Recent topics in singular Hermitian metrics

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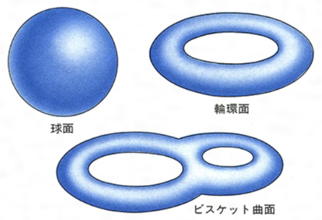
Problem

How many smooth projective varieties?
How to classify smooth projective varieties?

In dimension 1, X is a Riemann surface. We can classify a genus.



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The key is...

There exists a smooth surface X_{min} such that

$$f: \quad X \quad \to \quad X_{min}$$

where f is a composition of blow-ups and X_{min} is "minimal" (X_{min} does not have blow-ups).

Enriques and Kodaira classified X_{min}.

Higher dimensional case

Definition

- L is ample $\Leftrightarrow \exists m \in \mathbb{N}_{>0}$ and $\exists s_0(x), \ldots, s_N(x) \in H^0(X, L^{\otimes m})$ s.t.
$$\begin{array}{cccc} \varphi_{|L^{\otimes m}|} : & X & \to & \mathbb{CP}^N \\ & x & \to & (s_0(x) : \cdots : s_N(x)) \end{array}$$
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- L is nef $\overset{def}{\Leftrightarrow} \forall C \subset X$: curve,$L.C \geq 0$
- L is big $\overset{def}{\Leftrightarrow} \limsup_{k \to +\infty} \frac{\dim H^0(X, L^{\otimes k})}{k^n} > 0$

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- L is big $\overset{def}{\Leftrightarrow} \limsup_{k\to+\infty} \frac{\dim H^0(X, L^{\otimes k})}{k^n} > 0$
- L is pseudo-effective $\overset{def}{\Leftrightarrow} \exists A$: ample and $\forall m \in \mathbb{N}_{>0}$, $L^{\otimes m} \otimes A$ is big.

	Algebraic Geometry
Ample	$\exists m \in \mathbb{N}_{>0}$ s.t. $\varphi_{\lvert L^{\otimes m}\rvert}$ is embedding
Nef	$\forall C \subset X$:curve $L.C \geq 0$
Big	$\limsup_{k\to+\infty} \dim H^0(X, L^{\otimes k})/(k^n) > 0$
Pseudo effective	$\exists\ A$: ample and $\forall\ m \in \mathbb{N}_{>0}$, s.t. $L^{\otimes m} \otimes A$ is big.

Minimal model conjecture

We consider a canonical bundle $K_X := \det(T_X)^{-1}$.

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Conjecture (Minimal model conjecture (a little modified))

If K_X is pseudo-effective, does X have minimal model?

X has minimal model if there exists $f := X \dashrightarrow X_{min}$ such that X_{min} is minimal model. (f is a composition of flip and divisorial contraction)

Minimal model conjecture

If $\dim X \leq 2$ then X has minimal model. (By Riemann, Enriqus, Kodaira and so on.)

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If $\dim X = 3$*, Minimal model conjecture is true.*

Theorem (Birkar-Casini-Hacon-Mckernan 2010)

If K_X *is big , then* X *has a minimal model.*

Theorem (Kodaira 53)

Then L is ample if and only if L has a smooth metric with positive curvature.

It is so-called Kodaira embedding Theorem.
The same things hold if L is nef, big, or pseudo-effective.

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- h is a singular Hermitian metric (sHm) on L
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Before the next results...

- h is a singular Hermitian metric (sHm) on L
  $\overset{def}{\Leftrightarrow}$ $\exists$ a smooth metric h_0 and $\varphi \in L^1_{loc}(X)$ s.t. $h = h_0 e^{-\varphi}$
- The curvature current $\sqrt{-1}\Theta_{L,h} := \sqrt{-1}\Theta_{L,h_0} + \sqrt{-1}\partial\bar{\partial}\varphi$ for any sHm h.

Theorem (Demailly 92)

Let ω be a Kähler form.

- *L is nef* $\Leftrightarrow \forall \epsilon > 0, \exists h_\epsilon$ *smooth metric, s.t.* $\sqrt{-1}\Theta_{L,h_\epsilon} + \epsilon\omega \geq 0$.

