{\otimes b}$ is globally generated at x, and consequently $L_i^{\otimes ab}\otimes A^b$ is globally generated at x for any $1\le i\le r$, . Therefore L_i is pseudo-effective.

(2) ($\Leftarrow$) Since L_i is pseudo-effective, L_i has a singular hermitian metric h_i with semipositive curvature current. We put $h = \oplus_{i=1}^r h_i$, which is a Griffiths semipositive singular hermitian metric on E. Therefore by Corollary 4.3.3, E is pseudo-effective. □

4.4. On the case of torsion-free coherent sheaves

THEOREM 4.4.1. Let X be a smooth projective variety and $\mathcal{F} \neq 0$ be a torsion-free coherent sheaf on X.

(1) $\mathcal{F}$ is pseudo-effective iff there exists an ample line bundle A such that $\mathrm{Sym}^k(\mathcal{F})^{\vee\vee} \otimes A$ has a Griffiths semipositive singular hermitian metric for any $k \in \mathbb{N}_{>0}$.
(2) $\mathcal{F}$ is weakly positive iff there exist an ample line bundle A and a Zariski open set $U \subset X$ such that $\mathrm{Sym}^k(\mathcal{F})^{\vee\vee} \otimes A$ has a Griffiths semipositive singular hermitian metric h_k for any $k \in \mathbb{N}_{>0}$ and the Lelong number of h_k at x is less than 2 for any $x \in U$ and any $k \in \mathbb{N}_{>0}$.
(3) $\mathcal{F}$ is big iff there exist an ample line bundle A and $k \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^k(\mathcal{F})^{\vee\vee} \otimes A^{-1}$ has a Griffiths semipositive singular hermitian metric.

PROOF. We put $E := \mathcal{F}|_{X_\mathcal{F}}$, which is a vector bundle on $X_\mathcal{F}$. Since $\mathrm{Sym}^k(\mathcal{F})^{\vee\vee} \otimes A$ is reflexive for any $k \in \mathbb{N}_{>0}$, we have

$$H^0(X_\mathcal{F}, \mathrm{Sym}^k(E) \otimes A) \simeq H^0(X, \mathrm{Sym}^k(\mathcal{F})^{\vee\vee} \otimes A). \tag{4.4.1}$$

(1)($\Rightarrow$). We assume that $\mathcal{F}$ is pseudo-effective We take a point $x \in X_\mathcal{F}$ such that $\mathcal{F}$ is weakly positive at x and take an ample line bundle A such that $A \otimes (\det \mathcal{F})^\vee$ is ample. For any $k \in \mathbb{N}_{>0}$, there exists $b \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^{kb}(\mathcal{F})^{\vee\vee} \otimes (A \otimes (\det \mathcal{F})^\vee)^b$ is globally generated at x. Therefore by 4.4.1, the vector bundle $\mathrm{Sym}^{kb}(E) \otimes (A \otimes \det E^\vee)^b$ on $X_\mathcal{F}$ is globally generated at x.

By the argument of Corollary 4.2.3, there exists a Griffiths semipositive singular hermitian metric h on $(\mathrm{Sym}^k(\mathcal{F})^{\vee\vee} \otimes A)|_{X_\mathcal{F}} = \mathrm{Sym}^k(E) \otimes A$ (h is smooth outside a countable union of proper Zariski closed sets). From $\mathrm{codim}(X \setminus X_\mathcal{F}) \geq 2$, h extends to $X_{(\mathrm{Sym}^k(\mathcal{F})^{\vee\vee} \otimes A)}$. Therefore $\mathrm{Sym}^k(\mathcal{F})^{\vee\vee} \otimes A$ has a Griffiths semipositive singular hermitian metric h.

(1)($\Leftarrow$). From $X_\mathcal{F} \subset X_{(\mathrm{Sym}^k(\mathcal{F})^{\vee\vee} \otimes A)}$ for any $k \in \mathbb{N}_{>0}$, the vector bundle $\mathrm{Sym}^k(E) \otimes A$ on $X_\mathcal{F}$ has a Griffiths semipositive singular hermitian metric on for any $k \in \mathbb{N}_{>0}$. By using the argument of the proof of (C) $\Rightarrow$ (A) in Theorem 4.3.1, there exists a point $x \in X_\mathcal{F}$ such that E is weakly positive at x. (We use Demailly's L^2 estimate on a complete Kähler manifold [**Dem82**, Theorem 5.1] instead of the injectivity theorem in [**CDM17**] since $X_\mathcal{F}$ may not be a weakly 1-complete. See also [**PT18**, Theorem 2.5.3]). Hence $\mathcal{F}$ is pseudo-effective.

(2) The proof is similar to (1) and the proof of Theorem 4.3.2. (We can take h_k such that h_k is smooth on $U \cap X_\mathcal{F}$ if $\mathcal{F}$ is weakly positive at any point $x \in U$.)

(3)($\Rightarrow$). The proof is similar to the proof of ($\Rightarrow$) in (1).

($\Leftarrow$). By the Corollary 4.3.3 and (1), $\mathrm{Sym}^{2k}(\mathcal{F})^{\vee\vee} \otimes A^{-1}$ is pseudo-effective. □

We give an application of Theorem 4.4.1. If a torsion-free coherent sheaf $\mathcal{F}$ has a Griffiths semipositive singular hermitian metric h, then $\mathcal{F}$ is pseudo-effective. Moreover if there exists a Zariski open set U such that h is continuous on $U \cap X_{\mathcal{F}}$, then $\mathcal{F}$ is weakly positive at any $x \in U \cap X_{\mathcal{F}}$. Let $f\colon X \to Y$ be a surjective morphism between smooth projective varieties. Then for any $m \in \mathbb{N}_{>0}$, $f_*(mK_{X/Y})$ has a Griffiths semipositive singular hermitian metric h_{mNS} such that h_{mNS} is continuous over the regular locus of f by [**PT18**, Theorem 1.1] or [**HPS18**, Theorem 27.1]. Therefore, $f_*(mK_{X/Y})$ is weakly positive at any point in the regular locus of f. This fact was already proved in [**PT18**, Theorem 5.1.2].

CHAPTER 5

On projective manifolds with pseudo-effective tangent bundle

Abstract. In this paper, we develop the theory of singular hermitian metrics on vector bundles. As an application, we give a structure theorem of a projective manifold X with pseudo-effective tangent bundle: X admits a smooth fibration $X \to Y$ to a flat projective manifold Y such that its general fiber is rationally connected. Moreover, by applying this structure theorem, we classify all the minimal surfaces with pseudo-effective tangent bundle and study general non-minimal surfaces, which provide examples of (possibly singular) positively curved tangent bundles. This is a joint work with Genki Hosono and Shin-ichi Matsumura.

5.1. Introduction

The structure theorem for compact Kähler manifolds with semi-positive bisectional curvature was established by Howard-Smyth-Wu and Mok in [**HSW81**] and [**Mok88**] after the Frankel conjecture (resp. the Hartshorne conjecture) had been solved by Siu-Yau (resp. Mori) in [**SY80**] (resp. [**Mor79**]). As an algebraic analog of semi-positive bisectional curvature, Campana-Peternell and Demailly-Peternell-Schneider generalized the structure theorem of Howard-Smyth-Wu to nef tangent bundles in [**CP91**] and [**DPS94**], and further they classified the surfaces and the 3-folds with nef tangent bundle. (see [**CP91**] and [**MOS+15**] for the Campana-Peternell conjecture).

It is of interest to consider pseudo-effective tangent bundles as a natural generalization of the above structure results. The theory of singular hermitian metrics on vector bundles, which has been rapidly developed, is a crucial tool to understand pseudo-effective vector bundles. Therefore, in this paper, we first develop the theory of singular hermitian metrics on vector bundles (more generally torsion free sheaves). As one of the main applications, we obtain the following structure theorem for projective manifolds with pseudo-effective tangent bundle (and also for compact Kähler manifolds, see Theorem 5.2.12).

Theorem 5.1.1. Let X be a projective manifold with pseudo-effective tangent bundle. Then X admits a (surjective) morphism $\phi : X \to Y$ with connected fiber to a smooth manifold Y with the following properties:

(1) The morphism $\phi : X \to Y$ is smooth (that is, all the fibers are smooth).

(2) The image Y admits a finite étale cover $A \to Y$ by an abelian variety A.
(3) A general fiber F of ϕ is rationally connected.
(4) A general fiber F of ϕ also has the pseudo-effective tangent bundle.

Moreover, if we further assume that T_X admits a positively curved singular hermitian metric, then we have:

(5) The standard exact sequence of tangent bundles
$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow \phi^* T_Y \longrightarrow 0$$
splits.
(6) The morphism $\phi : X \to Y$ is locally trivial (that is, all the fibers are smooth and isomorphic).

Theorem 5.1.1 is based on the argument in [**Mat18b**] and the theory of singular hermitian metrics on vector bundles developed in this paper. In particular, Theorem 5.1.2, Theorem 5.1.3, and Theorem 5.1.4 play an important role in the proof. Theorem 5.1.2, which can be seen as a generalization of [**CM**], gives a characterization of numerically flat vector bundles in terms of pseudo-effectivity. The proof depends on the theory of admissible hermitian-Einstein metrics in [**BS94**]. Theorem 5.1.3 and Theorem 5.1.4 were proved in [**HPS18**] under the stronger assumption of the minimal extension property. Our contribution is to remove this assumption, which enables us to use the notion of singular hermitian metrics flexibly.

Theorem 5.1.2. Let X be a projective manifold and let $\mathcal{E}$ be a reflexive coherent sheaf on X. If $\mathcal{E}$ is pseudo-effective and the first Chen class $c_1(\mathcal{E})$ is zero, then $\mathcal{E}$ is locally free on X and numerically flat.

Theorem 5.1.3. Let E be a vector bundle with positively curved (singular) hermitian metric on a (not necessarily compact) complex manifold X. Let
$$0 \to S \to E \to Q \to 0$$
be an exact sequence by vector bundles S and Q on X. If the first Chern class $c_1(Q)$ is zero, the above exact sequence splits.

Theorem 5.1.4. Let X be a compact Kähler manifold and let
$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0$$
be an exact sequence of reflexive coherent sheaves $\mathcal{S}$, $\mathcal{E}$, and $\mathcal{Q}$ on X. If $\mathcal{E}$ admits a positively curved (singular) hermitian metric and the first Chen class $c_1(\mathcal{Q}) = 0$, then we have:

(1) $\mathcal{Q}$ is locally free and hermitian flat.
(2) $\mathcal{E} \to \mathcal{Q}$ is a surjective bundle morphism on $X_{\mathcal{E}}$.
(3) The above exact sequence splits on X.

Here $X_{\mathcal{E}}$ is the maximal Zariski open set where $\mathcal{E}$ is locally free.

It is natural to attempt to classify all the surfaces X with pseudo-effective tangent bundle, as an application of Theorem 5.1.1. In the case of the tangent bundle being nef, a surface X has no curve with negative self-intersection, and thus X is always minimal. However, a surface X with pseudo-effective tangent bundle may not be minimal, which is one of the difficulties to classify them. In this paper, we classify all the minimal surfaces (see subsection 5.3.1 for more detail):

THEOREM 5.1.5. We have:

(1) If a (not necessarily minimal) ruled surface $X \to C$ has the pseudo-effective tangent bundle T_X, then the base C is the projective line $\mathbb{P}^1$ or an elliptic curve.
(2) Further, in the case of C being an elliptic curve, the surface X is a minimal ruled surface (that is, the ruling $X \to C$ is a smooth morphism).
(3) Conversely, any minimal ruled surfaces $X \to C$ over an elliptic curve and over the projective line $C = \mathbb{P}^1$ have the pseudo-effective tangent bundle T_X.

Moreover, we study the remaining problem (that is, the classification for blow-ups of Hirzebruch surfaces) in detail. These studies provide interesting examples of pseudo-effective or singular positively curved vector bundles.

5.2. Proof of the main results

DEFINITION 5.2.1. A torsion free coherent sheaf $\mathcal{E}$ on a compact complex manifold X is said to be *pseudo-effective* if for any integer $m > 0$ there exists a singular hermitian metric h_m on $\mathrm{Sym}^m \mathcal{E}$ such that

$$\sqrt{-1}\partial\overline{\partial} \log |u|^2_{h_m^\vee} \geq -\omega \text{ on } X_{\mathcal{E}}$$

for any local holomorphic section u of $\mathrm{Sym}^m \mathcal{E}$. Here ω is a fixed hermitian form on X.

The above definition is equivalent to the definition (5) below when X is a projective manifold (see [**Iwa18b**, Theorem 1.3]). This section is devoted to the proof of the main results.

5.2.1. Numerically flat vector bundles.

In this subsection, we give a proof for Theorem 5.1.2 after we prove Lemma 5.2.2 and Lemma 5.2.4 for preparation. Lemma 5.2.2, which easily follows from the result of [**DPS94**, Proposition 1.16], is quite useful and often used in this paper.

LEMMA 5.2.2. Let X be a projective manifold and let $\mathcal{E}$ be an almost nef torsion free coherent sheaf on X.

(1) Any non-zero section $\tau \in H^0(X, \mathcal{E}^\vee)$ is non-vanishing on $X_{\mathcal{E}}$.
(2) Let $\mathcal{S}$ be a reflexive coherent sheaf such that $\det \mathcal{S}$ is pseudo-effective and let $0 \to \mathcal{S} \to \mathcal{E}^\vee$ be an injective sheaf morphism. Then $\mathcal{S}$ is locally free on $X_{\mathcal{E}}$ and the above morphism is an injective bundle morphism on $X_{\mathcal{E}}$.

PROOF. In [**DPS94**], the same conclusion was proved for nef vector bundles. We denote by Z a countable union of proper subvarieties of X satisfying the definition of almost nef sheaves. We may assume that $X \setminus X_{\mathcal{E}} \subset Z$ by adding the subvariety $X \setminus X_{\mathcal{E}}$ into Z.

(1) Let $\tau \in H^0(X, \mathcal{E}^\vee)$ be a non-zero section. For an arbitrary point $p \in X_{\mathcal{E}}$, by taking a complete intersection of ample hypersurfaces, we construct a curve C passing through p such that $C \not\subset Z$. We may assume that $C \subset X_{\mathcal{E}}$ by $\operatorname{codim}(X \setminus X_{\mathcal{E}}) \geq 2$. Then $\mathcal{E}|_C$ is a nef vector bundle thanks to $C \subset X_{\mathcal{E}}$, and thus it follows that the non-zero section $\tau|_C$ is non-vanishing from [**DPS94**, Proposition 1.16]. In particular, the section τ is non-vanishing at p.

(2) Following the argument in [**DPS94**], we obtain the non-zero section

$$\tau \in H^0(X, \Lambda^p \mathcal{E}^\vee \otimes \det \mathcal{S}^\vee)$$

from the induced morphism $\det \mathcal{S} \to \Lambda^p \mathcal{E}^\vee$. Here $p := \operatorname{rank} \mathcal{S}$. We remark that $\Lambda^p \mathcal{E} \otimes \det \mathcal{S}$ is also almost nef by the assumption on $\mathcal{S}$. Hence, by applying the first conclusion and [**DPS94**, Lemma 1.20] to τ, we can obtain the desired conclusion. □

LEMMA 5.2.3. Let X be a compact complex manifold and let $\mathcal{E}$ be a pseudo-effective torsion free coherent sheaf on X. Then the same conclusion as in Lemma 5.2.2 holds.

