

OpenAI Math Digest

CATALOGUE 357

Bi-Lipschitz coordinates at every regular RCD point

Catalogue guide & one proof overview

Independent reading guide · English edition

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Contents

The catalogue guide is followed by the article. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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CATALOGUE 357

Bi-Lipschitz coordinates at every regular RCD point

Official catalogue 357 concerns open bi-Lipschitz coordinates at every prescribed regular point of a noncollapsed $\mathbf{RCD}(K, n)$ space. It contains one manuscript. The proposed proof compensates for the scale-dependent degeneration of harmonic coordinates in their value space, obtaining all-pairs metric bounds and an open image.

The article explains how the two main inputs combine, summable directional pinching, iteration in value space, and all pairs, the limit and openness. The distinction from known a.e. rectifiability on measurable pieces is the ambient-open domain and same-dimensional open image at every prescribed regular point.

[Theorem 1.1 · p. 1](#)

Paper in this catalogue

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 357 · 2026-10-19 checked source snapshot

#

01

Manuscript and overview

Bi-Lipschitz Coordinates at Regular Points of Noncollapsed RCD Spaces →

Short name: Bi-Lipschitz Coordinates at Regular RCD Points

September 25, 2026 · 43 pages

Original paper (GitHub) ↗

Principal claim

For integer $n \geq 2$, noncollapsed full $\text{RCD}(K, n)$ with reference measure H^n , every point whose pointed GH tangents are all R^n has an ambient-open neighborhood L_n -bi-Lipschitz to an open subset of R^n . L_n depends only on n ; charts use restricted ambient distance. A full-measure open region containing all regular points has a countable atlas.

Theorem 1.1; Corollary 1.2 · pp. 1–2 ↗

Role and available overview

Open harmonic coordinates feed cumulative matrix control of Bochner mass and summable directional defects into inverse-transpose updates in value space. Finite-scale derivative estimates then yield both distance bounds for every pair and openness of the limit.

Read the proof overview →

Sources and checking scope

The official subject is **Differential geometry**, with catalogue range 333–361; catalogue 357 appears on overview page 38. The reference snapshot is official repository commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a. The manuscript is the 43-page version by OpenAI dated September 25, 2026.

OpenAI Research Catalog · Differential geometry, Catalogue 357 · p. 38

Theorem 1.1 in the manuscript states that L_n depends **only on dimension**, while the neighborhood radius may depend on the space and point. This is distinguished from the official overview's abbreviated description of a point-dependent constant; the article and table follow the manuscript's quantifiers. Corollary 1.2 gives a full-measure open region containing all regular points and a countable atlas, without asserting that the exact regular locus is open.

Theorem 1.1 · p. 1

Corollary 1.2 · pp. 1–2

The principal statements and proof connections were compared with the manuscript, together with statements and uses of the main external inputs in harmonic approximation, heat-kernel estimates, noncollapsed volume theory and invariance of domain. Exhaustive external comparison for local Poisson regularity, Hessian calculus, the Bochner inequality and distance Laplacian comparison, as well as independent verification of all proofs, remains incomplete. See the [article's checking scope](#). Relations with other catalogues are uninvestigated; having one manuscript does not imply absence of external dependencies.

Bi-Lipschitz Coordinates at Regular Points of Noncollapsed RCD Spaces

Integrating potentially unbounded inverse Gram matrices through summable directional errors

The manuscript constructs open bi-Lipschitz coordinates around each prescribed regular point of a noncollapsed RCD space, using the restricted ambient distance. It does not assume uniform nondegeneracy of the harmonic coordinate differential: smooth maps on coordinate value space compensate for degeneration across scales. This article explains the proposed argument; it does not certify the correctness of the entire proof.

Manuscript author: OpenAI · September 25, 2026 · 43 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Open bi-Lipschitz coordinates at every regular point

For every integer $n \geq 2$ there is $L_n \geq 1$ such that the following holds. For every $K \in \mathbb{R}$, every noncollapsed $\text{RCD}(K, n)$ space $(X, d, \mathcal{H}_d^n)$, and every $p \in X$, if every pointed Gromov–Hausdorff tangent at p is pointed-isometric to $(\mathbb{R}^n, |\cdot|, 0)$, there are an open neighborhood $U \subset X$ of p , an open set $V \subset \mathbb{R}^n$, and a homeomorphism $F : U \rightarrow V$ satisfying

$$L_n^{-1}d(x, z) \leq |F(x) - F(z)| \leq L_n d(x, z) \quad (x, z \in U).$$

Here RCD means **full unreduced** $\text{CD}(K, n)$ with **quadratic Cheeger energy**, and the reference measure is exactly $\mathcal{H}_d^n$. Spaces are complete, separable and geodesic, and the measure has full support and is finite on bounded sets. The distance is $d|_{U \times U}$, rather than an intrinsic distance on U . The distortion depends only on dimension; the neighborhood radius may depend on the space and the point.

