On the structure of a log smooth pair in the equality case of the Bogomolov-Gieseker inequality

Masataka Iwai

Research Alliance Center for Mathematical Sciences,
Tohoku University (RACMaS)

July 13th, 2021.
Geometry Seminar in Tohoku University.

Notations

- X: n-dimensional smooth projective variety over $\mathbb{C}$.
- T_X: holomorphic tangent bundle of X.
- $-K_X := \det T_X$: anti-canonical line bundle (divisor).
- $K_X := \det \Omega_X$: canonical line bundle (divisor).
- D: simple normal crossing divisor on X.
- $\Omega_X(\log D)$: logarithmic cotangent bundle.
- $T_X(-\log D) := \Omega_X(\log D)^\vee$: logarithmic tangent bundle.

Introduction

> If the equality holds in "some good inequality",
> then the structure of X is restricted.

Theorem (Chen-Ogiue 75)

Let (X, ω) be a compact Kähler-Einstein manifold, and $\alpha := \{\omega\} \in H^{1,1}(X, \mathbb{R})$. Then,

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2 \right) \alpha^{n-2} \geq 0.$$

Equality holds iff $X_{univ} \cong \mathbb{CP}^n, \mathbb{C}^n$, or $\mathbb{B}^n$.

If $c_1(T_X) = 0$ or K_X is ample, then X has a Kähler-Einstein metric. ([Yau78][Aubin-Yau78])

Theorem (Miyaoka 77, Yau 78)

1. If $-K_X$ is ample and X has a Kähler-Einstein metric, then

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2 \right) (-K_X)^{n-2} \geq 0.$$

   Equality holds iff $X \cong \mathbb{CP}^n$.

2. If $c_1(T_X) = 0$, then $c_2(T_X) H^{n-2} \geq 0$ for any ample H.
   Equality holds iff X is a torus up to a finite étale cover.

3. If K_X is ample, then

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2 \right) (K_X)^{n-2} \geq 0.$$

   Equality holds iff X is a ball quotient.

Miyaoka-Yau inequality is a kind of Bogomolov-Gieseker inequality for "some vector bundle".

Theorem (Bogomolov, Gieseker)

Let E be a rank r vector bundle and H be an ample line bundle. If E is H-semistable, then

$$\left(c_2(E) - \frac{r-1}{2r} c_1(E)^2 \right) H^{n-2} \geq 0.$$

Why does Miyaoka-Yau inequality hold?

Definition (Tian 92)

Let L be a line bundle on X. From

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^1(X, \Omega_X^1) = \mathsf{Ext}^1(\mathcal{O}_X, \Omega_X^1),$$

there exist a vector bundle W_L induced by $c_1(L)$ and

$$0 \to \Omega_X^1 \to W_L \to \mathcal{O}_X \to 0.$$

Let $\mathcal{E}_L$ be a dual bundle of W_L. Then

$$0 \to \mathcal{O}_X \to \mathcal{E}_L \to T_X \to 0.$$

$\mathcal{E}_L$ is extension sheaf of T_X by $\mathcal{O}_X$ with the extension class $c_1(L)$.
$\mathcal{E}_{-K_X}$ is called the canonical extension sheaf of T_X by $\mathcal{O}_X$.

Why does Miyaoka-Yau inequality hold?

Propoerty

$\mathsf{rk}(\mathcal{E}_L) = n + 1, c_1(\mathcal{E}_L) = c_1(T_X), c_2(\mathcal{E}_L) = c_2(T_X)$

Theorem (Tian 92)

If $-K_X$ is ample and X has a Kähler-Einstein metric, then the canonical extension sheaf $\mathcal{E}_{-K_X}$ is $-K_X$-semistable.

Corollary

If $-K_X$ is ample and X has a Kähler-Einstein metric, then

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2 \right) (-K_X)^{n-2} \geq 0.$$

Theorem (Greb-Kebekus-Peternell 20)

Let X be a projective klt variety. Assume that $-K_X$ is nef. Then the following are equivalent.

