

OpenAI Math Digest

CATALOGUE 356

Gigli's characterization of Alexandrov curvature

Catalogue synthesis & 2 proof overviews

Independent reading guide · English edition

Edition: 2026-10-19

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AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 2 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

Catalogue 356 · Synthesis

[Read on the website ↗](#)

CATALOGUE 356

Gigli's characterization of Alexandrov curvature

The two papers establish the equivalence between Gigli's distributional curvature lower bound and an Alexandrov lower curvature bound, together with the weak-Hessian theorem used in the converse.

The two papers in official catalogue order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 356 · 2026-10-19 checked source snapshot

#

01

Manuscript and overview

Gigli's distributional curvature characterization of Alexandrov spaces →

Short name: Distributional curvature characterization

September 24, 2026 · 53 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For integer $n \geq 2$ and $\kappa \in \mathbb{R}$, characterizes n -dimensional $\text{CBB}(\kappa)$ by full-support unreduced $\text{RCD}((n-1)\kappa, n)$ with the exact measure $\mathcal{H}^n$ and Gigli's distributional curvature lower bound on global TestV .

[Theorem 1.1 · p. 2](#) ↗

Role and available overview

The forward implication uses integrated short-flow expansions and local error removal. The converse passes through a heat index inequality, noncollapsed frames and weak distance-Hessian comparison, then uses the companion theorem to obtain triangle comparison.

[Read the proof overview](#) →

#

02

Manuscript and overview

Weak Hessian bounds along every geodesic in RCD spaces →

Short name: Weak Hessian bounds

September 24, 2026 · 11 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

On a full-support $\text{RCD}(K, N)$ space, $1 < N < \infty$, a weak Hessian upper bound G for bounded globally Lipschitz F , with G bounded and continuous, yields $(F \cdot \sigma)'' \leq \ell^2 G \cdot \sigma$ in distributions along every constant-speed minimizing geodesic σ .

[Theorem 1.1 · p. 2](#) ↗

Role and available overview

Bounded measure changes and generator-domain closure yield variable-curvature CD; nonbranching and concentration of endpoint measures localize it to a prescribed geodesic. Direct input to Theorem 6.1 of the main paper, allowing collapse and requiring neither $F \in L^2$ nor an L^2 Hessian.

[Read the proof overview →](#)

References on the arrows

- **OpenAI26a:** OpenAI, Gigli’s distributional curvature characterization of Alexandrov spaces, September 24, 2026. Both directions of the characterization and the application of the weak-Hessian input. [\[OpenAI26a, Theorem 1.1, p. 2\]](#)
- **OpenAI26b:** OpenAI, Weak Hessian bounds along every geodesic in RCD spaces, September 24, 2026. Passage from a weak Hessian bound to a distributional inequality along every geodesic. [\[OpenAI26b, Theorem 1.1, p. 2\]](#)

Both directions and the companion input

A $\Rightarrow$ B Forward: geometry to distributional curvature

$\text{CBB}(\kappa), \dim_H M = n \implies m = \mathcal{H}^n, \text{supp } m = M, \text{RCD}((n-1)\kappa, n), \mathcal{R} \geq \kappa$
(global Test/TestV)

Obtain the RCD structure, then use integrated configurations, error removal and global approximation.

[OpenAI26a, Theorem 3.1, p. 9]

B $\Rightarrow$ A Converse: global distributional curvature lower bound

$m = \mathcal{H}^n, \text{supp } m = M, \text{RCD}((n-1)\kappa, n), \mathcal{R} \geq \kappa$
Integer $n \geq 2, \kappa \in \mathbb{R}$; RCD is unreduced.

Insert curvature into the Riccati identity and remove viscosity.

[OpenAI26a, Theorem 4.1, p. 24]

Heat index inequality

$$H_{\Phi_1}(Z, Z)(f) \leq \int f(|DY|^2 - \kappa|Y \wedge v|^2)$$

Common terminal paths; weighted strong convergence of velocities

Use noncollapsed frames and countable trials to bound Jacobi energy.

[OpenAI26a, Theorem 5.10, pp. 46–47]

Weak Hessian comparison for squared distance

$$H_q(Z, Z) \leq (|Z_\parallel|^2 + r_{s_\kappa(r)}^{s'_\kappa(r)} |Z_\perp|^2) m, \quad q = r^2/2$$

Signed Radon measure with nonpositive singular part

A monotone cutoff preserves the signs of the singular and radial terms.

[OpenAI26a, Lemma 6.2, pp. 48–49]

Make a bounded global distance modification

$$F_p = \chi(r^2/2), \quad H_{F_p} \leq G_p m$$

Inside a small ball: $F_p = m_\kappa(r), G_p = 1 - \kappa F_p$

Apply the companion theorem to F_p, G_p to reach every geodesic.

[OpenAI26a, Theorem 6.1, p. 48]

Model ODE along every minimizing geodesic

$$y'' + \kappa \ell^2 y \leq \ell^2, \quad y = m_\kappa(d(p, \sigma_t))$$

In distributions, along the sides of short triangles

One-dimensional comparison, local four-point criterion, and globalization for a complete length space.

[OpenAI26a, Theorem 6.3, pp. 49–50]

Triangle comparison and the global characterization

local point-on-side $\implies$ local four-point $\implies \text{CBB}(\kappa)$

$0 < \mathcal{H}^n(B) < \infty \implies \dim_H M = n$

OpenAI26b Weak-Hessian theorem

General full-support

$\text{RCD}(K, N), 1 < N < \infty$

Bounded globally Lipschitz F ;

bounded continuous G

[OpenAI26b, Theorem 1.1, p. 2]

Weak Hessian bounds to every geodesic: direct input to the main paper.

Both directions of the characterization and the point where the weak-Hessian theorem enters

Open figure ↗ TeX source ↓

The forward implication first obtains the RCD structure from $\text{CBB}(\kappa)$. Integrated short-flow expansions, local removal of a square error, approximation and exhaustion then recover the curvature inequality on the original global **Test** / **TestV** classes. The converse builds a heat-regularized index inequality from the curvature lower bound. Frames defined at every time along almost every path of suitable test plans, together with countable trials and noncollapse, turn it into weak Hessian comparison for squared distance. The [characterization article](#) explains these steps.

The second paper enters after constructing a bounded global distance modification. Theorem 6.1 of the main paper is precisely Theorem 1.1 of the companion, supplying the model ODE along every minimizing geodesic. Comparison for short triangles yields the local four-point condition and then globalization; the exact measure $\mathcal{H}^n$ recovers dimension n .

[OpenAI26a, Theorem 6.1, p. 48]

[OpenAI26a, Theorem 6.3, pp. 49–50]

The companion itself applies to general full-support finite-dimensional RCD spaces, including collapsed spaces. It requires neither global L^2 integrability of F nor an L^2 Hessian. The [weak-Hessian article](#) explains bounded measure changes, closure in the generator domain and concentration of endpoint measures. Its proof does not depend on the characterization or its frame construction.

Sources and checking scope

The official subject is **Differential geometry (333–361)**. The catalogue and both manuscripts are fixed at commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a; both manuscript versions are dated September 24, 2026.

[OpenAICatalog, Catalogue 356, p. 38]

We compared the principal statements, the core reductions in both directions of the main paper, the companion’s proof text and its application in the main paper against the sources. Each article records the statements and uses of external inputs and the remaining unchecked scope. This does not independently verify every estimate or lemma of the main paper or the complete proofs of external references.

356 / 01

[Read on the website ↗](#)

CATALOGUE 356 · PAPER 01 / 2 Overview

Gigli's distributional curvature characterization of Alexandrov spaces

From fourth-order transport expansions to global distributional curvature, and from heat-regularized index forms to comparison along every geodesic

The manuscript characterizes Alexandrov lower curvature bounds in integer dimension $n \geq 2$ using the exact Hausdorff measure and Gigli's distributional curvature. The forward implication extracts curvature from integrated small configurations. The reverse implication passes through an index inequality, the weak Hessian of distance, and a model equation along every geodesic.

Manuscript author: OpenAI · September 24, 2026 · 53 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Gigli's characterization

Let $n \geq 2$ be an integer, $\kappa \in \mathbb{R}$, and (M, d) a complete separable metric space. The following are equivalent.

