

OpenAI Math Digest

CATALOGUE 305

Counterexamples to disk embedding and Wall's manifold conjecture

Catalogue synthesis & 3 proof overviews

Independent reading guide · English edition

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AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 3 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

Counterexamples to disk embedding and Wall's manifold conjecture

The catalogue treats two four-dimensional disk obstructions: one fixing only the boundary circles, and another also retaining induced normal framings. The latter paper's marked geometry, algebraic potentials, and finite tensor theorem feed directly into the realization problem for finitely presented integral PD4 groups as groups of closed aspherical topological four-manifolds. This guide follows the manuscripts' claims and the connections between their proofs.

1. Papers in official catalogue order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 305 · 2026-10-21 checked source snapshot

#

01

Manuscript and overview

A boundary-only obstruction to four-dimensional disk embedding →

Short name: A boundary-only obstruction to four-dimensional disk embedding

September 24, 2026 · 48 pages

Original paper (GitHub) ↗

Principal claim

Excludes disjoint locally flat fillings of specified boundary circles for a finite disk configuration with framed algebraic dual spheres, without output framing or relative homotopy constraints. Deduces nongoodness of F_2 and failure of the conjunction of unrestricted surgery and s-cobordism.

Theorem 1.1 · p. 3; Corollaries 1.2–1.3 · pp. 4–5 ↗

Role and available overview

Actual filling images give cut data and full defect ideals; algebraic replacement and specialization lead to a Frobenius–Koszul obstruction.

Read the proof overview →

#

02

[Manuscript and overview](#)

A marked tensor obstruction to four-dimensional disk embedding →

Short name: A marked tensor obstruction to four-dimensional disk embedding

September 24, 2026 · 53 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Asserts that a disk configuration with the prescribed algebraic dual conditions admits no disjoint locally flat replacements preserving boundary maps and induced normal framings, without prescribing their relative homotopy classes.

[Theorem 1.1 · p. 2](#) ↗

Role and available overview

Connects independently conjugated boundary laws, markings on nonreduced critical schemes, algebraic potentials, and a finite tensor obstruction for all active triples. Supplies intermediate inputs to the PD4 paper.

[Read the proof overview →](#)

#

03

Manuscript and overview

A PD4 group without an aspherical manifold model →

Short name: A PD4 group without an aspherical manifold model

September 24, 2026 · 32 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Constructs a finitely presented integral PD_4 group with a finite oriented four-dimensional BG and excludes realization as the group of a closed aspherical topological four-manifold.

