

On projective manifolds whose tangent bundles contain positive subbundles

Masataka Iwai

Osaka City University(OCAMI), Kyoto University(RIMS)

February 6th, 2021.

Grauert theory and recent complex geometry

- ① Introduction
- ② Main result
- ③ Proof
- ④ Remark: Fujita's decomposition

X : n -dimensional smooth projective variety over \mathbb{C} .

T_X : holomorphic tangent bundle of X .

$-K_X := \det T_X$ anti-canonical line bundle.

If T_X is 'positive', then the structure of X is restricted.

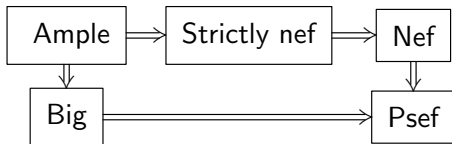
'positive' means ample, nef, and so on...

Positivity of vector bundles

E : vector bundle of X .

$\pi : \mathbb{P}(E) \rightarrow X$: projective bundle of X .

- E is ample (resp. strictly nef, nef)
 $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is ample (resp. strictly nef, nef) on $\mathbb{P}(E)$.
- E is big (big in the sense of Viehweg)
 $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is big and $\pi(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1))) \neq X$.
- E is pseudo-effective (weakly positive in the sense of Nakayama)
 $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is psef and $\pi(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))) \neq X$.



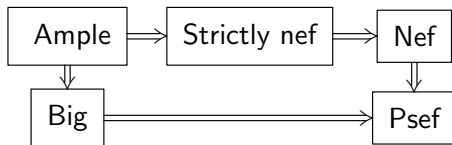
Remark and Example

- The definition of ample (resp. strictly nef, nef, big, psef) coincides with the usual one if E is a line bundle.
- $T_{\mathbb{CP}^n}$ is ample.
- L_1, \dots, L_r : line bundles on X . Set $E := \bigoplus_{i=1}^r L_i$.
 E is ample (resp. strictly nef, nef, big, psef)
iff any L_i is ample (resp. strictly nef, nef, big, psef).
- A : Abelian variety. Then T_A is nef.

If T_X is ample (nef, big, and so on...),
then the structure of X is restricted.

Theorem

- ① (Mori 78) If T_X is ample, then $X \cong \mathbb{CP}^n$.
- ② (Li-Ou-Yang 19) If T_X is strictly nef, then $X \cong \mathbb{CP}^n$.
- ③ (Fulger-Murayama 21) If T_X is big, then $X \cong \mathbb{CP}^n$.



Theorem (Campana-Peternell 91, Demailly -Peternell-Schneider 94)

If T_X is nef, then
there exist a finite étale morphism $\pi : \tilde{X} \rightarrow X$ and
a smooth surjective morphism $\alpha : \tilde{X} \rightarrow A$ s.t.

- A is an Abelian variety.
- Any fiber of α is Fano.

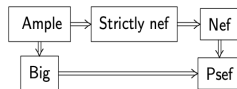
Theorem (Hosono-I.-Matsumura 21)

If T_X is psef, then
there exist a finite étale morphism $\pi : \tilde{X} \rightarrow X$ and
a smooth surjective morphism $\alpha : \tilde{X} \rightarrow A$ s.t.

- A is an Abelian variety.
- Any fiber of α is rationally connected.

Summary

[LOY 19]
 T_X is strictly nef $\Rightarrow X \cong \mathbb{CP}^n$



[Mori 78]
 T_X is ample $\Rightarrow X \cong \mathbb{CP}^n$

[CP91, DPS94] T_X is nef
 $\Rightarrow X \approx$ Abelian variety + Fano

[FM 21]
 T_X is big $\Rightarrow X \cong \mathbb{CP}^n$

[HIM 21] T_X is psef
 $\Rightarrow X \approx$ Abelian variety
+ rationally connected variety

If T_X is 'positive' (such as ample, nef, and so on),
then we know the structure of X .

Peternell's question and related researches

Peternell proposed the following question.

Question (Peternell 01)

Let \mathcal{F} be a locally free subsheaf of T_X .

What can be said on the structure of X if \mathcal{F} is nef or psef?

