# On projective manifolds whose tangent bundles contain positive subbundles

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Grauert theory and recent complex geometry

# Notations

$X$ : $n$-dimensional smooth projective variety over $\mathbb{C}$.
$T_X$ : holomorphic tangent bundle of $X$.
$-K_X := \det T_X$ anti-canonical line bundle.

If $T_X$ is 'positive', then the structure of $X$ is restricted.

'positive' means ample, nef, and so on...

# Positivity of vector bundles

$E$ : vector bundle of $X$.
$\pi : \mathbb{P}(E) \to X$ : projective bundle of $X$.

- $E$ is ample (resp. strictly nef, nef)
  $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is ample (resp. strictly nef, nef) on $\mathbb{P}(E)$.
- $E$ is big (big in the sense of Viehweg)
  $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is big and $\pi\left(\mathbb{B}_+(\mathcal{O}_{\mathbb{P}(E)}(1))\right) \neq X$.
- $E$ is pseudo-effective (weakly positive in the sense of Nakayama)
  $\Leftrightarrow \mathcal{O}_{\mathbb{P}(E)}(1)$ is psef and $\pi\left(\mathbb{B}_-(\mathcal{O}_{\mathbb{P}(E)}(1))\right) \neq X$.

$$\begin{tikzcd}
\boxed{\text{Ample}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{\text{Strictly nef}} \arrow[r, Rightarrow] & \boxed{\text{Nef}} \arrow[d, Rightarrow] \\
\boxed{\text{Big}} \arrow[rr, Rightarrow] & & \boxed{\text{Psef}}
\end{tikzcd}$$

# Remark and Example

- The definition of ample (resp. strictly nef, nef, big, psef) coincides with the usual one if $E$ is a line bundle.
- $T_{\mathbb{CP}^n}$ is ample.
- $L_1, \cdots, L_r$ : line bundles on $X$. Set $E := \oplus_{i=1}^r L_i$.
  $E$ is ample (resp. strictly nef, nef, big, psef)
  iff any $L_i$ is ample (resp. strictly nef, nef, big, psef).
- $A$ : Abelian variety. Then $T_A$ is nef.

# Previous researches

If $T_X$ is ample (nef, big, and so on...),
then the structure of $X$ is restricted.

**Theorem**

1. (*Mori 78*) *If $T_X$ is ample, then $X \cong \mathbb{CP}^n$.*
2. (*Li-Ou-Yang 19*) *If $T_X$ is strictly nef, then $X \cong \mathbb{CP}^n$.*
3. (*Fulger-Murayama 21*) *If $T_X$ is big, then $X \cong \mathbb{CP}^n$.*

$$\begin{tikzcd}
\boxed{\text{Ample}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{\text{Strictly nef}} \arrow[r, Rightarrow] & \boxed{\text{Nef}} \arrow[d, Rightarrow] \\
\boxed{\text{Big}} \arrow[rr, Rightarrow] & & \boxed{\text{Psef}}
\end{tikzcd}$$

**Theorem (Campana-Peternell 91, Demailly -Peternell-Schneider 94)**

*If $T_X$ is nef, then*
*there exist a finite étale morphism $\pi : \tilde{X} \to X$ and*
*a smooth surjective morphism $\alpha : \tilde{X} \to A$ s.t.*

- *$A$ is an Abelian variety.*
- *Any fiber of $\alpha$ is Fano.*

**Theorem (Hosono-I.-Matsumura 21)**

*If $T_X$ is psef, then*
*there exist a finite étale morphism $\pi : \tilde{X} \to X$ and*
*a smooth surjective morphism $\alpha : \tilde{X} \to A$ s.t.*

- *$A$ is an Abelian variety.*
- *Any fiber of $\alpha$ is rationally connected.*

# Summary

$$\begin{tikzcd} \text{Ample} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Strictly nef} \arrow[r, Rightarrow] & \text{Nef} \arrow[d, Rightarrow] \\ \text{Big} \arrow[rr, Rightarrow] & & \text{Psef} \end{tikzcd}$$

[LOY 19]
$T_X$ is strictly nef $\Rightarrow X \cong \mathbb{CP}^n$

[Mori 78]
$T_X$ is ample $\Rightarrow X \cong \mathbb{CP}^n$

[CP91, DPS94] $T_X$ is nef
$\Rightarrow X \approx$ Abelian variety + Fano

[FM 21]
$T_X$ is big $\Rightarrow X \cong \mathbb{CP}^n$

[HIM 21] $T_X$ is psef
$\Rightarrow X \approx$ Abelian variety
+ rationally connected variety

If $T_X$ is 'positive' (such as ample, nef, and so on),
then we know the structure of $X$.

# Peternell's question and related researches

Peternell proposed the following question.

**Question (Peternell 01)**

Let $\mathcal{F}$ be a locally free subsheaf of $T_X$.
What can be said on the structure of $X$ if $\mathcal{F}$ is nef or psef?

