REMARKS ON MIYAOKA'S INEQUALITY FOR COMPACT KÄHLER MANIFOLDS

MASATAKA IWAI

1. Miyaoka's inequality for compact Käler manifolds

In this note, we will prove Miyaoka's inequality for compact Kähler manifolds. A precise statement is as follows:

Theorem 1.1. *Let X be a compact Kähler manifold and ω be a Kähler form. If K_X is nef and $\nu(K_X) \geq 2$, then there exists ε_0 depending on (X, ω) such that*

$$\left(3c_2(\Omega_X^1) - c_1(\Omega_X^1)^2\right)(K_X + \varepsilon\omega)^{n-2} \geq 0. \tag{1.1}$$

holds for any $0 < \varepsilon < \varepsilon_0$.

Moreover, if

$$\left(3c_2(\Omega_X^1) - c_1(\Omega_X^1)^2\right)(K_X + \varepsilon\omega)^{n-2} = 0$$

holds for some $0 < \varepsilon < \varepsilon_0$, then there exists a finite étale cover $A \times S \to X$, where A is a torus and S is a smooth projective surface whose universal cover is an open ball.

In [Miy87], Miyaoka proved this type inequality for normal projective variety smooth in codimension 2. So, this type inequality (1.1) is called "*Miyaoka's inequality.*" There are many studies related to Miyaoka's inequality, for example, [Lan02], [RT23] and [RT22]. Anyway, for any KLT (Kawamata log terminal) projective variety, Miyaoka's Inequality holds by [IMM24]. But in Kähler case, we don't know whether Miyaoka's inequalities hold, even if compact Kähler manifold case. This is because, in Miyaoka's proof, he use some cutting argument by hypersurfaces. His argument can not be applied for compact Kähler manifold.

In [IMM24], we show Miyaoka's inequality by using Higgs bundle. So, by using this argument, we can prove Miyaoka's inequality like (1.1), because the argument about Higgs bundle can be applied for compact Kähler manifolds thanks to Simpson's results in [Sim88].

Hence the proof of Theorem 1.1 is not new. Indeed, we can prove this inequality only by using the argument of [Cao13], [IM22] and [IMM24]. However, it is better to prove it, so the author decided to write the proof here.

Proof of Theorem 1.1.

Step 1: Set up

Set $\nu := \nu(c_1(K_X))$ and $\alpha_\varepsilon := c_1(K_X) + \varepsilon\{\omega\}$ for any $\varepsilon > 0$. By [Cao13, Proposition 2.3], if $\varepsilon > 0$ is small enough, the α_ε^{n-1}-Harder Narasimhan filtration

$$0 =: \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l := \Omega_X^1$$

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is independent of ε. Set $\mathcal{G}_i := \mathcal{E}_i/\mathcal{E}_{i-1}$ and $r_i := \mathrm{rk}(\mathcal{G}_i)$. Since Ω^1_X is α_ε^{n-1}-generically nef for any positive ε small enough by [Cao13], we have $\mu_{\alpha_\varepsilon}(\mathcal{G}_i) \geq 0$. The sheaf $\mathcal{G}_i$ is an α_ε^{n-1}-semistable sheaf. By [IM22, Claim 6.2], we obtain

$$c_1(\mathcal{G}_i)c_1(K_X)^\nu\{\omega\}^{n-1-\nu} = 0.$$

The Bogomolov-Gieseker inequality shows that $c_2(\Omega^1_X)\alpha_\varepsilon^{n-2} > 0$ holds if $l = 1$. Hence, we may assume that $l \geq 2$. Set $a_i := c_1(\mathcal{G}_i)c_1(K_X)^{\nu-1}\{\omega\}^{n-\nu}$. Then, we have

$$\mu_{\alpha_\varepsilon}(\mathcal{G}_i) = \binom{n-1}{\nu-1}\frac{a_i}{r_i}\varepsilon^{n-\nu} + O(\varepsilon^{n-\nu+1}).$$

From $\mu_{\alpha_\varepsilon}(\mathcal{G}_1) > \cdots > \mu_{\alpha_\varepsilon}(\mathcal{G}_l) \geq 0$, we obtain $a_1/r_1 \geq a_2/r_2 \geq \cdots \geq a_l/r_l \geq 0$ for sufficiently small $\varepsilon > 0$.

