Abundance theorem for minimal compact Kähler manifolds with vanishing second Chern class

joint work with Shin-ichi Matsumura (Tohoku University)

Masataka Iwai

Osaka University

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Notations

- X: compact Kähler manifold over $\mathbb{C}$ of $n := \dim_{\mathbb{C}} X \geq 2$.
- Ω_X^1: holomorphic cotangent bundle of X.
- $K_X := \det \Omega_X^1$: canonical line bundle.

Abundance Conjecture

Let X be a projective manifold (or compact Kähler manifold). If K_X is nef, then K_X is semiample.

L : line bundle on X

1. L is nef
   $\Leftrightarrow L.C := \int_C \deg_C(L|_C) \geq 0$ for any (complex) curve $C \subset X$. (if X is projective).

   $\Leftrightarrow$ for any $\epsilon > 0$, there exists a smooth metric h_ϵ s.t. $\sqrt{-1}\Theta_h \geq -\epsilon\omega$.
2. L is semiample $\Leftrightarrow$ there exists $m \in \mathbb{N}_{>0}$ and a basis $s_0, \ldots, s_N \in H^0(X, L^{\otimes m})$ s.t. the following morphisim is well-defined:

$$\begin{array}{cccc} \Phi_{|L^{\otimes m}|}: & X & \to & \mathbb{CP}^N \\ & x & \longmapsto & (s_0(x) : s_1(x) : \cdots : s_N(x)) \end{array}$$

Abundance conjecture

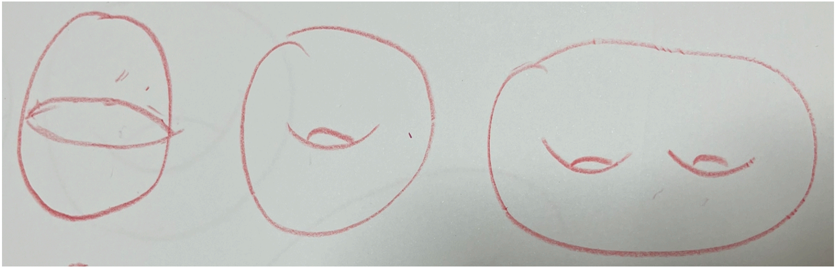
Conjecture (?)

Up to birational equivalence, any projective manifold can be decomposed into the followings varieties:

1. *Fano variety* (*variety with positive Ricci curvature*)*.*
2. *Calabi-Yau variety* (*variety with zero Ricci curvature*)*.*
3. *Canonically polarized variety* (*variety with negative Ricci curvature*)*.*

$\dim_{\mathbb{C}} X = 1$ case

	$\mathbb{CP}^1$	elliptic curve	curve with genus ≥ 2
genus	$g = 0$	$g = 1$	$g \geq 2$
Ricci curvature	$+$	0	$-$
curvature of K_X	$-$	0	$+$
	Fano	Calabi-Yau	Canonically polarized



$\dim_{\mathbb{C}} X = 2$ case

If $\dim_{\mathbb{C}} X = 2$, by Castelnuovo contraction theorem, there exists a sequence of birational morphisms ("Blow-down"):

$$X := X_0 \to X_1 \to X_2 \to \cdots \to X_k =: Y$$

satisfying one of the followings:

1. Y is $\mathbb{CP}^2$ or $\mathbb{CP}^1$-bundle. [Fano fibration]
2. K_Y is nef.

In $\dim_{\mathbb{C}} X = 2$, Abundance conjecture holds. So in the 2nd case, K_Y is semiample and there exists a fibration $\Phi_{|K_Y^{\otimes m}|} : Y \to Z$ s.t.

1. $(\dim Y > \dim Z)$ a general fiber of $\Phi_{|K_Y^{\otimes m}|}$ is Calabi-Yau. In particular, Y is torus, bielliptic, K3, Enriques, or elliptic surface. [Calabi-Yau fibration]
2. $(\dim Y = \dim Z)$ $\Phi_{|K_Y^{\otimes m}|}$ is birational and Z is canonically polarized.

$\therefore$ The conjecture holds if $\dim_{\mathbb{C}} X = 2$.

Higher dimensional case

Conjecture (Existence of Minimal model)

For any projective manifold X, there exists a projective variety Y birational to X such that

1. *Y has Fano fibration, i.e. $Y \to Z$ such that general fiber is Fano, or*
2. *K_Y is nef.*

Conjecture (Abundance Conjecture)

If K_Y is nef, then K_Y is semiample.

If two conjecture holds, then for any projective manifold X, there exists a projective variety Y birational to X such that

1. Y has Fano fibration,
2. Y has Calabi-Yau fibration, or
3. Y is canoincally polarized.

Main result

Theorem (I.-Matsumura 22)

Let X be a compact Kähler manifold of $n := \dim_{\mathbb{C}} X \geq 2$. If $c_2(X) = c_2(\Omega^1_X) = 0$ and K_X is nef, then K_X is semiample.

