Vanishing theorems of vector bundles with singular Hermitian metrics

Masataka Iwai

The University of Tokyo

Geometry seminar in Nagoya Univ. May 29, 2018

Introduction

Theorem (Kodaira 53)

Let X be a compact Kähler manifold and L be a holomorphic line bundle. Assume L has a smooth metric with positive curvature. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L) = 0.$$

Introduction

Theorem (Kodaira 53)

Let X be a compact Kähler manifold and L be a holomorphic line bundle. Assume L has a smooth metric with positive curvature. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L) = 0.$$

By Kodaira's vanishing theorem, L is ample if and only if L has a smooth metric with positive curvature.

Theorem (Kawamata 82 ,Viehweg 82)

Let X be smooth projective variety and L be a nef and big line bundle. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L) = 0.$$

Theorem (Kawamata 82 ,Viehweg 82)

Let X be smooth projective variety and L be a nef and big line bundle. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L) = 0.$$

- By Kawamata-Viehweg vanishing theorem (and so on), we have Kawamata-Shokurov base point free theorem, Mori's cone theorem and so on.

Theorem (Kawamata 82 ,Viehweg 82)

Let X be smooth projective variety and L be a nef and big line bundle. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L) = 0.$$

- By Kawamata-Viehweg vanishing theorem (and so on), we have Kawamata-Shokurov base point free theorem, Mori's cone theorem and so on.
- L is nef $\overset{def}{\Leftrightarrow} \forall C \subset X$: curve,$L.C \geq 0$
- L is big $\overset{def}{\Leftrightarrow} \limsup_{k \to +\infty} \frac{\dim H^0(X, L^{\otimes k})}{k^n} > 0$

Before the next results...

Before the next results...

- h is a singular Hermitian metric (sHm) on L
  $\overset{def}{\Leftrightarrow}$ $\exists$ a smooth metric h_0 and $\varphi \in L^1_{loc}(X)$ s.t. $h = h_0 e^{-\varphi}$
- $\sqrt{-1}\Theta_{L,h} := \sqrt{-1}\Theta_{L,h_0} + \sqrt{-1}\partial\overline{\partial}\varphi$ for any sHm h.

Before the next results...

- h is a singular Hermitian metric (sHm) on L
  $\overset{def}{\Leftrightarrow}$ $\exists$ a smooth metric h_0 and $\varphi \in L^1_{loc}(X)$ s.t. $h = h_0 e^{-\varphi}$
- $\sqrt{-1}\Theta_{L,h} := \sqrt{-1}\Theta_{L,h_0} + \sqrt{-1}\partial\overline{\partial}\varphi$ for any sHm h.
- The multiplier ideal sheaf $\mathcal{J}(h)$ of h

$$\mathcal{J}(h)_x := \{f \in \mathcal{O}_{X,x}; \exists U \ni x, \int_U |f|^2 e^{-\varphi} d\lambda < \infty\},$$

where $d\lambda$ is the standard Lesbegue measure.

Before the next results...

- h is a singular Hermitian metric (sHm) on L
  $\overset{def}{\Leftrightarrow}$ $\exists$ a smooth metric h_0 and $\varphi \in L^1_{loc}(X)$ s.t. $h = h_0 e^{-\varphi}$
- $\sqrt{-1}\Theta_{L,h} := \sqrt{-1}\Theta_{L,h_0} + \sqrt{-1}\partial\overline{\partial}\varphi$ for any sHm h.
- The multiplier ideal sheaf $\mathcal{J}(h)$ of h

$$\mathcal{J}(h)_x := \{f \in \mathcal{O}_{X,x}; \exists U \ni x, \int_U |f|^2 e^{-\varphi} d\lambda < \infty\},$$

  where $d\lambda$ is the standard Lesbegue measure.
- The multiplier ideal sheaf $\mathcal{J}(h)$ is coherent sheaf if $\sqrt{-1}\Theta_{L,h} \geq 0$ (this is NOT trivial).

Theorem (Nadel 89. (cf. Demailly 82))

Let (X, ω) be a compact Kähler manifold and L be a holomorphic line bundle. Assume h has a sHm on L such that $\sqrt{-1}\Theta_{L,h} \geq \epsilon\omega$ in the sense of current for some $\epsilon \in \mathbb{R}_{>0}$. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L \otimes \mathcal{J}(h)) = 0.$$

Theorem (Nadel 89. (cf. Demailly 82))

Let (X, ω) be a compact Kähler manifold and L be a holomorphic line bundle. Assume h has a sHm on L such that $\sqrt{-1}\Theta_{L,h} \geq \epsilon\omega$ in the sense of current for some $\epsilon \in \mathbb{R}_{>0}$. Then for any $q \geq 1$

$$H^q(X, K_X \otimes L \otimes \mathcal{J}(h)) = 0.$$

By Nadel vanishing theorem, we have Angehrn-Siu's theorem (if L is ample line bundle then $K_X \otimes L^{\otimes \frac{n^2+n}{2}+1}$ is globally generated) and so on.