Corollary 1.2 — An open manifold region of full measure

Under the hypotheses of Theorem 1.1 there is an open set $\mathcal{U} \subset X$ containing all n -regular points, with $\mathcal{H}_d^n(X \setminus \mathcal{U}) = 0$. It has a countable atlas of L_n -bi-Lipschitz charts from ambient-open subsets of $\mathcal{U}$ onto open subsets of $\mathbb{R}^n$. Each chart uses the restricted ambient distance, and transition maps are L_n^2 -bi-Lipschitz. The region may contain points outside the regular locus; the corollary does not assert that the regular locus itself is open.

Corollary 1.2 · pp. 1–2

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Bi-Lipschitz Coordinates at Regular RCD Points](#). External inputs use the following keys.

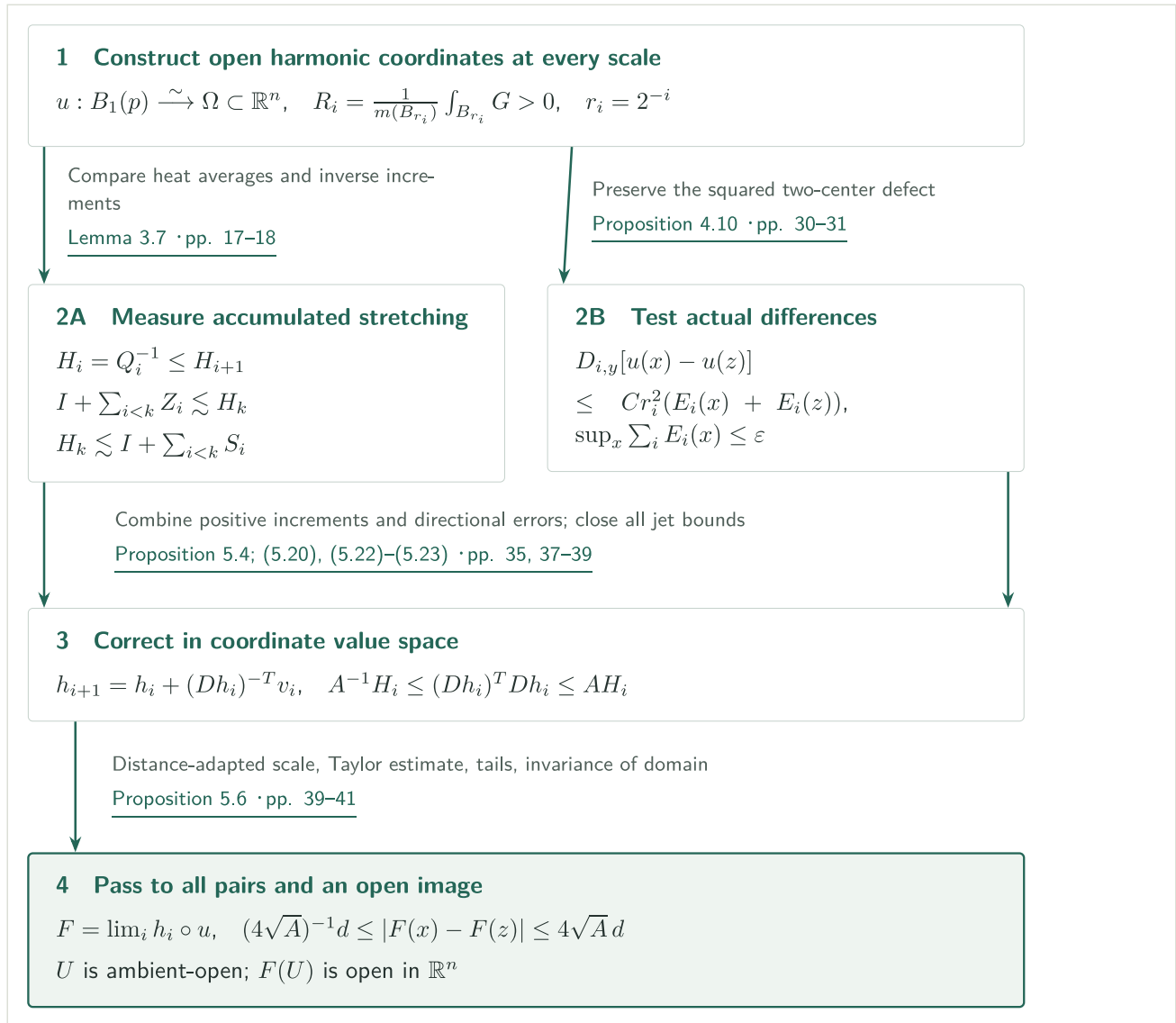
- [AH18](#) Luigi Ambrosio and Shouhei Honda — Local spectral convergence in $\text{RCD}_*(K, N)$ spaces: Interior harmonic approximation, directly used to construct the coordinates in Lemma 2.2. [arXiv:1703.04939v1](#) (2017); journal publication 2018.
- [JLZ16](#) Renjin Jiang, Huaiqian Li and Huichun Zhang — Heat kernel bounds on metric measure spaces and some applications: Gaussian heat-kernel and gradient bounds, direct inputs in §3.1. [arXiv:1407.5289v2](#); journal publication 2016.
- [DPG18](#) Guido De Philippis and Nicola Gigli — Non-collapsed spaces with Ricci curvature bounded from below: Noncollapsed volume convergence, almost rigidity, density bounds at every point, and full-measure regularity for the corollary. [arXiv:1708.02060v3](#), 23 June 2025; journal publication 2018.
- [Hat02](#) Allen Hatcher — Algebraic Topology: Invariance of domain for the continuous injection on the open value domain in Proposition 5.6. Cambridge University Press, 2002; author electronic edition retrieved 2026-10-19.
- [MN19](#) Andrea Mondino and Aaron Naber — Structure theory of metric measure spaces with lower Ricci curvature bounds: Background comparison: rectifiability on a.e. measurable pieces; not a direct input to the main theorem. [arXiv:1405.2222v3](#); journal publication 2019.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1 · p. 1](#)