PROOF OF LEMMA 5.2.3. We will prove only the conclusion (1). For the metric h_m on $\operatorname{Sym}^m \mathcal{E}$ satisfying the property in Definition 5.2.1, we consider the function f_m on X defined by

$$f_m := \frac{1}{m} \log |\tau^m|_{h_m^\vee}.$$

By the construction of h_m, we have

$$\sqrt{-1}\partial\overline{\partial} f_m \geq -\frac{1}{m}\omega,$$

and thus its weak limit (after we take a subsequence) should be zero. On the other hand, when we assume τ has the zero point at some point $p \in X_{\mathcal{E}}$, it can be shown that the Lelong number of f_m is greater than or equal to one. This is a contradiction to the fact that the weak limit is zero. Indeed, the section τ^m can be locally written as $\tau^m = \sum_I \tau_I e_I$. Here $\{e_i\}_{i=1}^r$ is a local frame of $\mathcal{E}$, I is a multi-index of degree m, and $e_I := \prod_{i \in I} e_i$. It follows that the holomorphic function τ_I has the multiplicity $\geq m$ at p from $\tau = 0$ at $p \in X_{\mathcal{E}}$. It can be seen that $|\langle e_I, e_J \rangle_{h_m^\vee}|$ is bounded since $\log |u|_{h_m^\vee}$ is almost psh for any local section u (for example see [**PT18**, Lemma 2.2.4]). Hence we can easily check that

$$|\tau^m|_{h_m^\vee} \leq C \sum_I |\tau_I|.$$

This implies that the Lelong number of f_m is greater than or equal to one. □

LEMMA 5.2.4. Let X be a projective manifold and E be a vector bundle on X. Let X_0 be a Zariski open set in X with $\operatorname{codim}(X \setminus X_0) \geq 2 + i$. Then the morphism induced by the restriction

$$H^j(X, E) \to H^j(X_0, E)$$

is an isomorphism for any $j \leq i$.

PROOF. The proof is given by the standard argument in terms of ample hypersurfaces and the induction on dimension. □

Theorem 5.2.5, which is a slight generalization of [**CM**], heavily depends on the theory of admissible hermitian-Einstein metrics developed in [**BS94**].

THEOREM 5.2.5 (=Theorem 5.1.2, cf. [**CM**]). Let X be a projective manifold and let $\mathcal{E}$ be a reflexive coherent sheaf. If $\mathcal{E}$ is pseudo-effective and the first Chen class $c_1(\mathcal{E})$ is zero, then $\mathcal{E}$ is locally free and numerically flat.

PROOF OF THEOREM 5.2.5. The induction on the rank r of $\mathcal{E}$ will give the proof. Reflexive coherent sheaves of rank one are always line bundles (see [**Har80**]), and thus the conclusion is obvious in the case of $r = 1$. It is not so difficult to check the numerical flatness of $\mathcal{E}$ if $\mathcal{E}$ is shown to be locally free (see the proof in [**DPS94**, Theorem 1.18] or the argument below). We will focus on the proof of local freeness.

In the proof, we fix an ample line bundle A on X. In the case of $r > 1$, we take a coherent subsheaf $\mathcal{S}$ with the minimal rank among coherent subsheaves of $\mathcal{E}$ satisfying that $\int_X c_1(\mathcal{S}) \cdot c_1(A)^{n-1} \geq 0$. We may assume that $\mathcal{S}$ is reflexive by taking the double dual if necessary. Now we consider the following exact sequence of sheaves:

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} := \mathcal{E}/\mathcal{S} \to 0. \tag{5.2.1}$$

The quotient sheaf $\mathcal{Q} := \mathcal{E}/\mathcal{S}$ is pseudo-effective. In particular, the first Chern class $c_1(\mathcal{Q})$ is also pseudo-effective. On the other hand, we have

$$0 = c_1(\mathcal{E}) = c_1(\mathcal{S}) + c_1(\mathcal{Q}).$$

Then it follows that $c_1(\mathcal{S}) = c_1(\mathcal{Q}) = 0$ since $c_1(\mathcal{Q})$ is pseudo-effective and we have

$$\int_X c_1(\mathcal{Q}) \cdot c_1(A)^{n-1} = -\int_X c_1(\mathcal{S}) \cdot c_1(A)^{n-1} \leq 0.$$

By applying Lemma 5.2.2 to $\mathcal{Q}^\vee \to \mathcal{E}^\vee$, we can see that $\mathcal{Q}$ (and thus $\mathcal{S}$) is a vector bundle on $X_\mathcal{E}$ and the above morphism is a bundle morphism on $X_\mathcal{E}$.

We first consider the case where the rank of $\mathcal{S}$ is equal to $r = \operatorname{rank} \mathcal{E}$. In this case, we obtain $\mathcal{S} = \mathcal{E}$. Indeed, it follows that $\mathcal{S} \cong \mathcal{E}$ on $X_\mathcal{E}$ since the bundle morphism $\mathcal{S} \to \mathcal{E}$ on $X_\mathcal{E}$ is an isomorphism. Then we can easily check $\mathcal{S} = \mathcal{E}$ by the reflexivity and $\operatorname{codim}(X \setminus X_\mathcal{E}) \geq 3$. Further we can prove that

$$\int_X c_2(\mathcal{E}) \cdot c_1(A)^{n-2} = 0.$$

Indeed, for a surface $S := H_1 \cap H_2 \cap \cdots \cap H_{n-2}$ in X constructed by general members H_i of a complete linear system A, it follows that $\mathcal{E}|_S$ is a pseudo-effective vector bundle from $\mathrm{codim}(X \setminus X_{\mathcal{E}}) \geq 3$. Hence $\mathcal{E}|_S$ is numerically flat on S, and thus $c_2(\mathcal{E}|_S) = 0$ (see [**DPS94**] or [**CH17**, Corollary 2.12]). We can easily check that

$$\int_X c_2(\mathcal{E}) \cdot c_1(A)^{n-2} = \int_S c_2(\mathcal{E}|_S) = 0.$$

By the assumption of $c_1(\mathcal{E}) = 0$ and the result of [**BS94**, Corollary 3], we can conclude that $\mathcal{E}$ is a hermitian flat vector bundle on X from the stability of the reflexive sheaf $\mathcal{S} = \mathcal{E}$. Therefore $\mathcal{E}$ is locally free and numerically flat.

It remains to consider the case of $\mathrm{rank}\, \mathcal{S} < \mathrm{rank}\, \mathcal{E}$. In this case, we consider the surjective bundle morphism

$$\Lambda^{m+1}\mathcal{E} \otimes \det \mathcal{Q}^{\vee} \to \mathcal{S}$$

on $X_{\mathcal{E}}$. By $\mathrm{codim}(X \setminus X_{\mathcal{E}}) \geq 3$ and $c_1(\mathcal{Q}) = 0$, the reflexive sheaf $\mathcal{S}$ is pseudo-effective. Therefore we can conclude that $\mathcal{S}$ is a numerically flat vector bundle on X by the induction hypothesis.

On the other hand, the sheaf $\mathcal{Q}$ itself may not be a vector bundle, but, the reflexive hull $\mathcal{Q}^{\vee\vee}$ is a vector bundle on X by the induction hypothesis. The extension class obtained from the exact sequence (5.2.1) on $X_{\mathcal{E}}$ can be extended to the extension class (defined on X) of $\mathcal{S}$ and $\mathcal{Q}^{\vee\vee}$ by Lemma 5.2.4. The extended class determines the vector bundle whose restriction to $X_{\mathcal{E}}$ corresponds to $\mathcal{E}$. This implies that $\mathcal{E}$ is a vector bundle by the reflexivity of $\mathcal{E}$. □

5.2.2. Splitting theorem for positively curved vector bundles. In this subsection, we prove Theorem 5.1.3 and Theorem 5.1.4.

LEMMA 5.2.6. *Let $\mathcal{Q}$ be a reflexive coherent sheaf on a compact complex manifold X. If $\mathcal{Q}$ admits a positively curved singular hermitian metric $g_{\mathcal{Q}}$ and $c_1(\mathcal{Q}) = 0$, then we have:*

(1) *$(\mathcal{Q}, g_{\mathcal{Q}})$ is hermitian flat on $X_{\mathcal{Q}}$.*
(2) *If we further assume that X is Kähler, then $\mathcal{Q}$ is a locally free sheaf on X and $g_{\mathcal{Q}}$ extends to a hermitian flat metric on X.*

PROOF. (1) We follow the argument in [**CP17**]. The following lemma proved by Raufi [**Rau15**] is essential:

LEMMA 5.2.7 ([**Rau15**, Thm 1.6]). *Let E be a holomorphic vector bundle and h_E be a positively curved singular hermitian metric on E. If the induced metric $\det h_E$ on the determinant bundle $\det E$ is non-singular (that is, smooth metric), then the curvature current $\sqrt{-1}\Theta_{h_E}$ of h_E is well-defined as an $\mathrm{End}(E)$-valued $(1,1)$-form with measure coefficients.*

In our situation $\det g_{\mathcal{Q}}$ is a positively curved singular hermitian metric on the determinant bundle $\det \mathcal{Q}$. By $c_1(\mathcal{Q}) = 0$, the curvature $\sqrt{-1}\Theta_{\det g_{\mathcal{Q}}}$ of $\det g_{\mathcal{Q}}$ is identically zero on $X_{\mathcal{Q}}$. In particular, it can be seen that $\det g_{\mathcal{Q}}$ is non-singular. Then, by Raufi's result, the curvature current $\sqrt{-1}\Theta = \sqrt{-1}\Theta_{g_{\mathcal{Q}}}$ of $g_{\mathcal{Q}}$ is well-defined on $X_{\mathcal{Q}}$.

We locally write the curvature $\sqrt{-1}\Theta$ as

$$\sqrt{-1}\Theta = \sum_{j,k,\alpha,\beta} \mu_{j\overline{k}\alpha\overline{\beta}} dz^j \wedge d\overline{z}^k e_\alpha \otimes e_\beta^\vee,$$

where $(z_1, \ldots, z_n)$ denotes a local coordinate and $e_1, \ldots, e_r$ denotes a local frame of $\mathcal{Q}$. Then, by $0 = \sqrt{-1}\Theta_{\det g_{\mathcal{Q}}} = \sqrt{-1}tr\Theta_{g_{\mathcal{Q}}}$, we obtain

$$\sum_{j,k}\sum_{\alpha} \mu_{j\overline{k}\alpha\overline{\alpha}} dz^j \wedge d\overline{z}^k = 0.$$

Since $g_{\mathcal{Q}}$ is positively curved,

$$\sum_{j,k} \mu_{j\overline{k}\alpha\overline{\alpha}} dz^j \wedge d\overline{z}^k \geq 0$$

for every α. Then we have that $\mu_{j\overline{k}\alpha\overline{\alpha}} = 0$ for every j, k, α.

For every α and β, we have that

$$\mathrm{Re}(\xi^\alpha \overline{\xi}^\beta \sum_{j,k} \mu_{j\overline{k}\alpha\overline{\beta}} v^j \overline{v}^k) \geq 0.$$

From this we can conclude that $\mu_{j\overline{k}\alpha\overline{\beta}} = 0$ for every j, k, α, β and thus $\sqrt{-1}\Theta_{g_{\mathcal{Q}}} = 0$.

(2) It follows that $\mathcal{Q}$ is polystable from (1) and [**BS94**, Theorem 3]. We have $\mathrm{codim}(X \setminus X_{\mathcal{Q}}) \geq 3$ and $(\mathcal{Q}, g_Q)$ is hermitian flat on $X_{\mathcal{Q}}$. Hence it can be shown that $c_1(\mathcal{Q}) = 0$ and $c_2(\mathcal{Q}) = 0$. We can see that $\mathcal{Q}$ is actually locally free and hermitian flat by [**BS94**, Theorem 4]. □

We prepare the following lemma for the proof of Theorem 5.1.3.

LEMMA 5.2.8. Let (E, h) be a hermitian flat vector bundle on a complex manifold X. Then for any point $x \in X$ and a basis $e_{1,x}, \ldots, e_{r,x}$ on the fiber E_x, there exists a local holomorphic frame $e_1, \ldots, e_r$ near x such that $e_j(x) = e_{j,x}$ and $\langle e_i, e_j \rangle_h$ is constant.

PROOF. Let D be the Chern connection associated to (E, h). Then, by flatness, we can take a local frame $\{e_j\}$ around x such that $De_j \equiv 0$. We can assume that $e_j(x) = e_{j,x}$. Since D is compatible with h, we have that $d\langle e_i, e_j \rangle_h = \{De_i, e_j\}_h + \{e_i, De_j\}_h = 0$, thus $\langle e_i, e_j \rangle_h$ is constant. Moreover, taking the $(0, 1)$-part of $De_j \equiv 0$, we obtain that $\bar{\partial} e_j \equiv 0$, which shows that e_j is holomorphic. □

THEOREM 5.2.9. (=Theorem 5.1.3) Let E be a vector bundle with positively curved (singular) hermitian metric g on a (not necessarily compact) complex manifold X. Let

$$0 \to S \to E \to Q \to 0$$

be an exact sequence of vector bundles on X. If the first Chern class $c_1(Q)$ is zero, the above exact sequence splits.

PROOF OF THEOREM 5.2.9. The following proof is a generalization of [**Hos17**, Theorem 5.1]. We will work on dual bundles. By taking the dual, we have the following exact sequence

$$0 \to Q^\vee \to E^\vee \to S^\vee \to 0. \tag{5.2.2}$$

Then we have a negatively curved singular hermitian metric $h^\vee$ whose restriction to $Q^\vee$ is flat by (the dual of) Lemma 5.2.6 (1). Therefore, by Lemma 5.2.8, we can take a holomorphic orthonormal frame $(\kappa_1^\alpha, \ldots, \kappa_q^\alpha)$ of $Q^\vee$ on a small open set U^α. Let ϵ_j^α be the image of κ_j^α in $E^\vee$. Take $\epsilon_{q+1}^\alpha, \ldots, \epsilon_{q+s}^\alpha$ such that $(\epsilon_1^\alpha, \ldots, \epsilon_{q+s}^\alpha)$ is a local frame of $E^\vee$. Let σ_j^α be the image of ϵ_j^α in $S^\vee$. We remark that $(\sigma_{q+1}^\alpha, \ldots, \sigma_{q+s}^\alpha)$ is a local frame of $S^\vee$. We will write the transition function of $Q^\vee$ and $S^\vee$ as follows:

$$\begin{aligned}
\kappa_1^\alpha &= \Phi_{1,1}^{Q^\vee,\alpha\beta}\kappa_1^\beta + \cdots + \Phi_{1,q}^{Q^\vee,\alpha\beta}\kappa_q^\beta, \\
&\vdots \\
\kappa_q^\alpha &= \Phi_{q,1}^{Q^\vee,\alpha\beta}\kappa_1^\beta + \cdots + \Phi_{q,q}^{Q^\vee,\alpha\beta}\kappa_q^\beta, \\
\\
\sigma_{q+1}^\alpha &= \Phi_{q+1,q+1}^{S^\vee,\alpha\beta}\sigma_{q+1}^\beta + \cdots + \Phi_{q+1,q+s}^{S^\vee,\alpha\beta}\sigma_{q+s}^\beta, \\
&\vdots \\
\sigma_{q+s}^\alpha &= \Phi_{q+s,q+1}^{S^\vee,\alpha\beta}\sigma_{q+1}^\beta + \cdots + \Phi_{q+s,q+s}^{S^\vee,\alpha\beta}\sigma_{q+s}^\beta.
\end{aligned}$$