[Theorem 1.1 · pp. 1–2](#) ↗

Role and available overview

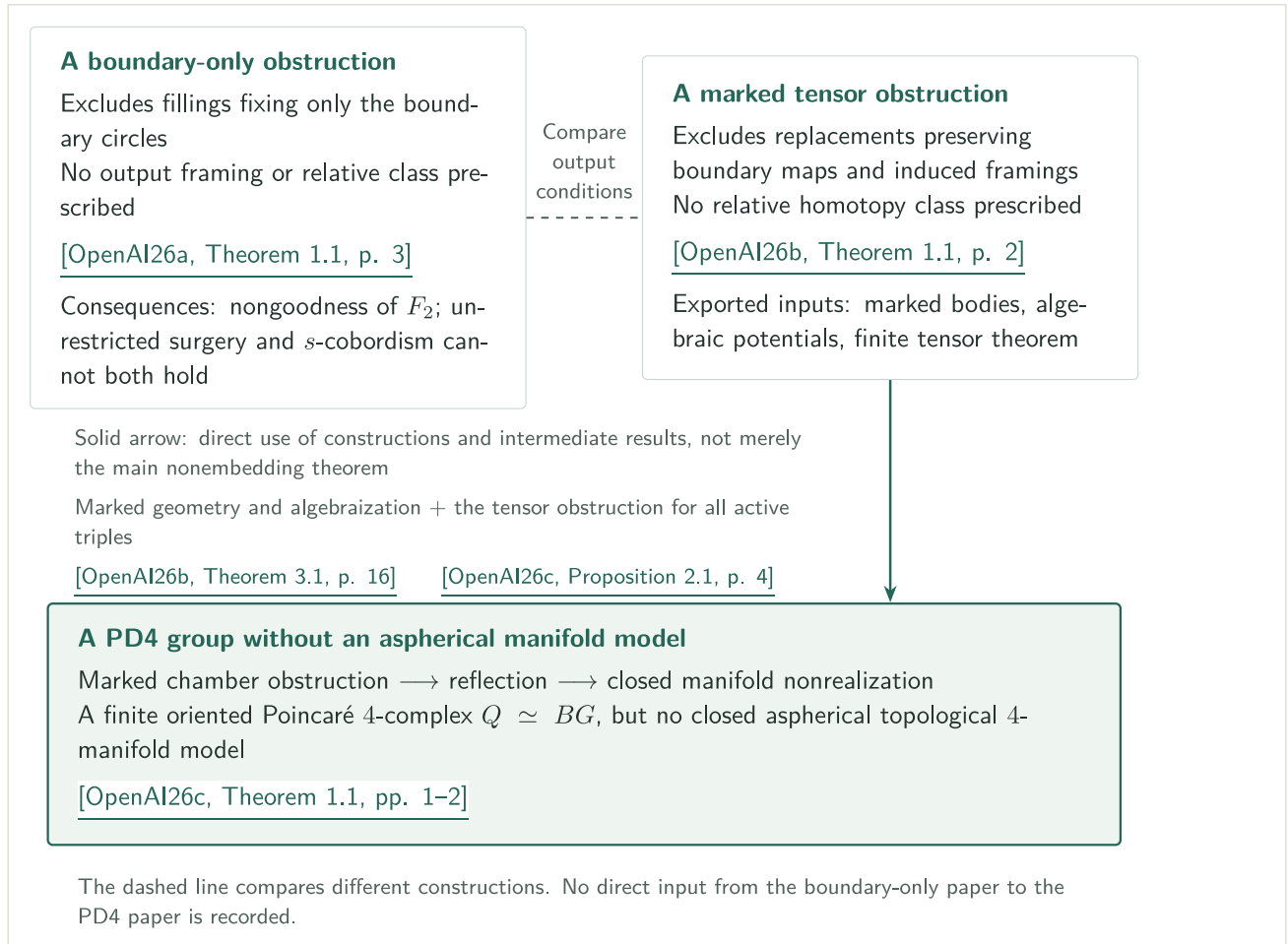
Transfers the marked paper's geometry and tensor interface to a marked chamber, then uses reflection, a marked stable product, and direction-controlled sphere removal.

[Read the proof overview](#) →

2. How the results fit together

All three keys refer to manuscripts by OpenAI dated September 24, 2026. Source citations in the diagram and text include result numbers and pages; article links use paper titles.

- [OpenAI26a](#) — A boundary-only obstruction to four-dimensional disk embedding. A construction excluding arbitrary locally flat fillings of the specified boundary circles.
- [OpenAI26b](#) — A marked tensor obstruction to four-dimensional disk embedding. A construction excluding replacements preserving boundary maps and induced framings, and supplying intermediate inputs to the next paper.
- [OpenAI26c](#) — A PD_4 group without an aspherical manifold model. Reflection and stabilized sphere removal transfer a marked relative obstruction to closed manifold nonrealization.



Two disk obstructions and the direct input to PD4 nonrealization

[Open figure ↗](#) [TeX source ↓](#)

1. The disk conclusions impose different output conditions

A [boundary-only obstruction](#) has framed algebraic dual spheres in its input but excludes boundary fillings without requiring output framings, geometric duals, or relative homotopy classes. The main theorem of [A marked tensor obstruction](#) excludes replacements preserving boundary maps and the input disks' induced normal framings. It also leaves the replacement disks' relative homotopy classes unspecified. These are compared as different constructions; their examples and ambient manifolds are not identified.

[\[OpenAI26a, Theorem 1.1, p. 3\]](#)

[\[OpenAI26a, Proposition 2.8; §7, pp. 16, 47\]](#)

[\[OpenAI26b, Theorem 1.1, p. 2\]](#)

2. The PD4 paper imports intermediate results from the marked paper

A PD4 group without an aspherical manifold model uses more than the marked paper's main nonembedding theorem. Marked grope bodies and exterior duals construct its chamber; algebraization and Vertex data apply to cuts extracted from a hypothetical filling; the finite tensor theorem gives the contradiction. An additional form algebra on punctured vertices transfers this interface to arbitrary topological fillings.

Reflection then constructs a finite oriented aspherical Poincaré four-complex. A hypothetical manifold model gives, through a marked stable product, a graph-type filling of the original boundary, contradicting the chamber obstruction. The solid arrow records direct use of the intermediate results. No direct dependency from the boundary-only paper to the PD4 paper is recorded in the proof passages examined.