Proof route. Construct open harmonic coordinates u on a rescaled space. Heat averages make the stretching corresponding to their inverse Gram matrices monotone, and accumulated Bochner mass controls that stretching. Independently, evaluation of the same mass on actual coordinate differences has summable defects. These two inputs drive a smooth iteration. The derivative estimates in Proposition 5.4 become all-pairs estimates in Proposition 5.6.



The two inputs and their convergence into open coordinates

[Open figure ↗](#) [TeX source ↓](#)

1. Normalization at every center and scale, and open harmonic coordinates

Rescale at the regular point by $d_\nu = s_\nu^{-1}d$ and $m_\nu = s_\nu^{-n}\mathcal{H}_d^n$. Noncollapsed volume convergence and Bishop–Gromov comparison keep volume ratios almost Euclidean at every center in a fixed large region and at all smaller radii. Lemma 2.2 combines this with almost-volume rigidity and

constructs harmonic columns $u = (u^1, \dots, u^n)^T$ on expanding domains. The direct input from [AH18] is strong $H^{1,2}$ harmonic approximation on **strictly interior balls**. Neither convergence of an entire boundary-value problem nor regularity of the neighboring points is assumed.

$$r_i = 2^{-i}, \quad G = (\langle \nabla u^j, \nabla u^k \rangle)_{j,k}, \quad R_i(y) = \frac{1}{m(B_{r_i}(y))} \int_{B_{r_i}(y)} G \, dm.$$

Lemma 2.3 proves $R_i(y) > 0$ at all centers and dyadic scales and Euclidean approximation by $R_i(y)^{-1/2}(u - u(y))/r_i$. Testing distinct points at a scale comparable to their separation gives injectivity. Lemma 2.5 iterates approximate surjectivity at smaller scales to reduce a target-value residual. The centers move by a convergent geometric series; completeness produces a preimage and hence inclusion of a central ellipsoid in the image. Thus $u : B_1(p) \rightarrow \Omega = u(B_1(p))$ is a homeomorphism onto an open set. Uniform bi-Lipschitz control is not yet concluded.

Lemmas 2.2–2.5 · pp. 6–9

[DPG18, Theorems 1.2, 1.3, 1.6, pp. 3–4]

[DPG18, Corollary 2.14, p. 17]

[AH18, Corollary 4.12, p. 20]

2. Separate cumulative stretching from directional error

The positive semidefinite matrix-valued measure $M = \frac{1}{2} \Delta G + \kappa G m$ retains its singular part. For a fixed enlargement L , put

$$D_{i,y} = R_i(y)^{-1} [r_i^{2-n} M(B_{Lr_i}(y))] R_i(y)^{-1}.$$

Corrected heat averages $Q_i = P_{r_i^2}(\chi G) + \lambda r_i^2 I$ and their inverses $H_i = Q_i^{-1}$ satisfy $Q_i \sim R_i$ and $H_i \leq H_{i+1}$. Gaussian heat-kernel bounds and the gradient estimate are direct inputs from [JLZ16]. Lemma 3.1 first turns the gradient estimate into a difference estimate, then uses continuity and full support to extend it to every source pole. This avoids integrating an arbitrary almost-everywhere gradient representative against a singular measure.