The transition functions for $E^\vee$ can be written as

$$\begin{aligned}
\epsilon_1^\alpha &= \Phi_{11}^{Q^\vee,\alpha\beta}\epsilon_1^\beta + \cdots + \Phi_{1q}^{Q^\vee,\alpha\beta}\epsilon_q^\beta, \\
&\vdots \\
\epsilon_q^\alpha &= \Phi_{q1}^{Q^\vee,\alpha\beta}\epsilon_1^\beta + \cdots + \Phi_{qq}^{Q^\vee,\alpha\beta}\epsilon_q^\beta, \\
\epsilon_{q+1}^\alpha &= \Phi_{q+1,1}^{E^\vee,\alpha\beta}\epsilon_1^\beta + \cdots + \Phi_{q+1,q}^{E^\vee,\alpha\beta}\epsilon_q^\beta + \Phi_{q+1,q+1}^{S^\vee,\alpha\beta}\epsilon_{q+1}^\beta + \cdots + \Phi_{q+1,q+s}^{S^\vee,\alpha\beta}\epsilon_r^\beta, \\
&\vdots \\
\epsilon_{q+s}^\alpha &= \Phi_{q+s,1}^{E^\vee,\alpha\beta}\epsilon_1^\beta + \cdots + \Phi_{q+s,q}^{E^\vee,\alpha\beta}\epsilon_q^\beta + \Phi_{q+s,q+1}^{S^\vee,\alpha\beta}\epsilon_{q+1}^\beta + \cdots + \Phi_{q+s,q+s}^{S^\vee,\alpha\beta}\epsilon_{q+s}^\beta.
\end{aligned}$$

For short we will write the coefficient matrix as

$$\Phi^{E^\vee,\alpha\beta} = \begin{pmatrix} \Phi^{Q^\vee,\alpha\beta} & 0 \\ \Psi^{\alpha\beta} & \Phi^{S^\vee,\alpha\beta} \end{pmatrix}.$$

Next, let h^α be the matrix

$$h^\alpha := \begin{pmatrix} \langle \epsilon_1^\alpha, \epsilon_1^\alpha \rangle_h & \langle \epsilon_1^\alpha, \epsilon_2^\alpha \rangle_h & \cdots & \langle \epsilon_1^\alpha, \epsilon_{q+s}^\alpha \rangle_h \\ \vdots & \ddots & & \vdots \\ \langle \epsilon_{q+s}^\alpha, \epsilon_1^\alpha \rangle_h & \cdots & & \langle \epsilon_{q+s}^\alpha, \epsilon_{q+s}^\alpha \rangle_h \end{pmatrix}.$$

Note that the upper-left $q \times q$ -matrix is constant by the choice of $\epsilon_1^\alpha, \ldots, \epsilon_q^\alpha$. Since h is negatively curved, by [**Hos17**, Proposition 5.2], coefficients of the lower-left $s \times q$-matrix is holomorphic (say ϕ^α). Then we can write as

$$h^\alpha = \begin{pmatrix} C^\alpha & \overline{\phi^\alpha} \\ \phi^\alpha & * \end{pmatrix},$$

where C^α is a $q \times q$-matrix whose coefficients are constant on U^α. By the equality

$$h^\alpha = \Phi^{E^\vee, \alpha\beta} h^\beta (\overline{{}^t\Phi^{E^\vee, \alpha\beta}}),$$

we have

$$\begin{aligned} C^\alpha &= \Phi^{Q^\vee, \alpha\beta} C^\beta (\overline{{}^t\Phi^{Q^\vee, \alpha\beta}}), \\ \phi^\alpha &= \Psi^{\alpha\beta} C^\beta (\overline{{}^t\Phi^{Q^\vee, \alpha\beta}}) + \Phi^{S^\vee, \alpha\beta} \phi^\beta (\overline{{}^t\Phi^{Q^\vee, \alpha\beta}}). \end{aligned}$$

From these equalities, it follows that

$$\phi^\alpha (C^\alpha)^{-1} = \Psi^{\alpha\beta} (\Phi^{Q^\vee, \alpha\beta})^{-1} + \Phi^{S^\vee, \alpha\beta} \phi^\beta (C^\beta)^{-1} (\Phi^{Q^\vee, \alpha\beta})^{-1}.$$

On the other hand, the extension class of the given exact sequence can be calculated as the cohomology class of the Čech 1-cocycle

$$\left\{ \sum_{\lambda=q+1}^{q+s} \sum_{\mu=1}^{q} \Psi_{\lambda,\mu}^{\alpha\beta} \kappa_\mu^\beta \otimes (\sigma_\lambda^\alpha)^\vee \in H^0(U_{\alpha\beta}, \mathcal{O}(Q^\vee \otimes S)) \right\}_{\alpha\beta}$$

$$= \left\{ \sum_{\lambda=q+1}^{q+s} \sum_{\mu=1}^{q} \sum_{\nu=1}^{q} \Psi_{\lambda,\mu}^{\alpha\beta} ((\Phi^{Q^\vee, \alpha\beta})^{-1})_{\mu\nu} \kappa_\nu^\alpha \otimes (\sigma_\lambda^\alpha)^\vee \right\}_{\alpha\beta}.$$

It is the differential of the following Čech 0-cochain

$$\left\{ \sum_{\nu=1}^{q} \sum_{\lambda=q+1}^{q+s} (\phi^\alpha (C^\alpha)^{-1})_{\lambda\nu} \kappa_\nu^\alpha \otimes (\sigma_\lambda^\alpha)^\vee \in H^0(U_\alpha, \mathcal{O}(Q^\vee \otimes S)) \right\}_\alpha,$$

thus the extension class is zero. Therefore the given sequence (5.2.2) splits. □

THEOREM 5.2.10 (=Theorem 5.1.4). *Let X be a compact complex manifold and let*

$$0 \to \mathcal{S} \to \mathcal{E} \to \mathcal{Q} \to 0$$

be an exact sequence of reflexive coherent sheaves $\mathcal{S}$, $\mathcal{E}$, and $\mathcal{Q}$ on X. If $\mathcal{E}$ admits a positively curved (singular) hermitian metric and the first Chen class $c_1(\mathcal{Q}) = 0$, then we have:

(1) $\mathcal{Q}$ is locally free and hermitian flat.
(2) $\mathcal{E} \to \mathcal{Q}$ is a surjective bundle morphism on $X_{\mathcal{E}}$.
(3) The above exact sequence splits on X.

Proof of Theorem 5.2.10. The conclusion (1) follows from Lemma 5.2.6 and the conclusion (2) follows from Lemma 5.2.2. Also, from Theorem 5.2.9, it follows that there exists a bundle morphism $j : \mathcal{Q} \to \mathcal{E}$ on $X_{\mathcal{E}}$ such that

$$\mathcal{E} = \mathcal{S} \oplus j(\mathcal{Q}) \text{ on } X_{\mathcal{E}}.$$

By taking the pushforward i_* by the natural inclusion $i : X_{\mathcal{E}} \to X$ and the double dual, we obtain

$$(i_*\mathcal{E})^{\vee\vee} = (i_*\mathcal{S})^{\vee\vee} \oplus (i_*j(\mathcal{Q}))^{\vee\vee} \text{ on } X.$$

By $\operatorname{codim}(X \setminus X_{\mathcal{E}}) \geq 3$ and the reflexivity, we have $\mathcal{E} \cong (i_*\mathcal{E})^{\vee\vee}$, $\mathcal{S} \cong (i_*\mathcal{S})^{\vee\vee}$, and $\mathcal{Q} \cong (i_*j(\mathcal{Q}))^{\vee\vee}$. This finishes the proof. □

5.2.3. Pseudo-effective tangent bundles. This subsection is devoted to the proof of Theorem 5.1.1.

Theorem 5.2.11 (=Theorem 5.1.1). Let X be a projective manifold with pseudo-effective tangent bundle. Then X admits a morphism $\phi : X \to Y$ with connected fiber to a smooth manifold Y with the following properties:

(1) The morphism $\phi : X \to Y$ is smooth (that is, all the fibers are smooth).
(2) The image Y admits a finite étale cover $A \to Y$ by an abelian variety A.
(3) A general fiber F of ϕ is rationally connected.
(4) A general fiber F of ϕ also has the pseudo-effective tangent bundle.

Moreover, if we further assume that T_X admits a positively curved singular hermitian metric, then

(5) The following exact sequence splits:

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow \phi^* T_Y \longrightarrow 0.$$

(6) The morphism $\phi : X \to Y$ is locally trivial (that is, all the fibers are smooth and isomorphic).

Proof of Theorem 5.2.11. For a projective manifold X with the pseudo-effective tangent bundle T_X, we consider an MRC fibration $\phi : X \dashrightarrow Y$ to a projective manifold Y, and take a resolution $\pi : \bar{X} \to X$ of the indeterminacy locus of ϕ. Here we have the following commutative diagram:

$$\begin{array}{ccc} \bar{X} & & \\ {\scriptstyle\pi}\downarrow & \searrow^{\bar{\phi}} & \\ X & \overset{\phi}{\dashrightarrow} & Y. \end{array}$$

(1) To prove the conclusion (1) (and also (3)) by using [**Hör07**, Corollary 2.11], we will construct a foliation on X (that is, an integrable subbundle of T_X) whose general leaf is rationally connected. We will show that the relative tangent bundle $T_{X/Y} \subset T_X$ (which is defined only on a Zariski open set of X) can be extended to a subbundle of T_X on X. If it can be shown, it is not so difficult to check that this subbundle is integrable and its general leaf is rationally connected (that is, all the assumptions in [**Hör07**, Corollary 2.11] are satisfied).

Now we have the exact sequence of coherent sheaves

$$0 \longrightarrow \bar{\phi}^*\Omega_Y \longrightarrow \Omega_{\bar{X}} \longrightarrow \Omega_{\bar{X}/Y} := \Omega_{\bar{X}}/\bar{\phi}^*\Omega_Y \longrightarrow 0.$$

Then we obtain the injective sheaf morphism $0 \to \pi_*\bar{\phi}^*\Omega_Y \to \Omega_X$ by taking the push-forward. Here we used the formula $\pi_*\Omega_{\bar{X}} = \Omega_X$. By taking the dual, we obtain the exact sequence

$$0 \longrightarrow \mathcal{S} := \operatorname{Ker} r \longrightarrow T_X \xrightarrow{\ r\ } \mathcal{Q} := (\pi_*\bar{\phi}^*\Omega_Y)^\vee. \tag{5.2.3}$$

We remark that the above sequence corresponds to the standard exact sequence of tangent bundles on a Zariski open set where ϕ is a smooth morphism.

The morphism r is generically surjective, and thus the reflexive sheaf Q is also pseudo-effective. In particular, the first Chern class $c_1(\mathcal{Q})$ is also pseudo-effective. On the other hand, it follows that the image Y of MRC fibrations has the pseudo-effective canonical bundle K_Y from [**BDPP13**] and [**GHS03**]. Further $\mathcal{Q}$ coincides with the usual pullback of T_Y on X_0. Here X_0 is the maximal Zariski open set where ϕ is a morphism. Hence, by $\operatorname{codim}(X \setminus X_0) \geq 2$, it can be shown that

$$-c_1(\mathcal{Q}) = c_1(\pi_*\bar{\phi}^*\Omega_Y) = c_1(\pi_*\bar{\phi}^*K_Y)$$

is pseudo-effective.

By the above argument, we can see that $\mathcal{Q}$ is a pseudo-effective reflexive sheaf with $c_1(\mathcal{Q}) = 0$, and thus we can conclude that $\mathcal{Q}$ is a numerically flat vector bundle on X by Theorem 5.1.2. By applying Lemma 5.2.2 to $0 \to \mathcal{Q}^\vee \to \Omega_X$ induced by (5.2.3), it can be seen that the sequence (5.2.3) is a bundle morphism on X. In particular, we can see that ϕ is smooth on X_0 (since the sequence (5.2.3) is not a bundle morphism on the non-smooth locus of ϕ). The subbundle $\mathcal{S}$ defined by the kernel corresponds to the relative tangent bundle $T_{X/Y}$ defined on X_0. Hence $\mathcal{S}$ determines the foliation on X since $T_{X/Y}$ is integrable on X_0 (for example, see [**Mat18b**, subsection 2.2]). Further, its general leaf is rationally connected. Indeed, there exists a Zariski open set Y_1 in Y

such that $\phi : X_1 := \phi^{-1}(Y_1) \to Y_1$ is a proper morphism since $\phi : X \dashrightarrow Y$ is an almost holomorphic map (that is, general fibers are compact). A general leaf of $\mathcal{S}$ corresponds to a general fiber of ϕ by $\mathcal{S} = T_{X/Y}$ on X_1, and thus it is rationally connected. Therefore we can choose an MRC fibration to be holomorphic and smooth by [**Hör07**, Corollary 2.11]. We use the same notation $\phi : X \to Y$ for the smooth MRC fibration.

(2) By (1), we have the standard exact sequence

$$0 \longrightarrow T_{X/Y} \longrightarrow T_X \longrightarrow \phi^* T_Y \longrightarrow 0,$$

and also we have already checked that $\phi^* T_Y$ is pseudo-effective and $c_1(\phi^* T_Y) = 0$. The pull-back $\phi^* T_Y$ is numerically flat by Theorem 5.1.2, and thus T_Y is also numerically flat. The Beauville-Bogomolov decomposition (see [**Bea83**]) asserts that there exists a finite étale cover $Y' \to Y$ such that Y' is the product of hyperkähler manifolds, Calabi-Yau manifolds, and abelian varieties. Let Z be a component of Y' of hyperkähler manifolds or Calabi-Yau manifolds. We remark that T_Z is also numerically flat. In general, numerically flat vector bundles are local systems (for example see [**DPS94**]). Hence T_Z should be a trivial vector bundle on Z since Z is simply connected and T_Z is also numerically flat. This is a contradiction to the definition of hyperkähler manifolds or Calabi-Yau manifolds. Hence the image Y admits a finite étale cover $A \to Y$ by an abelian variety A.

(4) We prove the conclusion (4). By considering the restriction of the standard exact sequence of the tangent bundle to a general fiber F, we obtain

$$0 \longrightarrow T_{X/Y}|_F = T_F \longrightarrow T_X|_F \longrightarrow \phi^* T_Y|_F = N_{F/X} = \mathcal{O}_F^{\oplus m} \longrightarrow 0.$$

When we consider the projective space bundle $f : \mathbb{P}(T_X) \to X$ and the non-nef locus $B \subset \mathbb{P}(T_X)$ of $\mathcal{O}_{\mathbb{P}(T_X)}(1)$, it can be seen that $f(B)$ is a proper subvariety of X by pseudo-effectivity of T_X. By considering the commutative diagram

$$\begin{array}{ccc} \mathbb{P}(T_X|_F) & \hookrightarrow & \mathbb{P}(T_X) \\ \downarrow f & & \downarrow f \\ F & \hookrightarrow & X \end{array}$$

we can see that the image of the non-nef locus of $\mathcal{O}_{\mathbb{P}(T_X|_F)}(1)$ is contained in $f(B \cap F)$. For a general fiber F, the image $f(B \cap F)$ is still a proper subvariety of F. Hence $T_X|_F$ is pseudo-effective. The surjective bundle morphism

$$\Lambda^{m+1}(T_X|_F) \to T_F$$

induced by the above exact sequence implies that T_F is pseudo-effective.