[OpenAI26c, §2.1; Proposition 2.1; §2.4, pp. 4–5, 8–10]

[OpenAI26c, Proposition 3.2, pp. 13–14]

[OpenAI26c, Proposition 4.5; Proposition 5.5, pp. 19–20, 29–30]

3. Follow an input into its application

Input from the marked paper

Lemmas 2.3–2.4

Use in the PD4 paper

§§2.1–2.3, Proposition 2.1(1)–(2)

Content and connecting hypotheses

Bodies realizing all conjugated boundary laws and algebraic duals; the tube-side marking is retained in constructing the hyperbolic basis

Input from the marked paper

Proposition 4.5, Theorem 5.1, Propositions 5.2 and 6.1

Use in the PD4 paper

§§2.1 and 2.4

Content and connecting hypotheses

Markings on nonreduced critical schemes and algebraic potentials in separate coordinate blocks; PD4 Lemma 2.4 supplies filled cohomology and Stokes for topological vertices

Input from the marked paper

Theorem 3.1

Use in the PD4 paper

Proposition 2.1(3), end of §2.4

Content and connecting hypotheses

The contradiction uses $m \geq 5$, characteristic zero, complementary tangents, all mixed nilpotent terms, and actual gradient-ideal memberships for every active triple

[OpenAI26b, Lemmas 2.3–2.4, pp. 9–12]

[OpenAI26b, Proposition 4.5; Theorem 5.1; Proposition 5.2; Proposition 6.1, pp. 29–30, 32, 37]

[OpenAI26b, Theorem 3.1, p. 16]

The marked chamber section of the PD4 article explains these conditions and stages. Comparing an input statement with its application is distinct from independently verifying its entire proof.

4. Scope of the consequences and sources

The boundary-only paper deduces nongoodness of F_2 and every group containing it as a subgroup; it does not identify the constructed manifold's group with F_2 . It also concludes that the unrestricted topological surgery and s -cobordism assertions **cannot both hold**, without deciding which fails separately. The PD4 paper asserts nonrealization in **dimension four** for a group with finite BG. These consequences should not be conflated into one implication.

[OpenAI26a, Corollaries 1.2–1.3, pp. 4–5]

[OpenAI26c, Theorem 1.1, pp. 1–2]

The official subject is Topology (304–321). The displayed title follows item 305 in the official overview; CONTENTS.md instead uses Four-dimensional disk embedding and Wall's conjecture. The reference commit is 1581d4cbdd69a0faf34b2089614fe2a814316a6a; both printed and PDF overview pagination place the item on page 33. All three manuscripts are dated September 24, 2026.

OpenAI Research Catalog · Catalogue 305, Topology · p. 33

Each article connects all conclusions of its main results to a proof overview and records direct inputs, application passages, and unchecked points. The original text of Quinn's relative stable product theorem used by the PD4 paper was not acquired, so that external connection remains

unchecked. The original statements of the external $\text{CAT}(0)$, Farrell–Jones, and punctured-smoothing inputs were also not compared in this pass. The work is not a complete independent proof verification or completed expert peer review.

A boundary-only obstruction to four-dimensional disk embedding

Turn a boundary filling into deformation data of cuts, then force two cubic laws to conflict in a Frobenius–Koszul complex.

The manuscript constructs a finite configuration satisfying the algebraic input for four-dimensional disc embedding and claims that its boundary link has no disjoint locally flat disc filling. Neither relative homotopy classes nor output framings are prescribed. This article describes the manuscript's argument; it is not an independent verification of the entire proof.

Manuscript author: OpenAI · September 24, 2026 · 48 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Nonfilling determined by the boundary alone

There exist a compact connected oriented smooth four-manifold M with nonempty boundary, an integer $k \geq 1$, proper generic disc immersions $f_i : (D^2, S^1) \rightarrow (M, \partial M)$, and framed sphere maps $g_i : S^2 \rightarrow M$ such that the union of the boundary restrictions is an embedding and, over $\mathbb{Z}[\pi_1(M)]$,

$$\lambda(f_i, g_j) = \delta_{ij}1, \quad \lambda(g_i, g_j) = 0, \quad \tilde{\mu}(g_i) = 0,$$

but the specified boundary circles do not bound pairwise disjoint proper locally flat embedded discs in M . Each g_i can be chosen smoothly embedded; distinct spheres may intersect. Here $\tilde{\mu}$ is the reduced Wall invariant valued in $\mathbb{Z}[\pi_1(M)]/\langle h - h^{-1}, \mathbb{Z} \cdot 1 \rangle_{\text{add}}$. No vanishing of the intersection invariants among the input discs is assumed.