At each scale choose a bounded number of centers within $O(r_i)$ of x , and write $Z_i = \sum_y D_{i,y}$ and $S_i = \sum_y \theta_{i,y} D_{i,y}$. When $0 \leq \theta_{i,y} \leq 1$ and one weight-one mass ball contains $B_{r_i}(x)$, Lemma 3.7 gives

$$Z_i \leq \varepsilon H_i, \quad I + \sum_{i < k} Z_i \leq C_* H_k, \quad H_k \leq C_* \left(I + \sum_{i < k} S_i \right).$$

The matrices H_k need not remain bounded. In addition to telescoping inverse increments, the argument bounds the cost of switching from ball to heat normalization by the final H_k ; no commutativity is assumed. Independently, the next section evaluates $D_{i,y}$ on actual differences $u(x) - u(z)$ and obtains summable defects.

Lemmas 3.2–3.7 · pp. 11–18

[JLZ16, Theorem 1.2; Corollary 1.2, pp. 2–3]

3. Fix dimensional constants before choosing the rescaling

First fix the enlargements, the Gevrey order of the cutoff, C_* , A , B , and the final pair-comparison scale ratio D , using only dimension. Then choose a sufficiently late rescaling. Returning to the original metric multiplies coordinate values by s_ν as well, preserving $L_n = 4\sqrt{A(n)}$. The required rescaling and the original neighborhood radius may depend on the point and the space.

Proof of Theorem 1.1 · p. 41

3. Proposition 4.10 — Summable control along two-center differences

Proof target : Proposition 4.10; (4.18), (4.26) · pp. 24–26, 30–31

Proposition 4.10 — Summable directional pinching

Use the rescaling sequence and harmonic columns of Lemma 2.2, and fix any $L, C_{\text{sep}} > 1$. For all sufficiently late rescalings, nonnegative functions $E_i(a)$, $a \in B_4(p)$ and $i \geq 0$, can be chosen simultaneously with

$$\sup_{a \in B_4(p)} \sum_{i \geq 0} E_i(a) = o(1), \quad \inf_{a \in B_4(p)} E_i(a) > 0 \quad \text{for each fixed } i.$$

Here $P_i(a)$ is the defect $r_i^{-n} \mu_i^a(B_{8Sr_i}(a)) + \kappa r_i^2$ of Lemma 4.2, with positive measure $\mu_i^a = (n + C_0 \kappa r_i^2)m - \Delta(d_a^2/2)$ and fixed enlargement S . The construction (4.26) allows $E_i \geq P_i$. For every $x, z, y \in B_4(p)$ satisfying $d(x, y), d(z, y) \leq C_{\text{sep}} r_i$, the matrices defined above satisfy

$$D_{i,y}[u(x) - u(z)] \leq C r_i^2 \{E_i(x) + E_i(z)\}.$$

The same functions work at all centers and all dyadic scales; enlargement constants are fixed before the late-space threshold is chosen.

Proposition 4.10; (4.18), (4.26) · pp. 24–26, 30–31

1 Build defects from positive sources

$$\mu_i^a = (n + C_0 \kappa r_i^2) m - \Delta(d_a^2/2) \geq 0, \quad \sup_a \sum_i P_i(a) = o(1)$$

Common Laplacian, radial affine improvement, exponentially weighted convolution
[Lemmas 4.1–4.6 · pp. 20–26](#)

2 Make the two-center difference harmonic

$$v = (f_i^z - f_i^x)/r_i^2, \quad \Delta v = 0, \quad s^2 = E_i(x) + E_i(z)$$

Use r_i as the length unit

$$\sup_a \sum_i E_i(a) = o(1)$$

Weak radial transport and Young's inequality; contradiction with affine orthogonality
[Lemmas 4.8–4.9; Proposition 4.10 · pp. 28–31](#)

3 Control the residual before normalization

$$w = v - c_0 - b \cdot U, \quad \alpha = \|w\|_{L^2(B_R)}$$

$$\|w\|_{L^2(B_{2R})} \leq C(\alpha + s) \implies \alpha \leq Cs$$

Positivity squares the coefficient error; (4.50) restores the units
[Proposition 4.10; \(4.18\), \(4.26\) · pp. 24–26, 30–31](#)

4 Restore the squared mass bound to the original scale

$$b = \delta + O(s), \quad M'(\delta \cdot U)(B_L) \leq Cs^2$$

$$D_{i,y}[u(x) - u(z)] \leq Cr_i^2(E_i(x) + E_i(z))$$

Poisson replacement and radial transport yield two-center pinching

[Open figure ↗](#) [TeX source ↓](#)