By [IM22, Section 6], we obtain

(1.2)
$$\begin{aligned}
c_1(K_X)^2\alpha_\varepsilon^{n-2} &= \sum_{1\leq k\leq l} c_1(\mathcal{G}_k)c_1(K_X)\alpha_\varepsilon^{n-2} \\
&= \binom{n-2}{\nu-2}\left(\sum_{1\leq k\leq l} a_k\right)\varepsilon^{n-\nu} + O(\varepsilon^{n-\nu+1}), \\
\frac{(c_1(\mathcal{G}_i)c_1(\mathcal{E})\alpha_\varepsilon^{n-2})^2}{c_1(\mathcal{E})^2\alpha_\varepsilon^{n-2}} &= \left(\binom{n-2}{\nu-2}a_i\varepsilon^{n-\nu} + O(\varepsilon^{n-\nu+1})\right)^2 \cdot \left(\frac{\varepsilon^{-n+\nu}}{\binom{n-2}{\nu-2}\left(\sum_{1\leq k\leq l} a_k\right)} + O(\varepsilon^{-n+\nu+1})\right) \\
&= \binom{n-2}{\nu-2}\left(\sum_{1\leq k\leq l} a_k\right)^{-1} a_i^2\varepsilon^{-n+\nu} + O(\varepsilon^{n-\nu+1}).
\end{aligned}$$

Step 2: Estimate of $c_2(\mathcal{G}_i)\alpha_\varepsilon^{n-2}$.

Since $\mathcal{G}_i$ is α_ε^{n-1}-semistable for any $2 \leq i \leq l$, the Bogomolov-Gieseker inequality yields

$$\left(c_2(\mathcal{G}_i) - \frac{r_i - 1}{2r_i}c_1(\mathcal{G}_i)^2\right)\alpha_\varepsilon^{n-2} \geq 0 \tag{1.3}$$

To get a desired inequality, we need to estimate $c_2(\mathcal{G}_1)\alpha_\varepsilon^{n-2}$ more detail. We define the Higgs sheaf $(\mathcal{H}, \theta)$ by $\mathcal{H} := \mathcal{G}_1 \oplus \mathcal{O}_X$ and

$$\begin{array}{rccc}
\theta\colon & \mathcal{H} = \mathcal{G}_1 \oplus \mathcal{O}_X & \to & \mathcal{H} \otimes \Omega^1_X = (\mathcal{G}_1 \oplus \mathcal{O}_X) \otimes \Omega^1_X \\
& (a, b) & \mapsto & (0, a).
\end{array}$$

Since $\mathcal{G}_1 \subset \Omega^1_X$ is α_ε^{n-1}-semistable with $\mu_{\alpha_\varepsilon^{n-1}}(\mathcal{G}_1) > 0$, the Higgs sheaf $(\mathcal{H}, \theta)$ is α_ε^{n-1}-stable by the same argument in [IMM24, Proposition 2.8]. Hence the Bogomolov-Gieseker inequality in [Sim88] yields

$$\left(c_2(\mathcal{G}_1) - \frac{r_1}{2(r_1+1)}c_1(\mathcal{G}_1)^2\right)\alpha_\varepsilon^{n-2} \geq 0. \tag{1.4}$$

Step 3: Calculation of $\left(6c_2(\Omega^1_X) - 2c_1(\Omega^1_X)^2\right)\alpha_\varepsilon^{n-2}$.