Remarks

- L is nef $\overset{def}{\Leftrightarrow} \forall C \subset X$: curve, $L.C \geq 0$
  $\Leftrightarrow \forall \epsilon > 0, \exists h_\epsilon$ smooth metric, s.t. $\sqrt{-1}\Theta_{L,h} + \epsilon\omega \geq 0$.

Remarks

- L is nef $\overset{def}{\Leftrightarrow} \forall C \subset X$: curve,$L.C \geq 0$
  $\Leftrightarrow \forall \epsilon > 0, \exists h_\epsilon$ smooth metric, s.t. $\sqrt{-1}\Theta_{L,h} + \epsilon\omega \geq 0$.
- L is big $\overset{def}{\Leftrightarrow} \limsup_{k \to +\infty} \frac{\dim H^0(X, L^{\otimes k})}{k^n} > 0$
  $\Leftrightarrow \exists \epsilon > 0, \exists h$ sHm, s.t. $\sqrt{-1}\Theta_{L,h} - \epsilon\omega \geq 0$ in the sense of current.

Remarks

- L is nef $\overset{def}{\Leftrightarrow} \forall C \subset X$: curve,$L.C \geq 0$
  $\Leftrightarrow \forall \epsilon > 0, \exists h_\epsilon$ smooth metric, s.t. $\sqrt{-1}\Theta_{L,h} + \epsilon\omega \geq 0$.
- L is big $\overset{def}{\Leftrightarrow} \limsup_{k\to+\infty} \frac{\dim H^0(X,L^{\otimes k})}{k^n} > 0$
  $\Leftrightarrow \exists \epsilon > 0, \exists h$ sHm, s.t. $\sqrt{-1}\Theta_{L,h} - \epsilon\omega \geq 0$ in the sense of current.
- Nadel vanishing $\Rightarrow$ Kawamata-Viehweg vanishing $\Rightarrow$ Kodaira vanishing

We have many applications by using vanishing theorems.

We have many applications by using vanishing theorems.

Aim

Prove that vanishing theorems in more general setting.

In this talk, we show that vanishing theorems of vector bundles with sHm.

More explicitly...

Aim

Prove that for any $q \geq 1$ and any vector bundle E,

$$H^q(X, K_X \otimes E(h)) = 0$$

with some assumption (positivity of metric).

More explicitly...

Aim

Prove that for any $q \geq 1$ and any vector bundle E,

$$H^q(X, K_X \otimes E(h)) = 0$$

with some assumption (positivity of metric).

- h is a sHm of a vector bundle E.
- $E(h)$ is a higher rank analogy of multiplier ideal sheaf.
- h has "some positivity".

Positivity of metric -smooth case-

(X, ω) : a compact Kähler manifold with a Kähler form.
(E, h) :a holomorphic vector bundle of rank r with a smooth metric
$(V; z_1, \ldots, z_n)$: a system of local coordinate near $x \in X$.
$(e_1, \ldots, e_r)$: a local orthogonal holomorphic frame of E.

Positivity of metric -smooth case-

(X, ω) : a compact Kähler manifold with a Kähler form.
(E, h) :a holomorphic vector bundle of rank r with a smooth metric
$(V; z_1, \ldots, z_n)$: a system of local coordinate near $x \in X$.
$(e_1, \ldots, e_r)$: a local orthogonal holomorphic frame of E. The Chern curvature tensor of h:

$$\sqrt{-1}\Theta_{E,h} = \sqrt{-1} \sum_{1 \le j,k \le n,\, 1 \le \lambda,\mu \le r} c_{jk\lambda\mu}\, dz_j \wedge d\bar{z}_k \otimes e_\lambda^* \otimes e_\mu$$

Kähler form : $\omega = \sqrt{-1} \sum_{1 \le j,k \le n} \omega_{jk} dz_j \wedge d\bar{z}_k$

Definition and Notation

$\sqrt{-1}\Theta_{E,h} \geq_{Nak} (\leq_{Nak}) C\omega \otimes id_E \Leftrightarrow$

$$\sum_{1\leq j,k\leq n,\, 1\leq \lambda,\mu\leq r} c_{jk\lambda\mu} u_{j\lambda}\bar{u}_{k\mu} - C \sum_{1\leq j,k\leq n,\, 1\leq \lambda\leq r} \omega_{jk} u_{j\lambda}\bar{u}_{k\lambda} \geq (\leq) 0$$

for any $u = \sum_{1\leq j\leq n,\, 1\leq \lambda\leq r} u_{j\lambda} dz_j \otimes e_\lambda \in T_x X \otimes E_x$ and any $x \in X$.