Theorem (Andreatta-Wisńiewski 01)

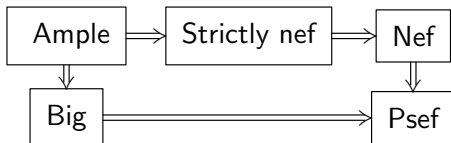
If \mathcal{F} is a rank r ample locally free subsheaf of T_X , then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus r}$ or $\mathcal{F} \cong T_{\mathbb{CP}^n}$.

Theorem (Liu-Ou-Yang 20)

If \mathcal{F} is a rank r strictly nef locally free subsheaf of T_X , then X admits a \mathbb{CP}^d -bundle structure $\phi : X \rightarrow Y$ for some $d \geq r$ s.t.

- $\mathcal{F} \cong T_{X/Y}$ or \mathcal{F} is projectively flat and $\mathcal{F}|_F \cong \mathcal{O}_{\mathbb{CP}^d}(1)^{\oplus r}$.
- Y is hyperbolic. (any holomorphic map $\mathbb{C} \rightarrow Y$ is constant.)

What can be said on the structure of X
if $\mathcal{F} \subset T_X$ is nef, big, or psef?



We give a partial answer to Peternell's question.

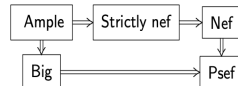
Theorem (I. 20)

Let \mathcal{F} be a subbundle of T_X . Assume that \mathcal{F} is a foliation. If \mathcal{F} is psef, then there exists a smooth morphism $f : X \rightarrow Y$ s.t.

- *Any fiber of f is rationally connected.*
- *There exists a numerically flat subbundle \mathcal{G} of T_Y s.t \mathcal{G} is a foliation.*
- *There exists an exact sequence of vector bundles:*

$$0 \rightarrow T_{X/Y} \rightarrow \mathcal{F} \rightarrow f^*\mathcal{G} \rightarrow 0.$$

Main results



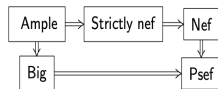
Corollary (I. 20)

Let \mathcal{F} be a rank r subbundle of T_X . Assume that \mathcal{F} is a foliation.

- ① If \mathcal{F} is ample, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.
- ② If \mathcal{F} is nef and big, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.
- ③ If \mathcal{F} is nef, then any fiber of f is Fano.
- ④ If \mathcal{F} is big, then any fiber of f is \mathbb{CP}^r and $\mathcal{F} \cong T_{X/Y}$.

Summary

[LOY 20] \mathcal{F} is strictly nef \Rightarrow
 X is \mathbb{CP}^d -bundle over a hyperbolic manifold.



[AW 01]
 \mathcal{F} is ample $\Rightarrow X \cong \mathbb{CP}^n$.

[I. 20] \mathcal{F} is nef \Rightarrow
 $X \approx$ variety with a flat foliation
+ Fano

[I. 20] \mathcal{F} is big \Rightarrow
 $X \approx$ variety with a flat foliation
+ \mathbb{CP}^r

[I. 20] \mathcal{F} is psef \Rightarrow
 $X \approx$ variety with a flat foliation
+ rationally connected variety

Remark on the structure of a variety with a flat foliation

What can be said on the structure of Y
if $\mathcal{G} \subset T_Y$ is a numerically flat subbundle and foliation ?

\mathcal{G} is numerically flat $\Rightarrow c_1(\mathcal{G}) = c_2(\mathcal{G}) = 0$.

Case 1. $\mathcal{G} = T_Y$.

By $c_1(T_Y) = c_2(T_Y) = 0$, \exists a finite étale $\pi : A \rightarrow Y$ s.t. A is an Abelian variety.

Case 2. $\text{rank} \mathcal{G} = \dim Y - 1$.

This foliation is classified by [Touzet 08], [Pereira-Touzet 13] and [Druel 17]. (3 types)

Case 3. $\text{rank} \mathcal{G} = \dim Y - 2$.

This foliation is classified by [Druel 18]. (3 types)

Sketch proof -from the viewpoint of slopes-

A : ample line bundle on X . \mathcal{E} : torsion free coherent sheaf on X .

$$\mu_A(\mathcal{E}) := \frac{c_1(\mathcal{E})A^{n-1}}{\text{rank}\mathcal{E}}$$

$$\mu_A^{\min}(\mathcal{E}) := \inf\{\mu_A(\mathcal{Q}) : \mathcal{E} \twoheadrightarrow \mathcal{Q}\}$$

The following theorem is obtained by combining [Miyaoka 87, Campana-Păun 19] with [Höring 07].

