On the structure of a log smooth pair in the equality case of the Bogomolov-Gieseker inequality

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Notations

- X: n-dimensional smooth projective variety over $\mathbb{C}$.
- T_X: holomorphic tangent bundle of X.
- $-K_X := \det T_X$: anti-canonical line bundle (divisor).
- $K_X := \det \Omega_X$: canonical line bundle (divisor).
- D: simple normal crossing divisor on X.
- $\Omega_X(\log D)$: logarithmic cotangent bundle.
- $T_X(-\log D) := \Omega_X(\log D)^\vee$: logarithmic tangent bundle.

If the equality holds in "some good inequality",
then the structure of X is restricted.

Theorem (Chen-Ogiue 75)

Let (X, ω) be a compact Kähler-Einstein manifold, and $\alpha := \{\omega\} \in H^{1,1}(X, \mathbb{R})$. Then,

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2 \right) \alpha^{n-2} \geq 0.$$

Equality holds iff $X_{univ} \cong \mathbb{CP}^n, \mathbb{C}^n$, or $\mathbb{B}^n$.

Corollary

If $-K_X$ is ample and X has a Kähler-Einstein metric, then

$$\left(c_2(T_X) - \frac{n}{2(n+1)}c_1(T_X)^2\right)(-K_X)^{n-2} \geq 0.$$

Equality holds iff $X \cong \mathbb{CP}^n$.

Why does Miyaoka-Yau inequality hold?

Miyaoka-Yau inequality is a kind of Bogomolov-Gieseker inequality for "some vector bundle".

Theorem (Bogomolov, Gieseker)

Let E be a rank r vector bundle and H be an ample line bundle. If E is H-semistable, then

$$\left(c_2(E) - \frac{r-1}{2r}c_1(E)^2\right) H^{n-2} \geq 0.$$

Why does Miyaoka-Yau inequality hold?

Definition (Tian 92)

Let L be a line bundle on X. From

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^1(X, \Omega_X^1) = \mathrm{Ext}^1(\mathcal{O}_X, \Omega_X^1),$$

there exist a vector bundle W_L induced by $c_1(L)$ and

$$0 \to \Omega_X^1 \to W_L \to \mathcal{O}_X \to 0.$$

Let $\mathcal{E}_L$ be a dual bundle of W_L. Then

$$0 \to \mathcal{O}_X \to \mathcal{E}_L \to T_X \to 0.$$

$\mathcal{E}_L$ is <u>extension sheaf of T_X by $\mathcal{O}_X$ with the extension class $c_1(L)$</u>.
$\mathcal{E}_{-K_X}$ is called the <u>canonical extension sheaf of T_X by $\mathcal{O}_X$.</u>

Why does Miyaoka-Yau inequality hold?

Propoerty

$\mathrm{rk}(\mathcal{E}_L) = n+1, c_1(\mathcal{E}_L) = c_1(T_X), c_2(\mathcal{E}_L) = c_2(T_X)$

Theorem (Tian 92)

If $-K_X$ is ample and X has a Kähler-Einstein metric, then the canonical extension sheaf $\mathcal{E}_{-K_X}$ is $-K_X$-semistable.

Corollary

If $-K_X$ is ample and X has a Kähler-Einstein metric, then

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2\right)(-K_X)^{n-2} \geq 0.$$

Greb-Kebekus-Peternell's work

Theorem (Greb-Kebekus-Peternell 20)

Let X be a projective klt variety. Assume that $-K_X$ is nef. Then the following are equivalent.