In particular, if $c_2(X) = 0$, then the abundance conjecture holds.

Corollary (Höring 13 (cf. I.-Matsumura 22))

Under the assumptions above, there exists a finite covering $X' \to X$ such that one of the followings holds:

1. *X' is a torus.*
2. *there exists a (holomorphic) submersion $X' \to Y$ such that any fiber is torus and Y is a curve with genus ≥ 2.*

In particular, if $c_2(X) = 0$, then X can be decomposed into a torus and a curve with genus ≥ 2.

Definition

Let L be a nef line bundle.

- The Iitaka dimension $\kappa(L)$ is defined by

$$\kappa(L) := \limsup_{m \to \infty} \frac{\log \dim_{\mathbb{C}} H^0(X, L^{\otimes m})}{\log m} \in \{-\infty, 0, \ldots, n\}.$$

- The numerical Iitaka dimension $\nu(L)$ is defined by

$$\nu(L) := \max\{k \,|\, c_1(L)^k \neq 0 \in H^{k,k}(X, \mathbb{R})\} \in \{0, \ldots, n\}.$$

- L is abundant iff $\kappa(L) = \nu(L)$.

In general $\kappa(L) \leq \nu(L)$ holds.

Theorem (Kawamata 85, Nakayama 92)

If K_X is nef and abundant, then K_X is semiample.

Abundance conjecture holds in the following case:

1. $\nu(K_X) = n$.
   For any nef line bundle L, if $\nu(L) = n$ then L is abundant.
2. $\nu(K_X) = 0$.
   By the Beauville-Bogomolov decomposition theorem.
3. $\dim_{\mathbb{C}} X \leq 2$.
   By an elementary (but not easy) argument.
4. $\dim_{\mathbb{C}} X = 3$.
   - X is projective and $\nu(K_X) = 1$. [Miyaoka 87][Miyaoka 88] (4 papers).
   - X is projective and $\nu(K_X) = 2$. [Kawamata 92].
   - X is Kähler. [Campana-Höring-Peternell 16].

Proof of main result

Sketch of the proof

1. If $c_2(X) = 0$ and K_X is nef, then $\nu(K_X) \leq 1$ and Ω^1_X is nef.
2. If Ω^1_X is nef and $\nu(K_X) \leq 1$, then K_X is semiample.

A vector bundle E is nef iff $\mathcal{O}_{\mathbb{P}(E)}(1)$ is nef on $\mathbb{P}(E)$.

Proof of Step 1 -generically nef-

Theorem (Miyaoka 87, Campana-Peternell 11, Campana-Păun 19, Enoki 88, Cao 13)

If K_X is nef, then $\int_X c_1(Q) \wedge \omega^{n-1} \geq 0$ holds for any Kähler form ω and any torsion-free sheaf $\Omega_X^1 \twoheadrightarrow Q$.

Definition

A vector bundle E is *generically nef* if $\int_X c_1(Q) \wedge \omega^{n-1} \geq 0$ holds for any Kähler form ω and any torsion-free sheaf $E \twoheadrightarrow Q$.

In general, generically nef does not imply nef.

E nef $+ \nu(\det(E)) \leq 1 \Rightarrow c_2(E) = 0$
E generically nef $+ \nu(\det(E)) \leq 1 \not\Rightarrow c_2(E) = 0$

Theorem (Miyaoka 87, Ou17, Cao13, I.-Matsumura 22)

Let E be a generically nef vector bundle. If $\det E$ is nef and $c_2(E) = 0$, then one of the followings holds:

1. *If $\nu(c_1(E)) \geq 2$, there exists a nef line bundle $L \subset E$ such that $L \cong \det E$ and $c_2(E/L) = 0$.*
2. *If $\nu(c_1(E)) = 1$, there exists a filtration of torsion-free sheaves*

   $$0 = E_0 \subset E_1 \subset \cdots \subset E_l = E$$

   such that the followings hold:
   - *For any small $\epsilon > 0$, this filtration is Harder-Narasimhan filtration with respect to $(c_1(E) + \epsilon\omega)^{n-1}$.*
   - *For any $1 \leq i \leq l$, the exists a nonnegative real number λ_i such that $c_1(E_i/E_{i-1}) = \lambda_i c_1(E)$ and $c_2(E_i/E_{i-1}) = 0$.*
3. *If $\nu(c_1(E)) = 0$, E is nef.*

Remark (Nakayama04, Wu22, Liu-Ou-Yang 20)

Each $G_i := (E_i/E_{i-1})^{**}$ is a rank r_i projectively flat vector bundle, that is, there exists a representation

$$\pi_1(X) \to \mathbb{P}GL(r_i, \mathbb{C})$$

such that $\mathbb{P}(G_i)$ is constructed by this represention.