**Theorem (Andreatta-Wiśniewski 01)**

*If $\mathcal{F}$ is a rank $r$ ample locally free subsheaf of $T_X$, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong \mathcal{O}_{\mathbb{CP}^n}(1)^{\oplus r}$ or $\mathcal{F} \cong T_{\mathbb{CP}^n}$.*

**Theorem (Liu-Ou-Yang 20)**

*If $\mathcal{F}$ is a rank $r$ strictly nef locally free subsheaf of $T_X$, then $X$ admits a $\mathbb{CP}^d$-bundle structure $\phi : X \to Y$ for some $d \geq r$ s.t.*

- $\mathcal{F} \cong T_{X/Y}$ *or* $\mathcal{F}$ *is projectively flat and* $\mathcal{F}|_F \cong \mathcal{O}_{\mathbb{CP}^d}(1)^{\oplus r}$.
- $Y$ *is hyperbolic. (any holomorphic map* $\mathbb{C} \to Y$ *is constant.)*

What can be said on the structure of $X$ if $\mathcal{F} \subset T_X$ is nef, big, or psef?

$$\begin{tikzcd}
\boxed{\text{Ample}} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \boxed{\text{Strictly nef}} \arrow[r, Rightarrow] & \boxed{\text{Nef}} \arrow[d, Rightarrow] \\
\boxed{\text{Big}} \arrow[rr, Rightarrow] & & \boxed{\text{Psef}}
\end{tikzcd}$$

# Main results

We give a partial answer to Peternell's question.

**Theorem (I. 20)**

*Let $\mathcal{F}$ be a subbundle of $T_X$. Assume that $\mathcal{F}$ is a foliation. If $\mathcal{F}$ is psef, then there exists a smooth morphism $f : X \to Y$ s.t.*

- *Any fiber of $f$ is rationally connected.*
- *There exists a numerically flat subbundle $\mathcal{G}$ of $T_Y$ s.t $\mathcal{G}$ is a foliation.*
- *There exists an exact sequence of vector bundles:*

$$0 \to T_{X/Y} \to \mathcal{F} \to f^*\mathcal{G} \to 0.$$

# Main results

$$\begin{tikzcd} \text{Ample} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Strictly nef} \arrow[r, Rightarrow] & \text{Nef} \arrow[d, Rightarrow] \\ \text{Big} \arrow[rr, Rightarrow] & & \text{Psef} \end{tikzcd}$$

## Corollary (I. 20)

*Let $\mathcal{F}$ be a rank $r$ subbundle of $T_X$. Assume that $\mathcal{F}$ is a foliation.*

1. *If $\mathcal{F}$ is ample, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.*
2. *If $\mathcal{F}$ is nef and big, then $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.*
3. *If $\mathcal{F}$ is nef, then any fiber of $f$ is Fano.*
4. *If $\mathcal{F}$ is big, then any fiber of $f$ is $\mathbb{CP}^r$ and $\mathcal{F} \cong T_{X/Y}$.*

# Summary

$$\begin{tikzcd} \text{Ample} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{Strictly nef} \arrow[r, Rightarrow] & \text{Nef} \arrow[d, Rightarrow] \\ \text{Big} \arrow[rr, Rightarrow] & & \text{Psef} \end{tikzcd}$$

[LOY 20] $\mathcal{F}$ is strictly nef $\Rightarrow$
$X$ is $\mathbb{CP}^d$-bundle over a hyperbolic manifold.

[AW 01]
$\mathcal{F}$ is ample $\Rightarrow X \cong \mathbb{CP}^n$.

[I. 20] $\mathcal{F}$ is nef $\Rightarrow$
$X \approx$ variety with a flat foliation
$+$ Fano

[I. 20] $\mathcal{F}$ is big $\Rightarrow$
$X \approx$ variety with a flat foliation
$+ \mathbb{CP}^r$

[I. 20] $\mathcal{F}$ is psef $\Rightarrow$
$X \approx$ variety with a flat foliation
$+$ rationally connected variety

# Remark on the structure of a variety with a flat foliation

What can be said on the structure of $Y$
if $\mathcal{G} \subset T_Y$ is a numerically flat subbundle and foliation ?

$\mathcal{G}$ is numerically flat $\Rightarrow c_1(\mathcal{G}) = c_2(\mathcal{G}) = 0$.

Case 1. $\mathcal{G} = T_Y$.
By $c_1(T_Y) = c_2(T_Y) = 0$, $\exists$ a finite étale $\pi : A \to Y$ s.t. $A$ is an Abelian variety.