1. *There exists an ample Cartier divisor H such that the canonical extension sheaf $\mathcal{E}_{-K_X}$ is H-semistable and*

$$\left(\hat{c}_2(\Omega_X^{[1]}) - \frac{n}{2(n+1)} \hat{c}_1(\Omega_X^{[1]})^2 \right) [H]^{n-2} = 0.$$

2. *X is a quotient of $\mathbb{CP}^n$ or an Abelian variety by the action of a finite group of automorphisms without fixed points in codimension one.*

Theorem (Greb-Kebekus-Peternell 20)

Let X be a projective klt variety of dimension $n \geq 2$ and H be an ample divisor. If $\Omega_X^{[1]}$ is H-semistable and

$$\left(\hat{c_2}(\Omega_X^{[1]}) - \frac{n-1}{2n} \hat{c_1}(\Omega_X^{[1]})^2 \right) [H]^{n-2} = 0,$$

then X is a quasi-Abelian variety, that is, there exists a quasi-étale cover $\widetilde{X} \to X$ from an Abelian variety $\widetilde{X}$ to X.

Toward a log smooth pair case

Definition (Li 20)

Let D be a simple normal crossing divisor and L be a line bundle. From

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^1(X, \Omega_X^1)$$

$$\xrightarrow{\Phi} H^1(X, \Omega_X^1(\log D)) = \mathsf{Ext}^1(\mathcal{O}_X, \Omega_X^1(\log D)),$$

there exist a vector bundle W_L induced by $\Phi(c_1(L))$ and

$$0 \to \Omega_X^1(\log D) \to W_L \to \mathcal{O}_X \to 0.$$

Let $\mathcal{E}_L$ be a dual bundle of W_L. Then

$$0 \to \mathcal{O}_X \to \mathcal{E}_L \to T_X(-\log D) \to 0.$$

$\mathcal{E}_L$ is called the *extension sheaf of* $T_X(-\log D)$ *by* $\mathcal{O}_X$ *with the extension class* $c_1(L)$.

Toward a log smooth pair case

Theorem (Li 20)

If $c_1(T_X(-\log D)) = -(K_X + D) = 0$, then $\mathcal{E}_H$ is H-semistable for any ample line bundle H.

We do not know when $\mathcal{E}_{-(K_X+D)}$ is H-semistable for a log Fano pair (X, D).

Main results

Theorem (I. 21)

Let X be a smooth projective variety of dimension $n \geq 2$, D be a simple normal crossing divisor, and H be an ample divisor. Assume that $-(K_X + D)$ is nef. If the extension sheaf $\mathcal{E}_L$ is H-semistable for some line bundle L and

$$\left(c_2\big(T_X(-\log D)\big) - \frac{n}{2(n+1)} c_1\big(T_X(-\log D)\big)^2 \right) H^{n-2} = 0,$$

then one of the following statements holds.

1. *(X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.*

2. *$(X, D) \cong (\mathbb{CP}^n, 0)$.*

Main results

Theorem (I. 21)

Let X be a smooth projective variety of dimension $n \geq 2$, D be a simple normal crossing divisor, and H be an ample divisor. Assume that $-(K_X + D)$ is nef. If $T_X(-\log D)$ is H-semistable and

$$\left(c_2\big(T_X(-\log D)\big) - \frac{n-1}{2n}c_1\big(T_X(-\log D)\big)^2 \right) H^{n-2} = 0,$$

then one of the following statements holds.

1. (X, D) *is a toric fiber bundle over a finite étale quotient of an Abelian variety.*

2. X *is rationally connected, $c_1(K_X + D) \neq 0$, and there exists a Cartier divisior B with $T_X(-\log D) \cong \mathcal{O}_X(B)^{\oplus n}$.*

Moreover, if (2) holds and (X, D) is a Mori fiber space, then $(X, D) \cong (\mathbb{CP}^n, H_{\mathbb{CP}^n})$, where $H_{\mathbb{CP}^n}$ is a hyperplane of $\mathbb{CP}^n$.

Remarks

Remark [I.21]

For any $m \in \mathbb{N}_{>0}$, $\exists$ an example (X_m, D_m) such that

1. X_m is an m-th Hirzebruch surface.
   (In particular, X_m is rationally connected.)

2. $c_1(K_{X_m} + D_m) \neq 0$.

3. $\exists$ a Cartier divisor B_m with $T_{X_m}(-\log D_m) \cong \mathcal{O}_{X_m}(B_m)^{\oplus 2}$.

4. $(X_m, D_m) \not\cong (\mathbb{CP}^2, H_{\mathbb{CP}^2})$.