Definition and Notation

$\sqrt{-1}\Theta_{E,h} \geq_{Nak} (\leq_{Nak}) C\omega \otimes id_E \Leftrightarrow$

$$\sum_{1\leq j,k\leq n,\, 1\leq \lambda,\mu\leq r} c_{jk\lambda\mu} u_{j\lambda}\bar{u}_{k\mu} - C \sum_{1\leq j,k\leq n,\, 1\leq \lambda\leq r} \omega_{jk} u_{j\lambda}\bar{u}_{k\lambda} \geq (\leq) 0$$

for any $u = \sum_{1\leq j\leq n,\, 1\leq \lambda\leq r} u_{j\lambda} dz_j \otimes e_\lambda \in T_x X \otimes E_x$ and any $x \in X$.

$\sqrt{-1}\Theta_{E,h} \geq_{Grif} (\leq_{Grif}) C\omega \otimes id_E \Leftrightarrow$

$$\sum_{1\leq j,k\leq n,\, 1\leq \lambda,\mu\leq r} c_{jk\lambda\mu} \xi_j a_\lambda \bar{\xi}_k \bar{a}_\mu - C \sum_{1\leq j,k\leq n,\, 1\leq \lambda\leq r} \omega_{jk} \xi_j \bar{\xi}_k |a_\lambda|^2 \geq (\leq) 0$$

for any $u = \sum_{1\leq j\leq n,\, 1\leq \lambda\leq r} \xi_j a_\lambda dz_j \otimes e_\lambda \in T_x X \otimes E_x$ and any $x \in X$

- $\sqrt{-1}\Theta_{E,h} \geq_{Nak} (\leq_{Nak}) 0 \overset{def}{\Leftrightarrow} h$ is Nakano semipositive (seminegative).
- $\sqrt{-1}\Theta_{E,h} \geq_{Grif} (\leq_{Grif}) 0 \overset{def}{\Leftrightarrow} h$ is Griffiths semipositive (seminegative).

- $\sqrt{-1}\Theta_{E,h} \geq_{Nak} (\leq_{Nak}) 0 \overset{def}{\Leftrightarrow} h$ is Nakano semipositive (seminegative).
- $\sqrt{-1}\Theta_{E,h} \geq_{Grif} (\leq_{Grif}) 0 \overset{def}{\Leftrightarrow} h$ is Griffiths semipositive (seminegative).
- Nakano semipositive $\Rightarrow$ Griffiths semipositive

- $\sqrt{-1}\Theta_{E,h} \geq_{Nak} (\leq_{Nak}) 0 \overset{def}{\Leftrightarrow} h$ is Nakano semipositive (seminegative).
- $\sqrt{-1}\Theta_{E,h} \geq_{Grif} (\leq_{Grif}) 0 \overset{def}{\Leftrightarrow} h$ is Griffiths semipositive (seminegative).
- Nakano semipositive $\Rightarrow$ Griffiths semipositive

Theorem (Nakano 55)

Let X be a compact Kähler manifold and E be a holomorphic vector bundle. Assume E has a smooth metric h such that $\sqrt{-1}\Theta_{E,h} \geq_{Nak} C\omega \otimes id_E$ for some $C \in \mathbb{R}_{>0}$.
Then for any $q \geq 1$

$$H^q(X, K_X \otimes E) = 0.$$

- $\sqrt{-1}\Theta_{E,h} \geq_{Nak} (\leq_{Nak}) 0 \overset{def}{\Leftrightarrow} h$ is Nakano semipositive (seminegative).
- $\sqrt{-1}\Theta_{E,h} \geq_{Grif} (\leq_{Grif}) 0 \overset{def}{\Leftrightarrow} h$ is Griffiths semipositive (seminegative).
- Nakano semipositive $\Rightarrow$ Griffiths semipositive

Theorem (Nakano 55)

Let X be a compact Kähler manifold and E be a holomorphic vector bundle. Assume E has a smooth metric h such that $\sqrt{-1}\Theta_{E,h} \geq_{Nak} C\omega \otimes id_E$ for some $C \in \mathbb{R}_{>0}$.
Then for any $q \geq 1$

$$H^q(X, K_X \otimes E) = 0.$$

Nakano positivity is useful of vanishing theorems.
(Griffiths positivity is too weak not to vanish the cohomology.)