We finally show that the MRC fibration $\phi : X \to Y$ is locally trivial if we further assume X admits a positively curved singular hermitian metric. Under the assumption of such a metric, the exact sequence of the tangent bundle splits (that is, $T_X \cong T_{X/Y} \oplus$

ϕ^*T_Y) by Theorem 5.1.4. Then, by Ehrensmann's theorem (see also [**Hör07**, Lemma 3.19]), we can see that $\phi : X \to Y$ is locally trivial. □

THEOREM 5.2.12. Let X be a compact Kähler manifold with pseudo-effective tangent bundle and $\phi : X \to Y := \mathrm{Alb}(X)$ be its Albanese map. Then the Albanese map ϕ is a surjective smooth morphism and satisfies all the conclusions in Theorem 5.2.11 except for (3) and (6) by replacing an abelian variety in (2) with a compact complex torus.

PROOF. In the proof of Theorem 5.1.1, the assumption of the projectivity was used only for the proof of (1) and (6). The other arguments except for (1) and (6) work even if we replace MRC fibrations with the Albanese map. Hence it is sufficient to prove that the Albanese map ϕ is a surjective smooth morphism. It is easy to check it. Indeed, for a basis $\{\eta_k\}_{k=1}^{q}$ of $H^0(X, \Omega_X)$, it follows that any non-trivial linear combination of them is non-vanishing by Lemma 5.2.3. This implies that ϕ is a surjective smooth morphism (for example see [**CP91**]). □

In [**DPS94**], it was proved that X is a Fano manifold when T_X is nef and X is rationally connected. As an analog of this result, we suggest the following problem. We remark that the geometry of a general fiber F in Theorem 5.1.1 can be determined if the problem can be affirmatively solved.

PROBLEM 5.2.13. If a projective manifold X is rationally connected and has the pseudo-effective tangent bundle, then is the anti-canonical bundle $-K_X$ big?

5.3. Surfaces with pseudo-effective tangent bundle

Toward the classification of surfaces with pseudo-effective tangent bundle, we study minimal ruled surfaces in subsection 5.3.1 and their blow-ups in subsection 5.3.2, which provide interesting examples of positively curved vector bundles.

5.3.1. On minimal ruled surfaces. In this subsection, we consider a ruled surface $\phi : X \to C$ over a smooth curve C. When T_X is pseudo-effective, the base C should be either the projective line or an elliptic curve by Theorem 5.1.1. Conversely, it follows that any minimal ruled surfaces $\phi : X \to \mathbb{P}^1$ over $\mathbb{P}^1$ (that is, Hirzebruch surfaces) have the pseudo-effective tangent bundle from the following proposition. However, they do not have nef tangent bundle except for the case of $X = \mathbb{P}^1 \times \mathbb{P}^1$, since they have a curve with negative self-intersection.

PROPOSITION 5.3.1. If X is a projective toric manifold, then T_X is generically globally generated. In particular, any Hirzebruch surfaces have pseudo-effective tangent bundle.

PROOF. For a toric manifold X, we have an inclusion $(\mathbb{C}^*)^n \subset X$ as a Zariski open dense subset and an action $(\mathbb{C}^*)^n \curvearrowright X$. Consider a family of actions $(e^{i\theta}, 1, \cdots, 1)$.

Differentiate it by θ at $\theta = 0$, we obtain a holomorphic vector field on X. Similarly, we can construct n vector fields which generate $T_X|_{(\mathbb{C}^*)^n}$, and thus T_X is generically globally generated. □

Now we consider a ruled surface $\phi : X \to C$ over an elliptic curve C. Thanks to Theorem 5.1.1, we can see that the ruling $\phi : X \to C$ should be a smooth morphism when X has the pseudo-effective tangent bundle. The minimal ruled surface X over C can be classified by [**Ati55**], [**Ati57**], and [**Suw69**]: X is isomorphic to S_n, A_0, A_{-1}, or a surface in $\mathcal{S}_0$. Here a surface X in $\mathcal{S}_0$ means the projective space bundle $\mathbb{P}(\mathcal{O}_C \oplus L)$ for some $L \in Pic^0(C)$ and A_0 (resp. A_{-1}) is the projective space bundle associated with a vector bundle of rank 2 that is the non-split extension of $\mathcal{O}_C$ by $\mathcal{O}_C$ (resp. $\mathcal{O}_C(p)$), where p is a point in C. It can be seen that A_0, A_{-1}, and surfaces in $\mathcal{S}_0$ have the nef tangent bundle by [**CP91**], and thus the remaining problem is the case of $X = S_n$. The ruled surface S_n is the projective space bundle associated with the vector bundle $\mathcal{O}_C \oplus \mathcal{O}_C(np)$. Note that the tangent bundle of $S_0 = \mathbb{P}^1 \times C$ is nef. By the above observation, it is enough for our purpose to investigate $X = S_n$ in the case of $n \geq 1$. By the following proposition, we can see that S_n has the pseudo-effective tangent bundle (which is not nef), and further that it admits no positively curved singular hermitian metric.

PROPOSITION 5.3.2. Let $\phi : X \to C$ be a minimal ruled surface over an elliptic curve C. Then we have:

(1) The tangent bundle of S_n is pseudo-effective, but it does not admit positively curved singular hermitian metrics when $n \geq 1$.
(2) The tangent bundle of S_0, A_0, A_{-1}, and a surface in $\mathcal{S}_0$ is nef.

PROOF. All the ruled surfaces with nef tangent bundle are classified in [**CP91**], which implies that the conclusion (2) holds and the tangent bundle of S_n is not nef for $n \geq 1$.

From now on, let X be the projective space bundle S_n associated with the vector bundle $E_n := \mathcal{O}_C \oplus \mathcal{O}_C(np)$. We first check the latter statement in the conclusion (1). If $X = S_n$ admits a positively curved singular hermitian metric, the exact sequence

$$0 \to T_{X/C} \to T_X \to \phi^* T_C \to 0$$

splits by Theorem 5.1.4, and thus we have

$$h^0(X, T_X) = h^0(X, T_{X/C}) + h^0(X, \phi^* T_C). \tag{5.3.1}$$

On the other hand, we have $h^0(X, T_X) = n + 1$ from [**Suw69**, Theorem 3]. Also we can easily check that

$$\phi_*(T_{X/C}) = \phi_*(-K_X) = \mathrm{Sym}^2(E_n) \otimes \det E_n^\vee.$$

This implies that

$$h^0(X, T_{X/C}) = h^0(C, \mathcal{O}_C(-np) \oplus \mathcal{O}_C \oplus \mathcal{O}_C(np)) = n + 1.$$

This is a contradiction to (5.3.1).

We will prove that T_X is pseudo-effective. For this purpose, it is sufficient to prove that $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$ is generically globally generated for any $m \geq 0$. Our strategy is to observe a gluing condition of $X = S_n$ carefully to construct holomorphic sections that generate $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$ at general points.

Let v be a local coordinate centered at p and let $V \subset C$ be a sufficiently small open neighborhood of p. Further, let U be the open set $U := C \setminus \{p\}$ and u be the standard coordinate of the universal cover $\mathbb{C} \to C$. The ruled surface X can be constructed by gluing $(u, \zeta) \in U \times \mathbb{P}^1$ and $(v, \eta) \in V \times \mathbb{P}^1$ with the following identification:

$$\zeta = v^n \eta \quad \text{and} \quad [u] = p + v, \tag{5.3.2}$$

where ζ and η are the inhomogeneous coordinates of $\mathbb{P}^1$.

Let θ be a meromorphic section of $\mathrm{Sym}^m(T_X)$ with pole along the fiber $\phi^{-1}(p)$ of p. Our strategy is as follows: We first look for a sufficient condition for the pole of θ being of order at most 2. Then we concretely construct θ satisfying this condition, which can be regarded as a holomorphic section of $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$, and we show that such sections generate $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$ on a Zariski open set.

Now θ is a meromorphic section of $\mathrm{Sym}^m(T_X)$ whose pole appears only along the fiber $\phi^{-1}(p)$. Hence, by expanding θ on $U \times \mathbb{P}^1$, we have the following equality

$$\theta = \sum_{p=0}^{m} a_p(u,\zeta)\Big(\frac{\partial}{\partial\zeta}\Big)^{m-p}\Big(\frac{\partial}{\partial u}\Big)^{p} \text{ on } U \times \mathbb{P}^1. \tag{5.3.3}$$

Here a_p is a meromorphic function on X. The gluing condition (5.3.2) yields that

$$\frac{\partial}{\partial\zeta} = \frac{1}{v^n}\frac{\partial}{\partial\eta} \quad \text{and} \quad \frac{\partial}{\partial u} = -n\frac{\eta}{v}\frac{\partial}{\partial\eta} + \frac{\partial}{\partial v}. \tag{5.3.4}$$

Then we can obtain the following expansion of θ on $V \times \mathbb{P}^1$

$$\theta = \sum_{\ell=0}^{m}\Big\{\sum_{p=\ell}^{m} d_{p,\ell}\, a_p(v,\eta)\frac{\eta^{p-\ell}}{v^{n(m-p)+p-\ell}}\Big\}\Big(\frac{\partial}{\partial\eta}\Big)^{m-\ell}\Big(\frac{\partial}{\partial v}\Big)^{\ell} \text{ on } V \times \mathbb{P}^1 \tag{5.3.5}$$

by an involved, but straightforward computation. Here $d_{p,\ell}$ is the non-zero constant defined by $d_{p,\ell} := (-n)^{p-\ell}\binom{p}{p-\ell}$. The ruling $X \to C$ is locally trivial and sections of $\mathrm{Sym}^p(T_F)$ on a fiber F are polynomials of degree (at most) $2p$. This implies that the meromorphic function $a_{m-k}(u,\zeta)$ is a polynomial of degree $2k$ with respect to ζ, and thus we can write a_{m-k} as

$$a_{m-k}(v,\eta) = \sum_{q=0}^{2k} a_{m-k}^{(q)}(v)\,\zeta^q = \sum_{q=0}^{2k} a_{m-k}^{(q)}(v)\,v^{nq}\eta^q \quad \text{for any } 0 \leq k \leq m \tag{5.3.6}$$

for some meromorphic function $a^{(q)}_{m-k}(v)$ on C with pole only at p. Here we used (5.3.2) again.

We will find a sufficient condition for $a^{(q)}_{m-k}(v)$ for guaranteeing that the coefficients in (5.3.5) have the pole of order at most 2. We remark that the section θ satisfying this condition determines the holomorphic section of $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$. By substituting (5.3.6) for (5.3.5) and rearranging it concerning the powers of η, a sufficient and necessary condition can be obtained, but this method needs so complicated computation that we want to avoid to write down it. Here, to improve our prospect, we focus only on a sufficient condition by considering the restricted situation where $a^{(q)}_{m-k}=0$ for $q\neq k$. In this situation, it is not so difficult to show that θ determines the holomorphic section of $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$ if $a^{(q)}_{m-q}$ satisfies that

$$(5.3.7)\quad \sum_{p=0}^{q} d_{m-p,m-q}a^{(p)}_{m-p}(v)\frac{1}{v^{q-p}} \text{ has the pole of order} \leq 2 \text{ at } p \text{ for any } 0\leq q\leq m.$$

For an explanation, we prepare the table where we write down them for $q=0,1,2$.

$q=q$	coeff.of $(\partial/\partial\eta)^q(\partial/\partial v)^{m-q}$	$\sum_{p=0}^{q} d_{m-p,m-q}a^{(p)}_{m-p}/v^{q-p}$
$q=0$	coeff. of $(\partial/\partial\eta)^0(\partial/\partial v)^{m}$	$d_{m,m}a^{(0)}_{m}$
$q=1$	coeff. of $(\partial/\partial\eta)^1(\partial/\partial v)^{m-1}$	$d_{m,m-1}a^{(0)}_{m}/v+d_{m-1,m-1}a^{(1)}_{m-1}$
$q=2$	coeff. of $(\partial/\partial\eta)^2(\partial/\partial v)^{m-2}$	$d_{m,m-2}a^{(0)}_{m}/v^2+d_{m-1,m-2}a^{(1)}_{m-1}/v+d_{m-2,m-2}a^{(2)}_{m-2}$

To construct meromorphic functions $a^{(p)}_{m-p}$ on C satisfying (5.3.7), for every $n\geq 2$, we take meromorphic functions P_n on the elliptic curve C such that P_n has the pole only at p and its Laurent expansion at p can be written as follows:

$$P_n(v)=\frac{1}{v^n}+\sum_{k\geq n+1}\frac{a_k}{v^k}.$$

Note that we can easily find them by using Weierstrass's elliptic functions and their differential.

We first put $a^{(0)}_{m}:=P_2/d_{m,m}$. Then the second line from the top in the table satisfies (5.3.7) (that is, it has the pole of order at most 2) if we define $a^{(1)}_{m-1}$ by $a^{(1)}_{m-1}:=-d_{m,m-1}/d_{m-1,m-1}P_3$. By the same way, the third line also satisfies (5.3.7) if we define $a^{(2)}_{m-2}$ by an appropriate linear combination of P_3 and P_4. By repeating this process, we can construct meromorphic functions $a^{(p)}_{m-p}$ on C satisfying (5.3.7) by a linear combination of $\{P_3\}_{k=3}^{p+2}$. We denote by θ_0 the holomorphic section of $\mathrm{Sym}^m(T_X)\otimes\phi^*\mathcal{O}(2p)$ obtained from the above construction. The section θ_0 generates the vector $(\partial/\partial\eta)^0(\partial/\partial v)^m$ on a Zariski open set, since $a^{(0)}_{m}=P_2/d_{m,m}$ is non-zero.

Now we put $a_m^{(0)} := 0$ and $a_{m-1}^{(1)} := P_2/d_{m-1,m-1}$, so that the first and the second line in the table have pole of order at most 2. Then, by the same argument as above, we can construct meromorphic functions $a_{m-p}^{(p)}$ satisfying (5.3.7) by defining them by an appropriate linear combination of $\{P_k\}_{k=3}^{p+2}$ (for example $a_{m-2}^{(2)} := -d_{m-1,m-2}/d_{m-2,m-2}P_3$). We denote by θ_1 the obtained holomorphic section of $\mathrm{Sym}^m(T_X) \otimes \phi^*\mathcal{O}(2p)$. By the construction, the function $a_m^{(0)}$ is zero and $a_{m-1}^{(1)}$ is non-zero. Hence it follows the sections θ_0 and θ_1 generate the vectors $(\partial/\partial\eta)^0(\partial/\partial v)^m$ and $(\partial/\partial\eta)^1(\partial/\partial v)^{m-1}$ on a Zariski open set.

By repeating this process, we can construct holomorphic sections $\{\theta_p\}_{p=0}^m$ of $\mathrm{Sym}^m(T_X)\otimes \phi^*\mathcal{O}(2p)$ generating $\mathrm{Sym}^m(T_X) \otimes \phi^*\mathcal{O}(2p)$ on a Zariski open set. □

In the rest of this subsection, we suggest the following problem to investigate a gap between almost nefness and pseudo-effectivity of vector bundles.

PROBLEM 5.3.3. We consider an exact sequence of vector bundles

$$0 \longrightarrow S \longrightarrow E \longrightarrow Q \longrightarrow 0.$$

When S and Q are pseudo-effective, then is E pseudo-effective?

REMARK 5.3.4. When S and Q are nef, its extension E is also nef (see [**DPS94**, Proposition 1.15]). Hence we can easily show that E is almost nef if S and Q are almost nef. In particular, it can be shown that $\mathcal{O}_E(1)$ is pseudo-effective by [**BDPP13**], but we do not know whether or not E itself is pseudo-effective. The difficulty is to show that the image of the non-nef locus $\mathcal{O}_E(1)$ to X is properly contained in X. If Problem 5.3.3 can be affirmatively solved, the pseudo-effectivity of the tangent bundle of $X = S_n$ is easily obtained, by applying it to the standard exact sequence of the tangent bundle. In fact, we tried some methods in [**Suw69**], [**DPS94**], and [**Har70**] to solve Problem 5.3.3, but it did not succeed. This problem seems to be subtle since we do not know whether there is a gap between almost nefness and pseudo-effectivity.