- (A) (M, d) is an n -dimensional Alexandrov space with curvature at least κ .
- (B) With exactly $m = \mathcal{H}_d^n$, (M, d, m) has full support, satisfies the **unreduced** $\text{RCD}((n-1)\kappa, n)$ condition, and for every $X, Y \in \text{TestV}(M)$ and every nonnegative $f \in \text{Test}(M)$,

$$\mathcal{R}(X, Y, Y, X)(f) \geq \kappa \int_M f(|X|^2|Y|^2 - \langle X, Y \rangle^2) dm.$$

Here Δ is the nonpositive L^2 generator, and

$$\text{Test}(M) = \{f \in D(\Delta) : f \text{ has a bounded globally Lipschitz representative, } \Delta f \in W^{1,2}(M)\}, \quad \text{TestV}(M) = \left\{ \sum_{j=1}^{\ell} a_j \nabla b_j : \ell < \infty, a_j, b_j \in \text{Test}(M) \right\}.$$

With $[X, Y] = \nabla_X Y - \nabla_Y X$, the curvature functional, including its sign convention, is

$$\mathcal{R}(X, Y, Z, W)(f) = \int_M \left[-\langle \nabla_X(fW), \nabla_Y Z \rangle - f \langle \nabla_Y Z, W \rangle \operatorname{div} X \right. \\ \left. + \langle \nabla_Y(fW), \nabla_X Z \rangle + f \langle \nabla_X Z, W \rangle \operatorname{div} Y - f \langle \nabla_{[X, Y]} Z, W \rangle \right] dm.$$

The measure is not replaced by an arbitrary equivalent measure, and neither fields nor weights are restricted to compact support. For positive curvature, Alexandrov comparison is understood at the usual model scale.

Theorem 1.1; (1.1)–(1.4) · p. 2

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to Gigli's distributional curvature characterization of Alexandrov spaces. External inputs use the following keys.

- [KKS23](#) Vitali Kapovitch, Christian Ketterer, Karl-Theodor Sturm — On gluing Alexandrov spaces with lower Ricci curvature bounds: Alexandrov-to-unreduced-RCD input. arXiv:2003.06242v1 (2020); published 2023.
- [Gig19](#) Nicola Gigli — Riemann curvature tensor on RCD spaces and possible applications: Curvature sign, symmetries, and tensoriality over test functions. arXiv:1902.02282v1 (2019).
- [GRS16](#) Nicola Gigli, Tapio Rajala, Karl-Theodor Sturm — Optimal maps and exponentiation on finite-dimensional spaces with Ricci curvature bounded from below: Uniqueness of optimal paths from an absolutely continuous endpoint. arXiv:1305.4849v1 (2013); published 2016.
- [GH15](#) Nicola Gigli, Bang-Xian Han — The continuity equation on metric measure spaces: Continuity equation and action for bounded-compression Wasserstein curves. arXiv:1406.6350v1 (2014); published 2015.
- [JLZ16](#) Renjin Jiang, Huaiqian Li, Huichun Zhang — Heat kernel bounds on metric measure spaces and some applications: Gaussian heat bounds after weakening the Ricci parameter to a negative value. arXiv:1407.5289v2 (2015); published 2016.
- [DPG18](#) Guido De Philippis, Nicola Gigli — Non-collapsed spaces with Ricci curvature bounded from below: Euclidean tangents, dimension, and volume control from noncollapse. arXiv:1708.02060v3 (2018).
- [BPS21](#) Elia Bruè, Enrico Pasqualetto, Daniele Semola — Rectifiability of $\operatorname{RCD}(K, N)$ spaces via δ -splitting maps: Harmonic splitting maps near Euclidean rescalings. arXiv:2001.07911v1 (2020); published 2021.
- [Deng25](#) Qin Deng — Hölder continuity of tangent cones in $\operatorname{RCD}(K, N)$ spaces and applications to non-branching: Continuity of tangents from the same rescaling sequence along every geodesic. arXiv:2009.07956v2; published 2025.
- [GT21](#) Nicola Gigli, Luca Tamanini — Second order differentiation formula on $\operatorname{RCD}_*(K, N)$ spaces: Second-order differentiation along bounded-support optimal geodesic test plans. arXiv:1802.02463v1 (2018); published 2021.

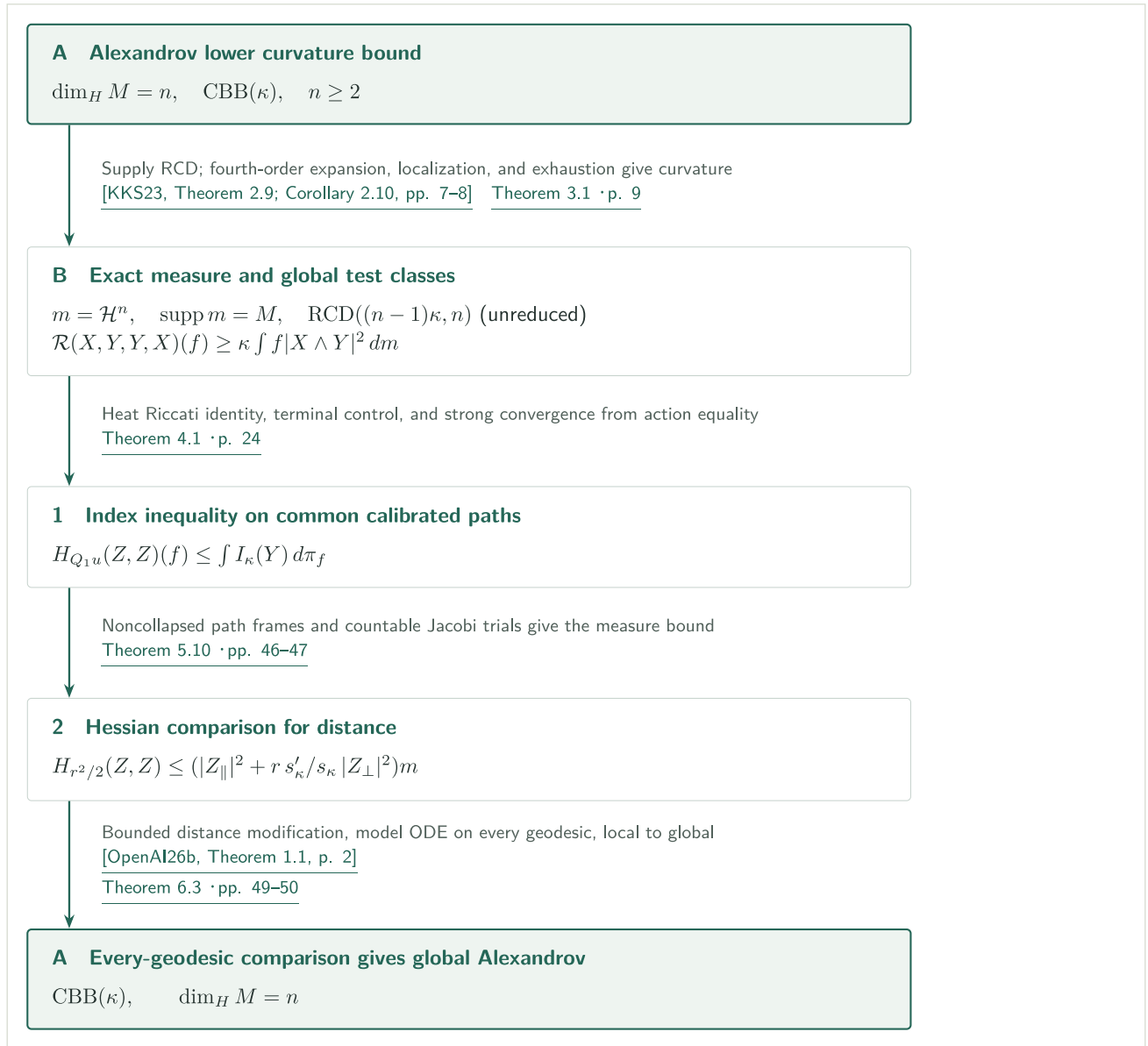
- [AKP24](#) Stephanie Alexander, Vitali Kapovitch, Anton Petrunin — Alexandrov Geometry: Foundations: Point-on-side comparison, local four-point criterion, and globalization for complete length spaces. Updated online PDF, retrieved 2026-10-19; book published 2024. Numbering matches the manuscript's cited 2026-09-24 online version..
- [OpenAI26b](#) OpenAI — Weak Hessian bounds along every geodesic in RCD spaces: From weak Hessian upper bounds for bounded functions to second-derivative inequalities along every geodesic. September 24, 2026.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1](#); (1.1)–(1.4) · p. 2

Proof route. The metric-measure part of the forward implication uses established Alexandrov theory. The new tensor inequality comes from removing the error in a fourth-order transport expansion by localization, followed by extension to the global test classes. In the reverse implication, (B) gives a heat-regularized index inequality. Noncollapse supplies frames along entire paths and a countable trial family, yielding Hessian comparison for distance. A bounded distance modification permits application of the companion weak-Hessian theorem; comparison along every geodesic then passes from local to global.