[Theorem 1.1 · p. 3](#)

Corollary 1.2 — Nongoodness of free groups

The free group F_2 is not good in the sense of Freedman–Quinn. Every group containing a subgroup isomorphic to F_2 is also not good. This does not identify the constructed manifold's fundamental group with F_2 .

Corollary 1.2 · p. 4

Corollary 1.3 — Incompatibility of two unrestricted assertions

The unrestricted four-dimensional topological surgery assertion and five-dimensional topological $\mathfrak{s}$ -cobordism assertion cannot both hold. The former asserts that a degree-one normal map from a compact four-manifold to a four-dimensional Poincaré pair $(Z, \partial Z)$, inducing a $\mathbb{Z}[\pi_1(Z)]$ -homology equivalence on the boundary and having zero Wall obstruction, is normally bordant relative to the boundary to a homotopy equivalence. The latter asserts that a compact five-dimensional topological $\mathfrak{s}$ -cobordism with a specified boundary product is homeomorphic to a product extending that structure. Neither assertion restricts the fundamental group. The corollary does not identify which assertion fails individually.

Corollary 1.3 · p. 5

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A boundary-only obstruction to four-dimensional disk embedding](#). External inputs use the following keys.

- [PRT25](#) Mark Powell, Arunima Ray, Peter Teichner — The 4-dimensional disc embedding theorem and dual spheres: Good-group disc embedding and closure under subgroups and quotients in Corollary 1.2. *Selecta Mathematica* 31 (2025), Article 80; publisher PDF.
- [KPT21](#) Min Hoon Kim, Mark Powell, Peter Teichner — The round handle problem: Reduction from surgery together with $\mathfrak{s}$ -cobordism to disc embedding in Corollary 1.3. Author Accepted Manuscript deposited 26 September 2022; published 2021.
- [DSGJ-v3](#) Alessandro De Stefani, Eloisa Grifo, Jack Jeffries — A Zariski–Nagata theorem for smooth $\mathbb{Z}$ -algebras: Differential-operator test for symbolic powers in Lemma 5.3. [arXiv:1709.01049v3](#).
- [Stacks](#) The Stacks Project Authors — The Stacks Project: Descent of the Taylor tuple from completion in Lemma 5.3; tags have no page numbering. Tags 00MA, 00MB, 00MC; web statements checked 2026-10-21.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — From geometry to a finite-complex contradiction

Proof target : [Theorem 1.1 · p. 3](#)

Proof route. Start with a graph-type four-manifold having $d = 5$ special edges. Construct algebraic duals in the exterior of capped-stage bases and extract cuts from a hypothetical boundary filling. Algebraize and correct the resulting representation-deformation potentials, specialize to positive characteristic, and use the incompatibility of two cubic laws in one finite differential-form complex.

1 Finite geometric configuration

$$\lambda(f_i, g_j) = \delta_{ij} 1, \quad \lambda(g_i, g_j) = 0, \quad \tilde{\mu}(g_i) = 0$$

Each g_i is framed and embedded. Assume a disjoint filling of the boundary circles.

Move the graph map with boundary fixed; impose the cubic end-holonomy law

Lemma 2.7; Proposition 2.8 · pp. 14–16

Lemmas 3.1–3.2; Proposition 3.8 · pp. 17–22

2 Cuts away from disc images and their deformations

$$G \simeq \Gamma, \quad J_i \subset G, \quad \partial J_i = T^2, \quad n^3 = 0$$

Paired restrictions in a double cover give complementary tangent spaces.

Duality gives gradient generation; Stokes gives the square-ideal identity

Propositions 4.6, 4.8; Corollary 4.11 · pp. 25–31

3 Retain the full defect ideal

$$v = (\partial_C \Psi)(a, Y), \quad (v) = \mathfrak{r}, \quad \Psi(a, Y) \in \mathfrak{r}^2$$

$$a_{i,1} a_{j,1} a_{k,1}, \quad p_i(Y_i^-) p_j(Y_j^-) p_k(Y_k^-) \in (v)$$

Algebraic replacement and invertible gradient change; no geometric realization needed

Lemma 5.3; Theorem 5.8 · pp. 33–38

Corollary 5.10 · p. 38

4 Correct to a zero graph and specialize

$$\Psi(a, \tilde{Y}) = 0, \quad \text{char } k = \ell > 2$$

Preserve complementary tangents, both cubic memberships, and their witnesses.