1. Turn volume defects into summable energies

The Poisson replacements f_i^a of $q^a = d_a^2/2$ have a common constant Laplacian. Incorporating the mass of the positive measure $\mu_i^a = (n + C_0 \kappa r_i^2) m - \Delta q^a$ into $P_i(a)$ and telescoping dyadic volume-ratio differences gives $\sup_a \sum_i P_i(a) = o(1)$. The positive-source L^1 gradient estimate in Lemma 4.1 is **linear in the mass**, allowing the shifted Bochner measure to be bounded by $P_i(a)$. Harmonic correction of radial derivatives and affine improvement, convolved with exponential decay, produce E_i in Lemma 4.6. Smallness at each separate scale would not by itself give this summability.

[Lemmas 4.1–4.6 · pp. 20–26](#)

2. Preserve squared error before normalizing the projection residual

In unit-scale coordinates, the difference v of the two Poisson replacements is harmonic because their constant Laplacians agree. Set $s^2 = E_i(x) + E_i(z)$, project v onto the affine harmonic coordinate span, and let α be the L^2 norm of the residual w . **Before** dividing by α , Lemma 4.8 tests a weak radial transport equation and obtains $\|w\|_{L^2(B_{2R})} \leq C(\alpha + s)$. The coefficient error uses the unnormalized $\|w\|_\infty$ bound and Young's inequality to retain Cs^2 .

If $\alpha/s \rightarrow \infty$, normalized residuals converge to a Euclidean harmonic limit. Its Euler-type equation forces it to be affine, while its projection orthogonality and unit norm give a contradiction. Thus $\alpha \leq Cs$. The resulting coefficient b differs from the actual normalized coordinate difference δ by $O(s)$; positivity of the Bochner measure makes its mass contribution $O(s^2)$. Restoring the powers of r_i using (4.50) gives the proposition.

Lemmas 4.8–4.9; Proposition 4.10 · pp. 28–31

4. Proposition 5.4 — Inverse-transpose updates and the metric induction

Proof target : Proposition 5.4; (5.20), (5.22)–(5.23) · pp. 35, 37–39

Proposition 5.4 — Uniform metric and jet bounds

Use sufficiently flat data from §§5.1–5.3: $\Omega = u(B_1(p))$, $r_i = 2^{-i}$, $Q_i = H_i^{-1}$, the preceding positive-matrix and summability bounds, and the bounded-overlap nets with Gevrey cutoffs $\phi_{i,y}$ of §5.2. Choose a fixed point in each patch with $E_i(z_{i,y}) < 2 \inf E_i$, put $c_{i,y} = u(z_{i,y})$, and define

$$v_i(P) = \sum_y \phi_{i,y}(P) D_{i,y}(P - c_{i,y}), \quad h_0(P) = P, \quad h_{i+1} = h_i + (Dh_i)^{-T} v_i.$$

Fix cutoff order $s > 1$ and $\sigma \geq \max\{1, s\}$. There are fixed A, B and a sufficiently small flatness threshold such that, at every finite i , h_i is smooth on all of Ω , $J_i = Dh_i$ is everywhere invertible, and for all $x \in B_1(p)$,

$$A^{-1}H_i(x) \leq J_i(u(x))^T J_i(u(x)) \leq AH_i(x),$$

$$r_i^{m-1} \|D^m h_i(u(x))\|_{H_i(x) \rightarrow \mathbb{R}^n} \leq B^{m-1} (m!)^{1+\sigma} \quad (m \geq 2).$$

The multilinear norm uses the $H_i(x)$ norm for every input. These matrices are frozen at the evaluation point and are not differentiated.

Proposition 5.4; (5.20), (5.22)–(5.23) · pp. 35, 37–39

1 Build a covector field with fixed patch centers

$$v_i(P) = \sum_y \phi_{i,y}(P) D_{i,y}(P - c_{i,y}), \quad Dv_i = S_i + e_i$$

Near-minimizers of the defect and quadratic-form Cauchy–Schwarz
[Lemma 5.1; \(5.9\)–\(5.10\) · pp. 34–35](#)

2 Use the inverse transpose for a positive metric increment

$$h_{i+1} = h_i + J_i^{-T} v_i, \quad J_i = Dh_i$$

$$|J_{i+1}\xi|^2 - |J_i\xi|^2 = 2S_i[\xi] + \text{directional errors}$$

Retain derivatives of the inverse; control them by ℓ^2 convolution
[\(5.21\), \(5.31\)–\(5.36\) · pp. 35, 38–39](#)

