On projective manifolds whose tangent bundles contain positive subbundles

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Notations

X : n-dimensional smooth projective variety over $\mathbb{C}$.
T_X : holomorphic tangent bundle of X.
$-K_X := \det T_X$ anti-canonical line bundle.

If T_X is 'positive', then the structure of X is restricted.

'positive' means ample, nef, and so on...

Positivity of vector bundles

E : vector bundle of X.
$\pi : \mathbb{P}(E) \to X$: projective bundle of X.

- E is ample (resp. strictly nef, nef)
  $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is ample (resp. strictly nef, nef) on $\mathbb{P}(E)$.
- E is big (big in the sense of Viehweg)
  $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is big and $\pi\left(\mathbb{B}_{+}(\mathcal{O}_{\mathbb{P}(E)}(1))\right) \neq X$.
- E is pseudo-effective (weakly positive in the sense of Nakayama)
  $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is psef and $\pi\left(\mathbb{B}_{-}(\mathcal{O}_{\mathbb{P}(E)}(1))\right) \neq X$.

$$
\begin{tikzcd}
\boxed{\text{Ample}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{\text{Strictly nef}} \arrow[r, Rightarrow] & \boxed{\text{Nef}} \arrow[d, Rightarrow] \\
\boxed{\text{Big}} \arrow[rr, Rightarrow] & & \boxed{\text{Psef}}
\end{tikzcd}
$$

Remark and Example

- The definition of ample (resp. strictly nef, nef, big, psef) coincides with the usual one if E is a line bundle.
- $T_{\mathbb{CP}^n}$ is ample.
- $L_1, \cdots, L_r$: line bundles on X. Set $E := \oplus_{i=1}^r L_i$.
  E is ample (resp. strictly nef, nef, big, psef)
  iff any L_i is ample (resp. strictly nef, nef, big, psef).
- A : Abelian variety. Then T_A is nef.

Previous researches

If T_X is ample (nef, big, and so on...),
then the structure of X is restricted.

Theorem

1. (*Mori 78*) *If T_X is ample, then $X \cong \mathbb{CP}^n$.*
2. (*Li-Ou-Yang 19*) *If T_X is strictly nef, then $X \cong \mathbb{CP}^n$.*
3. (*Fulger-Murayama 21*) *If T_X is big, then $X \cong \mathbb{CP}^n$.*

$$
\begin{tikzcd}
\boxed{\text{Ample}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{\text{Strictly nef}} \arrow[r, Rightarrow] & \boxed{\text{Nef}} \arrow[d, Rightarrow] \\
\boxed{\text{Big}} \arrow[rr, Rightarrow] & & \boxed{\text{Psef}}
\end{tikzcd}
$$

Theorem (Campana-Peternell 91, Demailly -Peternell-Schneider 94)

If T_X is nef, then
there exist a finite étale morphism $\pi : \tilde{X} \to X$ and
a smooth surjective morphism $\alpha : \tilde{X} \to A$ s.t.

- *A is an Abelian variety.*
- *Any fiber of α is Fano.*

Theorem (Hosono-I.-Matsumura 21)

If T_X is psef, then
there exist a finite étale morphism $\pi : \tilde{X} \to X$ and
a smooth surjective morphism $\alpha : \tilde{X} \to A$ s.t.

- *A is an Abelian variety.*
- *Any fiber of α is rationally connected.*

Summary

$$\begin{tikzcd}
\text{Ample} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Strictly nef} \arrow[r, Rightarrow] & \text{Nef} \arrow[d, Rightarrow] \\
\text{Big} \arrow[rr, Rightarrow] & & \text{Psef}
\end{tikzcd}$$

[LOY 19]
T_X is strictly nef $\Rightarrow X \cong \mathbb{CP}^n$

[Mori 78]
T_X is ample $\Rightarrow X \cong \mathbb{CP}^n$

[CP91, DPS94] T_X is nef
$\Rightarrow X \approx$ Abelian variety $+$ Fano

[FM 21]
T_X is big $\Rightarrow X \cong \mathbb{CP}^n$

[HIM 21] T_X is psef
$\Rightarrow X \approx$ Abelian variety
$+$ rationally connected variety

If T_X is 'positive' (such as ample, nef, and so on),
then we know the structure of X.

Peternell's question and related researches

Peternell proposed the following question.

Question (Peternell 01)

Let $\mathcal{F}$ be a locally free subsheaf of T_X.
What can be said on the structure of X if $\mathcal{F}$ is nef or psef?