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Nef	$\forall C \subset X$:curve $L.C \geq 0$	$\forall \epsilon > 0, \exists h_\epsilon$ smooth metric, s.t. $\sqrt{-1}\Theta_{L,h_\epsilon} + \epsilon\omega \geq 0$.
Big	$\limsup_{k\to+\infty}$ $\dim H^0(X, L^{\otimes k})/(k^n) > 0$	$\exists \epsilon > 0, \exists h$ sHm, s.t. $\sqrt{-1}\Theta_{L,h} - \epsilon\omega \geq 0$ in the sense of current.
Pseudo effective	$\exists$ A: ample and $\forall$ $m \in \mathbb{N}_{>0}$, s.t. $L^{\otimes m} \otimes A$ is big.	$\exists h$ sHm, s.t. $\sqrt{-1}\Theta_{L,h} \geq 0$ in the sense of current.

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Ample $\Rightarrow$ Nef, Ample $\Rightarrow$ Big, Big or Nef $\Rightarrow$ Pseudo-effective

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- $K_{X/Y} := K_X \otimes f^*(K_Y)^{-1}$ has a semipositive canonical singular Hermitian metric.
- $f_*(K_{X/Y})$ has a semipositive canonical singular Hermitian metric.

Theorem (Berndtsson-Paun 08)

Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties. Assume that there exists a regular value $y \in Y$ such that $H^0\Big(X_y, K_{X_y}^{\otimes a}\Big) \neq 0$.

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Then the bundle $K_{X/Y}^{\otimes a}$ admits a sHm h_a such that

- $\sqrt{-1}\Theta_{L,h_a} \geq 0$ *in the sense of current.*
- *For any regular value $w \in Y$ and any section $s \in H^0\big(X_w, K_{X_w}^{\otimes a}\big)$ we have*

$$|s|_{h_a}^{\frac{2}{a}}(z) \leq \int_{X_w} |s|^{\frac{2}{a}} < +\infty$$

for any $z \in X_w$. ($|s|^{\frac{2}{a}}$ as a semipositive continuous (m, m) form where $m = \dim X_w$.)

- h_a is a "canonical" singular Hermitian metric on $K_{X/Y}^{\otimes a}$.
- The condition $\sqrt{-1}\Theta_{L,h_a} \geq 0$ is useful.
- $|s|_{h_a}^{\frac{2}{a}}(z) < +\infty$ is also useful. (Maybe h_a has a singular point, that is $h_a(z) = +\infty$)

Application

Conjecture (Popa-Schnell 14)

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Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties, with Y of dimension n, and L be an ample line bundle on Y. For any $a \geq 1$, is the sheaf

$$f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$$

globally generated for all $b \geq a(n+1)$?

If $X = Y$ and f is an identity map, Popa-Schnell conjecture implies the following Fujita's freeness conjecture.

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Conjecture (Fujita's freeness conjecture)

Let X be a smooth projective n-dimensional variety and L be an ample line bundle on X. Is the line bundle $K_X \otimes L^{\otimes n+1}$ globally generated?

This conjecture is also open.

Partial result of Popa-Schnell's conjecture

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- (Popa-Schnell 14) If L is ample and globally generated, this conjecture holds.
- (Dutta 17) $f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$ is generated by the global sections at a **general point** $y \in Y$ for all $b \geq a\left(\frac{n(n+1)}{2} + 1\right)$.
- (Deng 17) $f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$ is generated by the global sections at a **general point** $y \in Y$ for all $b \geq n^2 - n + a(n+1)$.

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By using result, if f is smooth, $f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$ is globally generated on Y for all $b \geq \frac{n(n-1)}{2} + a(n+1)$.

Idea of proof

We show that for any regular value $y \in Y$, any section

$$s \in H^0\Big(X_y, K_{X_y}^{\otimes a} \otimes f^*(L)^{\otimes b}|_{X_y}\Big)$$

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3. We solve a $\overline{\partial}$-equation to extend the section S_V.

We explain (1) and (2).