1. *There exists an ample Cartier divisor H such that the canonical extension sheaf $\mathcal{E}_{-K_X}$ is H-semistable and*

$$\left(\hat{c_2}(\Omega_X^{[1]}) - \frac{n}{2(n+1)}\hat{c_1}(\Omega_X^{[1]})^2\right)[H]^{n-2} = 0.$$

2. *X is a quotient of $\mathbb{CP}^n$ or an Abelian variety by the action of a finite group of automorphisms without fixed points in codimension one.*

Theorem (Greb-Kebekus-Peternell 20)

Let X be a projective klt variety of dimension $n \geq 2$ and H be an ample divisor. If $\Omega_X^{[1]}$ is H-semistable and

$$\left(\hat{c}_2(\Omega_X^{[1]}) - \frac{n-1}{2n}\hat{c}_1(\Omega_X^{[1]})^2\right)[H]^{n-2} = 0,$$

then there exists a quasi-étale cover $\widetilde{X} \to X$ from an Abelian variety $\widetilde{X}$ to X.

Toward a log smooth pair case

Definition (Li 20)

Let D be a simple normal crossing divisor and L be a line bundle. From

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^1(X, \Omega_X^1)$$

$$\xrightarrow{\Phi} H^1(X, \Omega_X^1(\log D)) = \mathrm{Ext}^1(\mathcal{O}_X, \Omega_X^1(\log D)),$$

there exist a vector bundle W_L induced by $\Phi(c_1(L))$ and

$$0 \to \Omega_X^1(\log D) \to W_L \to \mathcal{O}_X \to 0.$$

Let $\mathcal{E}_L$ be a dual bundle of W_L. Then

$$0 \to \mathcal{O}_X \to \mathcal{E}_L \to T_X(-\log D) \to 0.$$

$\mathcal{E}_L$ is called the *extension sheaf of* $T_X(-\log D)$ *by* $\mathcal{O}_X$ *with the extension class* $c_1(L)$.

Toward a log smooth pair case

Theorem (Li 20)

If $c_1(T_X(-\log D)) = -(K_X + D) = 0$, then $\mathcal{E}_H$ is H-semistable for any ample line bundle H.

We do not know when $\mathcal{E}_{-(K_X+D)}$ is H-semistable for a log Fano pair (X, D).

Theorem (I. 21)

Let X be a smooth projective variety of dimension $n \geq 2$, D be a simple normal crossing divisor, and H be an ample divisor. Assume that $-(K_X + D)$ is nef. If the extension sheaf $\mathcal{E}_L$ is H-semistable for some line bundle L and

$$\left(c_2(T_X(-\log D)) - \frac{n}{2(n+1)} c_1(T_X(-\log D))^2 \right) H^{n-2} = 0,$$

then one of the following statements holds.

1. *(X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.*
2. *$(X, D) \cong (\mathbb{CP}^n, 0)$.*

Main results

Theorem (I. 21)

Let X be a smooth projective variety of dimension $n \geq 2$, D be a simple normal crossing divisor, and H be an ample divisor. Assume that $-(K_X + D)$ is nef. If $T_X(-\log D)$ is H-semistable and

$$\left(c_2(T_X(-\log D)) - \frac{n-1}{2n} c_1\left(T_X(-\log D)\right)^2\right) H^{n-2} = 0,$$

then one of the following statements holds.

1. *(X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.*
2. *X is rationally connected, $c_1(K_X + D) \neq 0$, and there exists a Cartier divisor B with $T_X(-\log D) \cong \mathcal{O}_X(B)^{\oplus n}$.*

Moreover, if (2) holds and (X, D) is a Mori fiber space, then $(X, D) \cong (\mathbb{CP}^n, H_{\mathbb{CP}^n})$, where $H_{\mathbb{CP}^n}$ is a hyperplane of $\mathbb{CP}^n$.

Remarks

Remark [I.21]

For any $m \in \mathbb{N}_{>0}$, $\exists$ an example (X_m, D_m) such that

1. X_m is an m-th Hirzebruch surface.
   (In particular, X_m is rationally connected.)
2. $c_1(K_{X_m} + D_m) \neq 0$.
3. $\exists$ a Cartier divisor B_m with $T_{X_m}(-\log D_m) \cong \mathcal{O}_{X_m}(B_m)^{\oplus 2}$.
4. $(X_m, D_m) \ncong (\mathbb{CP}^2, H_{\mathbb{CP}^2})$.