Definition of sHm on vector bundle

We adopt the definition by Hacon,Popa, and Schnell. (This definition is easy to understand)

Definition of sHm on vector bundle

We adopt the definition by Hacon,Popa, and Schnell. (This definition is easy to understand)

Definition (Hacon-Popa-Schnell 17)

A *singular Hermitian inner product* on a finite dimensional complex vector space V is a function $|-|_h \colon V \to [0, +\infty]$ with the following properties:

1. $|\alpha \cdot v|_h = |\alpha||v|_h : \forall \alpha \in \mathbb{C} \setminus 0, \forall v \in V$
2. $|0|_h = 0$
3. $|v + w|_h \leq |v|_h + |w|_h : \forall v, w \in V$
4. $|v + w|_h^2 + |v - w|_h^2 = |v|_h^2 + |w|_h^2 : \forall v, w \in V$

Definition (Hacon-Popa-Schnell 17)

Let X be a complex manifold E be a holomorphic vector bundle.
A *singular Hermitian metric* (sHm) on E is a function h that associates to any $x \in X$ a singular Hermitian inner product $|-|_{h,x} \colon E_x \to [0, +\infty]$ with the following properties:

1. $|v|_{h,x} = 0 \Leftrightarrow v = 0$ for almost everywhere x
2. $|v|_{h,x} < +\infty : \forall v \in E_x$ for almost everywhere x
3. For any open U and any $s \in H^0(U, E)$,

$$|s|_h \colon U \to [0, +\infty] \ \ ; \ \ x \to |s(x)|_{h,x}$$

is measurable function.

Definition (Păun-Takayama 14, Hacon-Popa-Schnell 17)

1. A sHm h on E is *Griffiths seminegative* if the function $\log |u|_h^2$ is plurisubharmonic for any local section u of E.
2. A sHm h on E is *Griffiths semipositive* if the dual metric $h^* = {}^t h^{-1}$ on the dual vector bundle E^* is Griffiths seminegative.

Definition (Păun-Takayama 14, Hacon-Popa-Schnell 17)

1. A sHm h on E is *Griffiths seminegative* if the function $\log |u|_h^2$ is plurisubharmonic for any local section u of E.
2. A sHm h on E is *Griffiths semipositive* if the dual metric $h^* = {}^t h^{-1}$ on the dual vector bundle E^* is Griffiths seminegative.

- $\varphi \in L^1_{loc}$ is plurisubharmonic iff $\sqrt{-1}\partial\overline{\partial}\varphi \geq 0$ in the sense of current.

Definition (Păun-Takayama 14, Hacon-Popa-Schnell 17)

1. A sHm h on E is *Griffiths seminegative* if the function $\log |u|_h^2$ is plurisubharmonic for any local section u of E.
2. A sHm h on E is *Griffiths semipositive* if the dual metric $h^* = {}^t h^{-1}$ on the dual vector bundle E^* is Griffiths seminegative.

- $\varphi \in L^1_{loc}$ is plurisubharmonic iff $\sqrt{-1}\partial\overline{\partial}\varphi \geq 0$ in the sense of current.
- If E is a line bundle, h is Griffiths semipositive sHm iff $\sqrt{-1}\Theta_{E,h} \geq 0$ in the sense of current.

Remarks

- Whenever h is smooth, h is Griffiths seminegative (in the usual sense) iff $\log |u|_h^2$ is plurisubharmonic for any local section u of E.

Remarks

- Whenever h is smooth, h is Griffiths seminegative (in the usual sense) iff $\log |u|_h^2$ is plurisubharmonic for any local section u of E.
- (Raufi 15) A Griffiths semipositive sHm DOES NOT always have a curvature current.

Remarks

- Whenever h is smooth, h is Griffiths seminegative (in the usual sense) iff $\log |u|_h^2$ is plurisubharmonic for any local section u of E.
- (Raufi 15) A Griffiths semipositive sHm DOES NOT always have a curvature current.
- Nowadays, Nakano positive sHm is NOT defined. (We don't have a "good" definition of Nakano positive sHm.)

History

- (deCataldo 98) deCataldo defined sHm on vector bundles by using approximation of smooth Hermitian metrics.
- (Berndtsson-Păun 08) Berndtsson and Păun defined sHm on vector bundles in another way.

History

- (deCataldo 98) deCataldo defined sHm on vector bundles by using approximation of smooth Hermitian metrics.
- (Berndtsson-Păun 08) Berndtsson and Păun defined sHm on vector bundles in another way.
- (Păun-Takayama 14)
  $f_*(mK_{X/Y})$ has a Griffith semipositive sHm called "Narasimhan-Simha" metric,
  for any surjective projective morphism $f\colon X \to Y$ between smooth complex manifold with connected fibers such that $f_*(mK_{X/Y}) \neq 0$.

History

Theorem (Cao-Păun 17)

Iitaka's $C_{n,m}$ conjecture holds if the base space is an Abelian variety,
i.e. for any surjective morphism $f\colon X \to Y$ with connected fiber between smooth projective variety such that Y is an Abelian variety,

$$\kappa(X) \geq \kappa(F)$$

holds where F is a generic fiber of f.

History

Theorem (Cao-Păun 17)

Iitaka's $C_{n,m}$ conjecture holds if the base space is an Abelian variety,
i.e. for any surjective morphism $f\colon X \to Y$ with connected fiber between smooth projective variety such that Y is an Abelian variety,

$$\kappa(X) \geq \kappa(F)$$

holds where F is a generic fiber of f.