Both directions and the intermediate results

[Open figure ↗](#) [TeX source ↓](#)

1. From (A) to (B)

The external Alexandrov-to-RCD theorem supplies unreduced $\text{RCD}((n-1)\kappa, n)$, without restricting to spaces without boundary. The additional content of Theorem 3.1 is the curvature inequality for all X, Y, f , obtained using weak derivatives alone.

[\[KKS23, Theorem 2.9; Corollary 2.10, pp. 7–8\]](#)

2. From (B) to (A)

Theorem 4.1 selects the same calibrated geodesic from each terminal point when the terminal weight changes. This permits the pointwise countable infimum of trial energies in Theorem 5.10. Theorem 6.3 then obtains both comparison along every geodesic and $\dim_H M = n$. The companion theorem supplies the passage from almost-everywhere path information to an arbitrary prescribed geodesic.

§7, proof of Theorem 1.1 · p. 50

3. Theorem 3.1 — From configuration comparison to the global test classes

Proof target : Theorem 3.1 · p. 9

Theorem 3.1 — The forward implication

Let (M, d) be a complete n -dimensional Alexandrov space, $n \geq 2$, with curvature bounded below by κ , and put $m = \mathcal{H}_d^n$. Then it has full support, is unreduced $\text{RCD}((n-1)\kappa, n)$, and satisfies the curvature inequality of Theorem 1.1 for every $X, Y \in \text{TestV}(M)$ and every nonnegative $f \in \text{Test}(M)$.

Theorem 3.1 · p. 9

1 Compare small configurations

$$\liminf_{h \downarrow 0} h^{-4} (2\mathbb{E}J_{0A}^h - \mathbb{E}J_{AB}^h) \geq \frac{\kappa}{6} \int f \mathbb{E}|A \wedge B|^2 dm$$

Sign averaging cancels lower orders; first $h \downarrow 0$, then correct acceleration
[Lemma 3.4 · pp. 13–15](#) [Proposition 3.7 · p. 19](#)

2 Identify the density variation and fourth-order term

$$\frac{1}{6} \mathbb{E} \mathcal{R}(A, B, B, A)(f) + \frac{1}{2} \mathbb{E} \int f |\nabla q_{AB} + \nabla_A(B - A)|^2 dm$$

Linearity recombines differentiated weights; use the four-direction finite law
[Lemmas 3.8–3.9; Proposition 3.10 · pp. 19–21](#)

3 Remove the square with local gradients

$$\sum_{\ell} \alpha_{\ell} = 1, \quad \nabla q_{AB, \ell} \simeq -\nabla_A(B - A)$$

$$\mathcal{R}(U, V, V, U)(f) \geq \kappa \int f |U \wedge V|^2 dm$$

Positive coefficient measures, strong L^2 field approximation, finally $f\chi_R \rightarrow f$
[Lemmas 3.11–3.12; Remark 3.13 · pp. 21–23](#)

4 Recover the original global quantifiers

$$X, Y \in \text{TestV}(M), \quad 0 \leq f \in \text{Test}(M)$$

Removing the fourth-order error and recovering the global classes

[Open figure ↗](#) [TeX source ↓](#)

1. Geometric and analytic bounds for the same energy difference

Choose gradients A, B of compact test functions with bounded Laplacian independently from a finite sign-symmetric law. Use short flows with velocity $hA + h^2 t S_A$. Write J_{0A}^h for their radial action and J_{AB}^h for the optimal transport action between endpoint measures. Alexandrov cosine comparison gives

$$\liminf_{h \downarrow 0} \frac{2\mathbb{E}J_{0A}^h - \mathbb{E}J_{AB}^h}{h^4} \geq \frac{\kappa}{6} \int f \mathbb{E}|A \wedge B|^2 dm.$$

For the other bound, the weak L^2 first variation of interpolation densities and weighted strong convergence of velocities permit an expansion with Sobolev dual potentials. Sign symmetry removes odd terms. First send $h \downarrow 0$; then let $S_A \rightarrow -\nabla_A A$ to remove the radial acceleration defect. The remaining difference is controlled by curvature and a nonnegative square:

$$\frac{\kappa}{6} \int f \mathbb{E}|A \wedge B|^2 dm \leq \frac{1}{6} \mathbb{E} \mathcal{R}(A, B, B, A)(f) + \frac{1}{2} \mathbb{E} \int f |\nabla q_{AB} + \nabla_A(B - A)|^2 dm.$$

This is an integrated weak-calculus argument. Pairing the first density variation with fixed L^2 Hessian coefficients is what justifies the fourth-order term.

Lemmas 3.3–3.6; Proposition 3.7 · pp. 12–19

2. Local gradient choices remove the square

A general L^2 vector field need not be approximable by one gradient, but it can be approximated using separate gradients on a finite partition. Apply Lemma 3.9 to $-\nabla_A(B - A)$ and split the weight into $f\alpha_\ell$. Linearity in the weight and $\sum \alpha_\ell = 1$ recombine even the differentiated-weight terms exactly, so no uniform bound on the partition derivatives is needed. The square error tends to zero. For the uniform law on $\{U, -U, V, -V\}$, only mixed pairs remain, giving the sectional inequality.

Lemmas 3.8–3.9; Proposition 3.10 · pp. 19–21

3. Globalization comes last

For a fixed finite gradient family, constant-coefficient nonnegative curvature functionals are positive Radon measures. Positivity of the resulting coefficient polynomial on rational vectors extends the inequality to variable test coefficients. Local approximation by potentials with bounded Laplacian then gives strong L^2 convergence of the fields, their divergences, and their covariant derivatives, which passes the curvature functional to the limit. Finally exhaust the support of the weight using $f\chi_R$. The original tensor integrands are globally integrable, and terms involving $d\chi_R$ are controlled by L^1 tails. This order recovers the original global quantifiers of Theorem 1.1.

Lemmas 3.11–3.12; Remark 3.13 · pp. 21–23

[Gig19, Definition 2.6; Proposition 2.7, pp. 5–6]

4. Theorem 4.1 — From heat regularization to the index inequality

Proof target : Theorem 4.1 · p. 24

Theorem 4.1 — The index inequality along calibrated paths

Assume (B). Let u be bounded globally Lipschitz and constant outside a compact set, and put $\Phi_t = Q_t u = \inf_y \{u(y) + d(\cdot, y)^2/(2t)\}$. Outside one m -null set there is a Borel path prescription $S(x)_1 = x$ satisfying the calibration identity

$$\Phi_1(S(x)_1) = u(S(x)_0) + d(S(x)_0, S(x)_1)^2/2.$$

For every nonnegative compactly supported Lipschitz weight f , $\pi_f = S_\#(fm)$ has terminal measure fm , bounded speed, and bounded compression on each $[\delta, 1]$, $\delta > 0$.

For every compact test gradient Z , $0 < \delta < 1$, and finite trial $Y_t = \sum_j a_j(t) \nabla g_j$, where $a_j \in C^\infty([\delta, 1])$ and g_j are compact test functions, satisfying the **global field identities** $Y_\delta = 0, Y_1 = Z$,

$$H_{\Phi_1}(Z, Z)(f) \leq \int \int_\delta^1 (|\partial_t Y + \nabla_v Y|^2 - \kappa |Y \wedge v|^2)(\gamma_t) dt d\pi_f(\gamma).$$

The prescription S and velocity representative are independent of Z, Y, f . The weak Hessian is the functional

$$H_F(Z, Z)(f) = - \int \operatorname{div}(fZ) \langle \nabla F, Z \rangle dm - \int f \langle \nabla F, \nabla_Z Z \rangle dm.$$

Theorem 4.1 · p. 24

1 Heat factors with prescribed terminal density

$$H_t^c = P_{ct}(e^{-u/(2c)}), \quad J_t^c = P_{c(1-t)}(f/H_1^c) \\ \rho^c = H^c J^c, \quad \rho_1^c = f, \quad v^c = \nabla \theta^c, \quad w^c = c \nabla \log \rho^c$$

Cancel inverse-viscosity terms and bound the positive Bochner measure from below
[Lemmas 4.7–4.8 · pp. 30–33](#)

2 The viscous index inequality

$$H_{\varphi_1^c + c \log f}(Z, Z)(f) \leq \int_\delta^1 \int \rho^c (|D_c Y|^2 - \kappa |Y \wedge v^c|^2) dm dt + \varepsilon_c$$

After W_2 continuity and tail control, action and duality estimates become equalities
[Lemma 4.12 · pp. 35–36](#)

3 Action equality gives strong velocity convergence

$$\varphi_t^c \rightarrow Q_t u, \quad \int \rho^c |v^c - V|^2 dm dt \rightarrow 0$$

Uniqueness from the terminal density, fixed ∇Y products, then remove the positive perturbation
[\[GRS16, Theorems 1.1, 1.3, p. 2\]](#) [Lemma 4.13 · pp. 36–37](#)

4 Common path prescription and the index inequality

$$\pi_f = S_\#(fm), \quad Y_\delta = 0, \quad Y_1 = Z \\ H_{Q_1 u}(Z, Z)(f) \leq \int \int_\delta^1 (|D_t Y|^2 - \kappa |Y \wedge v|^2) dt d\pi_f$$

Heat regularization with prescribed terminal density and the zero-viscosity limit

[Open figure ↗](#) [TeX source ↓](#)

1. Fix the terminal density and control the singular Bochner term

First take $f = f_0 + \lambda e^{-Dd(o, \cdot)} > 0$. For viscosity $c > 0$, construct $H_t^c = P_{ct}(e^{-u/(2c)})$, $J_t^c = P_{c(1-t)}(f/H_1^c)$, and $\rho_t^c = H_t^c J_t^c$. Put $\varphi = -2c \log H$, $\psi = 2c \log J$, $\theta = (\psi + \varphi)/2$, and $b = c \log \rho$. Then $v^c = \nabla \theta$ is the continuity-equation velocity and $w^c = \nabla b$ the density correction.