5.3.2. On rational surfaces. By the results in Subsection 5.3.1, it is enough for the classification of the surfaces to determine when the blow-up of the Hirzebruch surface has pseudo-effective tangent bundle. However, it seems to be a too hard problem to classify all the blow-ups completely since X delicately depends on the position and the number of blow-up points. In this subsection, we study only blow-ups along *general points*. The complete classification can not be achieved even in this case, but we obtain an interesting relation between positivity of tangent bundle and the geometry of Hirzebruch surfaces. The following proposition gives the requirement for the blow-up having pseudo-effective tangent bundle.

PROPOSITION 5.3.5. Let $\phi : \mathbb{F}_n \to \mathbb{P}^1$ be the Hirzebruch surface and let $\pi : X \to \mathbb{F}_n$ be the blow-up along the set Σ of general points on $\mathbb{F}_n$. Then we have:

(1) If the tangent bundle T_X of X is generically globally generated, then $\sharp\Sigma \leq 2$.

(2) If the tangent bundle T_X of X is pseudo-effective, then $\sharp\Sigma \leq 4$.

REMARK 5.3.6. The interesting point here is that the conclusion of $\sharp\Sigma \leq 2$ in (1) is optimal, and further the generic global generation and pseudo-effectivity differently behave for $\sharp\Sigma$. Indeed, it follows that the tangent bundle T_X in the case of $\sharp\Sigma \leq 3$ is pseudo-effective, but not generically globally generated from Proposition 5.3.8.

PROOF. (1) Fix a holomorphic vector field ξ on X. We shall define a holomorphic vector field θ_ξ on $\mathbb{P}^1$ as follows. Let t be a local holomorphic coordinate on $U \subset \mathbb{P}^1$. By pulling back dt, we obtain a holomorphic 1-form $\pi^*\phi^*dt$ on $\widetilde{U} := (\pi\circ\phi)^{-1}(U)$. Then $\langle \xi, \pi^*\phi^*dt\rangle$ is a holomorphic function on $\widetilde{U}$. Thus it is constant along each fiber and defines a holomorphic function on U. Now we define the holomorphic vector field θ_ξ on $\mathbb{P}^1$ to be

$$\theta_\xi := \langle \theta_\xi, dt\rangle \frac{\partial}{\partial t} \quad \text{and} \quad \langle \theta_\xi, dt\rangle := \langle \xi, \pi^*\phi^*dt\rangle.$$

Since we assumed that T_X is generically globally generated, we can choose ξ with $\theta_\xi \not\equiv 0$ on $\mathbb{P}^1$.

We claim that θ_ξ has zeros on the set $\phi(\Sigma) \subset \mathbb{P}^1$. To prove the claim, we take a local coordinate (t,s) on $\mathbb{F}_n$ centered at a point in Σ such that t is the pull-back of a local coordinate on $\mathbb{P}^1$. If we put $v := t/s$, then (v,s) is a coordinate on X. Then we have

$$\langle \xi, \pi^*\phi^*dt\rangle = \langle \xi, d(vs)\rangle = \langle \xi, sdv + vds\rangle.$$

The last term vanishes at $(v,s) = (0,0)$, and thus $\langle \theta_\xi, dt\rangle = 0$ at $t = 0$. This shows the claim.

In the case of $\sharp\Sigma \geq 3$, the vector field θ_ξ has at least three zeros on $\mathbb{P}^1$. It contradicts to the fact of $\deg T_{\mathbb{P}^1} = 2$, thus we have $\sharp\Sigma \leq 2$.

(2) Since T_X is pseudo-effective, we can choose an ample line bundle A and a sequence of positively curved singular hermitian metrics h_m on $(\operatorname{Sym}^m T_X) \otimes A$. Fix a smooth hermitian metric h_A on A with positive curvature. Then $h_m \otimes h_A^{-1}$ is a (possibly not positively curved) singular hermitian metric on $\operatorname{Sym}^m T_X$. Define a singular hermitian metric g_m on $\pi^*\phi^*T_{\mathbb{P}^1}$ by the m-th root of the quotient metric of $h_m \otimes h_A^{-1}$ induced by the morphism $\operatorname{Sym}^m T_X \to (\pi^*\phi^*T_{\mathbb{P}^1})^{\otimes m}$. Since $(h_m \otimes h_A^{-1}) \otimes h_A$ is positively curved, the metric $g_m^m \otimes h_A$ is also positively curved. The curvature current $\sqrt{-1}\Theta_{g_m}$ of g_m satisfies that

$$\sqrt{-1}\Theta_{g_m} \geq -\frac{1}{m}\omega_A.$$

Then by taking a subsequence (if necessary), we can assume that $\sqrt{-1}\Theta_{g_m}$ weakly converges to a positive current $T \in c_1(\pi^*\phi^*T_{\mathbb{P}^1})$. By the argument similar to (1), we obtain a d-closed positive $(1,1)$-current S in $c_1(T_{\mathbb{P}^1})$ such that $T = \phi^*\pi^*S$. Hence we have

$$\sqrt{-1}\Theta_{g_m} \to \pi^*\phi^*S = T \in c_1(\phi^*\pi^*T_{\mathbb{P}^1}).$$

We take a point $p \in \Sigma$ and put $p_0 := \phi(p)$. We claim that the following bound of the Lelong number

$$\nu(S, p_0) \geq \frac{1}{2}. \tag{5.3.8}$$

We fix a local coordinate t near $p_0 \in \mathbb{P}^1$. Let (t, s) be a coordinate on $\mathbb{F}_n$ centered at p. As before, by putting $v = t/s$, (v, s) is a coordinate on X. Let $p' \in X$ be a point defined by $(v, s) = (0, 0)$. Let C be a (local) holomorphic curve on X defined by $\{v = s\}$. We will denote $\overline{C} := \pi(C)$. The defining equation of $\overline{C}$ is $\{t/s = s\} = \{t = s^2\}$. Then we have

$$\nu(S, p_0) = \frac{1}{2}\nu(\phi^* S|_{\overline{C}}, p). \tag{5.3.9}$$

Indeed, the function $\phi^*\gamma$ is a local potential of $\phi^* S$ for a local potential γ of S. Note that s is a local coordinate on $\overline{C}$ while $t = s^2$ is a local coordinate on $\mathbb{P}^1$. We can calculate each Lelong number by the formula

$$\nu(S, p_0) = \liminf_{t \to 0} \frac{\gamma(t)}{\log |t|},$$

and thus

$$\nu(\phi^* S|_{\overline{C}}, p) = \liminf_{s \to 0} \frac{\phi^*\gamma(s^2, s)}{\log |s|} = \liminf_{s \to 0} \frac{\gamma(s^2)}{\log |s|} = 2\nu(S, p_0).$$

This proves (5.3.9). Since the Lelong number will increase after taking restriction, we have

$$\nu(\phi^* S|_{\overline{C}}, p) = \nu(T|_C, p') \geq \nu(T, p').$$

Lelong numbers will also increase after taking a weak limit of currents, thus we obtain

$$\nu(T, p') \geq \limsup_{m \to +\infty} \nu(\sqrt{-1}\Theta_{g_m}, p').$$

The local weight of g_m is written as

$$\frac{1}{2m} \log |(\pi^*\phi^*(dt))^m|^2_{h_m^{-1} \otimes h_A}.$$

Since $t = vs$ on X, we can calculate as follows:

$$|(\pi^*\phi^*(dt))^m|^2_{h_m^{-1} \otimes h_A} = |(vds + sdv)^m|^2_{h_m^{-1} \otimes h_A}. \tag{5.3.10}$$

Since h_m^{-1} is negatively curved and h_A is smooth, it follows that

$$|\cdot|^2_{h_m^{-1} \otimes h_A} \leq C_0 |\cdot|^2_{h_{\mathrm{sm}}}$$

for a smooth hermitian metric h_{sm} and some constant $C_0 > 0$ (both depending on m). Then the right-hand side of (5.3.10) is bounded as

$$\begin{aligned} &\leq C_0 |(vds + sdv)^m|^2_{h_{\mathrm{sm}}} \\ &\leq C_0 |(v, s)|^{2m}. \end{aligned}$$

Thus, the Lelong number of $\sqrt{-1}\Theta_{g_m}$ is bounded as

$$\nu(\sqrt{-1}\Theta_{g_m}, p') \geq \frac{1}{2m} \liminf_{(v,s)\to 0} \frac{C_0|(v,s)|^{2m}}{\log|(v,s)|} = 1.$$

This proves (5.3.8). Since $\deg T_{\mathbb{P}^1} = 2$, there must be at most four points where S has the Lelong number greater than or equal to $1/2$. Therefore $\sharp\Sigma \leq 4$. □

We finally prove Proposition 5.3.8 by applying the following lemma. The lemma is useful when we compare a vector field on a given manifold with its blow-up.

Lemma 5.3.7. Let $\pi : Y \to \mathbb{C}^2$ be the blow-up at $(\alpha, \beta) \in \mathbb{C}^2$ with the exceptional divisor E, and let (x, y) be the standard coordinate of $\mathbb{C}^2$. We consider a holomorphic section θ of $\mathrm{Sym}^m T_{\mathbb{C}^2}$ and its expansion

$$\theta = \sum_{k=0}^{m} f_k(x,y)\Big(\frac{\partial}{\partial x}\Big)^k\Big(\frac{\partial}{\partial y}\Big)^{m-k}.$$

Then the pull-back $(\pi|_{Y\setminus E})^*\theta$ by the isomorphism $\pi|_{Y\setminus E}$ on $Y \setminus E$ can be extended to the holomorphic section of $\mathrm{Sym}^m T_Y$ if and only if

$$\sum_{k=0}^{m} f_k(s+\alpha, st+\beta)\Big(\frac{\partial}{\partial s} - \frac{t}{s}\frac{\partial}{\partial t}\Big)^k\Big(\frac{1}{s}\frac{\partial}{\partial t}\Big)^{m-k}.$$

is holomorphic with respect to $(s,t) \in \mathbb{C}^2$.

Proof. We first remark that any holomorphic section ξ of $\mathrm{Sym}^m T_Y$ determines the section θ_ξ of $\mathrm{Sym}^m T_{\mathbb{C}^2}$. Indeed, a given section ξ induces the section θ_ξ of $\mathrm{Sym}^m T_{\mathbb{C}^2}$ on $\mathbb{C}^2 \setminus \{(\alpha,\beta)\}$ via the isomorphism $\pi|_{Y\setminus E}$, which can be extended on $\mathbb{C}^2$ since the blow-up center has codimension two.

We consider the descriptions:

$$Y = \{(x, y, [z:w]) \in \mathbb{C}^2 \times \mathbb{P}^1 \,|\, (x-\alpha)w = (y-\beta)z\},$$
$$E = \{(\alpha, \beta, [z:w]) \,|\, [z:w] \in \mathbb{P}^1\}.$$

and put the Zariski open set $Y' := Y \cap \{w \neq 0\}$. The following map r gives a coordinate of Y' and $\pi|_{Y'}$ can be written as follows:

$$\begin{array}{cccccc} r:\mathbb{C}^2 & \to & Y' & \pi|_{Y'}: Y' & \to & \mathbb{C}^2 \\ (s,t) & \mapsto & (s+\alpha, st+\beta, [1:t]) & (x,y,[z:w]) & \mapsto & (x,y) \end{array}$$

If $(\pi \circ r)^*\theta$ is holomorphic on $\mathbb{C}^2$, then $(\pi|_{Y\setminus E})^*\theta$ can be extended to the holomorphic section of $\mathrm{Sym}^m T_Y$. Indeed, in this case, the section $(\pi|_{Y\setminus E})^*\theta$ can be extended to the holomorphic section of $\mathrm{Sym}^m T_{Y'}$. Hence it can also be extended on Y since the codimension of $E \cap \{w = 0\}$ is two.

By calculation, we obtain

$$(\pi\circ r)^*\theta=\sum_{k=0}^{m} f_k(s+\alpha, st+\beta)\Big(\frac{\partial}{\partial s}-\frac{t}{s}\frac{\partial}{\partial t}\Big)^k\Big(\frac{1}{s}\frac{\partial}{\partial t}\Big)^{m-k}.$$

Hence $(\pi\circ r)^*\theta$ is holomorphic on $\mathbb{C}^2$ if and only if the right hand side is holomorphic in $(s,t)\in\mathbb{C}^2$, which completes the proof. □

Proposition 5.3.8. We have:

(1) The blow-up of the Hirzebruch surface $\mathbb{F}_n$ along general one or two points has the generically globally generated tangent bundle.
(2) The blow-up of the Hirzebruch surface $\mathbb{F}_n$ along general three points has the pseudo-effective tangent bundle.

Because the general case is tedious, we first show Proposition 5.3.8 in the simplest case $n=0$.

Proof of (1) for $\mathbb{F}_0$. In general, for a birational morphism $f: Y\to Z$ between projective manifolds, we have the natural inclusion $f_*T_Y\subset T_Z$. Since the natural inclusion is of course generically isomorphism, T_Z is generically globally generated if the tangent bundle T_Y is so. Therefore it is sufficient for the proof of (1) to treat only the blow-up $\pi: X\to\mathbb{F}_0$ along general two points p_1, p_2.

We take a Zariski open set $\mathbb{C}\times\mathbb{C}=W_0\subset\mathbb{F}_0$ with the local coordinate (x,y). We may assume that $p_1=(0,0)$ and $p_2=(1,1)$ by using the action of the automorphism group of $\mathbb{F}_0$. We define the set of holomorphic vector fields on W_0

$$\mathcal{T}:=\Big\{\sum_{k=0}^{2} a_k x^k\frac{\partial}{\partial x}+\sum_{l=0}^{2} b_l y^l\frac{\partial}{\partial y}\,\Big|\, a_k, b_l\in\mathbb{C}\Big\}.$$

We remark that any $\theta\in\mathcal{T}$ can be extended to a global holomorphic section of $T_{\mathbb{F}_0}$. From Lemma 5.3.7, for a holomorphic vector field

$$\theta:=a(x)\partial/\partial x+b(y)\partial/\partial y\in\mathcal{T}$$

it follows that θ can be lifted to the holomorphic section of T_Y if and only if

$$\frac{1}{s}\big(-a(s+\alpha)t+b(st+\beta)\big)\text{ is holomorphic with respect to }(s,t)$$

for $(\alpha,\beta)=(0,0)$ and $(\alpha,\beta)=(1,1)$. We choose θ_1 and θ_2 in $\mathcal{T}$ as follows:

$$\theta_1=(x^2-x)\frac{\partial}{\partial x}\text{ and }\theta_2=(y^2-y)\frac{\partial}{\partial y}.$$

Then we can easily see that $\pi^*\theta_1$ and $\pi^*\theta_2$ can be extended to the global holomorphic sections of T_X. For a point $q=(x,y)\in W_0$ such that $x\neq 0,1$ and $y\neq 0,1$, the vectors $\theta_1(q)$ and $\theta_2(q)$ at q give a basis of $T_{W_0,q}$. Therefore T_X is generically globally generated. □

Proof of (2) for $\mathbb{F}_0$. We use the same notations as in the proof of (1). Let $\pi : X \to \mathbb{F}_0$ be a blow-up of $\mathbb{F}_0$ along general three points p_1, p_2, p_3. Our goal in this proof is to show that $\mathrm{Sym}^2(T_X)$ is generically globally generated. Since p_1, p_2, p_3 are in general position, we may assume $p_1, p_2, p_3 \in W_0$, $p_1 = (0,0)$, $p_2 = (1,1)$, and $p_3 = (-1,-1)$ by the action of the automorphism group of $\mathbb{F}_0$.