If $H^0(X, K_X^{\otimes k}) = 0$ for all k, $\kappa(X) := -\infty$. Otherwise,
$\kappa(X) := \max\{a \in \{0, 1, \cdots n\} \colon \limsup_{k\to+\infty} \frac{\dim H^0(X, K_X^{\otimes k})}{k^a} > 0\}$.

History

- (Hacon-Popa-Schnell 17) Hacon Popa and Schnell took a survey of sHm of vector bundles and Păun-Takayama's result.
  They gave a simplified proof of Cao and Păun's result by using GV sheaf and sHm of vector bundles.

Higher rank analogy of multiplier ideal sheaf

Definition

Let (E, h) be a vector bundle with sHm.
The sheaf of locally square integrable holomorphic sections of E with respect to h is defined by

$$E(h)_x = \{f_x \in E_{(x)} : |f_x|_h^2 \in L^1_{loc}\} \;\; x \in X,$$

where $E_{(x)}$ the stalk of E at x, defined by $\varinjlim_{x \in U} H^0(U, E)$.

- $E(h)$ is a higher rank analogy of a multiplier ideal sheaf.

Higher rank analogy of multiplier ideal sheaf

Definition

Let (E, h) be a vector bundle with sHm.
The sheaf of locally square integrable holomorphic sections of E with respect to h is defined by

$$E(h)_x = \{f_x \in E_{(x)} : |f_x|_h^2 \in L^1_{loc}\} \ \ x \in X,$$

where $E_{(x)}$ the stalk of E at x, defined by $\varinjlim_{x \in U} H^0(U, E)$.

- $E(h)$ is a higher rank analogy of a multiplier ideal sheaf.
- We don't know whether this sheaf is coherent.

Toward vanishing theorems

Problems

Let (E, h) be a vector bundle with sHm.

1. Is $E(h)$ a coherent sheaf ?
2. Prove that for any $q \geq 1$ and any vector bundle E,

$$H^q(X, K_X \otimes E(h)) = 0$$

if h has some positivity.

Previous research

Theorem (deCataldo 98)

Let (X, ω) be a compact Kähler manifold with a Kähler form and (E, h) be a vector bundle with a sHm. We assume the following conditions:

Previous research

Theorem (deCataldo 98)

Let (X, ω) be a compact Kähler manifold with a Kähler form and (E, h) be a vector bundle with a sHm. We assume the following conditions:

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *There exists an approximate sequence of smooth Hermitian metrics $\{h_\mu\}_{\mu=1}^{\infty}$. such that $h_\mu \uparrow h$ pointwise.*
3. $\sqrt{-1}\Theta_{E,h_\mu} - \epsilon\omega \otimes Id_E \geq_{Nak} 0$ *for some positive constant ϵ.*

Previous research

Theorem (deCataldo 98)

Let (X, ω) be a compact Kähler manifold with a Kähler form and (E, h) be a vector bundle with a sHm. We assume the following conditions:

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *There exists an approximate sequence of smooth Hermitian metrics $\{h_\mu\}_{\mu=1}^{\infty}$. such that $h_\mu \uparrow h$ pointwise.*
3. $\sqrt{-1}\Theta_{E,h_\mu} - \epsilon\omega \otimes Id_E \geq_{Nak} 0$ *for some positive constant ϵ.*

Then

- *$E(h)$ is a coherent sheaf.*
- *For $q \geq 1$, $H^q(X, K_X \otimes E(h)) = 0$ holds.*

- (Hosono 17) A Griffiths semipositive sHm h DOES NOT always have an approximation $\{h_\mu\}_{\mu=1}^{\infty}$ such that
  1. h_μ is smooth and $h_\mu \uparrow h$
  2. $\sqrt{-1}\Theta_{E,h_\mu} \geq_{Nak} C\omega \otimes Id_E$ for some constant C.

- (Hosono 17) A Griffiths semipositive sHm h DOES NOT always have an approximation $\{h_\mu\}_{\mu=1}^{\infty}$ such that
  1. h_μ is smooth and $h_\mu \uparrow h$
  2. $\sqrt{-1}\Theta_{E,h_\mu} \geq_{Nak} C\omega \otimes Id_E$ for some constant C.
- About vanishing theorems of vector bundles, there exists few results.
  (Recently Inayama proved some vanishing theorems of $H^n(X, K_X \otimes E)$ and $H^q(X, K_X \otimes E \otimes \det E)$)

Main results

Theorem (I. 18)

Let (X, ω) be a Kähler manifold and (E, h) be a holomorphic vector bundle on X with a sHm. We assume the following conditions.

Main results

Theorem (I. 18)

Let (X, ω) be a Kähler manifold and (E, h) be a holomorphic vector bundle on X with a sHm. We assume the following conditions.