By the same calculation as in [IM22, Section 6], we obtain

(1.5)

$$
\begin{aligned}
&\left(6c_2(\Omega^1_X) - 2c_1(\Omega^1_X)^2\right)\alpha_\varepsilon^{n-2} \\
&= \left(c_1(K_X)^2 + \sum_{2\le i\le l}\left(6c_2(\mathcal{G}_i) - 3c_1(\mathcal{G}_i)^2\right) + 6c_2(\mathcal{G}_1) - 3c_1(\mathcal{G}_1)^2\right)\alpha_\varepsilon^{n-2} \\
&\underset{(1.3)}{\ge} \left(c_1(K_X)^2 + \sum_{2\le i\le l}\left(\frac{3(r_i-1)}{r_i}c_2(\mathcal{G}_i) - 3c_1(\mathcal{G}_i)^2\right) + 6c_2(\mathcal{G}_1) - 3c_1(\mathcal{G}_1)^2\right)\alpha_\varepsilon^{n-2} \\
&= \left(c_1(K_X)^2 - \sum_{2\le i\le l}\left(\frac{3}{r_i}c_1(\mathcal{G}_i)^2\right) + 6c_2(\mathcal{G}_1) - 3c_1(\mathcal{G}_1)^2\right)\alpha_\varepsilon^{n-2} \\
&\underset{\text{(by Hodge index Theorem)}}{\ge} c_1(K_X)^2\alpha_\varepsilon^{n-2} - 3\sum_{2\le i\le l}\frac{(c_1(\mathcal{G}_i)c_1(K_X)\alpha_\varepsilon^{n-2})^2}{r_ic_1(K_X)^2\alpha_\varepsilon^{n-2}} + \left(6c_2(\mathcal{G}_1) - 3c_1(\mathcal{G}_1)^2\right)\alpha_\varepsilon^{n-2}
\end{aligned}
$$

If $r_1 = 1$, the equation (1.4) implies $c_1(\mathcal{G}_1)^2 \le 0$. Thus, as in the estimates of [IM22, p.25], we obtain

$$
\begin{aligned}
&\left(6c_2(\Omega^1_X) - 2c_1(\Omega^1_X)^2\right)\alpha_\varepsilon^{n-2} \\
&\ge c_1(K_X)^2\alpha_\varepsilon^{n-2} - 3\sum_{2\le i\le l}\frac{(c_1(\mathcal{G}_i)c_1(K_X)\alpha_\varepsilon^{n-2})^2}{r_ic_1(K_X)^2\alpha_\varepsilon^{n-2}} \\
&\underset{(1.2)}{=} \binom{n-2}{\nu-2}\left(\sum_{1\le k\le l}a_k\right)^{-1}\left(\sum_{1\le i\le l}a_i\sum_{1\le j\le l}a_j - 3\sum_{2\le i\le l}\frac{a_i^2}{r_i}\right)\varepsilon^{n-\nu} + O(\varepsilon^{n-\nu+1}) \\
&\ge \binom{n-2}{\nu-2}\left(\sum_{1\le k\le l}a_k\right)^{-1}\left(\sum_{1\le i\le l}a_i\sum_{1\le j\le l}a_j - 3a_1\sum_{2\le i\le l}a_i\right)\varepsilon^{n-\nu} + O(\varepsilon^{n-\nu+1}) \quad \left(\text{by } \frac{a_1}{r_1} \ge \frac{a_i}{r_i}\right) \\
&= \binom{n-2}{\nu-2}\left(\sum_{1\le k\le l}a_k\right)^{-1}\left(\left(a_1 - \frac{1}{2}\sum_{2\le i\le l}a_i\right)^2 + \frac{3}{4}\left(\sum_{2\le i\le l}a_i\right)^2\right)\varepsilon^{n-\nu} + O(\varepsilon^{n-\nu+1}).
\end{aligned}
$$

Since $\sum_{1\le i\le l}a_i = 1$,

$$
\left(a_1 - \frac{1}{2}\sum_{2\le i\le l}a_i\right)^2 + \frac{3}{4}\left(\sum_{2\le i\le l}a_i\right)^2
$$

is always positive, hence $\left(3c_2(\Omega^1_X) - c_1(\Omega^1_X)^2\right)\alpha_\varepsilon^{n-2}$ is also positive for any $0 < \varepsilon \ll 1$.

