On the structure of projective manifolds whose tangent bundles are positive

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Notations

- X : n-dimensional smooth projective variety over $\mathbb{C}$.
- $T_X := (\Omega^1_X)^\vee$: holomorphic tangent bundle of X.
- $-K_X := \det T_X$ anti-canonical line bundle (divisor).
- $K_X := \det \Omega^1_X$: canonical line bundle (divisor).

Theme of this talk

If T_X is "positive", then the structure of X is restricted.

"positive" means ample, nef, big, and pseudo-effective.

Topics

1. Structure of X if T_X is "positive".
2. Structure of X if a subsheaf $\mathcal{F} \subset T_X$ is "positive".
3. Structure of a log smooth pair (X, D) if a logarithmic tangent bundle $T_X(-\log D)$ is "positive".

Positivity of vector bundles

E : vector bundle. $\pi : \mathbb{P}(E) \to X$: projective bundle of X.

- E is ample (resp. nef) $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{O}_{\mathbb{P}(E)}(1)$ is ample (resp. nef).
- E is big (big in the sense of Viehweg)
  $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{O}_{\mathbb{P}(E)}(1)$ is big and $\pi\left(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1))\right) \neq X$.
- E is pseudo-effective (in short. psef)
  $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{O}_{\mathbb{P}(E)}(1)$ is psef and $\pi\left(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))\right) \neq X$.

Caution!

This definitions of big and psef are different from the definitions in Lazarsfeld's textbook.

Example

A ample line bundle. $E := A \oplus A^\vee$ is big in the sense of Lazarsfeld. But E is Not big in the sense of Viehweg. (Not psef).

Remark. $L_1, \ldots, L_r$: line bundles. $E := \oplus_{i=1}^r L_i$ is ample (resp. nef, big, psef) iff any L_i is ample (resp. nef, big, psef).

1st topic -Structure of variety if T_X is "positive"-

Theorem

1. [*Mori 78*] *If* T_X *is ample, then* $X \cong \mathbb{CP}^n$.
2. [*Fulger-Murayama 21*] *If* T_X *is big, then* $X \cong \mathbb{CP}^n$.

Theorem (Campana-Peternell 91, Demailly -Peternell-Schneider 94)

If T_X *is nef, then* $\exists \pi : X' \to X$ *finite étale morphism and* $\exists \alpha : X' \to A$ *smooth surjective morphism s.t.*

- A *is an Abelian variety (in short, AV).*
- *Any fiber of* α *is Fano.*

Theorem (Hosono-I.-Matsumura 21)

If T_X is psef, then $\exists \pi : X' \to X$ finite étale morphism and $\exists \alpha : \to A$ smooth surjective morphism s.t.

- *A is an Abelian variety.*
- *Any fiber of α is rationally connected (in short, RC).*

$$\begin{tikzcd}
\boxed{\begin{array}{c}\text{[Mori 78]}\\ T_X \text{ is ample} \Rightarrow X \cong \mathbb{CP}^n\end{array}} \arrow[r] \arrow[d] & \boxed{\begin{array}{c}\text{[CP 91, DPS 94] } T_X \text{ is nef}\\ \Rightarrow X = \text{AV} + \text{Fano}\end{array}} \arrow[d] \\
\boxed{\begin{array}{c}\text{[FM 21]}\\ T_X \text{ is big} \Rightarrow X \cong \mathbb{CP}^n\end{array}} \arrow[r] & \boxed{\begin{array}{c}\text{[HIM 21] } T_X \text{ is psef}\\ \Rightarrow X = \text{AV} + \text{RC}\end{array}}
\end{tikzcd}$$

We give an another short proof of [DPS 94] and [HIM 21] in 2nd topic later.