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *$he^{-\zeta}$ is a Griffiths semipositive sHm on E for some continuous function ζ on X.*
3. *There exists $C \in \mathbb{R}$ such that $\sqrt{-1}\Theta_{E,h} - C\omega \otimes Id_E \geq_{Nak} 0$ on $X \setminus Z$.*

Main results

Theorem (I. 18)

Let (X, ω) be a Kähler manifold and (E, h) be a holomorphic vector bundle on X with a sHm. We assume the following conditions.

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *$he^{-\zeta}$ is a Griffiths semipositive sHm on E for some continuous function ζ on X.*
3. *There exists $C \in \mathbb{R}$ such that $\sqrt{-1}\Theta_{E,h} - C\omega \otimes Id_E \geq_{Nak} 0$ on $X \setminus Z$.*

Then the sheaf $E(h)$ is coherent.

Nadel-Nakano type vanishing theorems

Theorem (I. 18)

Let (X, ω) be a compact Kähler manifold and (E, h) be a holomorphic vector bundle on X with a sHm. We assume the following conditions.

Nadel-Nakano type vanishing theorems

Theorem (I. 18)

Let (X, ω) be a compact Kähler manifold and (E, h) be a holomorphic vector bundle on X with a sHm. We assume the following conditions.

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *$he^{-\zeta}$ is a Griffiths semipositive sHm on E for some continuous function ζ on X.*
3. *There exists $\epsilon \in \mathbb{R}_{>0}$ such that $\sqrt{-1}\Theta_{E,h} - \epsilon\omega \otimes Id_E \geq_{Nak} 0$ on $X \setminus Z$.*

Nadel-Nakano type vanishing theorems

Theorem (I. 18)

Let (X, ω) be a compact Kähler manifold and (E, h) be a holomorphic vector bundle on X with a sHm. We assume the following conditions.

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *$he^{-\zeta}$ is a Griffiths semipositive sHm on E for some continuous function ζ on X.*
3. *There exists $\epsilon \in \mathbb{R}_{>0}$ such that $\sqrt{-1}\Theta_{E,h} - \epsilon\omega \otimes Id_E \geq_{Nak} 0$ on $X \setminus Z$.*

Then $H^q(X, K_X \otimes E(h)) = 0$ holds for any $q \geq 1$.

Kollár type injectivity theorems

Theorem

Let (X, ω) be a compact Kähler manifold, (E, h) be a holomorphic vector bundle on X with a sHm and (L, h_L) be a holomorphic line bundle with a smooth metric. We assume the following conditions.

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *$he^{-\zeta}$ is a Griffiths semipositive sHm on E for some continuous function ζ on X.*
3. *$\sqrt{-1}\Theta_{E,h} \geq_{Nak} 0$ on $X \setminus Z$.*
4. *There exists $\epsilon \in \mathbb{R}_{>0}$ such that $\sqrt{-1}\Theta_{E,h} - \epsilon\sqrt{-1}\Theta_{L,h_L} \otimes Id_E \geq_{Nak} 0$ on $X \setminus Z$.*

Theorem (continue...)

Let s be a non zero section of L. Then for any $q \geq 0$, the multiplication homomorphism

$$\times s : H^q(X, K_X \otimes E(h)) \to H^q(X, K_X \otimes L \otimes E(h))$$

is injective.

Kollár-Ohsawa type vanishing theorems

Theorem (I. 18)

Let (X, ω) be a compact Kähler manifold and (E, h) be a holomorphic vector bundle on X with a sHm. Let $\pi \colon X \to W$ be a proper surjective holomorphic map to an analytic space with a Kähler form σ. We assume the following conditions.

1. *There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.*
2. *$he^{-\zeta}$ is a Griffiths semipositive sHm on E for some continuous function ζ on X.*
3. *$\sqrt{-1}\Theta_{E,h} - \pi^*\sigma \otimes \mathrm{Id}_E \geq_{Nak} 0$ on $X \setminus Z$.*

Then $H^q\big(W, \pi_*(K_X \otimes E(h))\big) = 0$ holds for any $q \geq 1$.

Idea of proof -general-

Idea

Idea of proof -general-

Idea

1. We DO NOT use approximation methods.

Idea of proof -general-

Idea

1. We DO NOT use approximation methods.
2. We solve $\bar{\partial}$-equation by ignoring analytic set Z.

Idea of proof -general-

Idea

1. We DO NOT use approximation methods.
2. We solve $\bar{\partial}$-equation by ignoring analytic set Z.

After all, we solve $\bar{\partial}$-equation

$$\bar{\partial} g = f$$

with some weight (sHm) on X for $\bar{\partial}$-closed (n, q) form f.

Idea of proof -general-

Idea

1. We DO NOT use approximation methods.
2. We solve $\bar{\partial}$-equation by ignoring analytic set Z.

After all, we solve $\bar{\partial}$-equation

$$\bar{\partial} g = f$$

with some weight (sHm) on X for $\bar{\partial}$-closed (n, q) form f.
However we do not treat this equation on X.