Theorem (relative version of Ohsawa-Takegoshi L^2 extension)

Let (M, h) be a line bundle on X with sHm h such that $\sqrt{-1}\Theta_{M,h} \geq 0$.
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We take a small open neighborhood $V \ni y$. Then for any $s \in H^0(X_y, K_{X_y} \otimes M|_{X_y})$ *such that* $\int_{X_y} |s|_h^2 < +\infty$, *there exists* $S_V \in H^0(f^{-1}(V), K_X \otimes M)$ *such that*
$S_V|_{X_y} = s$ *and* $\int_{f^{-1}(V)} |S_V|_h^2 < +\infty$

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We have $K_X \otimes M = K_X^{\otimes a} \otimes f^* L^{\otimes b}$ and

$$M = \underbrace{K_{X/Y}^{\otimes a-1}}_{\textit{has sHm } h_a^{a-1}} \otimes \underbrace{f^*(K_Y \otimes L^{\otimes n+1})^{\otimes a-1}}_{\textit{has metric } h_{semiposi}} \otimes \underbrace{f^* L^{\otimes \tilde{b}}}_{\textit{has metric } h_{posi}}$$

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Then M has sHm h such that $\sqrt{-1}\Theta_{M,h} \geq 0$ and

$$\int_{X_y} |s|_h^2 < +\infty.$$

We can use OT-extension theorem. ($|S_V|_h$ is bounded above for the extension S_V of s)

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For any surjective projective morphism $f\colon X \to Y$ between smooth complex manifold with connected fibers, $f_*(K_{X/Y}^{\otimes m})$ can be endowed a Griffith semipositive singular Hermitian metric h_{NS} called "Narasimhan-Simha" metric (if $f_*(K_{X/Y}^{\otimes m}) \neq 0$).

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$f_*(K_{X/Y}^{\otimes m})$ is NOT line bundle. So, we must define "Griffith semipositive" "singular Hermitian metrics on vector bundles and torsion free coherent sheaves"

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- (Cao-Paun 17, Hacon-Popa-Schnell 17) If $\det f_*(K_{X/Y}^{\otimes m})$ is numerically zero ($c_1(f_*(K_{X/Y}^{\otimes m})) = 0$) then $f_*(K_{X/Y}^{\otimes m})$ is flat vector bundle and h_{NS} is smooth.

Definition of sHm on vector bundle

We adopt the definition by Hacon,Popa, and Schnell. (This definition is easy to understand)

Definition (Hacon-Popa-Schnell 17)

A *singular Hermitian inner product* on a finite dimensional complex vector space V is a function $|-|_h : V \to [0, +\infty]$ with the following properties:

1. $|\alpha \cdot v|_h = |\alpha||v|_h : \forall \alpha \in \mathbb{C} \setminus 0, \forall v \in V$
2. $|0|_h = 0$
3. $|v + w|_h \leq |v|_h + |w|_h : \forall v, w \in V$
4. $|v + w|_h^2 + |v - w|_h^2 = |v|_h^2 + |w|_h^2 : \forall v, w \in V$

Definition (deCataldo 98, Berndtsson-Păun 08, Hacon-Popa-Schnell 17)

Let X be a complex manifold and E be a holomorphic vector bundle.

A *singular Hermitian metric* (sHm) on E is a function h that associates to any $x \in X$ a singular Hermitian inner product $|-|_{h,x} : E_x \to [0, +\infty]$ with the following properties:

1. $|v|_{h,x} = 0 \Leftrightarrow v = 0$ for almost everywhere x
2. $|v|_{h,x} < +\infty : \forall v \in E_x$ for almost everywhere x
3. For any open U and any $s \in H^0(U, E)$,

$$|s|_h : U \to [0, +\infty] \;\; ; \;\; x \to |s(x)|_{h,x}$$

is measurable function.

Definition (Berndtsson-Păun 08, Păun-Takayama 14, Hacon-Popa-Schnell 17)

1. A sHm h on E is *Griffiths seminegative* if the function $\log|u|_h^2$ is plurisubharmonic for any local section u of E.
2. A sHm h on E is *Griffiths semipositive* if the dual metric $h^* = {}^t h^{-1}$ on the dual vector bundle E^* is Griffiths seminegative.