By the above classification, we obtain the following theorem.

Theorem (I.-Matsumura 22)

For any generically nef vector bundle E, if $\det E$ is nef and $c_2(E) = 0$, then E is nef.

Step1. If $c_2(X) = 0$ and K_X is nef, then Ω^1_X is nef and $\nu(K_X) \leq 1$.

Assumptions ($c_2(X) = 0$ and K_X is nef)
$\Rightarrow$ Ω^1_X generically nef $\Rightarrow$ Ω^1_X is nef.

To obtain a contradiction, suppose $\nu(K_X) \geq 2$.
$\Rightarrow$ there exists a nef line bundle $L \subset \Omega^1_X$ such that $\nu(L) \geq 2$.
$\Rightarrow$ a contradiction by the Bogomolov-Sommese vanishing. □

Remark (Bogomolov-Sommese vanishing theorem)

for any $1 \leq p \leq n$ and line bundle $L \subset \Omega^p_X$, $\kappa(L) \leq p$ holds.
By [Mourougane 95] we can replace $\kappa(L)$ by $\nu(L)$.

Definition

X is special in the sense of Campana iff $\kappa(L) < p$ holds for any $1 \leq p \leq n$ and any line bundle $L \subset \Omega_X^p$.

Why X is "special"?

Definition

X is general type $\Leftrightarrow \kappa(K_X) = \nu(K_X) = \dim_{\mathbb{C}} X$.

Any variety X can be decomposed into"special variety" and "(log) general type".

Theorem (Cam04)

If X is not special, then there exists a non trivial dominant rational map $f : X \dashrightarrow Y$ such that the followings hold:

- *There exists a Zariski open set $Y_0 \subset Y$ such that $f : f^{-1}(Y_0) \to Y_0$ is proper.*
- *There exists a bimeromorphic morphism $\pi : X' \to X$ and surjective holomorphic morphism $f' : X' \to Y$ with $f' = f \circ \pi$.*

$$\begin{tikzcd} \tilde{X} \arrow[r, "f'"] \arrow[d, "\pi"'] & Y \\ X \arrow[ur, dashed, "f"'] & \end{tikzcd}$$

- *Any general fiber F of f is special.*
- *The effective $\mathbb{Q}$-divisor $\Delta_{f'}$ is canonically defined by f' on Y and $\kappa(K_Y + \Delta_{f'}) = \dim Y$. In particular, $(Y, \Delta_{f'})$ is log general type.*
- $\kappa(K_X) \geq \kappa(K_F) + \dim Y$.

Criterion of non-speciality

Theorem (Pereira-Rousseau-Touzet 21)

If there exists a line bundle $L \subset \Omega_X^1$ with $\nu(L) = 1$, then X is not special.

Even if there exists a line bundle $L \subset \Omega_X^1$ with $\nu(L) = 1$, $\kappa(L)$ is not necessarily 1. (This theorem is based on the theory of codimension 1 foliations.)

Theorem (I.-Matsumura 22)

If there exists a projectively flat vector bundle $E \subset \Omega_X^1$ with $\nu(c_1(E)) = 1$, then X is not special.

By the above theorem, we can easily show that "If Ω_X^1 is nef and $\nu(K_X) \leq 1$, then K_X is semiample".

Step2. If Ω^1_X is nef and $\nu(K_X) \leq 1$, then K_X is semiample.

We may assume that $\nu(K_X) = 1$. Use the induction argument.

$\nu(K_X) = 1$, Ω^1_X nef. (In particular, $c_2(\Omega^1_X) = 0$.)
$\Rightarrow$ there exists a filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_l = \Omega^1_X$ s.t. $(E_i/E_{i-1})^{**}$ is projectively flat, $c_1(E_i/E_{i-1}) = \lambda_i c_1(K_X)$.
$\Rightarrow$ $E_1 \subset \Omega^1_X$, E_1 is projectively flat, and $\nu(c_1(E_1)) = 1$.
$\Rightarrow$ X is not special.
$\Rightarrow$ There exists a nontrivial rational map $f : X \dashrightarrow Y$ s.t.

$$\kappa(K_X) \geq \kappa(K_F) + \dim Y$$

for any general fiber F.
$\Rightarrow$ $\kappa(K_F) = \nu(K_F) \geq 0$ since Ω^1_F is nef and $\nu(K_F) \leq \nu(K_X) \leq 1$.
$\Rightarrow$ $1 = \nu(K_X) \geq \kappa(K_X) \geq \kappa(K_F) + \dim Y \geq \dim Y \geq 1$.
$\Rightarrow$ K_X is abundant $\Rightarrow$ K_X is semiample. $\square$

Thank you for your attention!

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