Theorem (Andreatta-Wiśniewski 01)

If $\mathcal{F}$ is a rank r ample locally free subsheaf of T_X, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus r}$ or $\mathcal{F} \cong T_{\mathbb{CP}^n}$.

Theorem (Liu-Ou-Yang 20)

If $\mathcal{F}$ is a rank r strictly nef locally free subsheaf of T_X, then X admits a $\mathbb{CP}^d$-bundle structure $\phi : X \to Y$ for some $d \geq r$ s.t.

- $\mathcal{F} \cong T_{X/Y}$ *or* $\mathcal{F}$ *is projectively flat and* $\mathcal{F}|_F \cong \mathcal{O}_{\mathbb{CP}^d}(1)^{\oplus r}$.
- Y *is hyperbolic.* (*any holomorphic map* $\mathbb{C} \to Y$ *is constant.*)

What can be said on the structure of X
if $\mathcal{F} \subset T_X$ is nef, big, or psef?

$$\begin{tikzcd}
\boxed{\text{Ample}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{\text{Strictly nef}} \arrow[r, Rightarrow] & \boxed{\text{Nef}} \arrow[d, Rightarrow] \\
\boxed{\text{Big}} \arrow[rr, Rightarrow] & & \boxed{\text{Psef}}
\end{tikzcd}$$

Main results

We give a partial answer to Peternell's question.

Theorem (I. 20)

Let $\mathcal{F}$ be a subbundle of T_X. Assume that $\mathcal{F}$ is a foliation. If $\mathcal{F}$ is psef, then there exists a smooth morphism $f : X \to Y$ s.t.

- *Any fiber of f is rationally connected.*
- *There exists a numerically flat subbundle $\mathcal{G}$ of T_Y s.t $\mathcal{G}$ is a foliation.*
- *There exists an exact sequence of vector bundles:*

$$0 \to T_{X/Y} \to \mathcal{F} \to f^*\mathcal{G} \to 0.$$

Main results

$$\begin{tikzcd} \text{Ample} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Strictly nef} \arrow[r, Rightarrow] & \text{Nef} \arrow[d, Rightarrow] \\ \text{Big} \arrow[rr, Rightarrow] & & \text{Psef} \end{tikzcd}$$

Corollary (I. 20)

Let $\mathcal{F}$ be a rank r subbundle of T_X. Assume that $\mathcal{F}$ is a foliation.

1. *If $\mathcal{F}$ is ample, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.*
2. *If $\mathcal{F}$ is nef and big, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.*
3. *If $\mathcal{F}$ is nef, then any fiber of f is Fano.*
4. *If $\mathcal{F}$ is big, then any fiber of f is $\mathbb{CP}^r$ and $\mathcal{F} \cong T_{X/Y}$.*

Remark on the structure of a variety with a flat foliation

What can be said on the structure of Y
if $\mathcal{G} \subset T_Y$ is a numerically flat subbundle and foliation ?

$\mathcal{G}$ is numerically flat $\Rightarrow c_1(\mathcal{G}) = c_2(\mathcal{G}) = 0$.

Case 1. $\mathcal{G} = T_Y$.
By $c_1(T_Y) = c_2(T_Y) = 0$, $\exists$ a finite étale $\pi : A \to Y$ s.t. A is an Abelian variety.

Case 2. rank$\mathcal{G} = \dim Y - 1$.
This foliation is classified by [Touzet 08], [Pereira-Touzet 13] and [Druel 17]. (3 types)

Case 3. rank$\mathcal{G} = \dim Y - 2$.
This foliation is classified by [Druel 18]. (3 types)

Summary

$$\begin{tikzcd}
\text{Ample} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Strictly nef} \arrow[r, Rightarrow] & \text{Nef} \arrow[d, Rightarrow] \\
\text{Big} \arrow[rr, Rightarrow] & & \text{Psef}
\end{tikzcd}$$

[LOY 20] $\mathcal{F}$ is strictly nef $\Rightarrow$
X is $\mathbb{CP}^d$-bundle over a hyperbolic manifold.

[AW 01]
$\mathcal{F}$ is ample $\Rightarrow X \cong \mathbb{CP}^n$.

[I. 20] $\mathcal{F}$ is nef $\Rightarrow$
$X \approx$ variety with a flat foliation
$+$ Fano

[I. 20] $\mathcal{F}$ is big $\Rightarrow$
$X \approx$ variety with a flat foliation
$+ \mathbb{CP}^r$

[I. 20] $\mathcal{F}$ is psef $\Rightarrow$
$X \approx$ variety with a flat foliation
$+$ rationally connected variety