Related studies: -Structure of varieties if $-K_X$ is nef-

Theorem

- [*Cao 19*][*Cao-Höring 19*]. *If* $-K_X$ *is nef, then* $X_{univ} \cong RC \times \mathbb{C}^r \times CY \times IHS$.
- [*Campana-Cao-Matsumura 21*] *If* $-K_X$ *is nef, then we can take a locally trivial MRC morphism* $X \to Y$ *with* $c_1(Y) = 0$.
- [*Matsumura-Wang 21*] *If* X *is a klt variety and* $-K_X$ *is nef, then* $\exists X' \to X$ *quasi-étale cover, we can take a locally trivial MRC morphism* $X' \to Y'$ *s.t.* $K_{Y'} \equiv 0$ *and* Y' *is klt.*

By Matsumura-Wang's work, we know the structure of a klt variety with a nef anticanonical divisor.

Remark (Singular Beauville-Bogomolov decomposition by Druel 19, Greb-Guenancia-Kebekus 19, and Höring-Peternell 19.)

If X is a klt variety and $K_X \equiv 0$, then $\exists X' \to X$ quasi-étale cover s.t. $X' \cong AV \times (singular)CY \times (singular)IHS$.

Related studies: -T_X or $-K_X$ is strictly nef-

A line bundle L is strictly nef $\overset{\text{def}}{\Leftrightarrow}$ $L.C > 0$ for any curve $C \subset X$.

A vector bundle E is strictly nef $\overset{\text{def}}{\Leftrightarrow}$ $\mathcal{O}_{\mathbb{P}(E)}(1)$ is strictly nef.

Theorem

- *[Li-Ou-Yang 19] If T_X is strictly nef, then $X \cong \mathbb{CP}^n$.*
- *[Li-Ou-Yang 19] If $-K_X$ is strictly nef, then X is RC.*
- *[Liu-Ou-Yang-Wang-Zhong 21] If X is klt and $-K_X$ is strictly nef, then X is RC. (This also holds for klt pairs.)*

Conjecture (Campana-Peternell 91)

If $-K_X$ is strictly nef, then X is Fano.

Conjecture holds in this following case:

- [Maeda 93] X is smooth and $\dim X = 2$.
- [Serrano 95] X is smooth and $\dim X = 3$.
- [Uehara 00] X is canonical and $\dim X = 3$.
- [Liu-Ou-Yang-Wang-Zhong 21] X is klt and $\dim X = 3$.

Theorem (Andreatta-Wiśniewski 01)

If $\mathcal{F}$ is a rank r ample locally free subsheaf of T_X, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus r}$ or $\mathcal{F} \cong T_{\mathbb{CP}^n}$.

Peternell proposed the following question.

Question (Peternell 01)

Let $\mathcal{F}$ be a locally free subsheaf of T_X.
What can be said on the structure of X if $\mathcal{F}$ is nef or psef?

We give a partial answer to Peternell's question in the case $\mathcal{F}$ is nef, big, or psef.

Theorem (I. 21)

Let $\mathcal{F}$ be a subbundle of T_X. Assume that $\mathcal{F}$ is a foliation. If $\mathcal{F}$ is psef, then $\exists f : X \to Y$ smooth surjective morphism s.t.

- *Any fiber of f is RC.*
- *$\exists \mathcal{G} \subset T_Y$ numerically flat subbundle s.t $\mathcal{G}$ is a foliation.*
- *There exists an exact sequence of vector bundles:*

$$0 \to T_{X/Y} \to \mathcal{F} \to f^*\mathcal{G} \to 0.$$

Corollary (I. 21)

Under the above assumptions, the followings hold.

1. *If $\mathcal{F}$ is ample, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.*
2. *If $\mathcal{F}$ is nef and big, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.*
3. *If $\mathcal{F}$ is nef, then any fiber of f is Fano.*
4. *If $\mathcal{F}$ is big, then any fiber of f is $\mathbb{CP}^r$ and $\mathcal{F} \cong T_{X/Y}$.*

[AW 01]
$\mathcal{F}$ is ample $\Rightarrow X \cong \mathbb{CP}^n$.