Applying (B) in the Riccati identity and completing a square gives the index integrand and the acceleration error $H_{|w^c|^2/2 + c\Delta b^c}$. Smallness of w^c alone cannot handle the $1/c$ terms introduced by differentiating the density. The manuscript cancels these terms exactly before decomposing the remaining Bochner measure into a nonnegative measure and an L^2 remainder. The bound $c\Delta b^c \geq -e_c(t)$ with $\|e_c\|_{L^1(\delta, 1)} \rightarrow 0$, applied to quasi-continuous representatives, controls the singular measure contribution uniformly as the terminal cutoff $T \uparrow 1$.

[Lemmas 4.5–4.8; Proposition 4.9 · pp. 28–33](#)

[\[JLZ16, Theorem 1.2, p. 2\]](#)

2. Equality of actions justifies the quadratic-energy limit

Convergence $\varphi_t^c \rightarrow Q_t u$ and W_2 compactness of the density curves alone do not pass the index form to the limit. The action identity and Hopf–Lax duality force equality throughout the intervening estimates. This simultaneously identifies the limit as a Wasserstein geodesic and gives

$$\int_{\delta}^1 \int \rho_t^c |v_t^c - V_t|^2 dm dt \rightarrow 0.$$

Truncation extends this convergence to multiplication by the fixed L^2 coefficient ∇Y .

Uniqueness of optimal transport viewed backward from the absolutely continuous terminal measure gives the same path prescription under changes of terminal weight. The limits are $T \uparrow 1$, then $c \downarrow 0$, then $\lambda \downarrow 0$. Bounded compression is never assumed at the point-source time zero.

Lemmas 4.10–4.13; proof of Theorem 4.1 · pp. 33–37

[GH15, Theorem 3.5, p. 11] [GRS16, Theorems 1.1, 1.3, p. 2]

5. Theorem 5.10 — Frames on entire paths and Hessian comparison

Proof target : Theorem 5.10 · pp. 46–47

Theorem 5.10 — Distributional Hessian comparison

Assume (B). Choose $R_\kappa > 0$, with $R_\kappa \sqrt{\kappa} < \pi/2$ when $\kappa > 0$. For every $p \in M$, put $r = d(p, \cdot)$ and $q = r^2/2$. Let $s''_\kappa + \kappa s_\kappa = 0$, $s_\kappa(0) = 0$, $s'_\kappa(0) = 1$. For every compact test gradient Z , $H_q(Z, Z)$ on $B(p, R_\kappa)$ is a signed Radon measure with nonpositive singular part, and

$$H_q(Z, Z) \leq \left(|Z_\parallel|^2 + r \frac{s'_\kappa(r)}{s_\kappa(r)} |Z_\perp|^2 \right) m.$$

Here $|\nabla q| = r$ a.e.; on $r > 0$, $Z_\parallel = \langle Z, \nabla q \rangle \nabla q / r^2$ and $Z_\perp = Z - Z_\parallel$. At $r = 0$, the coefficient is interpreted as $|Z|^2$. The inequality holds against every nonnegative compactly supported Lipschitz weight on the ball, and the radius is independent of p .

Theorem 5.10 · pp. 46–47

1 Noncollapse and countable harmonic frames

$$m = \mathcal{H}^n, \quad \text{Cap}_2(N) = 0, \quad \mathcal{H}^{n-1}(S) = 0$$

Compression avoids exceptional sets; tangent continuity and finite-dimensional ODEs give frames

[Deng25, Theorems 1.1–1.2, p. 2] [GT21, Theorem 5.13, p. 50]

Proposition 5.1; Lemmas 5.4–5.5 · pp. 38–42

2 Frames at every time on almost every path

$$\gamma([\delta, 1]) \cap (N \cup S) = \emptyset, \quad D_t v = 0$$

Common terminal paths, positivity for a fixed trial, and pathwise approximation by a countable family

Proposition 5.6; Lemma 5.7 · pp. 42–45 Lemma 5.9 · pp. 45–46

3 Signed measure and countable infimum

$$H_q(Z, Z) = h_Z m + \mu_s, \quad \mu_s \leq 0$$

$$h_Z(x) \leq \inf_{Y \in \mathcal{Y}_{\delta, Z}} E_Y(x), \quad v(1) = \nabla q$$

Linear parallel component, s_κ transverse component, then a countable $\delta \downarrow 0$

Theorem 5.10 · pp. 46–47

4 Evaluate the Jacobi trial energy

$$H_q(Z, Z) \leq \left(|Z_\parallel|^2 + r \frac{s'_\kappa(r)}{s_\kappa(r)} |Z_\perp|^2 \right) m$$

From exceptional-set avoidance to a countable Jacobi trial family

[Open figure ↗](#) [TeX source ↓](#)

1. Upgrade spatial a.e. frames to frames at every path time

Noncollapse, $m = \mathcal{H}^n$, gives n -dimensional Euclidean tangents almost everywhere. A countable selection of harmonic splitting maps and an energy estimate for Gram-matrix rank defects split the exceptional set into a set N of zero 2-capacity and an $\mathcal{H}^{n-1}$ -null set S . Almost every **entire path** of a bounded-speed, bounded-compression plan avoids both. Deng’s Hölder continuity of tangent cones from the same rescaling sequence extends regularity from a dense set of times to all interior times. Compression handles the endpoints.

Positive minors of the continuous Gram matrices give local time-dependent frames. Solving finite-dimensional ODEs with L^2 connection coefficients constructs parallel frames. Gigli–Tamanini’s second-order differentiation formula makes the path velocity parallel. These constructions use the noncollapsed RCD hypotheses; the sectional bound enters in the radial application below.

Proposition 5.1; Lemmas 5.4–5.5; Proposition 5.6 · pp. 38–44

[DPG18, Theorem 1.12; Corollary 2.14, pp. 7, 17]

[BPS21, Proposition 1.4, p. 5] [Deng25, Theorems 1.1–1.2, p. 2] [GT21, Theorem 5.13, p. 50]

2. The countable infimum and the Jacobi trial

For $u = \min\{2R_\kappa r, 2R_\kappa^2\}$, $Q_1 u = q$ on the ball and p is the unique minimizer. The calibrated paths are radial geodesics from p . An L^2 terminal difference-quotient estimate identifies $v(1) = \nabla q$.

One fixed trial gives $E_0 m - H_q(Z, Z) \geq 0$. Riesz representation yields $H_q(Z, Z) = h_Z m + \mu_s$ with $\mu_s \leq 0$. A countable trial family satisfying global endpoint identities then gives $h_Z(x) \leq \inf_Y E_Y(x)$ outside one common null set. Along each path, let $P_\parallel, P_\perp$ be the parallel translates of $Z_\parallel, Z_\perp$. The field

$$W_t = \frac{t - \delta}{1 - \delta} P_\parallel(t) + \frac{s_\kappa((t - \delta)r)}{s_\kappa((1 - \delta)r)} P_\perp(t)$$

is approximated by the countable family in the pathwise $W^{1,2}$ norm. Its index energy equals

$$\frac{|Z_\parallel|^2}{1 - \delta} + r \frac{s'_\kappa((1 - \delta)r)}{s_\kappa((1 - \delta)r)} |Z_\perp|^2.$$

A countable sequence $\delta \downarrow 0$ gives the stated measure inequality. No measurable selection of the pathwise approximating sequences, and no density in an integrated space of fields along the whole plan, is required.