We define $\mathcal{T}$ by

$$\mathcal{T} := \left\{ \sum_{k=0}^{4} a_k x^k \Big(\frac{\partial}{\partial x}\Big)^2 + \sum_{0 \le k,l \le 2} b_{kl} x^k y^l \frac{\partial}{\partial x}\frac{\partial}{\partial y} + \sum_{k=0}^{4} c_k y^k \Big(\frac{\partial}{\partial y}\Big)^2 \,\Big|\, a_k, b_{kl}, c_k \in \mathbb{C} \right\}.$$

It is easy to show that any $\theta \in \mathcal{T}$ can be extended to a holomorphic global section of $\mathrm{Sym}^2 T_{\mathbb{F}_0}$.

By Lemma 5.3.7, we can see that, for a holomorphic section

$$\theta = a(x)\Big(\frac{\partial}{\partial x}\Big)^2 + b(x,y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + c(y)\Big(\frac{\partial}{\partial y}\Big)^2 \in \mathcal{T},$$

the section θ can be lifted to the section of $\mathrm{Sym}^2 T_{\mathbb{F}_0}$ if and only if the followings are holomorphic with respect to $(s,t) \in \mathbb{C} \times \mathbb{C}$:

$$\frac{1}{s}\big(-2a(s+\alpha, st+\beta)t + b(s+\alpha, st+\beta)\big),$$
$$\frac{1}{s^2}\big(a(s+\alpha, st+\beta)t^2 - b(s+\alpha, st+\beta)t + c(s+\alpha, st+\beta)\big),$$

for $(\alpha, \beta) = (0,0), (1,1), (-1,-1)$.

Here we put

$$\theta_1 = y^2(x^2-1)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y^2(y^2-1)\Big(\frac{\partial}{\partial y}\Big)^2,$$
$$\theta_2 = x^2(x^2-1)\Big(\frac{\partial}{\partial x}\Big)^2 + x^2(y^2-1)\frac{\partial}{\partial x}\frac{\partial}{\partial y},$$
$$\theta_3 = (x-y)^2\frac{\partial}{\partial x}\frac{\partial}{\partial y}.$$

Then we can easily show that $\pi^*\theta_1$, $\pi^*\theta_2$, and $\pi^*\theta_3$ can be extended to global holomorphic sections of $\mathrm{Sym}^2 T_X$. For a general point $q \in W_0$, it is easy to see that $\theta_1(q)$, $\theta_2(q)$, and $\theta_3(q)$ give a basis of $\mathrm{Sym}^2 T_{W_0,q}$. Therefore $\mathrm{Sym}^2 T_X$ is generically globally generated. □

As a preliminary of the proof for $\mathbb{F}_n$, we prove the following claim. We regard the Hirzebruch surface $\mathbb{F}_n$ for $n \ge 1$ as the hypersurface in $\mathbb{P}^1 \times \mathbb{P}^2$

$$\mathbb{F}_n = \{([X_1 : X_2], [Y_0 : Y_1 : Y_2]) \in \mathbb{P}^1 \times \mathbb{P}^2 \,|\, Y_1 X_2^n = Y_2 X_1^n\}.$$

We set $U = \{Y_1 \neq 0 \text{ or } Y_2 \neq 0\}$. We first observe the automorphism group of $\mathbb{F}_n$ so that general three points move to specific points, which makes our computation not so hard.

Claim 5.3.9. General three points $p_1, p_2, p_3 \in U$ move to $([1:0],[1:1:0]),([1:1],[1:1:1]),([1:-1],[1:1:(-1)^n])$ by the action of the automorphism group of $\mathbb{F}_n$.

Proof. Let S, T be variables and P_n be a vector subspace of homogeneous polynomials of degree n in $\mathbb{C}[S,T]$. The linear group $\mathrm{GL}(2,\mathbb{C})$ acts on P_n as follows: For any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C})$ and any $\sum_{k=0}^n a_k S^k T^{n-k} \in P_n$, we define the action by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \bullet \Big(\sum_{k=0}^{n} a_k S^k T^{n-k} \Big) := \sum_{k=0}^{n} a_k (aS+bT)^k (cS+dT)^{n-k}.$$

This induces the semidirect product $G_n := P_n \rtimes \mathrm{GL}(2,\mathbb{C})$.

For any $g = (\sum_{k=0}^n a_k S^k T^{n-k}, \begin{pmatrix} a & b \\ c & d \end{pmatrix}) \in G_n$, we define the action of $\mathbb{F}_n$ as follows: For any $q = ([X_1 : X_2],[Y_0 : Y_1 : Y_2]) \in \mathbb{F}_n$, we define $g(q)$ by

$$([aX_1+bX_2 : cX_1+dX_2],[Y_0X_1^n + Y_1 \sum_{k=0}^{n} a_k X_1^k X_2^{n-k} : Y_1(aX_1+bX_2)^n : Y_1(cX_1+dX_2)^n]),$$

if $X_1 \neq 0$ and by

$$([aX_1+bX_2 : cX_1+dX_2],[Y_0X_2^n + Y_2 \sum_{k=0}^{n} a_k X_1^k X_2^{n-k} : Y_2(aX_1+bX_2)^n : Y_2(cX_1+dX_2)^n])$$

if $X_2 \neq 0$ (see [**DI09**, Theorem 4.10] or [**Bla12**, Section 6.1]).

Note that the ruling $\phi : \mathbb{F}_n \to \mathbb{P}^1$ coincides with the first projection. We may assume that p_1, p_2 and p_3 are in U and also that the images of them in $\mathbb{P}^1$ are different from each other. By the action of $g = (0, \begin{pmatrix} a & b \\ c & d \end{pmatrix})$, we obtain

$$\phi(g(p_1)) = [1:0], \quad \phi(g(p_2)) = [1:1], \quad \phi(g(p_3)) = [1:-1]$$

if we properly choose g. Therefore we may assume

$$p_1 = ([1:0],[x_1 : y_1 : 0]), p_2 = ([1:1],[x_2 : y_2 : y_2]), p_3 = ([1:-1],[x_3 : y_3 : (-1)^n y_3]).$$

It follows that $y_k \neq 0$ for $k = 1,2,3$ since we have $g \cdot U \subset U$ for any $g \in G_n$.

In the case of $n = 1$ we put

$$a = \frac{x_1}{y_1} - \frac{x_2}{2y_2} - \frac{x_3}{2y_3}, \quad a_0 = -\frac{x_2}{2y_2} - \frac{x_3}{2y_3}, \quad a_1 = \frac{x_1}{y_1} - \frac{x_2}{y_2}.$$

Then p_1, p_2, p_3 respectively move to $([1:0],[1:1:0])$, $([1:1],[1:1:1])$, $([1:-1],[1:1:(-1)^n])$ by the action of $(a_0S+a_1T, \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}) \in G_1$, since we may assume $x_1/y_1 - x_2/2y_2 - x_3/2y_3 \neq 0$ since p_1, p_2, p_3 are general points.

In the case of $n \geq 2$, we put $m = 2\lfloor n/2 \rfloor$,

$$a_0 = \frac{x_1-y_1}{y_1}, \quad a_1 = -\frac{x_2-y_2}{2y_2} + \frac{x_3+y_3}{2y_3}, \quad a_m = -\frac{x_1-y_1}{y_1} - \frac{x_2-y_2}{2y_2} - \frac{x_3+y_3}{2y_3},$$

and $a_k = 0$ for $k \neq 0, 1, m$. Then p_1, p_2, p_3 respectively move to $([1:0],[1:1:0])$, $([1:1],[1:1:1])$, $([1:-1],[1:1:(-1)^n])$ by the action of $(\sum_{k=0}^{n} a_k S^k T^{n-k}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) \in G_n$. □

Proof of (1) for $\mathbb{F}_n$. We define the Zariski open sets $W_k \cong \mathbb{C} \times \mathbb{C}$ in $\mathbb{F}_n$ for $k = 1, 2, 3$ as follows:

$$\begin{array}{rcl} i_1 : W_1 & \to & \mathbb{F}_n \\ (x,y) & \mapsto & ([1:x],[1:y:x^n y]), \end{array} \quad \begin{array}{rcl} i_2 : W_2 & \to & \mathbb{F}_n \\ (u,v) & \mapsto & ([1:u],[v:1:u^n]), \end{array}$$

$$\begin{array}{rcl} i_3 : W_3 & \to & \mathbb{F}_n \\ (\zeta,\eta) & \mapsto & ([\zeta:\eta],[1:\zeta^n\eta:\eta]). \end{array}$$

We take $\theta = a(x,y)\partial/\partial x + b(x,y)\partial/\partial y \in H^0(W_1, T_{W_1})$. The section θ extends to a holomorphic global section of $T_{\mathbb{F}_n}$ if and only if θ is holomorphic on W_2 and W_3, since the codimension of $\mathbb{F}_n \setminus \cup_{k=1,2,3} W_k$ is two. A straightforward computation yields

$$\theta = a(u,1/v)\frac{\partial}{\partial u} - v^2 b(u,1/v)\frac{\partial}{\partial v} \quad \text{on } W_1 \cap W_2,$$

$$\theta = -\zeta^2 a(1/\zeta, \zeta^n\eta)\frac{\partial}{\partial \zeta} + \left(n\zeta\eta a(1/\zeta,\zeta^n\eta) + \frac{b(1/\zeta,\zeta^n\eta)}{\zeta^n}\right)\frac{\partial}{\partial \eta} \quad \text{on } W_1 \cap W_3.$$

Hence it can be seen that the section θ can be extended to a global holomorphic section of $T_{\mathbb{F}_n}$ if and only if we define $a(x,y)$ and $b(x,y)$ to be

$$a(x,y) = a_0 + a_1x + a_2x^2 \quad \text{and} \quad b(x,y) = (b_0 - na_2x)y + b_1(x)y^2$$

for some $a_0, a_1, a_2, b_0 \in \mathbb{C}$ and for some $b_1(x) \in \mathbb{C}[x]$ with $\deg(b_1) \leq n$. We define

$$\mathcal{T} := \left\{ (a_0 + a_1x + a_2x^2)\frac{\partial}{\partial x} + (b_0y - na_2xy + b_1y^2 + b_2xy^2)\frac{\partial}{\partial y} \,\middle|\, a_0, a_1, a_2, b_0, b_1, b_2 \in \mathbb{C} \right\}.$$

Then, by the above observation, it can be seen that any $\theta \in \mathcal{T}$ extends to a holomorphic global section of $T_{\mathbb{F}_n}$.

Let $\pi : X \to \mathbb{F}_n$ be the blow-up of $\mathbb{F}_n$ along general two points p_1, p_2. By Claim 5.3.9, we may assume $p_1, p_2 \in W_1$, $p_1 = (0,1)$ and $p_1 = (1,1)$. We choose θ_1 and θ_2 in

$\mathcal{T}$ as follows:

$$\theta_1 = y(y-1)\frac{\partial}{\partial y} \quad \text{and} \quad \theta_2 = x(x-1)\frac{\partial}{\partial x} + nxy(y-1)\frac{\partial}{\partial y}.$$

By Lemma 5.3.7, the sections θ_1 and θ_2 can be lifted to holomorphic global sections of T_X. For any point $q = (x, y) \in W_1$ such that $x \neq 0, 1$ and $y \neq 0, 1$, $(\theta_1)_q$ and $(\theta_2)_q$ give a basis of $T_{W_1,q}$. Therefore T_X is generically globally generated. □

Proof of (2) for $\mathbb{F}_n$. Let $\pi : X \to \mathbb{F}_n$ be a blow-up of $\mathbb{F}_n$ along general three points p_1, p_2, p_3. We show that $\mathrm{Sym}^2(T_X)$ is generically globally generated. By Claim 5.3.9, we may assume $p_1, p_2, p_3 \in W_1$, $p_1 = (0, 1)$, $p_2 = (1, 1)$, and $p_3 = (-1, -1)$.

We take

$$\theta = a(x,y)\Big(\frac{\partial}{\partial x}\Big)^2 + b(x,y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + c(x,y)\Big(\frac{\partial}{\partial y}\Big)^2 \in H^0(W_1, \mathrm{Sym}^2 T_{W_1}).$$

First we investigate the condition when θ extends to a global holomorphic section of $\mathrm{Sym}^2 T_{\mathbb{F}_n}$. We have

$$\theta = a(u,1/v)\Big(\frac{\partial}{\partial u}\Big)^2 - v^2 b(u,1/v)\frac{\partial}{\partial u}\frac{\partial}{\partial v} + v^4 c(u,1/v)\Big(\frac{\partial}{\partial v}\Big)^2 \text{ on } W_1 \cap W_2 \text{ and,}$$

$$\begin{aligned}\theta = {}& \zeta^4 a(1/\zeta, \zeta^n\eta)\Big(\frac{\partial}{\partial \zeta}\Big)^2 + \Big(-2n\zeta^3\eta a(1/\zeta, \zeta^n\eta) - \frac{1}{\zeta^{n-2}}b(\zeta, \zeta^n\eta)\Big)\frac{\partial}{\partial \zeta}\frac{\partial}{\partial \eta} \\ & + \Big(n^2\zeta^2\eta^2 a(1/\zeta, \zeta^n\eta) + \frac{n\eta}{\zeta^{n-1}}b(1/\zeta, \zeta^n\eta) + \frac{1}{\zeta^{2n}}c(1/\zeta, \zeta^n\eta)\Big)\Big(\frac{\partial}{\partial \eta}\Big)^2 \text{ on } W_1 \cap W_3.\end{aligned}$$

In the case of $n = 1$, the section θ extends to a global holomorphic section of $\mathrm{Sym}^2 T_{\mathbb{F}_n}$ if we have

- $a(x, y) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$,
- $b(x, y) = (b_0 + b_1x + b_2x^2 - 2a_4x^3)y + (b_3 + b_4x + b_5x^2 + b_6x^3)y^2$,
- $c(x, y) = (c_0 - (a_3 + b_2)x + a_4x^2)y^2 + (c_1 + c_2x - b_6x^2)y^3 + (c_3 + c_4x + c_5x^2 + c_6x^3 + c_7x^4)y^4$,

where all coefficients are constant. Here we put

- $\theta_1 = x(x^2-1)\Big(\frac{\partial}{\partial x}\Big)^2 + y\Big(-3x^2 + y(x^3+x^2+x-1)+1\Big)\frac{\partial}{\partial x}\frac{\partial}{\partial y}$
  $+ y^2\Big(2x + y^2(x^2+1) - y(x+1)^2\Big)\Big(\frac{\partial}{\partial y}\Big)^2,$
- $\theta_2 = x^2(-x^2+1)\Big(\frac{\partial}{\partial x}\Big)^2 + 2x^2y(x-y)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + x^2y^2(y^2-1)\Big(\frac{\partial}{\partial y}\Big)^2,$
- $\theta_3 = x(-x^2+1)\Big(\frac{\partial}{\partial x}\Big)^2 + y\Big(3x^2 + y(-x^2-2x+1)-1\Big)\frac{\partial}{\partial x}\frac{\partial}{\partial y}$
  $+ y^2\Big(-2x + y^2(2x-1)+1\Big)\Big(\frac{\partial}{\partial y}\Big)^2.$

Then, by using Lemma 5.3.7 again, the sections $\pi^*\theta_1$, $\pi^*\theta_2$ and $\pi^*\theta_3$ extend to holomorphic global sections of $\mathrm{Sym}^2 T_X$. For a general point $q \in W_1$, $\theta_1(q)$, $\theta_2(q)$, and $\theta_3(q)$ give basis of $\mathrm{Sym}^2 T_{W_1,q}$. Therefore $\mathrm{Sym}^2 T_X$ is generically globally generated.