Theorem (Miyaoka 87, Campana-Păun 19 + Höring 07)

Let $\mathcal{E} \subset T_X$ be a subbundle and foliation.

If $\mu_A^{\min}(\mathcal{E}) > 0$, then there exists a smooth surjective morphism $f : X \rightarrow Y$ s.t.

- *Any fiber of f is rationally connected.*
- $\mathcal{E} = T_{X/Y}$.

We obtain the following decomposition theorem, which is a generalization of Fujita's decomposition. (We will discuss later.)

Theorem (I. 20)

Let \mathcal{F} be a psef vector bundle on X .

Then there exist vector bundles \mathcal{E}, \mathcal{Q} s.t.

- *\mathcal{E} is psef and $\mu_A^{\min}(\mathcal{E}) > 0$ unless $\mathcal{E} = 0$.*
- *\mathcal{Q} is numerically flat.*
- *There exists an exact sequence of vector bundles:*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

Proof of main theorem

Assume that $\mathcal{F} \subset T_X$ is a subbundle, foliation and psef.

\Rightarrow

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

- $\mathcal{E} \subset T_X$ is a subbundle and foliation with $\mu_A^{\min}(\mathcal{E}) > 0$.
- \mathcal{Q} is numerically flat.

$\Rightarrow \exists$ smooth surjective morphism $f : X \rightarrow Y$
s.t. any fiber of f is rationally connected and $\mathcal{E} = T_{X/Y}$

$\Rightarrow \exists \mathcal{G} \subset T_Y$ s.t. $\mathcal{Q} = f^*\mathcal{G}$.

\mathcal{G} is a numerically flat subbundle and foliation on Y .

F : a fiber of f .

- ① If \mathcal{F} is ample $\Rightarrow f^*\mathcal{G} = 0$, both $T_{X/Y}$ and T_F are ample
 $\Rightarrow \dim Y = 0$, $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.
- ② If \mathcal{F} is nef and big...(same argument!)
- ③ If \mathcal{F} is nef $\Rightarrow T_F$ is nef and F is rationally connected
 $\Rightarrow F$ is Fano
- ④ If \mathcal{F} is big $\Rightarrow f^*\mathcal{G} = 0$, T_F is big $\Rightarrow F \cong \mathbb{CP}^r$ and $\mathcal{F} \cong T_{X/Y}$.

Remark: Fujita's decomposition

Theorem (Fujita 77, Catanese-Dettweiler 17)

Let $f : X \rightarrow C$ be a surjective morphism from a compact Kähler manifold X to a smooth projective curve C .

Then there exist vector bundles \mathcal{E}, \mathcal{Q} s.t.

- \mathcal{E} is ample unless $\mathcal{E} = 0$.
- \mathcal{Q} is flat.
- $f_*(K_{X/C}) \cong \mathcal{E} \oplus \mathcal{Q}$

By a similar argument of the decomposition of psef vector bundles, we have the following theorem.

Remark: Fujita's decomposition

Theorem (I. 20)

Let $f : X \rightarrow Y$ be a surjective morphism with connected fibers from a compact Kähler manifold X to a smooth projective variety Y . Let A be an ample line bundle on Y .

For any $m \in \mathbb{N}$, there exist reflexive sheaves \mathcal{E}, \mathcal{Q} s.t.

- \mathcal{E} is psef and $\mu_A^{min}(\mathcal{E}) > 0$ unless $\mathcal{E} = 0$.*
- \mathcal{Q} is hermitian flat vector bundle.*
- $f_*(mK_{X/Y})^{\vee\vee} \cong \mathcal{E} \oplus \mathcal{Q}$*

In particular, if Y is a curve, for any $m \in \mathbb{N}$, we have

$$f_*(mK_{X/Y}) \cong \mathcal{E} \oplus \mathcal{Q},$$

where \mathcal{E} is an ample vector bundle and \mathcal{Q} is a hermitian flat vector bundle.

Thank you for your attention!