[I. 21] $\mathcal{F}$ is nef $\Rightarrow$
$X =$ variety with a flat foliation
$+$ Fano

[I. 21] $\mathcal{F}$ is big $\Rightarrow$
$X =$ variety with a flat foliation
$+ \mathbb{CP}^r$

[I. 21] $\mathcal{F}$ is psef $\Rightarrow$
$X =$ variety with a flat foliation
$+$ RC

Remark

We know the structure of "variety with a flat foliation" if rank$\mathcal{G}$ is large.

• rank$\mathcal{G} = \dim Y \Rightarrow Y$ is AV up to finite étale cover. (By Yau's theorem.)

• rank$\mathcal{G} = \dim Y - 1 \Rightarrow$ This foliation is classified by [Touzet 08], [Pereira-Touzet 13], and [Druel 17]. (3 types).

Sketch proof -from the viewpoint of slopes-

A : ample line bundle. $\mathcal{E}$: torsion free coherent sheaf.

$$\mu_A(\mathcal{E}) := \frac{c_1(\mathcal{E})A^{n-1}}{\mathrm{rank}\mathcal{E}}$$

$$\mu_A^{min}(\mathcal{E}) := \inf\{\mu_A(\mathcal{Q}) : \mathcal{E} \twoheadrightarrow \mathcal{Q}\}$$

The following theorem is obtained by combining [Miyaoka 87] with [Höring 07].

Theorem (Miyaoka 87+ Höring 07)

Let $\mathcal{E} \subset T_X$ be a subbundle and foliation. If $\mu_A^{min}(\mathcal{E}) > 0$, then $\exists f : X \to Y$ smooth surjective morphism s.t.

- *Any fiber of f is RC.*
- $\mathcal{E} = T_{X/Y}$.

On the other hand, we obtain the following decomposition theorem.

Theorem (I. 21)

Let $\mathcal{F}$ be a psef vector bundle on X.
Then there exist vector bundles $\mathcal{E}, \mathcal{Q}$ s.t.

- *$\mathcal{E}$ is psef and $\mu_A^{min}(\mathcal{E}) > 0$ unless $\mathcal{E} = 0$.*
- *$\mathcal{Q}$ is numerically flat.*
- *There exists an exact sequence of vector bundles:*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Proof of main theorem and corollary

Assume that $\mathcal{F} \subset T_X$ is a subbundle, foliation and psef.
$\Rightarrow$

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

- $\mathcal{E} \subset T_X$ is a subbundle and foliation with $\mu_A^{min}(\mathcal{E}) > 0$.
- $\mathcal{Q}$ is numerically flat.

$\Rightarrow \exists\ f : X \to Y$ smooth surjective morphism
s.t. any fiber of f is RC and $\mathcal{E} = T_{X/Y}$.
$\Rightarrow \exists\ \mathcal{G} \subset T_Y$ s.t. $\mathcal{Q} = f^*\mathcal{G}$, $\mathcal{G}$ is a numerically flat subbundle and foliation. $\square$

F : a fiber of f.

1. $\mathcal{F}$ is ample $\Rightarrow$ $f^*\mathcal{G} = 0$, both $T_{X/Y}$ and T_F are ample
   $\Rightarrow \dim Y = 0$, $X \cong \mathbb{CP}^n$, $\mathcal{F} \cong T_{\mathbb{CP}^n}$.
2. $\mathcal{F}$ is nef and big...(same argument!)
3. $\mathcal{F}$ is nef $\Rightarrow$ T_F is nef, F is RC. $\Rightarrow$ F is Fano by [DPS 94. Proposition 3.10]
4. $\mathcal{F}$ is big $\Rightarrow$ $f^*\mathcal{G} = 0$, T_F is big $\Rightarrow$ $F \cong \mathbb{CP}^r$, $\mathcal{F} \cong T_{X/Y}$.

• Proof of [HIM 21].

$\mathcal{F} = T_X$ is psef. $\Rightarrow \mathcal{G} = T_Y$ is numerically flat.

