

On projective manifolds whose tangent bundles contain positive subbundles

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X : n -dimensional smooth projective variety over \mathbb{C} .

T_X : holomorphic tangent bundle of X .

$-K_X := \det T_X$ anti-canonical line bundle.

If T_X is 'positive', then the structure of X is restricted.

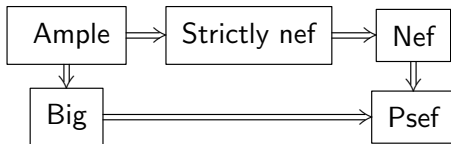
'positive' means ample, nef, and so on...

Positivity of vector bundles

E : vector bundle of X .

$\pi : \mathbb{P}(E) \rightarrow X$: projective bundle of X .

- E is ample (resp. strictly nef, nef)
 $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is ample (resp. strictly nef, nef) on $\mathbb{P}(E)$.
- E is big (big in the sense of Viehweg)
 $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is big and $\pi(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1))) \neq X$.
- E is pseudo-effective (weakly positive in the sense of Nakayama)
 $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is psef and $\pi(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))) \neq X$.



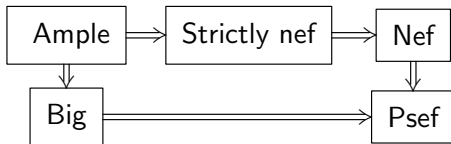
Remark and Example

- The definition of ample (resp. strictly nef, nef, big, psef) coincides with the usual one if E is a line bundle.
- $T_{\mathbb{CP}^n}$ is ample.
- L_1, \dots, L_r : line bundles on X . Set $E := \bigoplus_{i=1}^r L_i$.
 E is ample (resp. strictly nef, nef, big, psef)
iff any L_i is ample (resp. strictly nef, nef, big, psef).
- A : Abelian variety. Then T_A is nef.

If T_X is ample (nef, big, and so on...),
then the structure of X is restricted.

Theorem

- ① (Mori 78) If T_X is ample, then $X \cong \mathbb{CP}^n$.
- ② (Li-Ou-Yang 19) If T_X is strictly nef, then $X \cong \mathbb{CP}^n$.
- ③ (Fulger-Murayama 21) If T_X is big, then $X \cong \mathbb{CP}^n$.



Theorem (Campana-Peternell 91, Demailly -Peternell-Schneider 94)

*If T_X is nef, then
there exist a finite étale morphism $\pi : \tilde{X} \rightarrow X$ and
a smooth surjective morphism $\alpha : \tilde{X} \rightarrow A$ s.t.*

- *A is an Abelian variety.*
- *Any fiber of α is Fano.*

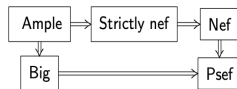
Theorem (Hosono-I.-Matsumura 21)

*If T_X is psef, then
there exist a finite étale morphism $\pi : \tilde{X} \rightarrow X$ and
a smooth surjective morphism $\alpha : \tilde{X} \rightarrow A$ s.t.*

- *A is an Abelian variety.*
- *Any fiber of α is rationally connected.*

Summary

[LOY 19]
 T_X is strictly nef $\Rightarrow X \cong \mathbb{CP}^n$



[Mori 78]
 T_X is ample $\Rightarrow X \cong \mathbb{CP}^n$

[CP91, DPS94] T_X is nef
 $\Rightarrow X \approx$ Abelian variety + Fano

[FM 21]
 T_X is big $\Rightarrow X \cong \mathbb{CP}^n$

[HIM 21] T_X is psef
 $\Rightarrow X \approx$ Abelian variety
+ rationally connected variety

If T_X is 'positive' (such as ample, nef, and so on),
then we know the structure of X .

Peternell's question and related researches

Peternell proposed the following question.

Question (Peternell 01)

Let \mathcal{F} be a locally free subsheaf of T_X .

What can be said on the structure of X if \mathcal{F} is nef or psef?