From now on, we may assume $r_1 \ge 2$.

Claim 1.2. The following estimate holds:

$$
\begin{aligned}
&\left(6c_2(\Omega^1_X) - 2c_1(\Omega^1_X)^2\right)\alpha_\varepsilon^{n-2} \\
&\ge \binom{n-2}{\nu-2}\left(\sum_{1\le k\le l}a_k\right)^{-1}\left(\left(1 - \frac{3}{r_1+1}\right)a_1^2 + \sum_{2\le i\le l}a_i\left(2 - \frac{3}{r_1}\right)a_1 + \left(\sum_{2\le i\le l}a_i\right)^2\right)\varepsilon^{n-\nu} + O(\varepsilon^{n-\nu+1})
\end{aligned}
$$

In particular, if $r_1 > 2$ or $\sum_{2\le i\le l} a_i > 0$, then $(3c_2(X) - c_1(X)^2)\,\alpha_\varepsilon^{n-2}$ is positive for any $0 < \varepsilon \ll 1$.

Proof of Claim 1.2. As in the estimates of [IM22, p. 25], we obtain

$$\begin{aligned}
&\left(6c_2(\Omega_X^1) - 2c_1(\Omega_X^1)^2\right)\alpha_\varepsilon^{n-2} \\
\underset{(1.4)}{\geq}\quad & c_1(K_X)^2\alpha_\varepsilon^{n-2} - 3\sum_{2\le i\le l}\frac{(c_1(\mathcal{G}_i)c_1(K_X)\alpha_\varepsilon^{n-2})^2}{r_ic_1(K_X)^2\alpha_\varepsilon^{n-2}} + \left(\frac{3r_1}{r_1+1}c_1(\mathcal{G}_1)^2 - 3c_1(\mathcal{G}_1)^2\right)\alpha_\varepsilon^{n-2} \\
\underset{\text{(by Hodge index Theorem)}}{\geq}\quad & c_1(K_X)^2\alpha_\varepsilon^{n-2} - 3\sum_{2\le i\le l}\frac{(c_1(\mathcal{G}_i)c_1(K_X)\alpha_\varepsilon^{n-2})^2}{r_ic_1(K_X)^2\alpha_\varepsilon^{n-2}} - 3\frac{(c_1(\mathcal{G}_1)c_1(K_X)\alpha_\varepsilon^{n-2})^2}{(r_1+1)c_1(K_X)^2\alpha_\varepsilon^{n-2}} \\
\underset{(1.2)}{=}\quad & \binom{n-2}{\nu-2}\left(\sum_{1\le k\le l}a_k\right)^{-1}\left(\sum_{1\le i\le l}a_i\sum_{1\le j\le l}a_j - 3\sum_{2\le i\le l}\frac{a_i^2}{r_i} - 3\frac{a_1^2}{r_1+1}\right)\varepsilon^{n-\nu} + O(\varepsilon^{n-\nu+1}) \\
\underset{\text{by } \frac{a_1}{r_1}\ge\frac{a_i}{r_i}}{\geq}\quad & \binom{n-2}{\nu-2}\left(\sum_{1\le k\le l}a_k\right)^{-1}\left(\sum_{1\le i\le l}a_i\sum_{1\le j\le l}a_j - 3\frac{a_1}{r_1}\sum_{2\le i\le l}a_i - 3\frac{a_1^2}{r_1+1}\right)\varepsilon^{n-\nu} + O(\varepsilon^{n-\nu+1}) \\
=\quad & \binom{n-2}{\nu-2}\left(\sum_{1\le k\le l}a_k\right)^{-1}\left(\left(1-\frac{3}{r_1+1}\right)a_1^2 + \sum_{2\le i\le l}a_i\left(2-\frac{3}{r_1}\right)a_1 + \left(\sum_{2\le i\le l}a_i\right)^2\right)\varepsilon^{n-\nu} \\
+\quad & O(\varepsilon^{n-\nu+1}).
\end{aligned}$$