Lemmas 5.7–5.9; Theorem 5.10 · pp. 44–47

6. Theorem 6.3 — From every-geodesic model inequalities to global comparison

Proof target : Theorem 6.3 · pp. 49–50

Theorem 6.3 — Alexandrov comparison

Assume (B) of Theorem 1.1: integer $n \geq 2$, $m = \mathcal{H}_d^n$, full-support unreduced $\text{RCD}((n - 1)\kappa, n)$, and the curvature lower bound for all global test fields and all nonnegative global test weights. Then (M, d) is an **n -dimensional** Alexandrov space with curvature at least κ .

Theorem 6.3 · pp. 49–50

1 Make a bounded distance modification

$$F_p = \chi(r^2/2), \quad \chi'(q) = \theta(q) \frac{s_\kappa(\sqrt{2q})}{\sqrt{2q}}$$

$$H_{F_p} \leq G_p m, \quad G_p = \theta(r^2/2) s'_\kappa(r)$$

The decreasing cutoff adds a nonpositive radial term; boundedness and continuity hold
[Theorem 6.1; Lemma 6.2 · pp. 48–49](#) [\[OpenAI26b, Theorem 1.1, p. 2\]](#)

2 The model inequality along every geodesic

$$y = m_\kappa(d(p, \sigma_t)), \quad y'' + \kappa \ell^2 y \leq \ell^2$$

For $\sigma \subset B_a(p)$, in distributions

Test the negative part; for $\kappa > 0$ use $\kappa \ell^2 < \pi^2$
[Theorem 6.3 · pp. 49–50](#)

3 Compare with the model triangle

$$y \geq \bar{y} \implies d(p, \sigma_t) \geq d(\bar{p}, \bar{\sigma}_t)$$

Apply globalization for complete length spaces; no ball convexity is assumed
[\[AKP24, Theorems 8.14, 8.30, 8.31, pp. 86, 93–95\]](#)

4 Local four-point comparison, globalization, dimension

$$\text{CBB}(\kappa), \quad 0 < \mathcal{H}^n(B) < \infty \implies \dim_H M = n$$

Bounded distance modifications and local-to-global comparison

[Open figure ↗](#) [TeX source ↓](#)

1. Construct bounded functions to which the companion theorem applies

Put $m_\kappa(r) = \int_0^r s_\kappa(s) ds$. Fix $0 < a < b < R_\kappa$ and a smooth nonincreasing θ equal to one on $[0, a^2/2]$ and zero on $[b^2/2, \infty)$. Define $\chi(0) = 0$, $\chi'(q) = \theta(q) s_\kappa(\sqrt{2q})/\sqrt{2q}$, and $F_p = \chi(r^2/2)$. Then F_p is bounded globally Lipschitz and constant outside the ball. In the Hessian chain rule, the additional radial term $r\theta' s_\kappa$ is nonpositive, and the singular part retains its sign. Thus the bounded continuous function $G_p = \theta(r^2/2) s'_\kappa(r)$ satisfies $H_{F_p}(Z, Z)(h) \leq \int h G_p |Z|^2 dm$ for every compact test gradient Z and nonnegative compact Lipschitz weight h . On $B_a(p)$, $F_p = m_\kappa(r)$ and $G_p = 1 - \kappa F_p$.

For a full-support $\text{RCD}(K, N)$ space with $1 < N < \infty$, the companion theorem turns precisely these boundedness, continuity, and weak-Hessian hypotheses into $(F_p \circ \sigma)'' \leq \ell^2 G_p \circ \sigma$ along **every** constant-speed minimizing geodesic σ of length ℓ , in distributions on $(0, 1)$. Neither $F_p \in L^2(m)$ nor an L^2 Hessian is required. The reference measure in this application remains unchanged.

[Theorem 6.1; Lemma 6.2 · pp. 48–49](#)

[\[OpenAI26b, Theorem 1.1, p. 2\]](#)

2. The model ODE, local four-point comparison, globalization, and dimension

For a triangle $[pxy]$ of perimeter less than a , its chosen side σ lies in $B_a(p)$. Hence $y = m_\kappa(d(p, \sigma_t))$ satisfies $y'' + \kappa \ell^2 y \leq \ell^2$. The corresponding function $\bar{y}$ on the model triangle solves the equality with the same endpoint values. Testing with the negative part of $y - \bar{y}$ gives $y \geq \bar{y}$ directly when $\kappa \leq 0$, and by the Dirichlet Poincaré inequality and $\kappa \ell^2 < \pi^2$ when $\kappa > 0$. Monotonicity of m_κ recovers point-on-side distance comparison.

Converting these short-triangle comparisons to four-point comparison in a small ball uses ambient minimizing segments; convexity of the ball is not assumed. The local CBB criterion and globalization for complete length spaces give $\text{CBB}(\kappa)$. Finally, every nonempty bounded ball has positive finite $\mathcal{H}^n$ measure. A countable exhaustion gives $\dim_H M = n$, completing both the curvature and dimension conclusions of (B) $\Rightarrow$ (A).

Theorem 6.3 · pp. 49–50

[AKP24, Theorems 8.14, 8.30, 8.31, pp. 86, 93–95]

7. Which papers supply which steps

Source

Kapovitch–Ketterer–Sturm

Use in this manuscript

Theorem 3.1

Role and hypotheses

Unreduced RCD for an n -dimensional Alexandrov space with the exact measure $\mathcal{H}^n$

Checking scope

Statements of Theorem 2.9 / Corollary 2.10 in the cited version and their application checked

Source

Bruè–Pasqualetto–Semola, Deng, Gigli–Tamanini

Use in this manuscript

Proposition 5.1, Lemma 5.5, Proposition 5.6

Role and hypotheses

Harmonic frames near Euclidean rescalings, tangent continuity along every geodesic, and parallel velocity. Splitting uses auxiliary dimension $N = n + 1$ and a point residual factor

Checking scope

Statements in the specified versions and their connections checked; external proofs are outside scope

Source

Weak Hessian bounds along every geodesic in RCD spaces

Use in this manuscript

Theorem 6.1, Lemma 6.2, Theorem 6.3

Role and hypotheses

From weak Hessians of bounded distance modifications to every-geodesic inequalities; a general RCD result without noncollapse

Checking scope

All hypotheses of Theorem 1.1 and this application checked; no circular input from the characterization

Source

Alexander–Kapovitch–Petrunin

Use in this manuscript

Theorem 6.3

Role and hypotheses

From point-on-side comparison to local CBB, then global CBB for a complete length space

Checking scope

Theorems 8.14 / 8.30 / 8.31 in the downloaded online version and their application checked

[KKS23, Theorem 2.9; Corollary 2.10, pp. 7–8] [BPS21, Proposition 1.4, p. 5]

[Deng25, Theorems 1.1–1.2, p. 2] [GT21, Theorem 5.13, p. 50] [OpenAI26b, Theorem 1.1, p. 2]

[AKP24, Theorems 8.14, 8.30, 8.31, pp. 86, 93–95]

8. Return to the sources

- Theorem 1.1; (1.1)–(1.4) · p. 2 — Measure, global classes, sign convention, and both implications
- Lemmas 3.3–3.6; Proposition 3.7 · pp. 12–19 — Compare configurations with the fourth-order transport expansion
- Lemmas 3.11–3.12; Remark 3.13 · pp. 21–23 — Recover the original global quantifiers
- Lemmas 4.5–4.8; Proposition 4.9 · pp. 28–33 — Uniform terminal control including singular Bochner measures
- Lemmas 4.10–4.13; proof of Theorem 4.1 · pp. 33–37 — Equality of actions, strong velocity convergence, and common path prescription
- Proposition 5.1; Lemmas 5.4–5.5; Proposition 5.6 · pp. 38–44 — Exceptional-set avoidance and frames at every path time
- Theorem 5.10 · pp. 46–47 — Countable infimum, signed measure, and Jacobi trial
- Theorem 6.1; Lemma 6.2 · pp. 48–49 — Bounded modifications meeting the companion hypotheses
- Theorem 6.3 · pp. 49–50 — Model ODE, local-to-global passage, and dimension

The reading covers the main statement; the configurations, density variation, fourth-order expansion, localization, and globalization in §3; the heat factors, singular Bochner terms, and action limit in §4; the exceptional sets, frames, countable trials, and Jacobi estimate in §5; and the distance modification and local-to-global argument in §§6–7. The external-input table records the scope of statement and application checks.

Limits of the check. We have not independently rederived every term of the long fourth-order expansion, every constant in the heat-kernel estimates, or all analytic lemmas, including the identification of capacities and Sobolev representatives. The foundational RCD references have not been recursively verified in full. The manuscript’s claimed characterization is distinguished from the statements and proof connections checked for this exposition.

The result numbering and statements in the retrieved AKP online PDF were checked; file identity with the 24 September 2026 snapshot cited by the manuscript has not been established.