p_i law gives a tensor cycle; a_i law gives an ambient primitive

Theorem 6.1; §§6.3–6.6; §7 · pp. 39–47

5 Contradiction from multiplication by a unit

$$B = k[s_1, s_2, s_3] / (s_1^2, s_2^2, s_3^2), \quad w = s_1 s_2 s_3 \neq 0$$

$$\delta_s \eta_s = w \alpha_s, \quad \delta_s \kappa_s = 0, \quad b_s = \text{Res}(\alpha_s \wedge \kappa_s) \in B^\times$$

$$0 = w b_s \implies w = 0$$

From boundary filling to the contradiction between two cubic laws

[Open figure ↗](#) [TeX source ↓](#)

1. Construct duals in the exterior and protect the actual disc images

Inside a graph-type manifold G , a finite family of capped stages encodes all independently conjugated triple commutators $[z_a^{c_1}, [z_b^{c_2}, z_c^{c_3}]]$. Delete only neighborhoods of the base surfaces to obtain M . Shallow caps give the input discs f_r , while compression of product tori using upper-stage caps gives g_r . Besides individual embeddedness and trivial normal bundles of the spheres, the retained unused caps kill the intersection-label comparison loops in $\pi_1(M)$. Opposite-sign intersections therefore cancel coefficient by coefficient in the group ring. Cancellation merely in $\pi_1(G)$ would not suffice.

Lemmas 2.4–2.5; Corollary 2.6 · pp. 12–14

Move the graph map on the actual images of hypothetical filling discs into the h -subgraph. Injectivity on fundamental groups and $\pi_2(\Gamma) = 0$ permit this change relative to the boundary. Smooth the map away from the protected disc images and take regular levels on the special edges. This produces smooth three-dimensional walls J_i , each with one boundary torus. The hypothetical discs themselves need not be smoothed or homotoped to the original caps. The cut manifold X may be disconnected, and closed wall components are retained.

Lemma 2.7; Proposition 2.8 · pp. 14–16

2. Retain the full ideal and both cubic laws

Mayer–Vietoris for a double cover gives complementary tangent images of the paired wall restrictions. For the adjoint representation of $SL_2(\mathbb{C})$, a finite set of conjugation operators spans $\text{End}(\mathfrak{sl}_2)$. Leading triple-commutator terms and Nakayama's lemma imply $\mathfrak{n}^3 = 0$ for the ideal of end-holonomy deviations in the deformation ring.

Lemmas 3.1–3.2; Proposition 3.8 · pp. 17–22

With wall coordinates (a_i, y_i) and restrictions (a_i, Y_i^+) and $(0, Y_i^-)$, boundary-polarized Chern–Simons potentials give

$$\Psi(a, C) = \sum_i (P_i(a_i, C_i^+) - P_i(0, C_i^-)), \quad v = (\partial_C \Psi)(a, Y).$$

Differentiate before substitution. For the full defect ideal $\mathfrak{r}$, the manuscript proves $(v) = \mathfrak{r}$ and $\Psi(a, Y) \in \mathfrak{r}^2$. The first uses duality and the wall-restriction isomorphism to kill any missed minimal obstruction, contradicting the impossibility of a flat lift. The second glues boundary connections and applies Stokes to express the value through curvature squared. Setting $p_i(y_i) = \partial_{a_{i,1}} P_i(0, y_i)$ yields, for every triple including repetitions,

$$a_{i,1} a_{j,1} a_{k,1} \in (v), \quad p_i(Y_i^-) p_j(Y_j^-) p_k(Y_k^-) \in (v).$$

The second membership comes from the boundary term in polarized variation. Vanishing on the common zero set would not provide the required input for the next stage.

Propositions 4.6, 4.8; Corollary 4.11 · pp. 25–31

3. Preserve witnesses through replacement, correction, and specialization

Lemma 5.3 replaces a formal potential, whose gradient jets are representable in one smooth algebraic local ring, by a single algebraic primitive modulo any prescribed ideal power. Faithful flatness of completion descends the whole Taylor tuple; differential tests for symbolic powers also control embedded components. After sufficiently high-order replacement, the square-ideal

identity allows a correction $\tilde{Y} = Y + Bv$ with $\Psi(a, \tilde{Y}) = 0$ exactly. An invertible change of gradient generators and difference factorizations transport the witnesses for both cubic memberships.

[DSGJ-v3, Theorem 3.6, p. 7]

Lemma 5.3; Theorem 5.8 · pp. 33–38

The potentials, corrected map, membership witnesses, and inverse tangent determinant all lie over one finitely generated $\mathbb{Z}$ -algebra. Specialization to a finite field of sufficiently large characteristic $\ell > 2$ preserves them. The corrected graph need not geometrically represent representations. Only the replacement data are specialized, not the original complex integrals.