Corollary (A characterization of a toric fiber bundle)

If $c_1(T_X(-\log D)) = 0$ *and* $c_2\big(T_X(-\log D)\big)H^{n-2} = 0$, *then* (X, D) *is a toric fiber bundle over a finite étale quotient of an Abelian variety.*

Related researches

- [Tsuji 88] and [Tian-Yau 87]. Assume $K_X + D$ is nef, big, and ample modulo D. If

$$\left(c_2(\Omega_X(\log D)) - \frac{n}{2(n+1)} c_1(\Omega_X(\log D))^2\right) c_1(\Omega_X(\log D))^{n-2} = 0,$$

then the universal cover of $X \setminus D$ is a unit ball in $\mathbb{C}^n$.
- [Deng20]. If the natural log Higgs bundle $(\Omega^1_X(\log D) \oplus \mathcal{O}_X, \theta)$ is H-polystable and

$$\left(c_2(\Omega_X(\log D)) - \frac{n}{2(n+1)} c_1(\Omega_X(\log D))^2\right) H^{n-2} = 0,$$

then the universal cover of $X \setminus D$ is a unit ball in $\mathbb{C}^n$.
- [Druel-LoBianco 20]. If $T_X(-\log D)$ is numerically flat, then (X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.

Key Idea

Theorem (Nakayama 04)

Let $\mathcal{E}$ be a rank r vector bundle. Then the following are equivalent.

1. $\mathrm{Sym}^r\mathcal{E} \otimes \det\mathcal{E}^\vee$ *is nef.* ($\Leftrightarrow \mathcal{E}\langle\frac{\det\mathcal{E}^\vee}{r}\rangle$ *is nef.*)
2. $\mathcal{E}$ *is H-semistable and*

$$\left(c_2(\mathcal{E}) - \frac{r-1}{2r}c_1(\mathcal{E})^2\right)H^{n-2} = 0.$$

If (1) or (2) holds, then $\mathcal{E}$ is called numerically projectively flat.

Key Idea

Theorem (Greb-Kebekus-Peternell 20, Liu-Ou-Yang 20)

If $\mathcal{E}$ is numerically projectively flat, then $\mathcal{E}$ is projectively flat.

Corollary

If X is simply connected and $\mathcal{E}$ is numerically projectively flat, then there exists a line bundle M such that $\mathcal{E} \cong M^{\oplus r}$.

$\mathcal{E}_L$ is H-semistable and $\left(c_2(\mathcal{E}_L) - \frac{n}{2(n+1)} c_1(\mathcal{E}_L)^2\right) H^{n-2} = 0$.

$\overset{\text{[Nak04]}}{\Longrightarrow} \mathcal{E}_L$ is numerically projectively flat. ($\mathcal{E}_L\langle\frac{K_X+D}{n+1}\rangle$ is nef.)

$\Longrightarrow T_X(-\log D)$ is nef.

Proofs - Main result 1-

$\mathcal{E}_L$ is H-semistable and $\left(c_2(\mathcal{E}_L) - \frac{n}{2(n+1)} c_1(\mathcal{E}_L)^2\right) H^{n-2} = 0.$
$\overset{\text{[Nak04]}}{\Longrightarrow} \mathcal{E}_L$ is numerically projectively flat. ($\mathcal{E}_L\langle \frac{K_X+D}{n+1}\rangle$ is nef.)
$\Longrightarrow T_X(-\log D)$ is nef. $\Longrightarrow T_X$ is pseudo-effective.
$\overset{\text{[HIM19]}}{\Longrightarrow}$ $\exists$ a finite étale quotient of an Abelian variety Y and $\exists$ a smooth morphism $f : X \to Y$ such that any fiber is rationally connected.