Hence, if $r_1 > 2$ or $\sum_{2\le i\le l} a_i > 0$, then

$$\left(1-\frac{3}{r_1+1}\right)a_1^2 + \sum_{2\le i\le l}a_i\left(2-\frac{3}{r_1}\right)a_1 + \left(\sum_{2\le i\le l}a_i\right)^2 > 0,$$

in particular, $(3c_2(\Omega_X^1) - c_1(\Omega_X^1)^2)\,\alpha_\varepsilon^{n-2}$ is positive for any $0 < \varepsilon \ll 1$. □

From now on, we assume that $r_1 = 2$ and $\sum_{2\le i\le l} a_i = 0$. Let us consider the case where $c_1(\mathcal{G}_2)c_1(K_X)^{\nu-t}\{\omega\}^{n-1-\nu+t} \neq 0$ holds for some $t \in \{2, \dots, \nu-1\}$. Then, we take the minimal number $s \in \{2, \dots, \nu-1\}$ such that $c_1(\mathcal{G}_2)c_1(K_X)^{\nu-s}\{\omega\}^{n-1-\nu+s} \neq 0$, and set $b_i := c_1(\mathcal{G}_i)c_1(K_X)^{\nu-s}\{\omega\}^{n-1-\nu+s}$ for any $i = 2, \dots, l$. Then, since we have

$$c_1(\mathcal{G}_i)\alpha_\varepsilon^{n-1} = \binom{n-1}{\nu-s}b_i\varepsilon^{n-\nu+s-1} + O(\varepsilon^{n-\nu+s}) \geq 0,$$

we obtain $b_i \geq 0$. Thus we conclude that $\sum_{2\leq i\leq l} b_i > 0$. Moreover, by the Hodge index Theorem, we can estimate as follows:

(1.6)

$$\begin{aligned} c_1(\mathcal{G}_i)c_1(K_X)\alpha_\varepsilon^{n-2} &= \binom{n-2}{\nu-s-1} b_i \varepsilon^{n-\nu+s-1} + O(\varepsilon^{n-\nu+s}) \\ c_1(\mathcal{G}_i)^2\alpha_\varepsilon^{n-2} &\leq \frac{(c_1(\mathcal{G}_i)c_1(K_X)\alpha_\varepsilon^{n-2})^2}{c_1(K_X)^2\alpha_\varepsilon^{n-2}} = \frac{\binom{n-2}{\nu-s-1}^2 b_i^2}{\binom{n-2}{\nu-2}a_1}\varepsilon^{n-\nu+2s-2} + O(\varepsilon^{n-\nu+2s-1}) \\ \left(\sum_{2\leq i\leq l} c_1(\mathcal{G}_i)\right)^2 \alpha_\varepsilon^{n-2} &\leq \frac{\left(\sum_{2\leq i\leq l} c_1(\mathcal{G}_i)c_1(K_X)\alpha_\varepsilon^{n-2}\right)^2}{c_1(K_X)^2\alpha_\varepsilon^{n-2}} \\ &= \frac{\binom{n-1}{\nu-s-1}^2 (\sum_{2\leq i\leq l} b_i)^2}{\binom{n-2}{\nu-2}a_1}\varepsilon^{n-\nu+2s-2} + O(\varepsilon^{n-\nu+2s-1}). \end{aligned}$$