In the case of $n \geq 2$, the section θ extends to a holomorphic global section of $\mathrm{Sym}^2 T_{\mathbb{F}_n}$ if

- $a(x,y) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$,
- $b(x,y) = (b_0 + b_1x + b_2x^2 - 2na_4x^3)y + (b_3 + b_4x + b_5x^2 + b_6x^3)y^2$,
- $c(x,y) = (c_0 - (n^2a_3 + nb_2)x + n^2a_4x^2)y^2 + (c_1 + c_2x + c_3x^2)y^3 + (c_4 + c_5x + c_6x^2 + c_7x^3 + c_8x^4)y^4$,

where all coefficients are constant. We put

- $\theta_1 = xy^2(x^2-1)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y^3\Big(-3x^2 + y(-x^4+2x^3+2x^2-1)+1\Big)\Big(\frac{\partial}{\partial y}\Big)^2,$
- $\theta_2 = xy^2(x^2-1)\frac{\partial}{\partial x}\frac{\partial}{\partial y} + y^2\Big(xy^2(x+2) - y(x+1)^2+1\Big)\Big(\frac{\partial}{\partial y}\Big)^2,$
- $\theta_3 = x(x^3-2x^2-x+2)\Big(\frac{\partial}{\partial x}\Big)^2$

$$+ y\Big(-2nx^3 + 6x^2 + 2x(n-1) - 2 + y\big(nx(n-6) + x^3(-n^2+6n-4)+2\big)\Big)\frac{\partial}{\partial x}\frac{\partial}{\partial y}$$

$$+ y^2\Big(nx(nx+2n-6) + 2n + 1 + y\big(-n^2(x+1)^2 + y(n^2+6nx-2n-1)\big)\Big)\Big(\frac{\partial}{\partial y}\Big)^2.$$

Then $\pi^*\theta_1$, $\pi^*\theta_2$ and $\pi^*\theta_3$ extend to holomorphic global sections of $\mathrm{Sym}^2 T_X$. For a general point $q \in W_1$, $\theta_1(q)$, $\theta_2(q)$ and $\theta_3(q)$ give basis of $\mathrm{Sym}^2 T_{W_1,q}$. Therefore $\mathrm{Sym}^2 T_X$ is generically globally generated. $\square$

CHAPTER 6

Miscellanies

6.1. Lelong number and non Kähler locus

The Lelong number of a singular hermitian metric on a vector bundle was defined by Berndtsson [**Ber17**]. Based on the Berndtsson work, we define a new Lelong number.

Let U be a unit ball in $\mathbb{C}^n$, $E = U \times \mathbb{C}^r$, h be a Griffiths semipositive singular hermitian metric. We take a standard frame $e_1, \cdots, e_r$ of E. Then we have $\mathbb{P}(E) = U \times \mathbb{P}^{r-1}$ and $\mathcal{O}_{\mathbb{P}(E)}(1)$ can be endowed with a singular hermitian metric g with semipositive curvature current induced by h. We have $g = e^{-\varphi(z,W)}$, where $\varphi(z, W)$ is a quasi-plurisubharmonic function on $U \times \mathbb{P}^{r-1}$.

Definition 6.1.1. We will denote by $\nu(\varphi, (0, W))$ the Lelong number φ at $(0, W)$. We define the folloing number.

$$\nu_{\sup}(h, 0) := \sup_{W \in \mathbb{P}^{r-1}} \nu(\varphi, (0, W)), \quad \nu_{\inf}(h, 0) := \inf_{W \in \mathbb{P}^{r-1}} \nu(\varphi, (0, W))$$

We explain more explicitly. Let $h^* = (h^*_{ij})$ be the dual metric of h on E. We take the chart $\{[W_1 : \cdots : W_r] \in \mathbb{P}^{r-1} \colon W_r \neq 0\}$ of $\mathbb{P}(E)$. As in Lemma 4.2.2, we define the isomorphism by

$$\begin{array}{ccc} U \times \{W_r \neq 0\} & \to & U \times \mathbb{C}^{r-1} \\ (z, [W_1 : \cdots : W_r]) & \to & (z, \frac{W_1}{W_r}, \cdots, \frac{W_{r-1}}{W_r}) \end{array}$$

and we may regard $U \times \{W_r \neq 0\}$ as $U \times \mathbb{C}^{r-1}$. Put $\eta_l := \frac{W_l}{W_r}$ for $1 \leq l \leq r-1$ and $\eta_r := 1$. In this setting, we have

$$\mathcal{O}_{\mathbb{P}(E)}(-1)|_{U \times \mathbb{C}^{r-1}} = \{(z, \eta, \xi) \in U \times \mathbb{C}^{r-1} \times \mathbb{C}^r \colon \eta_i \xi_j = \eta_j \xi_i\}$$

and the local section

$$e_{\mathcal{O}_{\mathbb{P}(E)}(-1)}(z, (\eta_1, \cdots, \eta_{r-1})) := (z, (\eta_1, \cdots, \eta_{r-1}), (\eta_1, \cdots, \eta_{r-1}, 1)).$$

Then the dual metric $g^* = e^{\varphi}$ on $\mathcal{O}_{\mathbb{P}(E)}(-1)$ is written by

$$g^*(z, \eta) = |(\eta_1, \cdots \eta_{r-1}, 1)|^2_{h^*} = \sum_{1 \leq i,j \leq r} h^*_{ij}(z) \eta_i \bar{\eta_j} = \frac{\sum_{1 \leq i,j \leq r} h^*_{ij}(z) W_i \bar{W}_j}{|W_r|^2}.$$

Therefore we have

$$\varphi(z,W)=\log(\sum_{1\le i,j\le r} h^*_{ij}(z)W_i\bar{W}_j)-2\log|W_r|$$

and

$$\nu(\varphi,(0,W))=\nu(\log(\sum_{1\le i,j\le r} h^*_{ij}(z)W_i\bar{W}_j),(0,W)).$$

for any $W\in\{W_r\neq 0\}$.

In this setting, we explain the relationship with the Lelong number defined by Berndtsson [**Ber17**]. For any $a=(a_1,\cdots,a_r)\in\mathbb{C}^r\setminus\{0\}$, we write $u_a=\sum_{1\le i\le r}a_ie^*_i$. The Lelong number of h^* at 0 in the direction u_a is defined by

$$\gamma_{h^*}(u_a,0)=\liminf_{z\to 0}\frac{\log|u_a|^2_{h^*}}{\log|z|}.$$

From $\log|u_a|^2_{h^*}=\log(\sum_{1\le i,j\le r}h^*_{ij}(z)a_i\bar{a}_j)$, we have the following corollary.

Corollary 6.1.2. In the above setting, the following hold.

(1) $\gamma_{h^*}(u_a,0)=\nu(\varphi|_{U\times\{[a_1:\cdots:a_r]\}},0)\ge\nu(\varphi,(0,[a_1:\cdots:a_r]))$ for any $a=(a_1,\cdots,a_r)\in\mathbb{C}^r\setminus\{0\}$.
(2) $\gamma_{h^*}(u_a,0)=\nu(\varphi,(0,[a_1:\cdots:a_r]))$ for general $a=(a_1,\cdots,a_r)\in\mathbb{C}^r\setminus\{0\}$.
(3) $\nu_{\inf}(h,0)=\inf_{(a_1,\cdots,a_r)\in\mathbb{C}^r\setminus\{0\}}\gamma_{h^*}(u_a,0)$.

Proof. The first equality of (1) is clear. By [**Dembook**, Theorem 7.13] we have the second inequality of (1) and (2).

We prove (3). By (1), we have $\nu_{\inf}(h,0)\le\inf_{(a_1,\cdots,a_r)\in\mathbb{C}^r\setminus\{0\}}\gamma_{h^*}(u_a,0)$. By [**Dem12**, Lemma 2.17], we have

$$\nu_{\inf}(h,0)=\nu(\varphi,\{0\}\times\mathbb{P}^{r-1})=\nu(\varphi,(0,[a_1:\cdots:a_r]))$$

for general $a=(a_1,\cdots,a_r)\in\mathbb{C}^r\setminus\{0\}$. Therefore by using (2), the proof is complete. □

We introduce the non-Kähler locus on vector bundles.

Definition 6.1.3. Let X be a smooth projective n-dimensional variety and E be a holomorphic vector bundle of rank r on X.

(1) For any $k\in\mathbb{N}_{>0}$ and any ample line buundle, we set

$$\mathcal{H}^+_{k,A}=\{h\colon h \text{ is a Griffiths semipositive singular hermitian metric of } \mathrm{Sym}^k(E)\otimes A^{-1}\}$$

(2) If E is V-big, the non-Kähler locus $L_{nK}(E)$ is defined by

$$L_{nK}(E):=\bigcap_{k,A}\bigcap_{h\in\mathcal{H}^+_{k,A}}\{x\in X\colon\nu_{\sup}(h,x)>0\},$$

where the cap is taken over all $k\in\mathbb{N}_{>0}$ and all ample line bundle A. By Theorem 4.1.2, this locus is well-defined.

This is a higher rank analogy of Boucksom's non-Kähler locus [**Bou04**]. In this section, we prove the following theorem.

THEOREM 6.1.4. If E is big, $L_{nK}(E) = \mathbb{B}_+(E)$ holds .

Therefore, we give a characterization of the augumented base locus by using singular hermitian metrics on vector bundles and the Lelong number.

Before the proof, we recall a singular hermitian metric induced by holomorphic sections, proposed by Hosono [**Hos17**, Chapter 4]. We assume that E is a globally generated at general point. Let $s_1, \dots, s_N \in H^0(X, E)$ be holomorphic sections. We take a local coordinate U and take a local holomorphic frame $e_1, \dots, e_r$ of E on U. Write $s_\alpha = \sum_{1 \le j \le r} f_{\alpha j} e_j$, where $f_{\alpha j}$ are holomorphic functions on U. A singular hermitian metric h_s induced by $s_1, \dots, s_N$ is given by

$$(h_s^*)_{jk} := \sum_{1 \le \alpha \le N} f_{\alpha j} \bar{f_{\alpha k}}.$$

By [**Hos17**, Example 3.6 and Proposition 4.1], h_s is Griffiths semipositive.

PROPOSITION 6.1.5. In this setting, $\{x \in X \colon \nu_{\sup}(h_s, x) > 0\} \subset Bs(E)$ holds.

PROOF. The $N \times r$ matrix A is defined by $A_{\alpha j} = f_{\alpha j}$ as in Lemma 4.2.2. By the standard linear algebra, we have $Bs(E) \cap U = \{x \in U \colon \text{rank } A(x) < r\}$.

Let $g = e^{-\varphi}$ be a singular hermitian metric with semipositive curvature current on $\mathcal{O}_{\mathbb{P}(E)}(1)$ induced by h_s. By the above argument, we have

$$\nu(\varphi, (z, W)) = \nu(\log(\sum_{1 \le j,k \le r, 1 \le \alpha \le N} f_{\alpha j} W_j \overline{f_{\alpha k} W_k}), (0, W)).$$

If $\nu_{\sup}(h_s, x) > 0$, there exists $a \in \mathbb{P}^{r-1}$ such that $\nu(\varphi, (x, a)) > 0$. We obtain

$$\sum_{1 \le j,k \le r, 1 \le \alpha \le N} f_{\alpha j}(x) a_j \overline{f_{\alpha k}(x) a_k} = 0,$$

and consequently we have $\sum_{1 \le j \le r} f_{\alpha j}(x) a_j = 0$ for any $1 \le \alpha \le N$. Hence we have rank $A(x) < r$, therefore $x \in Bs(E)$ holds. □

Now, we prove the Theorem 6.1.4.

PROOF. First, we show that $L_{nK}(E) \subset \mathbb{B}_+(E)$. We take a sufficiently ample line bundle A such that $\mathbb{B}_+(E) = \bigcap_{q \in \mathbb{N}_{>0}} \mathbb{B}(\mathrm{Sym}^q(E) \otimes A^{-1})$ by [**BKK+15**, Remark 2.7]. It is enough to show that $L_{nK}(E) \subset \mathbb{B}(\mathrm{Sym}^q(E) \otimes A^{-1})$ for any $q \in \mathbb{N}_{>0}$ such that $\mathbb{B}(\mathrm{Sym}^q(E) \otimes A^{-1}) \ne X$. We fix $q \in \mathbb{N}_{>0}$ and take $m \in \mathbb{N}_{>0}$ such that $\mathbb{B}(\mathrm{Sym}^q(E) \otimes A^{-1}) = Bs(\mathrm{Sym}^{qm}(E) \otimes A^{-m})$. From $Bs(\mathrm{Sym}^{qm}(E) \otimes A^{-m}) \ne X$, $\mathrm{Sym}^{qm}(E) \otimes A^{-m}$ can be endow with a Griffiths semipositive singular hermitian metric h induced by global sections and $\{x \in X \colon \nu_{\sup}(h, x) > 0\} \subset Bs(\mathrm{Sym}^{qm}(E) \otimes A^{-m})$ holds. By $h \in \mathcal{H}^+_{qm, A^m}$ and the definition of $L_{nK}(E)$, the proof is complete.

For inverse inclusion, we take a point $x \notin L_{nK}(E)$. There exist $k \in \mathbb{N}_{>0}$, an ample line bundle H, and a Griffiths semipositive singular hermitian metric h on $\mathrm{Sym}^k(E) \otimes H^{-1}$ such that $\nu_{\sup}(h,x)=0$. We will show that $x \notin \mathbb{B}_+(E)$, more precisely, there exists $q \in \mathbb{N}_{>0}$ such that $\mathrm{Sym}^q(E) \otimes A^{-1}$ is globally generated at x. This proof is similar to the proof of Theorem 4.3.1.

We take a local coordinate $(U; z_1, \cdots, z_n)$ near x. Let $\phi = \eta(n+1)\log|z-x|^2$, where η is a cut-off function such that $\eta \equiv 1$ near x. and we put $\psi := \frac{n}{n+1}\pi^*\phi$. Let h_H be a positive smooth hermitian metric on H. We take a positive integer m such that

(1) $m\sqrt{-1}\Theta_{h_H,H} + \sqrt{-1}\partial\overline{\partial}\eta \geq 0$ holds in the sense of current, and
(2) $\mathcal{O}_{\mathbb{P}(E)}(r) \otimes \pi^*(A^{-1} \otimes K_X^{-1} \otimes \det E^\vee \otimes H^m)$ is ample.

We will denote by g the singular hermitian metric with semipositive curvature current on $\mathcal{O}_{\mathbb{P}(E)}(k) \otimes H^{-1}$ induced by h. From $\nu_{\sup}(h,x)=0$, there exists a open set $x \in V \subset\subset U$such that g^{2m} is integrable on $\pi^{-1}(V)$ by Skoda's Theorem and the definition of $\nu_{\sup}$.