Idea of proof -general-

Idea

1. We DO NOT use approximation methods.
2. We solve $\bar{\partial}$-equation by ignoring analytic set Z.

After all, we solve $\bar{\partial}$-equation

$$\bar{\partial} g = f$$

with some weight (sHm) on X for $\bar{\partial}$-closed (n, q) form f.
However we do not treat this equation on X.
$\rightarrow$ We consider this equation on $X \setminus Z$.

Idea of proof -vanishing of cohomology-

$Y := X \setminus Z$. Y has a complete metric ω'.
$L^2_{n,q}(Y, E)_{\omega',h}$ is defined by

$$\{f\colon E\text{-valued } (n,q) \text{ form with measurable coefficients on } Y$$

$$s.t. \int_Y |f|^2_{\omega',h} dV_{\omega'} < +\infty\},$$

Idea of proof -vanishing of cohomology-

$Y := X \setminus Z$. Y has a complete metric ω'.
$L^2_{n,q}(Y, E)_{\omega',h}$ is defined by

$$\{f\colon E\text{-valued } (n,q) \text{ form with measurable coefficients on } Y$$

$$s.t. \int_Y |f|^2_{\omega',h} dV_{\omega'} < +\infty\},$$

Theorem

$$H^q(X, K_X \otimes E(h)) \cong \frac{L^2_{n,q}(Y, E)_{\omega',h} \cap \mathrm{Ker}\bar{\partial}}{\mathrm{Im}\bar{\partial}}$$

for any $q \geq 0$.

Sketch of proof -vanishing of cohomology-

We consider Cěch cohomology ($K_X \otimes E(h)$ is coherent.)

Sketch of proof -vanishing of cohomology-

We consider Cěch cohomology ($K_X \otimes E(h)$ is coherent.)
($\Rightarrow$)
We consider restriction to Y and only path together by using partition of unity.

Sketch of proof -vanishing of cohomology-

We consider Cěch cohomology ($K_X \otimes E(h)$ is coherent.)
($\Rightarrow$)
We consider restriction to Y and only path together by using partition of unity.
($\Leftarrow$)
By metric condition (1) and (3), we can restore to q-cochain outside Z.
By metric condition (2), it extend to X.

Applications

Question

How many sHm h on vector bundles satisfying the following conditions ($\star$):

Applications

Question

How many sHm h on vector bundles satisfying the following conditions ($\star$):

1. There exists a proper analytic subset Z such that h is smooth on $X \setminus Z$.
2. $he^{-\zeta}$ is a Griffiths semipositive sHm on E for some continuous function ζ on X.
3. There exists $\epsilon \in \mathbb{R}$ such that $\sqrt{-1}\Theta_{E,h} - \epsilon\omega \otimes Id_E \geq_{Nak} 0$ on $X \setminus Z$.

line bundle case

- L is big $\overset{def}{\Leftrightarrow} \limsup_{k\to+\infty} \frac{\dim H^0(X,L^{\otimes k})}{k^n} > 0$
  If L is big, then L has a sHm h satisfying condition $(\star)$

line bundle case

- L is big $\overset{def}{\Leftrightarrow} \limsup_{k\to+\infty} \frac{\dim H^0(X,L^{\otimes k})}{k^n} > 0$
  If L is big, then L has a sHm h satisfying condition $(\star)$
- L is pseudo-effective $\overset{def}{\Leftrightarrow}$ For any ample line bundle A and any $m \in \mathbb{N}_{>0}$, $L^{\otimes m} \otimes A$ is big.

line bundle case

- L is big $\overset{def}{\Leftrightarrow}$ $\limsup_{k\to+\infty} \frac{\dim H^0(X,L^{\otimes k})}{k^n} > 0$
  If L is big, then L has a sHm h satisfying condition $(\star)$
- L is pseudo-effective $\overset{def}{\Leftrightarrow}$ For any ample line bundle A and any $m \in \mathbb{N}_{>0}$, $L^{\otimes m} \otimes A$ is big.
  If L is pseudo-effective, then $L^{\otimes m} \otimes A$ has a sHm h satisfying condition $(\star)$ for any $m \in \mathbb{N}_{>0}$.