$\Rightarrow c_1(\Omega_Y^1) = 0$ and $c_2(\Omega_Y^1) = 0$.

$\Rightarrow \exists Y' \to Y$: finite étale s.t. Y' is AV.

$\Rightarrow \exists X' \to X$: finite étale s.t. $f' : X' \to Y'$ is smooth and any fiber of f' is RC.

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d] & Y' \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

• Proof of [DPS 94].

Any fiber F' of f' is RC, $T_{F'}$ is nef.

$\Rightarrow F'$ is Fano by [DPS 94. Proposition 3.10].

Related studies

Theorem

- [*Liu-Ou-Yang 20*] *If* $\mathcal{F} \subset T_X$ *is a rank* r *strictly nef locally free sheaf, then*
  - $\exists f : X \to Y$ *is a* $\mathbb{CP}^d$*-bundle s.t.* Y *is Brody hyperbolic.*
  - $\mathcal{F} \cong T_{X/Y}$ *or* $\mathcal{F}$ *is projectively flat and* $\mathcal{F}|_F \cong \mathcal{O}_{\mathbb{CP}^d}(1)^{\oplus r}$*.*
- [*Ou 21*] *If* $\mathcal{F} \subset T_X$ *is a subbundle and foliation s.t.* $-K_{\mathcal{F}} := \det(\mathcal{F})$ *is nef, then*
  - $\exists f : X \to Y$ *locally trivial with RC fibers.*
  - $\exists \mathcal{G} \subset T_Y$ *foliation with* $-K_{\mathcal{G}} \equiv 0$*.*
- [*Liu-Ou-Yang-Wang-Zhong 21*] *If* $\mathcal{F} \subset T_X$ *is subbundle and foliation and* $-K_{\mathcal{F}}$ *is strictly nef, then*
  - $\exists f : X \to Y$ *locally trivial with RC fibers s.t.* Y *is Brody hyperbolic and* K_Y *is ample.*
  - $\mathcal{F}$ *is induced by* f*. (In particular,* $\mathcal{F}$ *is algebraically integrable.)*

3rd case -Structure of variety s.t. $T_X(-\log D)$ is "positive"-

Conjecture (Campana-Cao-Matsumura 21)

Let (X, D) be a klt pair. If $-(K_X + D)$ is nef, then

1. $\exists \rho : (X, D) \to (R, D_R)$ *s.t.* ρ *is locally trivial.*
2. (R, D_R) *is a klt pair,* R *is smooth, and* $K_R + D_R \equiv 0$.
3. *Any general fiber* (X_r, D_r) *is "slope rationally connected".*

Lc pair (X, D) is slope rationally connected $\overset{\text{def}}{\Leftrightarrow}$ $\forall A$ ample, $\exists m(A)$ s.t. $\forall m > m(A)$, $H^0(X, \otimes^m \Omega^1(X, D) \otimes A) = 0$.

Observation

If $T(X, D)$ is "positive", then (X, D) maybe consists of

- a slope rationally connected manifold (Y, D_Y) and
- a log pair (R, D_R) with $K_R + D_R \equiv 0$.

The above observation is correct under certain conditions.

Introduction

Theorem (Greb-Kebekus-Peternell 20)

Let X be a projective klt variety. Assume that $-K_X$ is nef. Then the following are equivalent.

1. $\mathcal{E}_{-K_X}\langle\frac{K_X}{n+1}\rangle$ *is nef.* ($\mathcal{E}_{-K_X}$ *is the canonical extension sheaf. We defined this later.*)
2. X *is a quotient of* $\mathbb{CP}^n$ *or an Abelian variety by the action of a finite group of automorphisms without fixed points in codimension one.*

Theorem (Greb-Kebekus-Peternell 21)

Let X be a projective klt variety of dimension $n \geq 2$. If $T_X\langle\frac{K_X}{n}\rangle$ is nef, then $\exists\widetilde{X} \to X$ quasi-étale cover s.t. $\widetilde{X}$ is AV.

