Characterization of weakly positive torsion-free coherent sheaves by singular Hermitian metrics

Masataka Iwai

The University of Tokyo

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- L is big $\overset{def}{\Leftrightarrow}$ $\limsup_{k\to+\infty} \frac{\dim H^0(X, L^{\otimes k})}{k^n} > 0$
- L is pseudo-effective $\overset{def}{\Leftrightarrow}$ For any ample line bundle A and any $m \in \mathbb{N}_{>0}$, $L^{\otimes m} \otimes A$ is big.

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If K_X is nef, we call X is a minimal model.

Theorem (Demailly 92)

- *L is nef* $\Leftrightarrow \forall \epsilon > 0, \exists h_\epsilon$ *smooth metric, s.t.* $\sqrt{-1}\Theta_{L,h_\epsilon} + \epsilon\omega \geq 0$.
- *L is big* $\Leftrightarrow \exists \epsilon > 0, \exists h$ *sHm, s.t.* $\sqrt{-1}\Theta_{L,h} - \epsilon\omega \geq 0$ *in the sense of current.*
- *L is pseudo-effective* $\Leftrightarrow \exists h$ *sHm, s.t.* $\sqrt{-1}\Theta_{L,h} \geq 0$ *in the sense of current.*

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Aim

Prove that Kodaira's and Demailly's theorem in the case of vector bundles and torsion-free coherent sheaves with slight modification.

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- $X_{\mathcal{F}}$: the maximal Zariski open set where $\mathcal{F}$ is locally free.
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- we will denote by $S^k(\mathcal{F})$ the k-th symmetric power of $\mathcal{F}$ and denote by $\widehat{S}^k(\mathcal{F})$ the double dual of the sheaf of $S^k(\mathcal{F})$.

Definition

1. $\mathcal{F}$ is *weakly positive at* $x \in X$ if for any $a \in \mathbb{N}_{>0}$ and for any ample line bundle A, there exists $b \in \mathbb{N}_{>0}$ such that $\widehat{S}^{ab}(\mathcal{F}) \otimes A^b$ is globally generated at x.

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3. $\mathcal{F}$ is *weakly positive in the sense of Viehweg* if there exists a Zariski open set $U \subset X$ such that $\mathcal{F}$ is weakly positive at x for any $x \in U$.

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- If E is line bundle, E is weakly positive in the sense of Nakayama $\Leftrightarrow$ E is pseudo-effective.
- (Viehweg 83) $f_*(mK_{X/Y})$ is weakly positive in the sense of Viehweg for any fibration $f\colon X \to Y$.

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3. A singular Hermitian metric h on E is *Griffiths semipositive* (or *h is semipositively curved*) if there exists a Griffiths seminegative metric g on $\mathcal{F}^*|_{X_{\mathcal{F}}}$ such that $h|_{X_{\mathcal{F}}} = (g|_{X_{\mathcal{F}}})^*$

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(Hosono 17) There exists a weakly positive vector bundle E in the sense of Nakayama such that E does NOT have a Griffith semipositive sHm.

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   - *$\widehat{S}^k(\mathcal{F}) \otimes A$ has a Griffiths semipositive singular Hermitian metric h_k for any $k \in \mathbb{N}_{>0}$, and*
   - *the Lelong number of $\det h_k$ at x is less than 2 for any $x \in U$ and any $k \in \mathbb{N}_{>0}$.*

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- The same things hold when $\mathcal{F}$ is a nef or big vector bundle or a dd-ample sheaf.
- We concrete the basic theory of vector bundles and torsion-free coherent sheaves with sHm without using curvature current of sHm (c.f. chapter 5,6 in Demailly's book)
- Recently Deng studied Kobayashi hyperbolicity of moduli spaces by using Paun-Takayama's result.

Question

Find other applications of vector bundles and torsion-free coherent sheaves with sHm (such as the vanishing theorems about $\Omega(\log D)$, Seshadri constants, hyperbolicity, Hodge theory...).