Even if E is a vector bundle, do the same things hold?

vector bundle case -Viehweg's weakly positivity-

Definition (Viehweg 83, Nakayama 04)

1. E is *dd-ample at* $x \in X$ ("ample modulo double-duals") if there exist $b \in \mathbb{N}_{>0}$ and an ample line bundle A on X such that $Sym^b(E) \otimes A^{-1}$ is globally generated at x.

vector bundle case -Viehweg's weakly positivity-

Definition (Viehweg 83, Nakayama 04)

1. E is *dd-ample at* $x \in X$ ("ample modulo double-duals") if there exist $b \in \mathbb{N}_{>0}$ and an ample line bundle A on X such that $Sym^b(E) \otimes A^{-1}$ is globally generated at x.
2. E is *weakly positive at* $x \in X$ if for any $a \in \mathbb{N}_{>0}$ and for any ample line bundle A, there exists $b \in \mathbb{N}_{>0}$ such that $Sym^{ab}(E) \otimes A^b$ is globally generated at x.

vector bundle case -Viehweg's weakly positivity-

Definition (Viehweg 83, Nakayama 04)

1. E is *dd-ample at* $x \in X$ ("ample modulo double-duals") if there exist $b \in \mathbb{N}_{>0}$ and an ample line bundle A on X such that $Sym^b(E) \otimes A^{-1}$ is globally generated at x.
2. E is *weakly positive at* $x \in X$ if for any $a \in \mathbb{N}_{>0}$ and for any ample line bundle A, there exists $b \in \mathbb{N}_{>0}$ such that $Sym^{ab}(E) \otimes A^b$ is globally generated at x.
3. E is *weakly positive in the sense of Nakayama* if there exists a point $x \in X$ such that E is weakly positive at x.

If E is line bundle,

- E is dd ample at some point $\Leftrightarrow$ E is big
- E is weakly positive in the sense of Nakayama $\Leftrightarrow$ E is pseudo-effective

If E is line bundle,

- E is dd ample at some point $\Leftrightarrow$ E is big
- E is weakly positive in the sense of Nakayama $\Leftrightarrow$ E is pseudo-effective

dd ample at some point $\Rightarrow$ weakly positive in the sense of Nakayama.

If E is line bundle,

- E is dd ample at some point $\Leftrightarrow$ E is big
- E is weakly positive in the sense of Nakayama $\Leftrightarrow$ E is pseudo-effective

dd ample at some point $\Rightarrow$ weakly positive in the sense of Nakayama.

Theorem (Păun-Takayama 14)

If E has a Griffith semipositive sHm, then E is weakly positive in the sense of Nakayama.

Theorem (I. 18)

The following are equivalent.

(A) *E is weakly positive in the sense of Nakayama.*

Theorem (I. 18)

The following are equivalent.

(A) *E is weakly positive in the sense of Nakayama.*

(B) *There exists an ample line bundle A such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive sHm h_k for any $k \in \mathbb{N}_{>0}$. Moreover, for any $k \in \mathbb{N}_{>0}$, there exists a proper Zariski closed set $Z_k \subset X$ such that h_k is smooth and Nakano semipositive outside Z_k.*

Theorem (I. 18)

The following are equivalent.

(A) *E is weakly positive in the sense of Nakayama.*

(B) *There exists an ample line bundle A such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive sHm h_k for any $k \in \mathbb{N}_{>0}$. Moreover, for any $k \in \mathbb{N}_{>0}$, there exists a proper Zariski closed set $Z_k \subset X$ such that h_k is smooth and Nakano semipositive outside Z_k.*

(C) *There exists an ample line bundle A such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive sHm h_k for any $k \in \mathbb{N}_{>0}$.*

Theorem (I. 18)

The following are equivalent.

(A) *E is weakly positive in the sense of Nakayama.*

(B) *There exists an ample line bundle A such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive sHm h_k for any $k \in \mathbb{N}_{>0}$. Moreover, for any $k \in \mathbb{N}_{>0}$, there exists a proper Zariski closed set $Z_k \subset X$ such that h_k is smooth and Nakano semipositive outside Z_k.*

(C) *There exists an ample line bundle A such that $\mathrm{Sym}^k(E) \otimes A$ has a Griffiths semipositive sHm h_k for any $k \in \mathbb{N}_{>0}$.*

The similar things hold if E is dd ample at some point.

Outlook

- We concrete the basic theory of vector bundles with sHm without using curvature current of sHm (c.f. chapter 5,6 in Demailly's book)

Outlook

- We concrete the basic theory of vector bundles with sHm without using curvature current of sHm (c.f. chapter 5,6 in Demailly's book)
- On the other hand, Raufi et al. defined the Chern class of sHm by using curvature current.

Outlook

- We concrete the basic theory of vector bundles with sHm without using curvature current of sHm (c.f. chapter 5,6 in Demailly's book)
- On the other hand, Raufi et al. defined the Chern class of sHm by using curvature current.
- Cao and Höring proved the structure theorem with nef anticanonical divisor by using Paun-Takayama's result: $f_*(mK_{X/Y})$ has a Griffiths semipositive sHm.

Outlook

- We concrete the basic theory of vector bundles with sHm without using curvature current of sHm (c.f. chapter 5,6 in Demailly's book)
- On the other hand, Raufi et al. defined the Chern class of sHm by using curvature current.
- Cao and Höring proved the structure theorem with nef anticanonical divisor by using Paun-Takayama's result: $f_*(mK_{X/Y})$ has a Griffiths semipositive sHm.

Question

Find other applications of vector bundles with sHm.