On the global generation of direct images of pluri-adjoint line bundles

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Conjecture (Fujita's freeness conjecture)

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- if $n = 2$ (Reider 88)
- if $n = 3$ (Ein-Lazarsfeld 93)
- if $n = 4$ (Kawamata 97)
- if $n = 5$ (Ye-Zhu 15)

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(Angehrn-Siu 95) $K_X \otimes L^{\otimes b}$ is globally generated for any $b \geq \frac{n(n+1)}{2} + 1$.

Conjecture (Popa-Schnell 14)

Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties, with Y of dimension n, and L be an ample line bundle on Y. For any $a \geq 1$, is the sheaf

$$f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$$

globally generated for all $b \geq a(n+1)$?

If $X = Y$ and f is an identity map, Popa-Schnell conjecture implies Fujita's freeness conjecture.

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Even if (X, Δ) is a Kawamata log terminal $\mathbb{Q}$-pair of a normal projective variety and an effective divisor, their results hold, that is, $f_*\Big(\mathcal{O}_X\big(a(K_X + \Delta)\big)\Big) \otimes L^{\otimes b}$ is generated by the global sections at a **general point** $y \in Y$.

Theorem (I.17)

Let $f\colon X \to Y$ be a surjective morphism of smooth projective varieties, with Y of dimension n, and L be an ample line bundle on Y.
If y is a **regular value** *of f, then for any $a \geq 1$ the sheaf*

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is generated by the global sections at y for all $b \geq \frac{n(n-1)}{2} + a(n+1)$.

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By using result, if f is smooth, $f_*(K_X^{\otimes a}) \otimes L^{\otimes b}$ is globally generated on Y for all $b \geq \frac{n(n-1)}{2} + a(n+1)$.

Theorem (I.17, Dutta-Murayama 17)

Let (X, Δ) be a Kawamata log terminal $\mathbb{Q}$-pair of a normal projective variety and an effective divisor, and Y be a smooth projective n-dimensional variety.
Let $f\colon X \to Y$ be a surjective morphism, and L be an ample line bundle on Y.
For any $a \geq 1$ such that $a(K_X + \Delta)$ is an integral Cartier divisor, the sheaf

$$f_*\Big(\mathcal{O}_X\big(a(K_X + \Delta)\big)\Big) \otimes L^{\otimes b}$$

is generated by the global sections at a **general point** *$y \in Y$ for all $b \geq \frac{n(n-1)}{2} + a(n+1)$*

Idea of proof

We show that for any regular value $y \in Y$, any section

$$s \in H^0\Big(X_y, K_{X_y}^{\otimes a} \otimes f^*(L)^{\otimes b}|_{X_y}\Big)$$

can be extended to X.

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3. We solve a $\overline{\partial}$-equation to extend the section S_V.