Corollary (A characterization of a toric fiber bundle)

If $c_1(T_X(-\log D)) = 0$ and $c_2(T_X(-\log D))H^{n-2} = 0$, then (X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.

Related researches

• [Tsuji 88] and [Tian-Yau 87]. Assume $K_X + D$ is nef, big, and ample modulo D. If

$$\left(c_2\big(\Omega_X(\log D)\big) - \frac{n}{2(n+1)} c_1\big(\Omega_X(\log D)\big)^2 \right) c_1\big(\Omega_X(\log D)\big)^{n-2} = 0,$$

then the universal cover of $X \setminus D$ is a unit ball in $\mathbb{C}^n$.

• [Deng20]. If the natural log Higgs bundle $(\Omega_X^1(\log D) \oplus \mathcal{O}_X, \theta)$ is H-polystable and

$$\left(c_2\big(\Omega_X(-\log D)\big) - \frac{n}{2(n+1)} c_1\big(\Omega_X(-\log D)\big)^2 \right) H^{n-2} = 0,$$

then the universal cover of $X \setminus D$ is a unit ball in $\mathbb{C}^n$.

• [Druel-LoBianco 20]. If $T_X(-\log D)$ is numerically flat, then (X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.

Key Idea

Theorem (Nakayama 04)

Let $\mathcal{E}$ be a rank r vector bundle. Then the following are equivalent.

1. $\operatorname{Sym}^r \mathcal{E} \otimes \det \mathcal{E}^\vee$ *is nef.* ($\Leftrightarrow \mathcal{E}\langle \frac{\det \mathcal{E}^\vee}{r} \rangle$ *is nef.*)

2. $\mathcal{E}$ *is H-semistable and*

$$\left(c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2 \right) H^{n-2} = 0.$$

3. *There exists a filtration of subbundles:*

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_l = \mathcal{E}$$

   such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is projectively Hermitian flat and $\mu_H(\mathcal{E}_i/\mathcal{E}_{i-1}) = \mu_H(\mathcal{E})$ for any $1 \le i \le l$.

If (1), (2) or (3) holds, then $\mathcal{E}$ is called numerically projectively flat.

Theorem (Greb-Kebekus-Peternell 20, Liu-Ou-Yang 20)

If $\mathcal{E}$ is numerically projectively flat, then $\mathcal{E}$ is projectively flat.

Corollary

If X is simply connected and $\mathcal{E}$ is numerically projectively flat, then there exists a line bundle M such that $\mathcal{E} \cong M^{\oplus r}$.

Proofs - Main result 1-

$\mathcal{E}_L$ is H-semistable and $\left(c_2(\mathcal{E}_L) - \frac{n}{2(n+1)}c_1(\mathcal{E}_L)^2\right) H^{n-2} = 0$.

$\overset{[\text{Nak04}]}{\Longrightarrow} \mathcal{E}_L$ is numerically projectively flat.

$(\text{Sym}^{n+1}\mathcal{E}_L \otimes \mathcal{O}_X(K_X + D)$ is nef.$)$

$\Longrightarrow T_X(-\log D)$ is nef.

Proofs - Main result 1-

$\mathcal{E}_L$ is H-semistable and $\left(c_2(\mathcal{E}_L) - \frac{n}{2(n+1)} c_1(\mathcal{E}_L)^2 \right) H^{n-2} = 0.$

$\overset{[\text{Nak04}]}{\Longrightarrow} \mathcal{E}_L$ is numerically projectively flat.

$(\mathrm{Sym}^{n+1} \mathcal{E}_L \otimes \mathcal{O}_X(K_X + D)$ is nef.$)$

$\Longrightarrow T_X(-\log D)$ is nef. $\Longrightarrow T_X$ is pseudo-effective.

$\overset{[\text{HIM19}]}{\Longrightarrow} \exists$ a finite étale quotient of an Abelian variety Y and
$\exists$ a smooth morphism $f : X \to Y$ such that any fiber is rationally connected.

Proofs - Main result 1-

$\mathcal{E}_L$ is H-semistable and $\left(c_2(\mathcal{E}_L) - \frac{n}{2(n+1)} c_1(\mathcal{E}_L)^2 \right) H^{n-2} = 0$.

$\overset{[\text{Nak04}]}{\Longrightarrow} \mathcal{E}_L$ is numerically projectively flat.