Theorem (Andreatta-Wiśniewski 01)

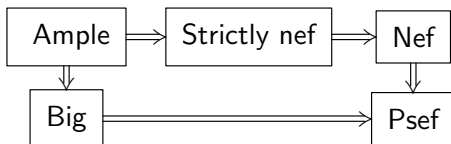
If \mathcal{F} is a rank r ample locally free subsheaf of T_X , then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus r}$ or $\mathcal{F} \cong T_{\mathbb{CP}^n}$.

Theorem (Liu-Ou-Yang 20)

If \mathcal{F} is a rank r strictly nef locally free subsheaf of T_X , then X admits a \mathbb{CP}^d -bundle structure $\phi : X \rightarrow Y$ for some $d \geq r$ s.t.

- $\mathcal{F} \cong T_{X/Y}$ or \mathcal{F} is projectively flat and $\mathcal{F}|_F \cong \mathcal{O}_{\mathbb{CP}^d}(1)^{\oplus r}$.
- Y is hyperbolic. (any holomorphic map $\mathbb{C} \rightarrow Y$ is constant.)

What can be said on the structure of X
if $\mathcal{F} \subset T_X$ is nef, big, or psef?



We give a partial answer to Peternell's question.

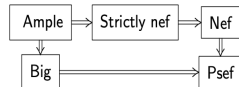
Theorem (I. 20)

Let \mathcal{F} be a subbundle of T_X . Assume that \mathcal{F} is a foliation. If \mathcal{F} is psef, then there exists a smooth morphism $f : X \rightarrow Y$ s.t.

- *Any fiber of f is rationally connected.*
- *There exists a numerically flat subbundle \mathcal{G} of T_Y s.t \mathcal{G} is a foliation.*
- *There exists an exact sequence of vector bundles:*

$$0 \rightarrow T_{X/Y} \rightarrow \mathcal{F} \rightarrow f^*\mathcal{G} \rightarrow 0.$$

Main results



Corollary (I. 20)

Let \mathcal{F} be a rank r subbundle of T_X . Assume that \mathcal{F} is a foliation.

- ① If \mathcal{F} is ample, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.
- ② If \mathcal{F} is nef and big, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.
- ③ If \mathcal{F} is nef, then any fiber of f is Fano.
- ④ If \mathcal{F} is big, then any fiber of f is \mathbb{CP}^r and $\mathcal{F} \cong T_{X/Y}$.

Remark on the structure of a variety with a flat foliation

What can be said on the structure of Y
if $\mathcal{G} \subset T_Y$ is a numerically flat subbundle and foliation ?

\mathcal{G} is numerically flat $\Rightarrow c_1(\mathcal{G}) = c_2(\mathcal{G}) = 0$.

Case 1. $\mathcal{G} = T_Y$.

By $c_1(T_Y) = c_2(T_Y) = 0$, \exists a finite étale $\pi : A \rightarrow Y$ s.t. A is an Abelian variety.

Case 2. $\text{rank} \mathcal{G} = \dim Y - 1$.

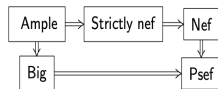
This foliation is classified by [Touzet 08], [Pereira-Touzet 13] and [Druel 17]. (3 types)

Case 3. $\text{rank} \mathcal{G} = \dim Y - 2$.

This foliation is classified by [Druel 18]. (3 types)

Summary

[LOY 20] \mathcal{F} is strictly nef \Rightarrow
 X is \mathbb{CP}^d -bundle over a hyperbolic manifold.



[AW 01]
 \mathcal{F} is ample $\Rightarrow X \cong \mathbb{CP}^n$.

[I. 20] \mathcal{F} is nef \Rightarrow
 $X \approx$ variety with a flat foliation
+ Fano

[I. 20] \mathcal{F} is big \Rightarrow
 $X \approx$ variety with a flat foliation
+ \mathbb{CP}^r

[I. 20] \mathcal{F} is psef \Rightarrow
 $X \approx$ variety with a flat foliation
+ rationally connected variety