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$\overset{T_X(-\log D)\ \text{nef}}{\Longrightarrow}$ (X, D) is a logarithmic deformation over Y.

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$\overset{\text{[HIM19]}}{\Longrightarrow}$ $\exists$ a finite étale quotient of an Abelian variety Y and $\exists$ a smooth morphism $f : X \to Y$ such that any fiber is rationally connected.

$\overset{T_X(-\log D) \text{ nef}}{\Longrightarrow}$ (X, D) is a logarithmic deformation over Y.

- Case 1: $c_1(K_F + D_F) = 0$ for any fiber F.

$\Rightarrow T_F(-\log D_F) \cong \mathcal{O}_F^{\oplus n}$. $\Rightarrow (F, D_F)$ is toric for any F.

$\Longrightarrow (X, D)$ is a toric fiber bundle over Y.

- Case 2: $c_1(K_F + D_F) \neq 0$ for some fiber F.

$\Rightarrow \dim Y = 0$. $\Rightarrow \exists$ a line bundle M with

$-(K_X + D) \sim (n+1)M \overset{\text{[FM20]}}{\Longrightarrow} (X, D) \cong (\mathbb{CP}^n, 0)$.

Remarks

Theorem (An easy consequence of Winkelman 04 or Brown-Mckernan-Svaldi-Zong 18)

If X is rationally connected, $-(K_X + D)$ is nef, and $T_X(-\log D) \cong \mathcal{O}_X^{\oplus n}$, then (X, D) is a toric variety with a boundary divisor D.

Theorem (Fujino-Miyamoto 20)

Let (X, D) be a projective log canonical pair. Assume that $K_X + D$ is not nef and $-(K_X + D) \equiv rM$ for some Cartier divisor M with $r > n$. Then $X \cong \mathbb{CP}^n$ and $\mathcal{O}_X(M) \cong \mathcal{O}_{\mathbb{CP}^n}(1)$.

Outlook

Question

Let (X, D) be a log smooth pair.
What is the structure of (X, D) if $T_X(-\log D)\langle \frac{K_X+D}{n} \rangle$ is nef ?

By [GKP 20], if $T_X\langle \frac{K_X}{n} \rangle$ is nef, then X is an Abelian variety up to a finite étale cover.

Cese 1 : $-(K_X + D)$ is nef

Proposition (I.21)

Assume that X is rationally connected, $-(K_X + D)$ is nef, $c_1(K_X + D) \neq 0$, and there exists a Cartier divisor B with $T_X(-\log D) \cong \mathcal{O}_X(B)^{\oplus n}$.
Let $f : (X, D) \to Y$ be a $(K_X + D)$-negative extremal contraction. Then one of the following statements holds.

- (a) $\dim Y = 0$ *and (X, D) is isomorphic to $(\mathbb{CP}^n, H_{\mathbb{CP}^n})$.*
- (b) *There exists an irreducible component C of D such that $f(C)$ is a point, Exc$(f) = C$, $C \cong \mathbb{CP}^{n-1}$, and C does not meet any other irreducible component of D.*

We do not know the structure of a $(K_Y + D_Y)$-negative extremal contraction.

Cese 2 : $K_X + D$ is nef

Conjecture

If both $K_X + D$ and $T_X(-\log D)\langle\frac{K_X+D}{n}\rangle$ are nef, then $K_X + D$ is semiample.

• By [GKP 20], if $T_X\langle\frac{K_X}{n}\rangle$ is nef, then K_X is nef and abundant $(\kappa(K_X) = \nu(K_X))$, so K_X is semiample.
• This conjecture may imply the first question. (Maybe...)
c.f. [Höring 13] If Ω_X is nef and K_X is semiample, then there exists a finite étale cover $X' \to X$ such that

- the Iitaka fibration $X' \to Y'$ is a smooth fibration onto a projective manifold with ample canonical divisor
- all the fibres are Abelian varieties.

Thank you for your attention!

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