Hence, by the same argument as in [IM22, p.26], we can get

$$\begin{aligned} &\left(6c_2(\Omega_X^1) - 2c_1(\Omega_X^1)^2\right)\alpha_\varepsilon^{n-2} \\ &\underset{(1.5)}{\geq} \left(c_1(K_X)^2 - 3\sum_{2\leq i\leq l}\frac{c_1(\mathcal{G}_i)^2}{r_i} + 6c_2(\mathcal{G}_1) - 3c_1(\mathcal{G}_1)^2\right)\alpha_\varepsilon^{n-2} \\ &\underset{(1.4)}{\geq} \left(c_1(K_X)^2 - 3\sum_{2\leq i\leq l}\frac{c_1(\mathcal{G}_i)^2}{r_i} - c_1(\mathcal{G}_1)^2\right)\alpha_\varepsilon^{n-2} \\ &= \left(2\sum_{2\leq i\leq l} c_1(\mathcal{G}_i)c_1(K_X) - \left(\sum_{2\leq i\leq l} c_1(\mathcal{G}_i)\right)^2 - \sum_{2\leq i\leq l}\frac{3}{r_i}c_1(\mathcal{G}_i)^2\right)\alpha_\varepsilon^{n-2} \\ &\underset{(1.6)}{\geq} 2\binom{n-2}{\nu-s-1}\left(\sum_{2\leq i\leq l} b_i\right)\varepsilon^{n-\nu+s-1} + O(\varepsilon^{n-\nu+s}). \end{aligned} \tag{1.7}$$

From $s-1>0$ and $\sum_{2\leq i\leq l} b_i > 0$ we obtain $(3c_2(\Omega_X^1) - c_1(\Omega_X^1)^2)\,\alpha_\varepsilon^{n-2} > 0$ for sufficiently small $\varepsilon > 0$.

Hence, from now on, we assume that $c_1(\mathcal{G}_i)c_1(K_X)^t\{\omega\}^{n-1-t} = 0$ for any $t = 1, \ldots, n-1$ and $i = 2, \ldots, l$. Then we have

$$c_1(\mathcal{G}_i)c_1(K_X)\alpha_\varepsilon^{n-2} = c_1(\mathcal{G}_i)c_1(K_X)(c_1(K_X) + \varepsilon\{\omega\})^{n-2} = 0. \tag{1.8}$$

From $c_1(K_X)^2\alpha_\varepsilon^{n-2} > 0$, we obtain

$$c_1(\mathcal{G}_i)^2\alpha_\varepsilon^{n-2} \leq 0 \quad \text{and} \quad \left(\sum_{2\leq i\leq l} c_1(\mathcal{G}_i)\right)^2 \alpha_\varepsilon^{n-2} \leq 0 \tag{1.9}$$

by the Hodge index Theorem in [IM22, Lemma 6.1]. Thus, it holds that

$$\begin{aligned}
&\left(6c_2(\Omega_X^1) - 2c_1(\Omega_X^1)^2\right)\alpha_\varepsilon^{n-2} \\
&\underset{(1.7)}{\geq} \left(2\sum_{2\leq i\leq l} c_1(\mathcal{G}_i)c_1(K_X) - \left(\sum_{2\leq i\leq l} c_1(\mathcal{G}_i)\right)^2 - \sum_{2\leq i\leq l}\frac{3}{r_i}c_1(\mathcal{G}_i)^2\right)\alpha_\varepsilon^{n-2} \\
&\underset{(1.8) \text{ and } (1.9)}{\geq} 0.
\end{aligned}$$

To summarize all the discussions in Step 3, we can say that $\left(3c_2(\Omega_X^1) - c_1(\Omega_X^1)^2\right)\alpha_\varepsilon^{n-2}$ is nonnegative for any $0 < \varepsilon \ll 1$.