Weak Hessian bounds along every geodesic in RCD spaces

Closing the weighted Bochner inequality and concentrating entropy on a prescribed geodesic

The manuscript passes from a distributional Hessian upper bound on a finite-dimensional RCD space to a second-derivative upper bound along every minimizing geodesic. Analytically, a resolvent built from the original heat flow controls multipliers for the weighted generator. Geometrically, nonbranching concentrates optimal transport on an arbitrary prescribed interior subsegment.

Manuscript author: OpenAI · September 24, 2026 · 11 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A distributional inequality along every minimizing geodesic

Let (M, d, m) be complete and separable, with $\text{supp } m = M$ and m finite on bounded sets, and assume $\text{RCD}(K, N)$ with $K \in \mathbb{R}$ and $1 < N < \infty$. The manuscript uses unreduced $\text{CD}(K, N)$ together with quadratic Cheeger energy. The Laplacian is the nonpositive L^2 generator; write $\Gamma(u, v) = \langle \nabla u, \nabla v \rangle$ and $\Gamma(u) = \Gamma(u, u)$.

Let $F : M \rightarrow \mathbb{R}$ be bounded and globally Lipschitz, and $G : M \rightarrow \mathbb{R}$ bounded and continuous. Set

$\text{Test}(M) = \{g \in D(\Delta) : g \text{ has a bounded globally Lipschitz representative, } \Delta g \in W^{1,2}(M)\}.$

For compactly supported $g \in \text{Test}(M)$, $Y = \nabla g$, and $h \in \text{Lip}_c(M)$, define

$$H_F(Y, Y)(h) = - \int_M \text{div}(hY) \langle \nabla F, Y \rangle dm - \int_M h \langle \nabla F, \nabla_Y Y \rangle dm,$$

where $\operatorname{div}(hY) = \Gamma(h, g) + h\Delta g$ and $\nabla_Y Y = \nabla(\Gamma(g)/2)$. Suppose that for every such g and every nonnegative $h \in \operatorname{Lip}_c(M)$,

$$H_F(\nabla g, \nabla g)(h) \leq \int_M h G \Gamma(g) \, dm.$$

Then every constant-speed minimizing geodesic $\sigma : [0, 1] \rightarrow M$, of length ℓ , satisfies

$$(F \circ \sigma)'' \leq \ell^2 G \circ \sigma \quad \text{in } \mathcal{D}'(0, 1).$$

Equivalently, for every $0 \leq \varphi \in C_c^\infty(0, 1)$,

$$\int_0^1 F(\sigma_t) \varphi''(t) \, dt \leq \ell^2 \int_0^1 G(\sigma_t) \varphi(t) \, dt.$$

When $G = c$ is constant, $F(\sigma_t) - c\ell^2 t^2/2$ is concave. Neither $m(M) < \infty$ nor $F \in L^2(m)$ is assumed: the weak Hessian definition only needs local first-order Sobolev regularity of F .

Theorem 1.1; (1.1)–(1.4) · p. 2

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Weak Hessian bounds along every geodesic in RCD spaces](#). External inputs use the following keys.

- [AGS15](#) Luigi Ambrosio, Nicola Gigli, Giuseppe Savaré — Bakry–Émery curvature-dimension condition and Riemannian Ricci curvature bounds: Original heat reverse-Poincaré estimate; intrinsic distance and constant BE–RCD equivalence. [arXiv:1209.5786v4](#).
- [BHS21](#) Mathias Braun, Karen Habermann, Karl-Theodor Sturm — Optimal transport, gradient estimates, and pathwise Brownian coupling on spaces with variable Ricci bounds: Variable-curvature BE₂–CD equivalence on an existing RCD space and the Green-kernel entropy formula. [arXiv:1906.09186v1](#).
- [Deng25](#) Qin Deng — Hölder continuity of tangent cones in RCD(K,N) spaces and applications to non-branching: Nonbranching of all geodesics in finite-dimensional RCD spaces. [arXiv:2009.07956v2](#).
- [Sav14](#) Giuseppe Savaré — Self-improvement of the Bakry–Émery condition and Wasserstein contraction of the heat flow in RCD(K,∞) metric measure spaces: Measure-valued Bochner inequality and multiplier representatives outside capacity-zero sets. [arXiv:1304.0643v2](#).
- [Gig18](#) Nicola Gigli — Nonsmooth differential geometry—an approach tailored for spaces with Ricci curvature bounded from below: Hessian-square estimate for the original test class. [arXiv:1407.0809v1](#).
- [GT21](#) Nicola Gigli, Luca Tamanini — Second order differentiation formula on RCD_{*}(K,N) spaces: Original heat regularization, good cutoffs, and the Hessian estimate on the Laplacian domain. [arXiv:1802.02463v1](#).

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : Theorem 1.1; (1.1)–(1.4) · p. 2

Proof route. The section on Proposition 4.1 establishes $\text{CD}(K - \lambda G, \infty)$ for $m_\lambda = e^{\lambda F} m$. The multiplier regularity needed there is proved in the section on Lemma 3.1. Here the overview includes the final passage from that curvature bound to one prescribed geodesic. The case $\ell = 0$ is immediate, so assume $\ell > 0$. Write $\mathbf{g}(s, t) = \min\{s, t\} - st$ for the Dirichlet Green kernel.

1 Distributional Hessian upper bound

$$H_F(\nabla g, \nabla g)(h) \leq \int h G \Gamma(g) dm$$

F bounded globally Lipschitz; G bounded continuous.

The original resolvent regularizes multipliers; isolate the Hessian term.

[Proposition 4.1 · pp. 7–8](#)

2 Curvature of the exponential weight

$$m_\lambda = e^{\lambda F} m, \quad \text{CD}(K - \lambda G, \infty)$$

Close the Bochner inequality in the common generator domain.

Finite-dimensional nonbranching and properness pin down all transport paths.

[\[Deng25, Theorem 1.3, p. 2\]](#)

3 Concentrate on a prescribed interior segment

$$\mu_i^j = \frac{m|_{B_{\varepsilon_j}(\tau_i)}}{m(B_{\varepsilon_j}(\tau_i))}, \quad \pi_j \rightharpoonup \delta_\tau$$

Nonbranching $\Rightarrow$ uniqueness of the strict subsegment τ .

Bounded path lengths and continuous F, G pass the Green term to the limit.

[\(5.1\)–\(5.2\) · pp. 9–10](#)

4 Eliminate the entropy error

$$\lambda_j \geq j(1 + |E_0^j| + |E_1^j|), \quad E_t^j \geq -\log m(Q)$$

$$F(\tau_t) \geq (1-t)F(\tau_0) + tF(\tau_1) - |\dot{\tau}|^2 \int_0^1 \mathbf{g}(s, t) G(\tau_s) ds$$

Use the Green formula on every interior interval to obtain concavity.

[§5 · p. 10](#)

5 Second derivative on every geodesic

$$a'' = \ell^2 G \circ \sigma, \quad F \circ \sigma - a \text{ is concave}$$

$$(F \circ \sigma)'' \leq \ell^2 G \circ \sigma \quad \text{in } \mathcal{D}'(0, 1)$$

Overview from the weak Hessian bound through weighted curvature and concentration to every geodesic

[Open figure ↗](#) [TeX source ↓](#)

1. Nonbranching fixes each interior subsegment

Deng's nonbranching theorem makes every **strict interior subsegment** of σ the unique minimizer between its endpoints. Otherwise, replacing it by a different minimizer and retaining the preceding and following portions of σ produces two minimizing geodesics of the same total length agreeing on a nontrivial initial interval. This uses nonbranching of all geodesics; essential nonbranching of transport plans alone would not justify the argument.

[Deng25, Theorem 1.3, p. 2]

Let $\tau : [0, 1] \rightarrow M$ be the constant-speed reparametrization of such a subsegment. For $\varepsilon_j \downarrow 0$, put

$$\mu_i^j = \frac{m|_{B_{\varepsilon_j}(\tau_i)}}{m(B_{\varepsilon_j}(\tau_i))}, \quad E_i^j = \text{Ent}_m(\mu_i^j) = -\log m(B_{\varepsilon_j}(\tau_i)), \quad i = 0, 1.$$

Full support and local finiteness make the denominators positive and finite. Choose $\lambda_j \geq j(1 + |E_0^j| + |E_1^j|)$ and use the curvature bound for m_{λ_j} to obtain a dynamical optimal plan π_j . Properness and the endpoint support constraints put all relevant paths inside one compact set Q , with lengths bounded by L . Arzelà–Ascoli and uniqueness of τ show that every limit of such paths is τ , hence $\pi_j \rightarrow \delta_\tau$. This argument depends only on the metric and the endpoints, although the weighted measures vary with j .

