

On the structure of a log smooth pair in the equality case of the Bogomolov-Gieseker inequality

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- ① Introduction
- ② Main results
- ③ Proofs

Notations

- X : n -dimensional smooth projective variety over \mathbb{C} .
- T_X : holomorphic tangent bundle of X .
- $-K_X := \det T_X$: anti-canonical line bundle (divisor).
- $K_X := \det \Omega_X$: canonical line bundle (divisor).
- D : simple normal crossing divisor on X .
- $\Omega_X(\log D)$: logarithmic cotangent bundle.
- $T_X(-\log D) := \Omega_X(\log D)^\vee$: logarithmic tangent bundle.

If the equality holds in "some good inequality",
then the structure of X is restricted.

Theorem (Chen-Ogiue 75)

Let (X, ω) be a compact Kähler-Einstein manifold, and $\alpha := \{\omega\} \in H^{1,1}(X, \mathbb{R})$. Then,

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2 \right) \alpha^{n-2} \geq 0.$$

Equality holds iff $X_{univ} \cong \mathbb{CP}^n, \mathbb{C}^n$, or \mathbb{B}^n .

If $c_1(T_X) = 0$ or K_X is ample, then X has a Kähler-Einstein metric. ([Yau78][Aubin-Yau78])

Theorem (Miyaoka 77, Yau 78)

- ① If $-K_X$ is ample and X has a Kähler-Einstein metric, then

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2 \right) (-K_X)^{n-2} \geq 0.$$

Equality holds iff $X \cong \mathbb{CP}^n$.

- ② If $c_1(T_X) = 0$, then $c_2(T_X)H^{n-2} \geq 0$ for any ample H .
Equality holds iff X is a torus up to a finite étale cover.

- ③ If K_X is ample, then

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2 \right) (K_X)^{n-2} \geq 0.$$

Equality holds iff X is a ball quotient.

Why does Miyaoka-Yau inequality hold?

Miyaoka-Yau inequality is a kind of Bogomolov-Gieseker inequality for "some vector bundle".

Theorem (Bogomolov, Gieseker)

Let E be a rank r vector bundle and H be an ample line bundle. If E is H -semistable, then

$$\left(c_2(E) - \frac{r-1}{2r} c_1(E)^2 \right) H^{n-2} \geq 0.$$

Why does Miyaoka-Yau inequality hold?

Definition (Tian 92)

Let L be a line bundle on X . From

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^1(X, \Omega_X^1) = \text{Ext}^1(\mathcal{O}_X, \Omega_X^1),$$

there exist a vector bundle W_L induced by $c_1(L)$ and

$$0 \rightarrow \Omega_X^1 \rightarrow W_L \rightarrow \mathcal{O}_X \rightarrow 0.$$

Let \mathcal{E}_L be a dual bundle of W_L . Then

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}_L \rightarrow T_X \rightarrow 0.$$

\mathcal{E}_L is extension sheaf of T_X by \mathcal{O}_X with the extension class $c_1(L)$.
 \mathcal{E}_{-K_X} is called the canonical extension sheaf of T_X by \mathcal{O}_X .

Why does Miyaoka-Yau inequality hold?

Propoerty

$$\mathrm{rk}(\mathcal{E}_L) = n + 1, c_1(\mathcal{E}_L) = c_1(T_X), c_2(\mathcal{E}_L) = c_2(T_X)$$

Theorem (Tian 92)

If $-K_X$ is ample and X has a Kähler-Einstein metric, then the canonical extension sheaf \mathcal{E}_{-K_X} is $-K_X$ -semistable.

Corollary

If $-K_X$ is ample and X has a Kähler-Einstein metric, then

$$\left(c_2(T_X) - \frac{n}{2(n+1)} c_1(T_X)^2 \right) (-K_X)^{n-2} \geq 0.$$

Theorem (Greb-Kebekus-Peternell 20)

Let X be a projective klt variety. Assume that $-K_X$ is nef. Then the following are equivalent.