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- When h is smooth, h is Griffiths seminegative (in the usual sense) iff $\log |u|_h^2$ is plurisubharmonic for any local section u of E.
- If E is a line bundle, h is Griffiths semipositive sHm iff $\sqrt{-1}\Theta_{E,h} \geq 0$ in the sense of current.

Applications -Viehweg's weakly positivity-

Definition

1. E is *dd-ample* if there exist $x \in X, a \in \mathbb{N}_{>0}$ and ample line bundle A such that
$Sym^a(E) \otimes A^{-1}$ is globally generated at x.

Applications -Viehweg's weakly positivity-

Definition

1. E is *dd-ample* if there exist $x \in X$,$a \in \mathbb{N}_{>0}$ and ample line bundle A such that $Sym^a(E) \otimes A^{-1}$ is globally generated at x.
2. E is pseudo-effective there exist $x \in X$ and ample line bundle A such that for any $a \in \mathbb{N}_{>0}$, there exists $b \in \mathbb{N}_{>0}$ such that $Sym^{ab}(E) \otimes A^b$ is globally generated at x

- (Viehweg 83) $f_*(mK_{X/Y})$ is pseudo-effective for any fibration $f\colon X \to Y$. (In fact $f_*(mK_{X/Y})$ is weakly-positive.)

	Algebraic Geometry	singular Hermitian metric
dd-ample	$\exists x \in X$, $\exists A$ ample, $\exists a, \in \mathbb{N}$ s.t. $Sym^a(E) \otimes A^{-1}$ is globally generated at x	
pseudo effective	$\exists x \in X$, $\exists A$ ample,s.t. $\forall a \in \mathbb{N}$, $\exists b, \in \mathbb{N}$, s.t. $Sym^{ab}(E) \otimes A^a$ is globally generated at x	

Theorem (I. 18)

1. *E is dd-ample iff there exist $a \in \mathbb{N}_{>0}$ and an ample line bundle A such that $Sym^a(E) \otimes A^{-1}$ has a Griffiths semipositive singular Hermitian metric h.*
   (Moreover h is smooth and Nakano semipositive on Zariski open set.)

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2. *E is pseudo-effective iff there exists an ample line bundle A such that $Sym^a(E) \otimes A$ has a Griffiths semipositive singular Hermitian metric for any $a \in \mathbb{N}_{>0}$.*

	Algebraic Geometry	singular Hermitian metric
dd-ample	$\exists x \in X$, $\exists A$ ample, $\exists a, \in \mathbb{N}$ s.t. $Sym^a(E) \otimes A^{-1}$ is globally generated at x	$\exists a, \in \mathbb{N}$,$\exists A$ ample s.t. $Sym^a(E) \otimes A^{-1}$ has Griffiths semipositive sHm
pseudo effective	$\exists x \in X$, $\exists A$ ample,s.t. $\forall a \in \mathbb{N}$, $\exists b, \in \mathbb{N}$, s.t. $Sym^{ab}(E) \otimes A^a$ is globally generated at x	$\exists A$ ample s.t.$\forall a \in \mathbb{N}$ $Sym^a(E) \otimes A$ has Griffith semipositive sHm

Theorem (Păun-Takayama 14)

If E has a Griffith semipositive sHm h, then E is pseudo-effective.

Proof.

$\forall a \in \mathbb{N}$, $\forall A$ ample, $Sym^a(E) \otimes A$ has Griffith semipositive $Sym^a(h) \otimes h_A$. $\square$

(Hosono 17) There exists a pseudo-effective vector bundle E such that E does NOT have a Griffith semipositive sHm.

Outlook

Outlook

Find other applications of vector bundles and torsion-free coherent sheaves with sHm.

- vanishing theorems about $\Omega(\log D)$ (Matsuki et al.)
- Seshadri constants (Lehman-Murayama.)
- Hodge theory ($f_*(K^m_{X/Y})$) is difficult for Hodge Theory but...)
- Application of degeneration problem ($f\colon X \to Y$ is morphism with general type fiber. Does $f_*(K^m_{X/Y})$ have Griffiths positive singular Hermitian metric? Actually $f_*(K^m_{X/Y})$ is dd-ample)
- singular Kobayashi Hitchin correspondence (like singular Kähler Einstein...)