Remark 1 -canonical extension sheaf-

Definition (Tian 92)

L: line bundle. From $c_1(L) \in H^1(X, \Omega_X^1) = \mathrm{Ext}^1(\mathcal{O}_X, \Omega_X^1)$, there exists a vector bundle W_L and

$$0 \to \Omega_X^1 \to W_L \to \mathcal{O}_X \to 0.$$

Set $\mathcal{E}_L := (W_L)^\vee$. Then

$$0 \to \mathcal{O}_X \to \mathcal{E}_L \to T_X \to 0.$$

If $L = -K_X$, $\mathcal{E}_{-K_X}$ is called the *canonical extension sheaf*.

Theorem (Tian 92)

If $-K_X$ is ample and X has a Kähler-Einstein metric, then the canonical extension sheaf $\mathcal{E}_{-K_X}$ is $-K_X$-semistable. In particular,

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2\right)(-K_X)^{n-2} \geq 0.$$

Remark 2 -Nakayama's theorem-

Theorem (Nakayama 04)

Let $\mathcal{E}$ be a rank r vector bundle. The following are equivalent.

1. $\mathcal{E}\langle\frac{\det\mathcal{E}^\vee}{r}\rangle$ *is nef.* ($\Leftrightarrow \mathrm{Sym}^r\mathcal{E}\otimes\det\mathcal{E}^\vee$ *is nef.*)
2. $\mathcal{E}$ *is H-semistable and*

$$\left(c_2(\mathcal{E})-\frac{r-1}{2r}c_1(\mathcal{E})^2\right)H^{n-2}=0$$

for some ample divisor H.

- $\mathcal{E}_{-K_X}\langle\frac{K_X}{n+1}\rangle$ is nef $\Leftrightarrow$ $\mathcal{E}_{-K_X}$ is H-semistable and

$$\left(c_2(T_X)-\frac{n}{2(n+1)}c_1(T_X)^2\right)H^{n-2}=0.$$

- $T_X\langle\frac{K_X}{n}\rangle$ is nef $\Leftrightarrow$ T_X is H-semistable and

$$\left(c_2(T_X)-\frac{n-1}{2n}c_1(T_X)^2\right)H^{n-2}=0.$$

Main results

Theorem (I. 21)

Let X be a smooth projective variety of dimension $n \geq 2$ and D be a simple normal crossing divisor. Assume that $-(K_X + D)$ is nef. If $\mathcal{E}_L\langle \frac{K_X+D}{n+1} \rangle$ is nef for some line bundle L, then one of the following statements holds.

1. *(X, D) is a toric fiber bundle over a finite étale quotient of AV.*
2. *$(X, D) \cong (\mathbb{CP}^n, 0)$.*

In this case, $T_X(-\log D)$ is nef.

Theorem (I. 21)

Let X be a smooth projective variety of dimension $n \geq 2$ and D be a simple normal crossing divisor. Assume that $-(K_X + D)$ is nef. If $T_X(-\log D)\langle \frac{K_X+D}{n} \rangle$ is nef, then one of the following statements holds.

1. *(X, D) is a toric fiber bundle over a finite étale quotient of AV.*
2. *X is RC, $c_1(K_X + D) \neq 0$, $\exists B$ Cartier divisor with $T_X(-\log D) \cong \mathcal{O}_X(B)^{\oplus n}$.*

Moreover, if (2) holds and (X, D) is a Mori fiber space, then $(X, D) \cong (\mathbb{CP}^n, H_{\mathbb{CP}^n})$, where $H_{\mathbb{CP}^n}$ is a hyperplane of $\mathbb{CP}^n$.

In this case, $T_X(-\log D)$ is nef.

Corollary (Characterizations of toric fiber bundles)

If $c_1(T_X(-\log D)) = 0$ and $c_2\big(T_X(-\log D)\big)H^{n-2} = 0$, then (X, D) is a toric fiber bundle over a finite étale quotient of AV.

Proof.

