Vanishing theorems of vector bundles with singular Hermitian metrics

Masataka Iwai

The University of Tokyo

MSJ Autumn Meeting 2018, at Okayama University.
September 25, 2018

Theorem (Kodaira 53)

Let X be a compact Kähler manifold and L be a holomorphic line bundle. Assume L has a smooth metric with positive curvature. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L) = 0.$$

Theorem (Kodaira 53)

Let X be a compact Kähler manifold and L be a holomorphic line bundle. Assume L has a smooth metric with positive curvature. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L) = 0.$$

By Kodaira's vanishing theorem, L is ample if and only if L has a smooth metric with positive curvature.

Before the next results...

Before the next results...

- h is a singular Hermitian metric (sHm) on L
  $\overset{def}{\Leftrightarrow}$ $\exists$ a smooth metric h_0 and $\varphi \in L^1_{loc}(X)$ s.t. $h = h_0 e^{-\varphi}$
- The curvature current $\sqrt{-1}\Theta_{L,h} := \sqrt{-1}\Theta_{L,h_0} + \sqrt{-1}\partial\overline{\partial}\varphi$ for any sHm h.

Before the next results...

- h is a singular Hermitian metric (sHm) on L
  $\overset{def}{\Leftrightarrow}$ $\exists$ a smooth metric h_0 and $\varphi \in L^1_{loc}(X)$ s.t. $h = h_0 e^{-\varphi}$
- The curvature current $\sqrt{-1}\Theta_{L,h} := \sqrt{-1}\Theta_{L,h_0} + \sqrt{-1}\partial\bar{\partial}\varphi$ for any sHm h.
- The multiplier ideal sheaf $\mathcal{J}(h)$ of h

$$\mathcal{J}(h)_x := \{f \in \mathcal{O}_{X,x}; \exists U \ni x, \int_U |f|^2 e^{-\varphi} d\lambda < \infty\},$$

  where $d\lambda$ is the standard Lesbegue measure.

Theorem (Nadel 89. (cf. Demailly 82))

Let (X, ω) *be a compact Kähler manifold and* L *be a holomorphic line bundle. Assume* h *has a sHm on* L *such that* $\sqrt{-1}\Theta_{L,h} \geq \epsilon\omega$ *in the sense of current for some* $\epsilon \in \mathbb{R}_{>0}$*. Then for any* $q \geq 1$

$$H^q(X, K_X \otimes L \otimes \mathcal{J}(h)) = 0.$$

Theorem (Nadel 89. (cf. Demailly 82))

Let (X, ω) be a compact Kähler manifold and L be a holomorphic line bundle. Assume h has a sHm on L such that $\sqrt{-1}\Theta_{L,h} \geq \epsilon\omega$ in the sense of current for some $\epsilon \in \mathbb{R}_{>0}$. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L \otimes \mathcal{J}(h)) = 0.$$

- By Nadel vanishing theorem, we have Angehrn-Siu's theorem (if L is ample line bundle then $K_X \otimes L^{\otimes \frac{n(n+1)}{2}+1}$ is globally generated) and so on.
- We can proved Kawamata-Viehweg vanishing theorem by Nadel vanishing theorem.

We have many applications by using vanishing theorems.

Aim

Prove that vanishing theorems in more general setting.

We have many applications by using vanishing theorems.

Aim

Prove that vanishing theorems in more general setting. More explicitly...
Prove that for any $q \geq 1$ and any vector bundle E,

$$H^q(X, K_X \otimes E(h)) = 0$$

with some assumption (positivity of metric).

We have many applications by using vanishing theorems.

Aim

Prove that vanishing theorems in more general setting. More explicitly...
Prove that for any $q \geq 1$ and any vector bundle E,

$$H^q(X, K_X \otimes E(h)) = 0$$

with some assumption (positivity of metric).

- h is a sHm of a vector bundle E.
- $E(h)$ is a higher rank analogy of multiplier ideal sheaf.
- h has "some positivity".

Definition of sHm on vector bundle

We adopt the definition by Hacon,Popa, and Schnell. (This definition is easy to understand)

Definition (Hacon-Popa-Schnell 17)

A *singular Hermitian inner product* on a finite dimensional complex vector space V is a function $|-|_h : V \to [0, +\infty]$ with the following properties:

1. $|\alpha \cdot v|_h = |\alpha||v|_h : \forall \alpha \in \mathbb{C} \setminus 0, \forall v \in V$
2. $|0|_h = 0$
3. $|v + w|_h \leq |v|_h + |w|_h : \forall v, w \in V$
4. $|v + w|_h^2 + |v - w|_h^2 = |v|_h^2 + |w|_h^2 : \forall v, w \in V$

Definition (deCataldo 98, Berndtsson-Păun 08, Hacon-Popa-Schnell 17)

Let X be a complex manifold and E be a holomorphic vector bundle.