3 Sum errors through diagonal second derivatives

$$b_{i+1} \leq \frac{1}{2}b_i + C_A(\sqrt{Z_i[\xi]} + \sqrt{E_i}q_i)$$

$$\sum_{i < k} b_i^2 \lesssim H_k[\xi], \quad \sum_i E_i \leq \varepsilon$$

Absorb metric errors by cumulative comparison; retain jets via factorial bounds and 2^{1-m}
[Proposition 5.4; \(5.20\), \(5.22\)–\(5.23\) · pp. 35, 37–39](#) [Lemmas 5.2–5.3 · pp. 36–37](#)

4 Close the metric and all-jet induction

$$A^{-1}H_i \leq J_i^T J_i \leq AH_i, \quad J_i \text{ is invertible}$$

$$r_i^{m-1} \|D^m h_i\|_{H_i \rightarrow \mathbb{R}^n} \leq B^{m-1} (m!)^{1+\sigma} \quad (m \geq 2)$$

Positive metric increments and summation through diagonal second derivatives

[Open figure ↗](#) [TeX source ↓](#)

1. Add the leading stretching directly to the metric

Selecting a fixed nearly minimizing point for each patch bounds v_i and the cutoff derivative terms by the defect $E_i(x)$ at the testing point. No continuity of E_i is needed. For $W = J^{-T}v$, the identity

$$|(J + DW)\xi|^2 - |J\xi|^2 = 2\xi^T Dv \xi - 2\langle D^2 h(\xi, \xi), J^{-T}v \rangle + |DW\xi|^2$$

has leading term $2S_i[\xi]$. Keep the diagonal second derivative introduced by differentiating the inverse transpose. With $q_i = |\xi|_{H_i}$ and $b_i = r_i |D^2 h_i(\xi, \xi)|/q_i$, the argument proves

$$b_{i+1} \leq \frac{1}{2}b_i + C_A(\sqrt{Z_i[\xi]} + \sqrt{E_i(x)}q_i), \quad \sum_{i < k} b_i^2 \leq C_A H_k[\xi].$$

Cauchy–Schwarz bounds $\sum b_i q_i \sqrt{E_i}$ using $\sum E_i$, without requiring $\sum \sqrt{E_i}$. The cumulative metric error in (5.35) is $C_A(\sqrt{\varepsilon} + \varepsilon)H_k[\xi]$. Combined with the reverse comparison of Lemma 3.7, it closes invertibility.

[\(5.21\), \(5.31\)–\(5.36\) · pp. 35, 38–39](#)

2. Separate derivative loss from induction over scales

The update contains an inverse first derivative, so finitely many derivative orders do not close. The factorial convolution estimate in Lemma 5.2 preserves the exponential base B in derivatives of the inverse matrix. The factor 2^{1-m} gained at the next scale absorbs the new contribution in order m , allowing Lemma 5.3 to preserve all jet bounds simultaneously. New-level invertibility is not assumed to prove the new-level metric bound: old-level inverses and jets give both the next metric bound and all next jets. Global injectivity of each h_i and a finite limit of H_i are unnecessary.

Lemmas 5.2–5.3 · pp. 36–37

5. Proposition 5.6 — All pairs, the limit, and openness

Proof target : Proposition 5.6 · pp. 39–41

Proposition 5.6 — The limiting chart

For sufficiently flat data in the preceding construction, $F(x) = \lim_{i \rightarrow \infty} h_i(u(x))$ exists uniformly on $B_1(p)$. For some $\rho > 0$, its restriction is a bi-Lipschitz homeomorphism from $B_\rho(p)$ onto an open subset of $\mathbb{R}^n$. Distance comparisons use the restriction of the ambient distance d .

Proposition 5.6 · pp. 39–41

1 Choose a scale for every pair

$$t = d(x, z) > 0, \quad Dt \leq r_i < 2Dt, \quad x, z \in B_\rho(p)$$

$$\Delta = u(z) - u(x), \quad t/2 \leq |\Delta|_{H_i(x)} \leq 2t$$

Use central-ellipsoid inclusion and the second-jet estimate

[Lemma 2.5 · p. 9](#) [\(5.37\)–\(5.42\) · pp. 40–41](#)

2 Apply Taylor on an interior line segment

$$h_i(u(z)) - h_i(u(x)) = J_i(u(x))\Delta + R$$

$$|R| \leq t/(8\sqrt{A}), \quad |J_i\Delta| \geq t/(2\sqrt{A})$$

Make both tails smaller than the linear term; obtain injectivity for all pairs

[\(5.37\)–\(5.42\) · pp. 40–41](#)