We put $\widetilde{L} := \mathcal{O}_{\mathbb{P}(E)}(2km) \otimes \pi^* H^{-2m} \otimes \mathcal{O}_{\mathbb{P}(E)}(r) \otimes \pi^*(A \otimes K_X^{-1} \otimes \det E^\vee \otimes H^m)$. Then we have

$$\begin{aligned}\mathcal{O}_{\mathbb{P}(E)}(2km) \otimes A^{-1} &= K_{\mathbb{P}(E)} \otimes \mathcal{O}_{\mathbb{P}(E)}(2km) \otimes \pi^* H^{-2m} \otimes \mathcal{O}_{\mathbb{P}(E)}(r) \otimes \pi^*(A \otimes K_X^{-1} \otimes \det E^\vee \otimes H^m) \\ &= K_{\mathbb{P}(E)} \otimes \widetilde{L}\end{aligned}$$

By the same argument of Theorem 4.3.1, $\widetilde{L}$ has a singular hermitian metric $\widetilde{h}$ with semipositive curvature current such that

$$\sqrt{-1}\Theta_{\widetilde{L},\widetilde{h}} + (1+\frac{\alpha}{n})\sqrt{-1}\partial\overline{\partial}\psi \geq 0 \text{ in the sense of current}$$

for any $\alpha \in [0,1]$.

In this setting, the same proof as in Step 3 of Theorem 4.3.1 (C) $\Rightarrow$ (A) works. The details left to the reader.

□

Unlike the non-Kähler locus, the non-nef locus is difficult. For any $k \in \mathbb{N}_{>0}$ and any ample line bundle A, we set

$$\mathcal{H}_{k,A}^{-} = \{h \colon h \text{ is a Griffiths semipositive singular hermitian metric of } S^k(E) \otimes A\}.$$

and set $\mathcal{H}_k^- := \cup_A \mathcal{H}_{k,A}^-$. For any point x, we write

$$\alpha_k(x) := \inf_{h \in \mathcal{H}_k^-} (\nu_{\sup}(h,x)).$$

Since we have $h^l \in \mathcal{H}_{kl}^-$ for any $h \in \mathcal{H}_k^-$, we obtain $l\alpha_k(x) \geq \alpha_{kl}(x)$. Therefore we define

$$\nu_{met}(E,x) := \inf_k \frac{\alpha_k}{k},$$

which is a higher rank analogy of the minimal multiplicities $\nu(\gamma, x)$ at x for any pseudo-effective cohomology class γ in [**Bou04**, Definition 3.1].

In this setting, we can easily to show that

$$\{x \in X \colon \nu_{met}(E, x) > 0\} \subset \bigcup_k \{x \in X \colon \alpha_k(x) > 0\} \subset \mathbb{B}_-(E)$$

by using the method in Theorem 6.1.4. However the inverse inclusion is unknown. It is difficult since there is no canonical way to give a singular hermitian metric to E by using a metric h_k on $S^k(E)$.

Moreover it is unknown that there exists a minimal singular hermitian metric on a pseudo-effective vector bundle, which is a higher rank analogy of a minimal singular hermitian metric defined by Demailly, Peternell and Schneider [**DPS94**] (see also [**Dem12**, Chapter 6]). It is also an interesting question.

6.2. An example of a rationally connected manifold.

Conjecture 6.2.1. [**NZ18**, Conjecture 1.6] Any compact Kähler manifold with negative scalar curvature cannot be rationally connected.

We give a partial answer of this conjecture.

Theorem 6.2.2. *Let X be a blow up $\mathbb{P}^2$ at general 14 points. Then X has a hermitian metric with negative scalar curvature and X is rationally connected.*

We don't know whether X has a Kähler metric with negative scalar curvature.

Proof. We use the following theorem.

Theorem 6.2.3. [**Yan17**, Theorem 1.3] *Let Y be a compact complex manifold. The following are equivalent.*

(1) K_Y^{-1} *is not pseudo-effective.*
(2) Y *has a hermitian metric with negative scalar curvature.*

It is enough to give a example such that K_X^{-1} is not pseudo-effective and X is rationally connected. Since rationally conectedness is birational property, X is rationally connected. We show that K_X^{-1} is not pseudo-effective.

We denote by $H = \mathcal{O}_{\mathbb{P}^2}(1)$ and by $\pi \colon X \to \mathbb{P}^2$ the blow up morphism. Let $\{E_i\}_{i=1}^{14}$ be exceptional divisors. We have

$$K_X = \pi^*(K_{\mathbb{P}^2}) + \sum_{i=1}^{14} E_i = \pi^*(-3H) + \sum_{i=1}^{14} E_i$$

By [**Ku94**],

$$A := \pi^*(4H) - \sum_{i=1}^{14} E_i$$

is ample divisor on X.

To obtain a contradiction, suppose that K_X^{-1} is pseudo-effective. Then for any $m \in \mathbb{N}_{>0}$ there exists a $n \in \mathbb{N}_{>0}$ such that $n(-mK_X + A)$ has a section. Therefore for any $m \in \mathbb{N}_{>0}$, we have $A(-mK_X + A) > 0$. We point out $(-mK_X + A) = (3m+4)\pi^*(H) - (m+1)\sum_{i=1}^{14} E_i$.

However, we have

$$\begin{aligned} A(-mK_X + A) &= (\pi^*(4H) - \sum_{i=1}^{14} E_i)((3m+4)\pi^*(H) - (m+1)\sum_{i=1}^{14} E_i) \\ &= 12m + 16 - 14(m+1) = -2m + 2, \end{aligned}$$

this is a contradiction. Therefore, K_X^{-1} is not pseudo-effective. □

6.3. Higher Fujita's decomposition

Definition 6.3.1. [**KM98**] Let X be a smooth projective manifold.

(1) A *1-cycle* is a formal linear combination of irreducible reduced and proper curves $C = \sum a_i C_i$.
(2) Two 1-cycle C, C' is *numerically equivalent* if $D.C = D.C'$ for any Cartier divisor D.
(3) $N_1(X)_{\mathbb{R}}$ is a $\mathbb{R}$-vector space of 1-cycles with real coefficients modulo numerical equivalence.

Definition 6.3.2. [**Lazi**] A class $\alpha \in N_1(X)_{\mathbb{R}}$ is *movable* if $D.\alpha \geq 0$ for any effective Cartier divisor D. The set of movable classes forms a closed convex cone $Mov(X) \subset N_1(X)_{\mathbb{R}}$, called the movable cone.

C is a strongly movable curve if $C = \pi_*(A_1 \cap \cdots \cap A_{n-1})$ for some proper modification $\pi : X \to Y$ and some ample divisors $A_1 \cdots A_{n-1}$. By [**BDPP13**], $Mov(X)$ is the closure of the cone spanned by strongly movable curves.

Let X be a smooth projective manifold and $\mathcal{E} \neq 0$ be a torsion-free coherent sheaf on X. For any $\alpha \in Mov(X)$, the *slope of* $\mathcal{E}$ *with respect to* α is defined by

$$\mu_\alpha(\mathcal{E}) := \frac{c_1(\mathcal{E}).\alpha}{rk\mathcal{E}}$$

$\mathcal{E}$ is *α-semistable* if $\mu_\alpha(\mathcal{F}) \leq \mu_\alpha(\mathcal{E})$ for any nonzero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$. $\mathcal{E}$ is *α-stable* if $\mu_\alpha(\mathcal{F}) < \mu_\alpha(\mathcal{E})$ for any nonzero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$ and $\mathcal{F} \neq \mathcal{E}$.

$\mu_\alpha^{max}(\mathcal{E})$ is define by supremum of $\mu_\alpha(\mathcal{F})$ for nonzero coherent subsheaf $\mathcal{F} \subset \mathcal{E}$. $\mu_\alpha^{min}(\mathcal{E})$ is define by infimum of $\mu_\alpha(\mathcal{Q})$ for nonzero torsion-free coherent quotient sheaf $\mathcal{E} \to \mathcal{Q}$. By [**Lazi**], there exists a subsheaf $\mathcal{F}_{max} \subset \mathcal{E}$ and such that $\mu_\alpha^{max}(\mathcal{E}) = \mu_\alpha(\mathcal{F}_{max})$ and there exists a torsion-free coherent quotient sheaf $\mathcal{E} \to \mathcal{Q}_{min}$ such that $\mu_\alpha^{min}(\mathcal{E}) = \mu_\alpha(\mathcal{Q}_{min})$. $\mathcal{F}_{max}$ is called *maximal α-destabilizing subsheaf* of $\mathcal{E}$.

In this section, we prove the following theorems.

THEOREM 6.3.3. Let X be a smooth projective manifold, $\mathcal{E}$ be a reflexive coherent sheaf and $\alpha = A^{n-1}$ for some ample line bundles A. If $\mathcal{E}$ has a Griffiths semipositive singular hermitian metric, then there exists a decompostision $\mathcal{E} \cong \mathcal{Q} \oplus \mathcal{G}$ such that

- $\mathcal{Q}$ is a hermitian flat vector bundle.
- $\mathcal{G}$ is a reflexive coherent sheaf and $\mu_\alpha^{min}(\mathcal{G}) > 0$.

PROOF. We point out $\mu_\alpha^{min}(\mathcal{E}) \geq 0$ since $\mathcal{E}$ is pseudo-effective.

The proof is by induction. If $rk\mathcal{E} = 1$ and $\mu_\alpha^{min}(\mathcal{E}) = 0$, then $\mu_\alpha(\mathcal{E}) = 0$. Hence $c_1(\mathcal{E})A_1 \cdots A_{n-1} = 0$, we have $c_1(\mathcal{E}) = 0$. Since $\mathcal{E}$ is pseudo-effective, $\mathcal{E}$ is hermitian flat.

If $\mu_\alpha^{min}(\mathcal{E}) = 0$, then we have a torsion-free coherent quotient sheaf $\mathcal{E} \to \mathcal{Q}_{min}$ such that $\mu_\alpha^{min}(\mathcal{E}) = \mu_\alpha(\mathcal{Q}_{min}) = 0$. $\mathcal{Q}_{min}$ is a reflexive coherent sheaf such that $c_1(\mathcal{Q}_{min}) = 0$ and $\mathcal{Q}_{min}$ has a Griffiths semipositive singular hermitian metric. By Theorem 5.2.5, $\mathcal{Q}_{min}$ is a hermitian flat vector bundle. We put $r' := rk\mathcal{Q}_{min} < rk\mathcal{E}$. By taking duals, we have

$$0 \to \mathcal{Q}_{min}^\vee \xrightarrow{\phi} \mathcal{E}^\vee \to \mathcal{K} \to 0,$$

where $\mathcal{K}$ is a cokernel of $\phi : \mathcal{Q}_{min}^\vee \to \mathcal{E}^\vee$. We have $\wedge^{r'}\phi \in \mathrm{Hom}(\det(\mathcal{Q}_{min}^\vee), \wedge^{r'}\mathcal{E}^\vee) \cong H^0(X, (\det(\mathcal{Q}_{min}^\vee) \otimes \wedge^{r'}\mathcal{E})^\vee)$. Since $\det(\mathcal{Q}_{min}^\vee) \otimes \wedge^{r'}\mathcal{E}$ is pseudo-effective, $\wedge^{r'}\phi$ is non vanishing on $X_\mathcal{E}$. Therefore $\phi|_{X_\mathcal{E}} : \mathcal{Q}_{min}^\vee|_{X_\mathcal{E}} \to \mathcal{E}^\vee|_{X_\mathcal{E}}$ is an injective bundle morphism and we have $\mathcal{K}|_{X_\mathcal{E}}$ is a vector bundle. By Theorem 5.1.3 we have

$$\mathcal{E}^\vee|_{X_\mathcal{E}} \cong \mathcal{Q}_{min}^\vee|_{X_\mathcal{E}} \oplus \mathcal{K}|_{X_\mathcal{E}}$$

From $codim(X_\mathcal{E}) \geq 2$, we have

$$\mathcal{E} \cong \mathcal{Q}_{min} \oplus \mathcal{K}^\vee,$$

by taking duals. Since $rk\mathcal{K}^\vee < r$, $\mathcal{K}^\vee$ is a reflexive coherent sheaf and $\mathcal{K}^\vee$ has a Griffiths semipositive singular hermitian metric, by induction hypothesis, we have

$$\mathcal{K}^\vee \cong \mathcal{Q}' \oplus \mathcal{G},$$

where $\mathcal{Q}'$ is a hermitian flat vector bundle and $\mu_\alpha^{min}(\mathcal{G}) > 0$. Therefore we put $\mathcal{Q} := \mathcal{Q}_{min} \oplus \mathcal{Q}'$, which complete the proof. □

PROPOSITION 6.3.4. Let X be a smooth projective manifold, $\mathcal{G}$ be a reflexive coherent sheaf and $\alpha = A^{n-1}$ for some ample line bundles A. If $\mu_\alpha^{min}(\mathcal{G}) > 0$, then $\mathcal{G}$ is generically ample, i.e. $\mathcal{G}|_C$ is ample on C for a general curve $C = D_1 \cap \cdots \cap D_{n-1}$ for general $D_i \in |m_iA|$ and $m_i >> 0$.

PROOF. This is a well known to expert. We prove for the readers. We use the following Mehta-Ramanathan's theorem.

THEOREM 6.3.5. [**MR82**][**Miy87**] Let X be a smooth projective manifold, $\mathcal{E}$ be a reflexive coherent sheaf and $\alpha = A^{n-1}$ for some ample line bundles A For large integer m and general $Y \in |mA|$, the maximal α_Y-destabilizing subsheaf of $\mathcal{E}|_Y$ extends to a saturated subsheaf of $\mathcal{E}$, where $\alpha_Y := (A|_Y)^{n-2}$. In particular $\mathcal{E}$ is α-semistable iff $\mathcal{E}|_Y$ is α_Y-semistable.

For any $1 \leq i \leq n-1$, we define $C_i := D_1 \cap \cdots \cap D_i$. We have $C_{n-1} = C$ and we put $C_0 = X$. For any $0 \leq i \leq n-1$, we put $\alpha_{C_i} := (A|_{C_i})^{n-1-i}$ and $\mathcal{F}_i$ is defined by a maximal α_{C_i}-destabilizing subsheaf of $\mathcal{G}^\vee|_{C_i}$. By Mehta-Ramanathan's theorem, we have $\mu_{\alpha_{C_i}}(\mathcal{F}_i) \leq \mu_{\alpha_{C_{i-1}}}(\mathcal{F}_{i-1})$ for any $1 \leq i \leq n-1$. Hence we have

$$\mu_\alpha^{max}(\mathcal{G}^\vee|_C) = \mu_{\alpha_{C_{n-1}}}(\mathcal{F}_{n-1}) \leq \mu_{\alpha_{C_0}}(\mathcal{F}_0) = \mu_\alpha^{max}(\mathcal{G}^\vee) < 0.$$

Therefore we have $\mu_\alpha^{min}(\mathcal{G}|_C) > 0$, $\mathcal{G}|_C$ is ample. □

In particular, we obtain a higher Fujita's decomposition of a direct image sheaf of relative pluricanonical line bundle.

THEOREM 6.3.6. (cf. [**Fuj77**][**CK19**]) Let X be a compact Kähler manifold, Y be a smooth projective manifold and $f : X \to Y$ be a proper surjective morphism with connected fibres. For any $m \in \mathbb{N}_{>0}$. we have a higher Fujita's decomposition

$$(f_*(mK_{X/Y}))^{\vee\vee} \cong Q \oplus G,$$

where Q is a hermitian flat vector bundle and G is a generically ample reflexive coherent sheaf.

In particular if Y is a curve, for any $m \in \mathbb{N}_{>0}$. we have a Fujita's decomposition

$$f_*(mK_{X/Y}) \cong Q \oplus G,$$

where Q is a hermitian flat vector bundle and G is an ample vector bundle.

PROOF. By [**Wang19**, Theorem B](or [**PT18**] ,[**HPS18**] in case when f is projective), $f_*(mK_{X/Y})$ has a Griffiths semipositive singular hermitian metric, which completes the proof by Theorem 6.3.3 and Proposition 6.3.4 □

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