- 1 *There exists an ample Cartier divisor H such that the canonical extension sheaf \mathcal{E}_{-K_X} is H -semistable and*

$$\left(\hat{c}_2(\Omega_X^{[1]}) - \frac{n}{2(n+1)} \hat{c}_1(\Omega_X^{[1]})^2 \right) [H]^{n-2} = 0.$$

- 2 *X is a quotient of \mathbb{CP}^n or an Abelian variety by the action of a finite group of automorphisms without fixed points in codimension one.*

Theorem (Greb-Kebekus-Peternell 20)

Let X be a projective klt variety of dimension $n \geq 2$ and H be an ample divisor. If $\Omega_X^{[1]}$ is H -semistable and

$$\left(\hat{c}_2(\Omega_X^{[1]}) - \frac{n-1}{2n} \hat{c}_1(\Omega_X^{[1]})^2 \right) [H]^{n-2} = 0,$$

then X is a quasi-Abelian variety, that is, there exists a quasi-étale cover $\tilde{X} \rightarrow X$ from an Abelian variety \tilde{X} to X .

Toward a log smooth pair case

Definition (Li 20)

Let D be a simple normal crossing divisor and L be a line bundle. From

$$H^1(X, \mathcal{O}_X^*) \xrightarrow{c_1} H^1(X, \Omega_X^1)$$

$$\xrightarrow{\Phi} H^1(X, \Omega_X^1(\log D)) = \mathrm{Ext}^1(\mathcal{O}_X, \Omega_X^1(\log D)),$$

there exist a vector bundle W_L induced by $\Phi(c_1(L))$ and

$$0 \rightarrow \Omega_X^1(\log D) \rightarrow W_L \rightarrow \mathcal{O}_X \rightarrow 0.$$

Let \mathcal{E}_L be a dual bundle of W_L . Then

$$0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}_L \rightarrow T_X(-\log D) \rightarrow 0.$$

\mathcal{E}_L is called the *extension sheaf of $T_X(-\log D)$ by \mathcal{O}_X with the extension class $c_1(L)$* .

Theorem (Li 20)

If $c_1(T_X(-\log D)) = -(K_X + D) = 0$, then \mathcal{E}_H is H -semistable for any ample line bundle H .

We do not know when $\mathcal{E}_{-(K_X+D)}$ is H -semistable for a log Fano pair (X, D) .

Theorem (I. 21)

Let X be a smooth projective variety of dimension $n \geq 2$, D be a simple normal crossing divisor, and H be an ample divisor. Assume that $-(K_X + D)$ is nef. If the extension sheaf \mathcal{E}_L is H -semistable for some line bundle L and

$$\left(c_2(T_X(-\log D)) - \frac{n}{2(n+1)} c_1(T_X(-\log D))^2 \right) H^{n-2} = 0,$$

then one of the following statements holds.

- ❶ (X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.
- ❷ $(X, D) \cong (\mathbb{CP}^n, 0)$.

Theorem (I. 21)

Let X be a smooth projective variety of dimension $n \geq 2$, D be a simple normal crossing divisor, and H be an ample divisor.

Assume that $-(K_X + D)$ is nef. If $T_X(-\log D)$ is H -semistable and

$$\left(c_2(T_X(-\log D)) - \frac{n-1}{2n} c_1(T_X(-\log D))^2 \right) H^{n-2} = 0,$$

then one of the following statements holds.

- ❶ *(X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.*
- ❷ *X is rationally connected, $c_1(K_X + D) \neq 0$, and there exists a Cartier divisor B with $T_X(-\log D) \cong \mathcal{O}_X(B)^{\oplus n}$.*

Moreover, if (2) holds and (X, D) is a Mori fiber space, then $(X, D) \cong (\mathbb{CP}^n, H_{\mathbb{CP}^n})$, where $H_{\mathbb{CP}^n}$ is a hyperplane of \mathbb{CP}^n .