3 Preserve both bounds in the uniform limit

$$\sup_x |F(x) - h_i(u(x))| \leq \tau r_i, \quad 4D\tau \leq (8\sqrt{A})^{-1}$$

$$(4\sqrt{A})^{-1}d(x, z) \leq |F(x) - F(z)| \leq 4\sqrt{A}d(x, z)$$

Apply the theorem through the previously constructed open harmonic coordinates

[\[Hat02, Theorem 2B.3, print p. 172 / PDF p. 181\]](#)

4 Make the image open by invariance of domain

$$F \circ u^{-1} : u(B_\rho(p)) \longrightarrow \mathbb{R}^n$$

Continuous injection on an open set $\implies F(B_\rho(p))$ is open

Finite-scale Taylor control yields all-pairs bounds and an open image

[Open figure ↗](#) [TeX source ↓](#)

1. Choose a scale for each pair and obtain both bounds

The increments satisfy $|h_{i+1}(u(x)) - h_i(u(x))| \leq C_A \varepsilon r_i$, giving uniform convergence and the tail bound $\sup_x |F(x) - h_i(u(x))| \leq \tau r_i$ with $\tau \rightarrow 0$. After fixing the second-derivative constant, choose a large fixed D and put $\rho = (16D + 4)^{-1}$. For any distinct $x, z \in B_\rho(p)$ set $t = d(x, z)$ and choose a dyadic scale with $Dt \leq r_i < 2Dt$.

Then $B_{r_i}(x) \subset B_1(p)$, and $\Delta = u(z) - u(x)$ satisfies $t/2 \leq |\Delta|_{H_i(x)} \leq 2t$. Central-ellipsoid inclusion puts the entire value-space line segment in Ω , with interior preimages for all derivative evaluation points. The linear Taylor term lies between $t/(2\sqrt{A})$ and $2\sqrt{A}t$, and the remainder is at most $t/(8\sqrt{A})$. A sufficiently late rescaling makes the two tails together at most $t/(8\sqrt{A})$. Consequently, for all pairs,

$$\frac{1}{4\sqrt{A}}d(x, z) \leq |F(x) - F(z)| \leq 4\sqrt{A}d(x, z).$$

This is a direct comparison obtained by choosing a scale from each pair's ambient distance, not an extension from almost-everywhere pairs or a path-length estimate.

(5.37)–(5.42) · pp. 40–41

2. Why the injective limit has open image

The lower bound makes F injective and its inverse Lipschitz. Lemma 2.5 already makes

$u(B_\rho(p))$ open, so invariance of domain applies to the continuous injection

$F \circ u^{-1} : u(B_\rho(p)) \rightarrow \mathbb{R}^n$. This gives openness of the image. The argument neither assumes that a general continuous injection has open image nor uses global injectivity of the smooth approximations h_i .

Proposition 5.6, final paragraph · p. 41

[Hat02, Theorem 2B.3, print p. 172 / PDF p. 181]

6. Corollary 1.2 — The open region and a countable atlas

Proof target : Corollary 1.2 · pp. 1–2

1 Place an open chart at every regular point

$$p \in \mathcal{R}_n \implies F_p : U_p \xrightarrow{\sim} V_p \subset \mathbb{R}^n$$

U_p, V_p are open; F_p is L_n -bi-Lipschitz

Transfer full-measure regularity to the union of chart domains
[\[DPG18, Theorem 1.12\(ii\),\(vi\), p. 7\]](#)

2 Take an open region of full measure

$$\mathcal{U} = \bigcup_{p \in \mathcal{R}_n} U_p \supset \mathcal{R}_n, \quad \mathcal{H}^n(X \setminus \mathcal{U}) = 0$$

Use second countability of a separable metric space; retain open domains
[Corollary 1.2 · pp. 1–2](#)

3 Choose a countable subcover

$$\mathcal{U} = \bigcup_{j \geq 1} U_{p_j}, \quad \{F_{p_j} : U_{p_j} \rightarrow V_{p_j}\}_{j \geq 1}$$

Compose the two-sided bounds for a chart and an inverse chart
[Theorem 1.1 · p. 1](#)

4 Multiply constants for transition maps

$$F_{p_k} \circ F_{p_j}^{-1} \text{ is } L_n^2\text{-bi-Lipschitz}$$

Use ambient distance; no assertion that the exact regular locus is open

Union of local charts, countable subcover, and transition bounds

[Open figure ↗](#) [TeX source ↓](#)

1. A full-measure open region containing every regular point

Apply Theorem 1.1 at every regular point and let $\mathcal{U}$ be the union of the chart domains. This is ambient-open and contains all regular points. Theorem 1.12(ii),(vi) of [DPG18], applied to the reference measure $\mathcal{H}_d^n$, makes the n -regular set full measure, hence $\mathcal{H}_d^n(X \setminus \mathcal{U}) = 0$. No step removes nonregular points from the union.