§5 · p. 9

2. Remove the entropy error while retaining variable G

Set $\mu_t^j = (e_t)_\# \pi_j$, $A_t^j = \int F d\mu_t^j$, $E_t^j = \text{Ent}_m(\mu_t^j)$, and

$$C_t^j = \int \int_0^1 \mathbf{g}(s, t) |\dot{\gamma}|^2 ds d\pi_j(\gamma), \quad D_t^j = \int \int_0^1 \mathbf{g}(s, t) G(\gamma_s) |\dot{\gamma}|^2 ds d\pi_j(\gamma).$$

Substituting $\text{Ent}_{m_{\lambda_j}}(\mu) = \text{Ent}_m(\mu) - \lambda_j \int F d\mu$ into the curvature inequality gives

$$A_t^j \geq (1 - t)A_0^j + tA_1^j - D_t^j + \frac{E_t^j - (1 - t)E_0^j - tE_1^j + KC_t^j}{\lambda_j}.$$

Jensen's inequality gives $E_t^j \geq -\log m(Q)$, and $0 \leq C_t^j \leq L^2/8$. The final term therefore has nonnegative lower limit, uniformly in t . Continuity of F, G on Q and uniform convergence of the paths yield

$$F(\tau_t) \geq (1 - t)F(\tau_0) + tF(\tau_1) - |\dot{\tau}|^2 \int_0^1 \mathbf{g}(s, t) G(\tau_s) ds.$$

Continuity preserves the spatial variation of G along the limiting path, rather than replacing it by $\sup G$.

(5.1)–(5.2) · pp. 9–10

[BHS21, Definition 1.2, p. 3]

3. Chord inequalities on all interior intervals give the distributional derivative

On an arbitrary compact interval inside $(0, 1)$ choose $a \in C^2$ with $a''(t) = \ell^2 G(\sigma_t)$. Apply the preceding inequality on every subinterval, restoring the squared-length factor from reparametrization. The Green formula says exactly that $F \circ \sigma - a$ lies above each of its chords. It is concave, so its second distributional derivative is nonpositive. For constant $G = c$, take $a(t) = c\ell^2 t^2/2$. No global L^2 integrability of F enters this argument.

§5 · p. 10

3. Lemma 3.1 — The common generator domain and multiplier regularity

Proof target : Lemma 3.1; (3.1)–(3.3) · p. 6

Lemma 3.1 — The generator under a bounded Lipschitz weight

On a full-support $\text{RCD}(K, N)$ space as in the main result, let V be bounded and globally Lipschitz. Set $m_V = e^V m$, and let Δ_V be the nonpositive generator of its Cheeger energy. The Sobolev spaces and weak gradients agree, and

$$D(\Delta_V) = D(\Delta), \quad \Delta_V u = \Delta u + \Gamma(V, u).$$

The two generator graph norms are equivalent. Moreover, every $\xi \in D(\Delta_V)$ with $\xi, \Delta_V \xi \in L^\infty$ has a globally Lipschitz representative.

Lemma 3.1; (3.1)–(3.3) · p. 6

1 A bounded change of measure

$$V \in \text{Lip}(M) \cap L^\infty, \quad m_V = e^V m$$

$$D(\Delta_V) = D(\Delta), \quad \Delta_V u = \Delta u + \Gamma(V, u)$$

Common weak gradients and equivalent generator graph norms.

Integrate the original reverse-Poincaré bound without assuming weighted curvature.

[\[AGS15, Corollary 2.3, \(2.34\), p. 18\]](#)

2 Gradient bounds for the original resolvent

$$R_a = (a - \Delta)^{-1} = \int_0^\infty e^{-at} P_t dt$$

$$\|R_a h\|_p \leq a^{-1} \|h\|_p, \quad \|\nabla R_a h\|_p \leq C_p a^{-1/2} \|h\|_p \quad (p = 2, \infty)$$

Bounded drift and the $a^{-1/2}$ decay give a contraction.

[Lemma 3.1; \(3.1\)–\(3.3\) · p. 6](#)

3 A contraction yields a bounded gradient

$$E = \{z \in W^{1,2} : z, |\nabla z| \in L^\infty\}$$

$$z \mapsto R_a(h + \Gamma(V, z)), \quad (a - \Delta_V)z = h$$

Large a controls both the L^2 and L^∞ gradient norms.

Uniqueness by weighted energy, followed by original Sobolev-to-Lipschitz.

[Lemma 3.1; \(3.1\)–\(3.3\) · p. 6](#)

4 A Lipschitz representative for Bochner multipliers

$$\xi \in D(\Delta_V), \quad \xi, \Delta_V \xi \in L^\infty$$

$$h = (a - \Delta_V)\xi \implies z = \xi \in \text{Lip}(M)$$

The original heat resolvent produces Lipschitz multipliers for the weighted Bochner inequality

[Open figure ↗](#) [TeX source ↓](#)

1. Equivalent measures give a common graph domain

Bounds above and below on e^V identify the compression conditions on test plans and hence weak gradients; the $W^{1,2}$ norms are equivalent. Multiplying test functions in the weak generator definition by $e^{\pm V}$ gives the drift formula. Since $|\nabla V| \in L^\infty$, the drift lies in L^2 . The identity $\|\nabla u\|_2^2 = -\int u \Delta u dm$ and its weighted counterpart then give equivalence of graph norms.

[Lemma 3.1; \(3.1\)–\(3.3\) · p. 6](#)

2. A contraction built from the original resolvent

Put $R_a = (a - \Delta)^{-1} = \int_0^\infty e^{-at} P_t dt$. The Markov property and reverse-Poincaré estimate for the original heat semigroup give, for sufficiently large a and $p = 2, \infty$,

$$\|R_a h\|_p \leq a^{-1} \|h\|_p, \quad \|\nabla R_a h\|_p \leq C_p a^{-1/2} \|h\|_p.$$

Equip $E = \{z \in W^{1,2} : z, |\nabla z| \in L^\infty\}$ with the sum of these four norms. For large a , $z \mapsto R_a(h + \Gamma(V, z))$ is a contraction on E . Its fixed point solves $(a - \Delta_V)z = h$. Taking $h = (a - \Delta_V)\xi \in L^2 \cap L^\infty$, the weighted energy identity gives uniqueness and identifies $z = \xi$. Apply Sobolev-to-Lipschitz on the original space. No weighted curvature bound has yet been assumed, so the argument is not circular.

[AGS15, Corollary 2.3, (2.34), p. 18]

Lemma 3.1; (3.1)–(3.3) · p. 6

4. Proposition 4.1 — From the weak Hessian bound to variable curvature

Proof target : Proposition 4.1 · pp. 7–8

Proposition 4.1 — Exponential reweighting changes curvature

Assume the same space, F, G , and weak Hessian inequality as in Theorem 1.1, quantified over every compactly supported $g \in \text{Test}(M)$ and every nonnegative $h \in \text{Lip}_c(M)$. Then, for every $\lambda > 0$, keeping the metric d unchanged,

$$(M, d, m_\lambda), \quad m_\lambda = e^{\lambda F} m,$$

satisfies $\text{CD}(K - \lambda G, \infty)$.

Proposition 4.1 · pp. 7–8

1 Use original Bochner with a Lipschitz multiplier

$$V = \lambda F, \quad w = e^V, \quad k = K - \lambda G$$

$$B_0(g, w\xi) \geq K \int w\xi\Gamma(g) dm$$

g is an original compact test function; ξ is a Lemma 3.1 multiplier.

Use the capacity-aware Bochner measure and product rule; apply the weak upper bound.

[§2.2; \(2.2\) · p. 5](#)

[\[Sav14, Lemma 3.2; \(3.4\), \(3.7\)–\(3.9\), pp. 11–12\]](#) [\(4.2\)–\(4.5\) · pp. 7–8](#)

2 Isolate the signed weak Hessian

$$B_V(g, \xi) - B_0(g, w\xi) = -H_V(\nabla g, \nabla g)(w\xi)$$

$$B_V(g, \xi) \geq \int k\xi\Gamma(g) dm_V$$

Bounded $\xi, \Delta_V \xi, |\nabla \xi|, k$ make each integral graph-continuous.

[\(4.6\)–\(4.7\) · p. 8](#)

[Remark 4.2 · p. 9](#)

3 Graph closure without differentiating the drift

$$B_V = \frac{1}{2} \int (\Delta_V \xi) \Gamma(g) dm_V + \int (\Delta_V g) \Gamma(\xi, g) dm_V + \int \xi (\Delta_V g)^2 dm_V$$

From the common compact core to all $D(\Delta_V)$; only then impose $\Delta_V g \in W^{1,2}$.