$(\mathrm{Sym}^{n+1} \mathcal{E}_L \otimes \mathcal{O}_X(K_X + D)$ is nef.)

$\Longrightarrow T_X(-\log D)$ is nef. $\Longrightarrow T_X$ is pseudo-effective.

$\overset{[\text{HIM19}]}{\Longrightarrow} \exists$ a finite étale quotient of an Abelian variety Y and $\exists$ a smooth morphism $f : X \to Y$ such that any fiber is rationally connected.

$\overset{T_X(-\log D) \text{ nef}}{\Longrightarrow} (X, D)$ is a logarithmic deformation over Y.

Proofs - Main result 1-

$\mathcal{E}_L$ is H-semistable and $\left(c_2(\mathcal{E}_L) - \frac{n}{2(n+1)} c_1(\mathcal{E}_L)^2 \right) H^{n-2} = 0$.

$\overset{[\text{Nak04}]}{\Longrightarrow} \mathcal{E}_L$ is numerically projectively flat.

$(\mathrm{Sym}^{n+1} \mathcal{E}_L \otimes \mathcal{O}_X(K_X + D)$ is nef.)

$\Longrightarrow T_X(-\log D)$ is nef. $\Longrightarrow T_X$ is pseudo-effective.

$\overset{[\text{HIM19}]}{\Longrightarrow} \exists$ a finite étale quotient of an Abelian variety Y and
$\exists$ a smooth morphism $f : X \to Y$ such that any fiber is rationally connected.

$\overset{T_X(-\log D) \text{ nef}}{\Longrightarrow} (X, D)$ is a logarithmic deformation over Y.

• Case 1: $c_1(K_F + D_F) = 0$ for any fiber F.

$\Rightarrow T_F(-\log D_F) \cong \mathcal{O}_F^{\oplus n}$. $\Rightarrow (F, D_F)$ is toric for any F.

$\Longrightarrow (X, D)$ is a toric fiber bundle over Y.

• Case 2: $c_1(K_F + D_F) \neq 0$ for some fiber F.

$\Rightarrow \dim Y = 0$. $\Rightarrow \exists$ a line bundle M with

$-(K_X + D) \sim (n+1)M \overset{[\text{FM20}]}{\Longrightarrow} (X, D) \cong (\mathbb{CP}^n, 0)$.

Remarks

Theorem (An easy consequence of Winkelman 04 or Brown-Mckernan-Svaldi-Zong 18)

If X is rationally connected, $-(K_X + D)$ is nef, and $T_X(-\log D) \cong \mathcal{O}_X^{\oplus n}$, then (X, D) is a toric variety with a boundary divisor D.

Theorem (Fujino-Miyamoto 20)

Let (X, D) be a projective log canonical pair. Assume that $K_X + D$ is not nef and $-(K_X + D) \equiv rM$ for some Cartier divisor M with $r > n$. Then $X \cong \mathbb{CP}^n$ with $\mathcal{O}_X(M) \cong \mathcal{O}_{\mathbb{CP}^n}(1)$.

Proofs - Main result 2 -

X is rationally connected, $c_1(K_X + D) \neq 0$, $\exists$ a Cartier divisior B with $T_X(-\log D) \cong \mathcal{O}_X(B)^{\oplus n}$, and $\phi : (X, D) \to Z$ is a Mori fiber space.

• Case 1: $\dim Z \neq 0$.

$\overset{\text{[FM20]}}{\Longrightarrow} (F, D_F) \cong (\mathbb{CP}^{n-1}, 0)$ for a general fiber F.

$\Longrightarrow \exists$ a long exact sequence:

$$H^1(F, \Omega_X^1(\log D)|_F) \to H^1(F, \Omega_F^1) \to H^2(F, \mathcal{O}_F)^{\oplus \dim Z}.$$

$\Longrightarrow H^1(F, \Omega_F^1) = 0.$ $\Longrightarrow$ Contradiction.

• Case 2: $\dim Z = 0$.

$\Longrightarrow (X, D)$ is log Fano, $-(K_X + D) \sim nB$, and $D \neq 0$.

$\overset{\text{[Fjt14]}}{\Longrightarrow} (X, D) \cong (\mathbb{CP}^n, H_{\mathbb{CP}^n}).$

Thank you for your attention!