Remark [I.21]

For any $m \in \mathbb{N}_{>0}$, \exists an example (X_m, D_m) such that

- ① X_m is an m -th Hirzebruch surface.
(In particular, X_m is rationally connected.)
- ② $c_1(K_{X_m} + D_m) \neq 0$.
- ③ \exists a Cartier divisor B_m with $T_{X_m}(-\log D_m) \cong \mathcal{O}_{X_m}(B_m)^{\oplus 2}$.
- ④ $(X_m, D_m) \not\cong (\mathbb{CP}^2, H_{\mathbb{CP}^2})$.

Corollary (A characterization of a toric fiber bundle)

If $c_1(T_X(-\log D)) = 0$ and $c_2(T_X(-\log D))H^{n-2} = 0$, then (X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.

- [Tsuji 88] and [Tian-Yau 87]. Assume $K_X + D$ is nef, big, and ample modulo D . If

$$\left(c_2(\Omega_X(\log D)) - \frac{n}{2(n+1)} c_1(\Omega_X(\log D))^2 \right) c_1(\Omega_X(\log D))^{n-2} = 0,$$

then the universal cover of $X \setminus D$ is a unit ball in \mathbb{C}^n .

- [Deng20]. If the natural log Higgs bundle $(\Omega_X^1(\log D) \oplus \mathcal{O}_X, \theta)$ is H -polystable and

$$\left(c_2(\Omega_X(-\log D)) - \frac{n}{2(n+1)} c_1(\Omega_X(-\log D))^2 \right) H^{n-2} = 0,$$

then the universal cover of $X \setminus D$ is a unit ball in \mathbb{C}^n .

- [Druel-LoBianco 20]. If $T_X(-\log D)$ is numerically flat, then (X, D) is a toric fiber bundle over a finite étale quotient of an Abelian variety.

Proofs - simple case $D = 0$ - (GKP's proof)

\mathcal{E}_L is H -semistable and $\left(c_2(\mathcal{E}_L) - \frac{n}{2(n+1)}c_1(\mathcal{E}_L)^2\right) H^{n-2} = 0$.

$\xRightarrow{[\text{Nak}^{04}]}$ $\mathcal{E}_L \langle \frac{K_X}{n+1} \rangle$ is nef. ($\Leftrightarrow \text{Sym}^{n+1} \mathcal{E}_L \otimes K_X$ is nef.)

$\xRightarrow{\mathcal{E}_L \rightarrow T_X}$ $T_X \langle \frac{K_X}{n+1} \rangle$ is nef.

$\xRightarrow{-K_X \text{ nef}}$ T_X is nef.

$\xRightarrow{[\text{DPS94}]}$ \exists a finite étale quotient of an Abelian variety Y and
 \exists a smooth morphism $f : X \rightarrow Y$ such that any fiber is Fano.
 $\implies \dim X = \dim Y$ or $\dim Y = 0$.

- Case 1: $\dim X = \dim Y$.

X is a finite étale quotient of an Abelian variety.

- Case 2: $\dim Y = 0$.

\exists a line bundle M with $\mathcal{E}_L \cong M^{\oplus n+1}$ by [LOY20, GKP20].

$\implies -K_X \sim (n+1)M \xRightarrow{[\text{KO73}]} X \cong \mathbb{CP}^n$.

Theorem (Nakayama 04)

Let \mathcal{E} be a rank r vector bundle. Then the following are equivalent.

- ❶ $\mathcal{E}(\frac{\det \mathcal{E}^\vee}{r})$ is nef. ($\Leftrightarrow \text{Sym}^r \mathcal{E} \otimes \det \mathcal{E}^\vee$ is nef.)
- ❷ \mathcal{E} is H -semistable and

$$\left(c_2(\mathcal{E}) - \frac{r-1}{2r} c_1(\mathcal{E})^2 \right) H^{n-2} = 0.$$

If (1) or (2) holds, then \mathcal{E} is called numerically projectively flat.