A *singular Hermitian metric* (sHm) on E is a function h that associates to any $x \in X$ a singular Hermitian inner product $|-|_{h,x} : E_x \to [0, +\infty]$ with the following properties:

1. $|v|_{h,x} = 0 \Leftrightarrow v = 0$ for almost everywhere x
2. $|v|_{h,x} < +\infty : \forall v \in E_x$ for almost everywhere x
3. For any open U and any $s \in H^0(U, E)$,

$$|s|_h : U \to [0, +\infty] \;\; ; \;\; x \to |s(x)|_{h,x}$$

is measurable function.

Definition (Berndtsson-Păun 08, Păun-Takayama 14, Hacon-Popa-Schnell 17)

1. A sHm h on E is *Griffiths seminegative* if the function $\log |u|_h^2$ is plurisubharmonic for any local section u of E.
2. A sHm h on E is *Griffiths semipositive* if the dual metric $h^* = {}^t h^{-1}$ on the dual vector bundle E^* is Griffiths seminegative.

Definition (Berndtsson-Păun 08, Păun-Takayama 14, Hacon-Popa-Schnell 17)

1. A sHm h on E is *Griffiths seminegative* if the function $\log |u|_h^2$ is plurisubharmonic for any local section u of E.
2. A sHm h on E is *Griffiths semipositive* if the dual metric $h^* = {}^t h^{-1}$ on the dual vector bundle E^* is Griffiths seminegative.

- When h is smooth, h is Griffiths seminegative (in the usual sense) iff $\log |u|_h^2$ is plurisubharmonic for any local section u of E.
- If E is a line bundle, h is Griffiths semipositive sHm iff $\sqrt{-1}\Theta_{E,h} \geq 0$ in the sense of current.

History

- (Păun-Takayama 14)
  $f_*(mK_{X/Y})$ has a Griffith semipositive sHm called "Narasimhan-Simha" metric for any fibration $f\colon X \to Y$.

History

- (Păun-Takayama 14)
  $f_*(mK_{X/Y})$ has a Griffith semipositive sHm called "Narasimhan-Simha" metric for any fibration $f\colon X \to Y$.
- (Cao-Păun 17) Iitaka's $C_{n,m}$ conjecture holds if the base space is an Abelian variety,

History

- (Păun-Takayama 14)
  $f_*(mK_{X/Y})$ has a Griffith semipositive sHm called "Narasimhan-Simha" metric for any fibration $f\colon X \to Y$.
- (Cao-Păun 17) Iitaka's $C_{n,m}$ conjecture holds if the base space is an Abelian variety,
- (Cao-Höring 17) The structure theorem with nef anticanonical divisor.

Higher rank analogy of multiplier ideal sheaf

Definition

Let (E, h) be a vector bundle with sHm.
The sheaf of locally square integrable holomorphic sections of E with respect to h is defined by

$$E(h)_x = \{f_x \in E_{(x)} : |f_x|_h^2 \in L^1_{loc}\} \;\; x \in X,$$

where $E_{(x)}$ the stalk of E at x, defined by $\varinjlim_{x \in U} H^0(U, E)$.

- $E(h)$ is a higher rank analogy of a multiplier ideal sheaf.
- We don't know whether this sheaf is coherent.

Main results

Theorem (I. 18)

Let (X, ω) be a Kähler manifold and (E, h) be a holomorphic vector bundle on X with a sHm. We assume the following conditions.

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *$he^{-\zeta}$ is a Griffiths semipositive sHm on E for some continuous function ζ on X.*
3. *There exists $C \in \mathbb{R}$ such that $\sqrt{-1}\Theta_{E,h} - C\omega \otimes Id_E \geq_{Nak} 0$ on $X \setminus Z$.*

Then the sheaf $E(h)$ is coherent.

Nadel-Nakano type vanishing theorems

Theorem (I. 18)

Let (X, ω) be a compact Kähler manifold and (E, h) be a holomorphic vector bundle on X with a sHm. We assume the following conditions.

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *$he^{-\zeta}$ is a Griffiths semipositive sHm on E for some continuous function ζ on X.*
3. *There exists $\epsilon \in \mathbb{R}_{>0}$ such that $\sqrt{-1}\Theta_{E,h} - \epsilon\omega \otimes Id_E \geq_{Nak} 0$ on $X \setminus Z$.*

Then $H^q(X, K_X \otimes E(h)) = 0$ holds for any $q \geq 1$.

Nadel-Nakano type vanishing theorems

Theorem (I. 18)

Let (X, ω) be a compact Kähler manifold and (E, h) be a holomorphic vector bundle on X with a sHm. We assume the following conditions.

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *$he^{-\zeta}$ is a Griffiths semipositive sHm on E for some continuous function ζ on X.*
3. *There exists $\epsilon \in \mathbb{R}_{>0}$ such that $\sqrt{-1}\Theta_{E,h} - \epsilon\omega \otimes Id_E \geq_{Nak} 0$ on $X \setminus Z$.*

Then $H^q(X, K_X \otimes E(h)) = 0$ holds for any $q \geq 1$.

Moreover, we proved Kollár type injectivity theorems and Kollár-Ohsawa type vanishing theorems of vector bundles.