

# REMARKS ON MINIMAL COMPACT KÄHLER MANIFOLDS WITH VANISHING SECOND CHERN CLASS

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## 1. REMARKS ON MINIMAL COMPACT KÄHLER MANIFOLDS WITH VANISHING SECOND CHERN CLASS

**Theorem 1.1.** *Let  $X \rightarrow C$  be a smooth torus fibration onto a curve  $C$  of genus  $\geq 2$ . Then, the numerical Kodaira dimension  $\nu(K_X)$  is equal to one and the cotangent bundle  $\Omega_X$  is nef. In particular,  $c_2(\Omega_X) = 0$ .*

The following proof is based on an idea by Xiaojun Wu.

*Proof.* Since  $f$  is a smooth torus fibration, the curvature of  $K_{X/C}$  is equal to the pull-back of the Weil-Peterson form as follows (c.f. [BCS20, Equation 1.1]):

$$2\pi c_1(K_{X/C}) = \frac{1}{\text{vol}X_c} f^* \omega_{\text{WP}}.$$

The dimension of  $C$  is one, so we conclude that  $c_1(\Omega_X)^2 = 0$ .

Since  $f^*\Omega_C$  is nef, it is enough to show that  $\Omega_{X/C}$  is also nef. This directly follows from [Gri70, Theorem 5.2 and Corollary 7.8] and [Kra97, Proposition 5]. Indeed, by [Gri70],  $R^1 f_* \mathcal{O}_X$  is Griffith seminegative.<sup>1</sup> Thus  $f_* \Omega_{X/C}$  is Griffith semipositive, so is  $\Omega_{X/C} = f^* f_* \Omega_{X/C}$ .  $\square$

Since the condition " $c_2(\Omega_X) = 0$ " is invariant under finite étale covers, we obtain the following corollary from [IM22].

**Corollary 1.2** (cf. [IM22]). *Let  $X$  be a compact Kähler manifold with nef canonical divisor  $K_X$ . Then  $c_2(\Omega_X) = 0$  in  $H^{2,2}(X, \mathbb{R})$  if and only if there exists a finite étale cover  $X' \rightarrow X$  such that one of the following holds depending on the Kodaira dimension:*

- (i) *In the case where  $\nu(K_X) = \kappa(K_X) = 0$ , the variety  $X'$  is isomorphic to a complex torus.*
- (ii) *In the case where  $\nu(K_X) = \kappa(K_X) = 1$ , the variety  $X'$  admits a smooth torus fibration  $X' \rightarrow C$  onto a curve of genus  $\geq 2$ .*

Similarly, we obtain the following corollary.

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<sup>1</sup>In Griffith's paper, the fibration is assumed to be projective. However, this result holds even in the case when  $f$  is proper Kähler submersion. For more details, please refer to [MT07, Section 2.3]. Using the same notation as in [MT07, Section 2.3], setting  $d = 1$  and  $p = 0$ , it follows that  $R^1 f_* \mathcal{O}_X$  is Griffith seminegative.

**Corollary 1.3** (cf. [IMM24]). *Let  $X$  be a projective klt variety of dimension  $n$  with nef canonical divisor  $K_X$ . Then the Miyaoka's equality holds for some ample divisors  $H_i$  on  $X$ :*

$$\left(3\widehat{c}_2(\Omega_X^{[1]}) - \widehat{c}_1(\Omega_X^{[1]})^2\right) H_1 \cdots H_{n-2} = 0$$

*if and only if there exists a finite quasi-étale cover  $X' \rightarrow X$  such that one of the following holds depending on the Kodaira dimension:*

- (i) *In the case where  $\nu(K_X) = \kappa(K_X) = 0$ , the variety  $X'$  is isomorphic to an abelian variety.*
- (ii) *In the case where  $\nu(K_X) = \kappa(K_X) = 1$ , the variety  $X'$  admits the structure of an abelian group scheme  $X' \rightarrow C$  over a curve  $C$  of general type.*
- (iii) *In the case where  $\nu(K_X) = \kappa(K_X) = 2$ , the variety  $X'$  is isomorphic to the product  $A \times S$  of an abelian variety  $A$  and a smooth surface  $S$  whose universal cover is an open ball in  $\mathbb{C}^2$ .*

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