Corollary 5.10 · p. 38

4. Kill a nonzero product in the same Frobenius quotient

Put $F(C) = \Psi(0, C)$ and use $\delta = dF \wedge$ on differential forms over $T = k[[C]]/(C_\nu^\ell)_\nu$. The complementary zero graphs give normal cycles α, β with nonzero residue pairing. The cubic law for the p_i and a tensor-kernel calculation select a tensor with at least $d - 2 \geq 3$ suitable positive factors.

Over $B = k[s_1, s_2, s_3]/(s_1^2, s_2^2, s_3^2)$, lift that tensor to a cycle κ_s and construct the moving graph's normal cycle α_s in the same coordinate Frobenius quotient. The mixed terms $s_i s_j$ and $s_1 s_2 s_3$ are retained. The cubic law for the a coordinates supplies an ambient primitive with $\delta_s \eta_s = w \alpha_s$, where $w = s_1 s_2 s_3$. Meanwhile $b_s = \text{Res}(\alpha_s \wedge \kappa_s)$ has nonzero constant term and is a unit. Thus $0 = w b_s$ forces $w = 0$, contrary to the squarefree monomial basis of B . This excludes the hypothetical boundary filling and, together with the constructed algebraic input, proves the theorem.

Theorem 6.1; §§6.3–6.6; §7 · pp. 39–47

3. Proof of Corollary 1.2 — Pass goodness through subgroups and quotients

Proof target : Corollary 1.2 · p. 4

1 Assume goodness
 F_2 good, $F_r \hookrightarrow F_2$, $F_r \twoheadrightarrow \pi_1(M)$
Compact smooth M has finitely generated fundamental group.

Subgroups and quotients of good groups are good
[\[PRT25, §1.1, p. 4\]](#)

2 Pass to the constructed group $\pi_1(M)$ good

Apply disc embedding to the discs and dual spheres of Theorem 1.1.

Algebraic dual data supply a boundary filling
[\[PRT25, Theorem A, p. 3\]](#)
[Theorem 1.1 · p. 3](#)

3 The forbidden filling

Disjoint locally flat filling discs contradict Theorem 1.1.

Thus F_2 is not good; neither is any overgroup, by subgroup closure.

From goodness to the forbidden boundary filling

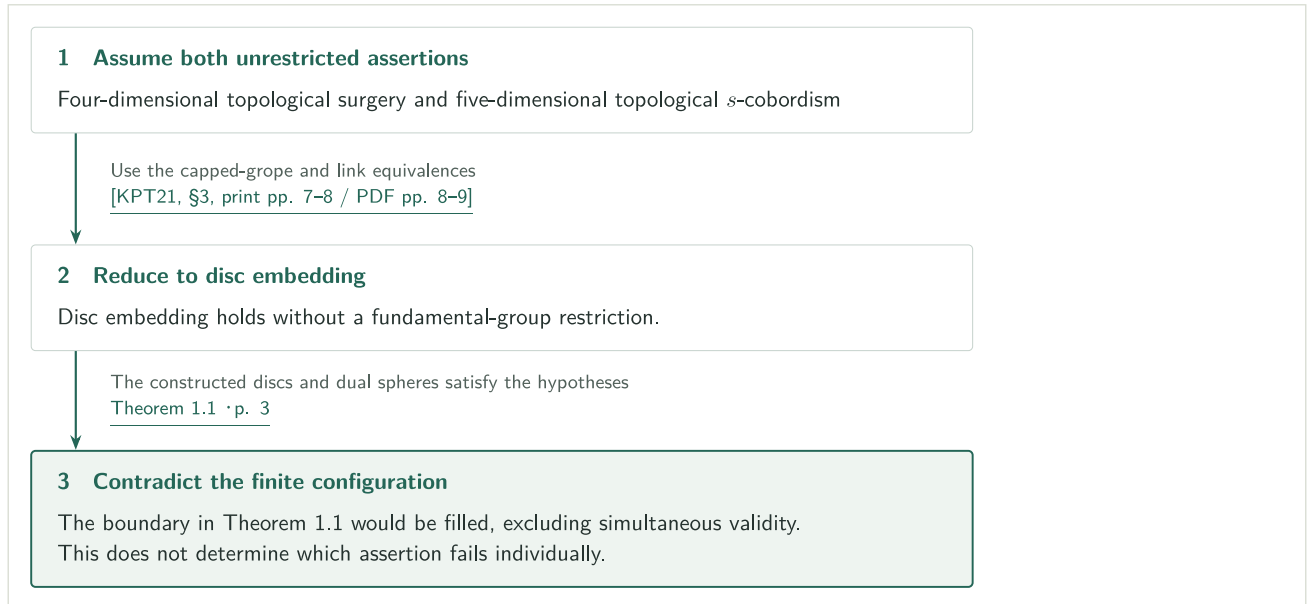
[Open figure ↗](#) [TeX source ↓](#)
1. Goodness of the constructed group would give a filling

Since M is compact and smooth, $\pi_1(M)$ is finitely presented, hence finitely generated. If it were good, the disc embedding theorem would apply to the intersection data and framed algebraic duals and supply the boundary filling forbidden by Theorem 1.1.