Check common gradients, Sobolev-to-Lipschitz, and Gaussian volume growth.

[\[AGS15, Theorem 4.17, p. 57\]](#)

4 Obtain the underlying RCD structure

$$K_V = \inf_M k > -\infty, \quad d_{\mathcal{E}_V} = d$$

$$\text{BE}(K_V, \infty) \implies \text{RCD}(K_V, \infty)$$

k is bounded continuous; graph closure supplies the actual BHS domain.

[\[BHS21, Theorem 1.1, p. 3\]](#) [\[BHS21, Definition 1.4, p. 4\]](#)

5 Variable-curvature entropy inequality

$$\text{BE}_2(k, \infty) \implies \text{CD}(k, \infty)$$

$$k = K - \lambda G, \quad m_\lambda = e^{\lambda F} m \quad (\lambda > 0)$$

The original Bochner measure, Hessian identity, graph closure, and curvature-dimension equivalences

[Open figure ↗](#) [TeX source ↓](#)

1. Lipschitz multipliers extract precisely the weak Hessian

Write $V = \lambda F$, $w = e^V$, and $k = K - \lambda G$. Fix nonnegative $\xi \in D(\Delta_V)$ with $\xi, \Delta_V \xi \in L^\infty$;

Lemma 3.1 makes ξ Lipschitz. For a compactly supported original test function g , set $q = \Gamma(g)/2$ and use

$$B_V(g, \xi) = - \int w \Gamma(\xi, q) dm + \int w [\xi \Delta_V g + \Gamma(\xi, g)] \Delta_V g dm.$$

The original Bochner measure charges no capacity-zero set. Test approximation with common compact support and a uniform Lipschitz bound therefore allows integration against the Lipschitz multiplier $w\xi$. A cutoff equal to one near $\text{supp } g$ handles the multiplier's possible noncompact support.

The product rule and the very definition of the weak Hessian give

$$B_V(g, \xi) - B_0(g, w\xi) = -H_V(\nabla g, \nabla g)(w\xi).$$

Combining the original Bochner lower bound with the upper bound on $H_V = \lambda H_F$ yields $B_V(g, \xi) \geq \int k\xi\Gamma(g) dm_V$. The minus sign in this Hessian identity explains the curvature term $K - \lambda G$.

§2.2; (2.2) · p. 5; (4.2)–(4.5) · pp. 7–8

[Sav14, Lemma 3.2; (3.4), (3.7)–(3.9), pp. 11–12]

[Gig18, Theorem 3.3.8, p. 110]

2. Close in the graph norm before using the weighted test class

The drift in $\Delta_V g = \Delta g + \Gamma(V, g)$ need not be differentiable. In Remark 4.2, taking $V = -|x_1|$ and $g = x_1$ locally near the origin gives $\Delta_V g = -\text{sign}(x_1)$ there, which is not even locally $W^{1,2}$.

First rewrite B_V as

$$\frac{1}{2} \int (\Delta_V \xi) \Gamma(g) dm_V + \int (\Delta_V g) \Gamma(\xi, g) dm_V + \int \xi (\Delta_V g)^2 dm_V.$$

Boundedness of $\xi, \Delta_V \xi, |\nabla \xi|, k$ makes this expression continuous under Δ_V graph convergence. The original heat flow and good cutoffs supply a compact test core; graph-norm equivalence in Lemma 3.1 transfers it to the weighted generator. Close the inequality for all $g \in D(\Delta_V)$ first. Only then impose $\Delta_V g \in W^{1,2}$ and integrate by parts to obtain

$$\frac{1}{2} \int (\Delta_V \xi) \Gamma(g) dm_V - \int \xi \Gamma(g, \Delta_V g) dm_V \geq \int k \xi \Gamma(g) dm_V.$$

In particular it holds on the actual BHS domain for $\text{BE}_2(k, \infty)$, which additionally requires $\Gamma(g) \in L^\infty$.

(4.6)–(4.7) · p. 8; Remark 4.2 · p. 9

[GT21, Appendix A, (A.7), Lemma A.2, (A.13), pp. 54–56]

[BHS21, Definition 1.4, p. 4]

3. Establish a constant-curvature RCD structure first

The BHS variable-curvature equivalence requires the underlying space already to satisfy an RCD condition with some constant lower bound. First lower k to $K_V = \inf k > -\infty$. The common weak gradients and Sobolev-to-Lipschitz property identify the intrinsic distance of $\mathcal{E}_V = 2 \mathbf{Ch}_V$ with d . AGS Theorem 3.9 gives an energy measure space, Theorem 3.14 gives upper regularity and conservativity, and Definition 3.16 identifies a Riemannian energy measure space. Gaussian volume growth survives bounded equivalence of measures, supplying the remaining hypothesis of Theorem 4.17. This yields $\mathbf{RCD}(K_V, \infty)$. Now k is bounded and continuous, hence lower semicontinuous and lower bounded; the preceding $\mathbf{BE}_2$ inequality gives $\mathbf{CD}(k, \infty)$.

[AGS15, Theorem 3.9, p. 33; Theorem 3.14, p. 38; Definition 3.16, p. 40]

[AGS15, Theorem 4.17, p. 57]

[BHS21, Theorem 1.1, p. 3]

5. Which papers supply which steps

Source and result

[Deng25, Theorem 1.3, p. 2]

Used at

§5, Theorem 1.1

Role and required hypotheses

Finite-dimensional nonbranching for $1 < N < \infty$ gives uniqueness of strict subsegments

Checking scope

Statement on p.2 and application on pp.9–10 compared

Source and result

[AGS15, Corollary 2.3, (2.34), p. 18] / [AGS15, Theorem 3.9, p. 33; Theorem 3.14, p. 38; Definition 3.16, p. 40] / [AGS15, Theorem 4.17, p. 57]

Used at

Lemma 3.1, Proposition 4.1

Role and required hypotheses

Original heat gradient regularization and constant $\text{BE} \rightarrow \text{RCD}$ after identifying the intrinsic distance

Checking scope

Input statements and hypotheses on pp.18, 33, 38, 40, 57 compared

Source and result

[BHS21, Definition 1.2, p. 3] / [BHS21, Definition 1.4, p. 4] / [BHS21, Theorem 1.1, p. 3]

Used at

Proposition 4.1, §5

Role and required hypotheses

$\text{BE}_2 \rightarrow \text{CD}$ for bounded continuous k on an existing RCD space; Green-kernel entropy formula

Checking scope

Statements, domains, and signs on pp.3–4 compared

Source and result

[Sav14, Lemma 3.2; (3.4), (3.7)–(3.9), pp. 11–12] / [Gig18, Theorem 3.3.8, p. 110]

Used at

§2.2, Proposition 4.1

Role and required hypotheses

Bochner measures, capacity-zero sets, and the Hessian-square term in the original inequality

Checking scope

Specified-version statements and Lipschitz multiplier approximation compared

Source and result

[GT21, Appendix A, (A.7), Lemma A.2, (A.13), pp. 54–56]

Used at

Lemma 2.1, Proposition 4.1

Role and required hypotheses

Bounded-Laplacian heat regularization, good cutoffs, graph core, and Hessian approximation

Checking scope

v1 pp.54–56 compared with manuscript pp.4–5

This table records direct inputs to the proof. The earlier Hessian-to-convexity approaches discussed in the introduction are distinguished from the external theorems used here. Connections with the other manuscript in catalogue 356 are recorded in the catalogue overview.

6. Return to the sources

- Theorem 1.1; (1.1)–(1.4) · p. 2 — Read the test class, weak Hessian, and quantification over every geodesic.
- Lemma 2.1 · pp. 4–5 — Construct the compact core and Lipschitz multiplier approximation from the original heat flow.
- Lemma 3.1; (3.1)–(3.3) · p. 6 — Follow the common weighted generator domain and resolvent regularity.
- Proposition 4.1 · pp. 7–8 — Read the Hessian identity, graph closure, and two stages of curvature equivalence.
- (5.1)–(5.2) · pp. 9–10 — Check the choice of lambda dominating endpoint entropy and the Green-kernel limit.
- §5 · p. 10 — Pass from every interior interval to the distributional second-derivative bound.

The statements and proof passages of Theorem 1.1, Lemmas 2.1 and 3.1, Proposition 4.1, and §5 were read, and the external input statements, application hypotheses, versions, and pages listed above were compared. The checks addressed the difference between original test functions and the weighted domain, graph closure, uniqueness of strict subsegments, and the simultaneous limit involving λ_j and endpoint entropy. Complete proofs of the external results and their

dependencies were not recursively verified; no expert peer review or formal verification is claimed. In particular, the full foundations of measure-valued Bochner calculus and heat-flow estimates remain external inputs.