$K_X + D \equiv 0 \Rightarrow \mathcal{E}_H$ is H-semistable for any ample H by [Li 20].

By assumption,

$$\left(c_2(T_X(-\log D)) - \frac{n}{2(n+1)}c_1(T_X(-\log D))^2\right) H^{n-2} = 0.$$

$\Rightarrow \mathcal{E}_H\langle \frac{K_X+D}{n+1} \rangle$ is nef.

$\Rightarrow (X, D)$ is a toric fiber bundle.

Proofs - Main result 1-

$\mathcal{E}_L\langle \frac{K_X+D}{n+1}\rangle$ is nef. $\Rightarrow T_X(-\log D)$ is nef. $\Rightarrow T_X$ is psef.

$\overset{\text{[HIM 21]}}{\Longrightarrow} \exists\, Y$ finite étale quotient of AV and

$\exists\, f : X \to Y$ smooth morphism s.t. any fiber of f is RC.

$\overset{T_X(-\log D)\text{ nef}}{\Longrightarrow} (X, D)$ is a logarithmic deformation over Y.

- Case 1: $c_1(K_F + D_F) = 0$ for any fiber F.

$\Rightarrow T_F(-\log D_F) \cong \mathcal{O}_F^{\oplus n}$. ($\because T_F(-\log D_F)$ is numerically flat)

$\overset{\text{[Winkelmann 04]}}{\Longrightarrow} (F, D_F)$ is toric for any F.

$\Longrightarrow (X, D)$ is a toric fiber bundle over Y.

- Case 2: $c_1(K_F + D_F) \neq 0$ for some fiber F.

$\Longrightarrow \dim Y = 0$. ($\because T_X(-\log D)\langle \frac{K_X+D}{n+1}\rangle$ is nef.)

$\Rightarrow \exists$ a line bundle M with $-(K_X + D) \sim (n+1)M$.

$\overset{\text{[Fujino-Miyamoto 20]}}{\Longrightarrow} (X, D) \cong (\mathbb{CP}^n, 0)$.

Related studies

In December 2021, Druel established the structure theorem of a log smooth pair s.t. $T_X(-\log D)\langle\frac{K_X+D}{n}\rangle$ is nef.

Theorem (Druel 21)

Let (X, D) be a log smooth pair of dimension $n \geq 2$. If $T_X(-\log D)\langle\frac{K_X+D}{n}\rangle$ is nef, then $\exists\gamma : Y \to X$ finite cover and $\exists\beta : Y \to Z$ birational projective morphism s.t.

- *$\exists B \subset Z$ s.t. $\gamma^{-1}(D) = \beta^{-1}(B) \cup \mathrm{Exc}\beta$.*
- *β is a blow up of finitely many points in $Z \setminus B$.*

Moreover, one of the followings holds.

1. *$T_Z(-\log B)$ is numeically flat.*
2. *$Z \cong \mathbb{CP}^n$ and $B \cong \mathbb{CP}^{n-1}$.*

Conjecture (Campana-Cao-Matsumura 21)

Let (X, D) be a klt pair. If $-(K_X + D)$ is nef, then

1. $\exists \rho : (X, D) \to (R, D_R)$ *s.t.* ρ *is locally trivial.*
2. (R, D_R) *is a klt pair,* R *is smooth, and* $K_R + D_R \equiv 0$.
3. *Any general fiber* (X_r, D_r) *is "slope rationally connected".*

This conjecture is open even if $T(X, D)$ is nef. (I think this conjecture maybe holds if $T(X, D)$ is psef.)
If (X, D) is an lc pair, this conjecture is not true even if $T(X, D)$ is nef.

Example (I. 21)

Let H_1, H_2 be a distinct two lines in $\mathbb{CP}^2$. Set $D := H_1 + H_2$. $T(\mathbb{CP}^2, D) = T_{\mathbb{CP}^2}(-\log D)$ is nef. However we can not take ρ as in the above conjecture.

Hence the first observation is false in general...

Thank you for your attention!

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