[\[PRT25, Theorem A, p. 3\]](#)**2. Transfer the conclusion to F_2 and its overgroups**

Choose $r \geq 2$ generators to obtain $F_r \twoheadrightarrow \pi_1(M)$. The elements $a^j b a^{-j}$ ($0 \leq j < r$) freely generate a copy of F_r in $F_2 = \langle a, b \rangle$. Closure of goodness under subgroups and quotients would make $\pi_1(M)$ good if F_2 were good. Subgroup closure also excludes goodness for every group containing F_2 .

[\[PRT25, §1.1, p. 4\]](#)**4. Proof of Corollary 1.3 — Assume surgery and s-cobordism together**Proof target : [Corollary 1.3 · p. 5](#)



Reduction from the two unrestricted assertions to disc embedding

[Open figure ↗](#) [TeX source ↓](#)

1. Simultaneous validity gives unrestricted disc embedding

The manuscript uses the reduction in Round handle problem from the two assertions together to disc embedding without a fundamental-group restriction. The finite configuration of Theorem 1.1 satisfies its algebraic and boundary hypotheses, so the forbidden filling would exist. Nongoodness of F_2 alone does not replace this implication.

[KPT21, §3, print pp. 7–8 / PDF pp. 8–9]

5. Which papers supply which steps

The cuts, deformations, and positive-characteristic contradiction form an internal chain. The manuscript does not use the marked-tensor or PD4 manuscript in this catalog as an input. The central external inputs are the good-group disc embedding theorem in Corollary 1.2, the surgery reduction in Corollary 1.3, and completion and differential symbolic-power tests in Lemma 5.3. The manuscript also identifies standard inputs such as duality and the regular-value theorem.

Lemma 5.3; Theorem 5.8 · pp. 33–38

6. Return to the sources

- Lemmas 2.4–2.5; Corollary 2.6 · pp. 12–14 — Check exterior group-ring labels and framings

- Lemma 2.7; Proposition 2.8 · pp. 14–16 — Read the construction of cuts using only boundary images
- Propositions 4.6, 4.8; Corollary 4.11 · pp. 25–31 — Track the two ideal identities and cubic laws
- Lemma 5.3; Theorem 5.8 · pp. 33–38 — Follow coefficient control through replacement and correction
- Theorem 6.1; §§6.3–6.6; §7 · pp. 39–47 — Read the contradiction in one finite complex and its application

The reading covered the main statements; framings, group-ring cancellation, and cuts in §2; complementarity and the two ideal identities in §§3–4; replacement and correction in §5; tensor selection, mixed terms, the common Frobenius quotient, and the residue contradiction in §6; and their assembly in §7. This does not independently reprove every local construction, identity, or external input. External checking covered the statements and application conditions of PRT25 Theorem A and §1.1, KPT21 §3, DSGJ-v3 Theorem 3.6 and Lemma 2.3, and Stacks Tags 00MA–00MC. Other excellence tags and the original texts for duality and regular values were not rechecked here.

A marked tensor obstruction to four-dimensional disk embedding

Algebraize while preserving based holonomy, then compare two vertex triple laws in a finite tensor complex.

The manuscript claims nonexistence of locally flat disc replacements preserving the input boundary maps and induced normal framings. It prescribes no relative homotopy classes for the replacements. The markings specify based curves and paths in the construction, not additional conditions on replacement discs. This article describes the manuscript's argument.