[\[DPG18, Theorem 1.12\(ii\),\(vi\), p. 7\]](#)

2. Countability and transition constants

A separable metric space is second countable, so the chart cover of its open subspace $\mathcal{U}$ has a countable subcover. Domains and images remain open, and each chart has distortion L_n . On overlaps, $F_\beta \circ F_\alpha^{-1}$ and its inverse both have Lipschitz constant at most L_n^2 . The manuscript does not assert openness of the exact regular locus or bounded Laplacians for the limiting coordinate components.

[Corollary 1.2 · pp. 1–2](#)

7. External inputs and where they are used

Input

[AH18] Corollary 4.12

Use in the manuscript

Lemma 2.2 and §2.2

Role and hypotheses

Interior harmonic approximation on a converging RCD^* sequence; strong $H^{1,2}$ approximation of Euclidean coordinates

Checking scope

Fixed-version statement and application compared; complete external proof not checked

Input

[JLZ16] Theorem 1.2 / Corollary 1.2

Use in the manuscript

§3.1, Lemmas 3.1–3.3

Role and hypotheses

A fixed negative curvature bound and volume comparison yield uniform Gaussian estimates.

Extension of a.e. gradient estimates to all-pole differences is done in this manuscript

Checking scope

Fixed-version statements and application compared; complete external proofs not checked

Input

[DPG18] Theorems 1.2, 1.3, 1.6 / Corollary 2.14

Use in the manuscript

Lemmas 2.2 and 4.2

Role and hypotheses

Noncollapsed volume convergence, volume almost rigidity, density bound at every point. Almost rigidity is applied on an interior ball after a fixed enlargement

Checking scope

Independent reviewer compared fixed-version statements and uses; full proofs not checked

Input

[DPG18] Theorem 1.12(ii),(vi)

Use in the manuscript

Corollary 1.2

Role and hypotheses

The noncollapsed measure hypothesis gives full-measure n -regularity

Checking scope

See the independent statement/application comparison record

Input

[Hat02] Theorem 2B.3

Use in the manuscript

Proposition 5.6

Role and hypotheses

Invariance of domain for a continuous injection from a Euclidean open set

Checking scope

See the independent statement/application comparison record

Lemma 2.1 and Lemma 4.2 also directly use local Poisson regularity, Hessian calculus, the dimensional Bochner inequality and Laplacian comparison for distance. Their uses within the manuscript were read, but comparison of every external statement and hypothesis is incomplete. The independent reviewer compared the statements and uses of De Philippis–Gigli volume convergence, almost rigidity and the density bound. The works of Cheeger–Colding, Cheeger–Jiang–Naber and other Reifenberg constructions are cited as methodological antecedents; that alone does not create direct dependency arrows in this article.

Theorem 1.1 of [MN19] covers a general $\mathbf{RCD}^*(K, N)$ space, outside a null set, by countably many measurable pieces bi-Lipschitz to measurable subsets of Euclidean spaces. The present manuscript assumes noncollapse but obtains ambient-open domains and same-dimensional open images at every prescribed regular point. This is a background comparison; measurable pieces are not being treated as open charts.

[MN19, Theorem 1.1, p. 3]

8. Return to the sources

- Lemmas 2.2–2.5 · pp. 6–9 — All-center, all-scale normalization and open harmonic coordinates
- Lemma 3.7 · pp. 17–18 — Cumulative comparisons without commuting inverse matrices
- Proposition 4.10; (4.18), (4.26) · pp. 24–26, 30–31 — Two-center directional defects and restoration of scale
- Proposition 5.4; (5.20), (5.22)–(5.23) · pp. 35, 37–39 — Simultaneous induction for the metric and all jets
- Proposition 5.6 · pp. 39–41 — Uniform limit, all-pairs bounds and open image
- Corollary 1.2 · pp. 1–2 — Full-measure region, countable atlas and transition constants

We read the statements and main connections in Theorem 1.1, Corollary 1.2, Lemmas 2.2–2.5, 3.5–3.7, 4.2, 4.8–4.9, and Propositions 4.10, 5.4 and 5.6 in the source text. In particular, we tracked the order of quantifiers, all centers and scales, squared defects, the inverse-transpose update, all-pairs bounds and invariance of domain separately. This is not an independent reconstruction of every local calculation in §§3.1–3.2 and Lemmas 4.4–4.6. Recursive verification of external proofs, expert peer review and formal verification were not performed. The correctness of the whole manuscript is not guaranteed beyond the recorded scope of statement and application checks.