Theorem (Greb-Kebekus-Peternell 20, Liu-Ou-Yang 20)

If \mathcal{E} is numerically projectively flat, then \mathcal{E} is projectively flat.

Corollary

If X is simply connected and \mathcal{E} is numerically projectively flat, then there exists a line bundle M such that $\mathcal{E} \cong M^{\oplus r}$.

Proofs - Main result 1-

\mathcal{E}_L is H -semistable and $\left(c_2(\mathcal{E}_L) - \frac{n}{2(n+1)}c_1(\mathcal{E}_L)^2\right) H^{n-2} = 0$.

$\xRightarrow{[\text{Nak04}]}$ $\mathcal{E}_L \langle \frac{K_X + D}{n+1} \rangle$ is nef.

$\mathcal{E}_L \rightarrow T_X(-\log D) \xRightarrow{\quad} T_X(-\log D) \langle \frac{K_X + D}{n+1} \rangle$ is nef.

$-(K_X + D) \xRightarrow{\text{nef}} T_X(-\log D)$ is nef. $\implies T_X$ is pseudo-effective.

$\xRightarrow{[\text{HIM19}]}$ \exists a finite étale quotient of an Abelian variety Y and

\exists a smooth morphism $f : X \rightarrow Y$ such that any fiber is **rationally connected**.

$T_X(-\log D) \xRightarrow{\text{nef}} (X, D)$ is a logarithmic deformation over Y .

• **Case 1:** $c_1(K_F + D_F) = 0$ for any fiber F .

$\implies T_F(-\log D_F) \cong \mathcal{O}_F^{\oplus n}$. $\implies (F, D_F)$ is toric for any F .

$\implies (X, D)$ is a toric fiber bundle over Y .

• **Case 2:** $c_1(K_F + D_F) \neq 0$ for some fiber F .

$\implies \dim Y = 0$. $\implies \exists$ a line bundle M with

$-(K_X + D) \sim (n+1)M \xRightarrow{[\text{FM20}]} (X, D) \cong (\mathbb{CP}^n, 0)$.

Theorem (An easy consequence of Winkelman 04 or Brown-Mckernan-Svaldi-Zong 18)

If X is rationally connected, $-(K_X + D)$ is nef, and $T_X(-\log D) \cong \mathcal{O}_X^{\oplus n}$, then (X, D) is a toric variety with a boundary divisor D .

Theorem (Fujino-Miyamoto 20)

Let (X, D) be a projective log canonical pair. Assume that $K_X + D$ is not nef and $-(K_X + D) \equiv rM$ for some Cartier divisor M with $r > n$. Then $X \cong \mathbb{CP}^n$ with $\mathcal{O}_X(M) \cong \mathcal{O}_{\mathbb{CP}^n}(1)$.

X is rationally connected, $c_1(K_X + D) \neq 0$, \exists a Cartier divisor B with $T_X(-\log D) \cong \mathcal{O}_X(B)^{\oplus n}$, and $\phi : (X, D) \rightarrow Z$ is a Mori fiber space.

- Case 1: $\dim Z \neq 0$.

$\xRightarrow{[\text{FM20}]} (F, D_F) \cong (\mathbb{CP}^{n-1}, 0)$ for a general fiber F .

$\implies \exists$ a long exact sequence:

$$H^1(F, \Omega_X^1(\log D)|_F) \rightarrow H^1(F, \Omega_F^1) \rightarrow H^2(F, \mathcal{O}_F)^{\oplus \dim Z}.$$

$\implies H^1(F, \Omega_F^1) = 0. \implies$ Contradiction.

- Case 2: $\dim Z = 0$.

$\implies (X, D)$ is log Fano, $-(K_X + D) \sim nB$, and $D \neq 0$.

$\xRightarrow{[\text{Fjt14}]} (X, D) \cong (\mathbb{CP}^n, H_{\mathbb{CP}^n}).$

Thank you for your attention!