Step 4: The structure of X if equality holds in (1.1)

We consider the case of $\left(6c_2(\Omega_X^1) - 2c_1(\Omega_X^1)^2\right)\alpha_\varepsilon^{n-2} = 0$ for some small $0 < \varepsilon \ll 1$. By the argument in Step 3, we obtain

$$\operatorname{rk}\mathcal{E}_1 = 2 \quad \text{and} \quad c_1(\mathcal{G}_i)^2\alpha_\varepsilon^{n-2} = 0$$

for any $2 \leq i \leq l$. Hence the Hodge index Theorem in [IM22, Lemma 6.1] implies $c_1(\mathcal{G}_i) \equiv 0$, and finally we can get $c_1(\mathcal{G}_1) \equiv c_1(K_X)$ and $l = 2$.

Set $\mathcal{Q} := \Omega_X^1/\mathcal{G}_1$. Then we have

$$\left(3c_2(\Omega_X^1) - c_1(\Omega_X^1)^2\right)\alpha_\varepsilon^{n-2} = \left(3c_2(\mathcal{G}_1) - c_1(\mathcal{G}_1)^2\right)\alpha_\varepsilon^{n-2} + 3c_2(\mathcal{Q})\alpha_\varepsilon^{n-2}. \tag{1.10}$$

Since the two terms in RHS of (1.10) is nonnegative, hence we obtain

$$\left(3c_2(\mathcal{G}_1) - c_1(\mathcal{G}_1)^2\right)\alpha_\varepsilon^{n-2} = 3c_2(\mathcal{Q})\alpha_\varepsilon^{n-2} = 0.$$

Thus $\mathcal{Q}$ is a rank $n-2$ flat locally free sheaf. Hence, $\mathcal{Q}^\vee \subset T_X$ is a regular codimension 2 foliation with hermitian flat structure. Thus, by [PT13], there exists a finite étale cover $A \times S \to X$, where A is a torus and S is a smooth projective surface whose universal cover is an open ball (see also [IMM24, Theorem 4.12]).

□

By putting together [IM22] and Theorem 1.1, we obtain the following theorem.

Theorem 1.3. *Let X be a compact Kähler manifold and ω be a Kähler form. If K_X is nef, then there exists ε_0 depending on (X, ω) such that*

$$\left(3c_2(\Omega_X^1) - c_1(\Omega_X^1)^2\right)(K_X + \varepsilon\omega)^{n-2} \geq 0 \tag{1.11}$$

holds for any $0 < \varepsilon < \varepsilon_0$.

Moreover, if

$$\left(3c_2(\Omega_X^1) - c_1(\Omega_X^1)^2\right)(K_X + \varepsilon\omega)^{n-2} = 0$$

holds for some $0 < \varepsilon < \varepsilon_0$, then, the canonical divisor K_X is semi-ample and $\nu(K_X) = \kappa(K_X)$ is either 0, 1, or 2. Moreover, up to finite étale cover of X, one of the following holds depending on the Kodaira dimension:

- *(i) In the case where $\nu(K_X) = \kappa(K_X) = 0$, the variety X is isomorphic to a complex torus.*
- *(ii) In the case where $\nu(K_X) = \kappa(K_X) = 1$, the variety X admits a smooth torus fibration $X \to C$ over a curve C of genus ≥ 2.*

(iii) *In the case where $\nu(K_X) = \kappa(K_X) = 2$, the variety X is isomorphic to the product $A \times S$ of a complex torus A and a smooth projective surface S whose universal cover is an open ball in $\mathbb{C}^2$.*

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Department of Mathematics, Graduate School of Science, Osaka University, 1-1, Machikaneyama-cho, Toyonaka, Osaka 560-0043, Japan.

Email address: `masataka@math.sci.osaka-u.ac.jp`

Email address: `masataka.math@gmail.com`

[1]In this conference, the author mentioned this contents at his talk. Then one audience asked to him "who proved the Miyaoka's inequality for compact Kähler manifolds ?". By this question, he found that Miyaoka's inequality for compact Kähler manifold has never been known before. So he decided to write this note.