Case 2. rank$\mathcal{G} = \dim Y - 1$.
This foliation is classified by [Touzet 08], [Pereira-Touzet 13] and [Druel 17]. (3 types)

Case 3. rank$\mathcal{G} = \dim Y - 2$.
This foliation is classified by [Druel 18]. (3 types)

# Sketch proof -from the viewpoint of slopes-

$A$ : ample line bundle on $X$. $\mathcal{E}$ : torsion free coherent sheaf on $X$.

$$\mu_A(\mathcal{E}) := \frac{c_1(\mathcal{E})A^{n-1}}{\mathrm{rank}\mathcal{E}}$$

$$\mu_A^{min}(\mathcal{E}) := \inf\{\mu_A(\mathcal{Q}) : \mathcal{E} \twoheadrightarrow \mathcal{Q}\}$$

The following theorem is obtained by combining [Miyaoka 87, Campana-Păun 19] with [Höring 07].

**Theorem (Miyaoka 87, Campana-Păun 19 + Höring 07)**

*Let $\mathcal{E} \subset T_X$ be a subbundle and foliation.*
*If $\mu_A^{min}(\mathcal{E}) > 0$, then there exists a smooth surjective morphism $f : X \to Y$ s.t.*

- *Any fiber of $f$ is rationally connected.*
- $\mathcal{E} = T_{X/Y}$.

We obtain the following decomposition theorem, which is a generalization of Fujita's decomposition. (We will discuss later.)

**Theorem (I. 20)**

*Let $\mathcal{F}$ be a psef vector bundle on $X$.*
*Then there exist vector bundles $\mathcal{E}, \mathcal{Q}$ s.t.*

- *$\mathcal{E}$ is psef and $\mu_A^{min}(\mathcal{E}) > 0$ unless $\mathcal{E} = 0$.*
- *$\mathcal{Q}$ is numerically flat.*
- *There exists an exact sequence of vector bundles:*

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

# Proof of main theorem

Assume that $\mathcal{F} \subset T_X$ is a subbundle, foliation and psef.

$\Rightarrow$

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0.$$

- $\mathcal{E} \subset T_X$ is a subbundle and foliation with $\mu_A^{min}(\mathcal{E}) > 0$.
- $\mathcal{Q}$ is numerically flat.

$\Rightarrow \exists$ smooth surjective morphism $f : X \to Y$
s.t. any fiber of $f$ is rationally connected and $\mathcal{E} = T_{X/Y}$

$\Rightarrow \exists\, \mathcal{G} \subset T_Y$ s.t. $\mathcal{Q} = f^*\mathcal{G}$.
$\mathcal{G}$ is a numerically flat subbundle and foliation on $Y$.

# Proof of corollary

$F$ : a fiber of $f$.

1. If $\mathcal{F}$ is ample $\Rightarrow f^*\mathcal{G} = 0$, both $T_{X/Y}$ and $T_F$ are ample $\Rightarrow \dim Y = 0$, $X \cong \mathbb{CP}^n$ and $\mathcal{F} \cong T_{\mathbb{CP}^n}$.
2. If $\mathcal{F}$ is nef and big...(same argument!)
3. If $\mathcal{F}$ is nef $\Rightarrow T_F$ is nef and $F$ is rationally connected $\Rightarrow F$ is Fano
4. If $\mathcal{F}$ is big $\Rightarrow f^*\mathcal{G} = 0$, $T_F$ is big $\Rightarrow F \cong \mathbb{CP}^r$ and $\mathcal{F} \cong T_{X/Y}$.

# Remark: Fujita's decomposition

**Theorem (Fujita 77, Catanese-Dettweiler 17)**

*Let $f : X \to C$ be a surjective morphism from a compact Kähler manifold $X$ to a smooth projective curve $C$.*
*Then there exist vector bundles $\mathcal{E}, \mathcal{Q}$ s.t.*

- *$\mathcal{E}$ is ample unless $\mathcal{E} = 0$.*
- *$\mathcal{Q}$ is flat.*
- $f_*(K_{X/C}) \cong \mathcal{E} \oplus \mathcal{Q}$

By a similar argument of the decomposition of psef vector bundles, we have the following theorem.

### Theorem (I. 20)

*Let $f : X \to Y$ be a surjective morphism with connected fibers from a compact Kähler manifold $X$ to a smooth projective variety $Y$. Let $A$ be an ample line bundle on $Y$.*
*For any $m \in \mathbb{N}$, there exist reflexive sheaves $\mathcal{E}, \mathcal{Q}$ s.t.*

- *$\mathcal{E}$ is psef and $\mu_A^{min}(\mathcal{E}) > 0$ unless $\mathcal{E} = 0$.*
- *$\mathcal{Q}$ is hermitian flat vector bundle.*
- $f_*(mK_{X/Y})^{\vee\vee} \cong \mathcal{E} \oplus \mathcal{Q}$

*In particular, if $Y$ is a curve, for any $m \in \mathbb{N}$, we have*

$$f_*(mK_{X/Y}) \cong \mathcal{E} \oplus \mathcal{Q},$$

*where $\mathcal{E}$ is an ample vector bundle and $\mathcal{Q}$ is a hermitian flat vector bundle.*

Thank you for your attention!