Manuscript author: OpenAI · September 24, 2026 · 53 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — No replacement preserving boundary maps and induced framings

There exist a compact connected oriented smooth four-manifold M with boundary, an integer $k \geq 1$, proper generic disc immersions $f_i : (D^2, S^1) \rightarrow (M, \partial M)$ with disjoint embedded boundary circles, and generically immersed framed spheres $g_i : S^2 \rightarrow M$ such that, with fixed whiskers and values in $\mathbb{Z}[\pi_1 M]$,

$$\lambda(f_i, g_j) = \delta_{ij}, \quad \lambda(g_i, g_j) = 0, \quad \tilde{\mu}(g_i) = 0,$$

but there are no pairwise disjoint proper locally flat embedded discs with the same boundary maps and the boundary normal framings induced by the f_i . The diagonal equations $\lambda(g_i, g_i) = 0$ are included. The target of $\tilde{\mu}$ is the additive quotient $\mathbb{Z}[\pi_1 M] / \langle h - h^{-1}, \mathbb{Z} \cdot 1 \rangle_{\text{add}}$. Neither preservation of the relative homotopy classes of the discs nor output dual spheres are required.

Theorem 1.1 · p. 2

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A marked tensor obstruction to four-dimensional disk embedding](#). External inputs use the following keys.

- [PTVV13](#) Tony Pantev, Bertrand Toën, Michel Vaquié, Gabriele Vezzosi — Shifted symplectic structures: Betti transgression over the sphere and derived Lagrangian intersection; checked in the numbered arXiv version. [arXiv:1111.3209v4](#), 22 April 2013.
- [Cal](#) Damien Calaque — Lagrangian structures on mapping stacks and semi-classical TFTs: Relative orientation supplies a Lagrangian boundary restriction. [arXiv:1306.3235v2](#), 7 November 2014.
- [BBJ](#) Christopher Brav, Vittoria Bussi, Dominic Joyce — A Darboux theorem for derived schemes with shifted symplectic structure: A minimal algebraic Darboux chart in shift -1 . [arXiv:1305.6302v4](#), 29 August 2018.
- [Artin69](#) Michael Artin — Algebraic approximation of structures over complete local rings: Approximate the final finite polynomial systems for graphs and witnesses while retaining first jets. *Publications Mathématiques de l’IHÉS* 36 (1969), 23–58; Numdam scan.
- [Getzler09](#) Ezra Getzler — Lie theory for nilpotent L_∞ -algebras: Nilpotent Maurer–Cartan comparison used in Appendix A; the derived-coefficient extension is proved in the manuscript. *Annals of Mathematics* 170 (2009), 271–301; publisher PDF.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — From a fixed disc problem to two vertices

Proof target : [Theorem 1.1](#) · [p. 2](#)

Proof route. Before choosing any hypothetical solution, construct an exterior E with finitely many grope relations and algebraic dual spheres. A solution preserving framings allows bottom-surface compression and smooth cuts producing two vertices A, B and $2m$ edges Y_i . The next section explains the separate algebraic edge potentials and markings; Section 4 explains why the resulting data cannot exist.

1 Fix the disc problem before choosing a solution

$E = X \setminus \bigcup_j \text{int } \nu(S_j)$, $\lambda(f_i, g_j) = \delta_{ij}$, $\lambda(g_i, g_j) = \tilde{\mu}(g_i) = 0$
 Assume disjoint replacements with the same boundary maps and induced normal framings.

Retained caps cancel group labels; cut away from the actual images

[Lemma 2.4 · pp. 11–13](#)

[Proposition 2.2; Lemmas 2.5–2.6 · pp. 7, 13–15](#)

2 Two vertices and active edges

$A, B, Y_1, \dots, Y_{2m}, \quad m \geq 5$

Framed compression gives smooth cuts and complementary internal tangents.

Connect Chern–Simons to the full relative trace comparison

[Proposition 4.5 · pp. 28–29](#)

[Theorem 5.1 · p. 30](#)

3 Marked algebraic edge models

$f_i = F_i \circ \phi_i, \quad F_i \text{ algebraic}$

Preserve based holonomy on the entire nonreduced critical scheme.

Fixed-residual corrections and Brunnian laws feed the finite tensor obstruction

[Proposition 6.1 · p. 37](#)

[Lemmas 6.5–6.7; Corollary 6.8 · pp. 44–47](#)

[Theorem 3.1 · p. 16](#)

4 The two triple laws are incompatible

$S_t(H_L) = S_0(H_R) = 0, \quad t_i t_j t_k \in J_L, \quad (b_i b_j b_k)(H_R) \in J_R$

Here J_L, J_R are full restricted gradient ideals; mixed parameter terms remain.

Overview from the fixed disc problem to the marked tensor obstruction

[Open figure ↗](#) [TeX source ↓](#)

1. Fix the input first and cut using a framed filling

Choose $m \geq 5$ and $p \geq 9$. Install triple-commutator-type grope words on three distinct ports for every finite independent choice of conjugations and signs. Each word has the Brunnian deletion property and cubic leading term $[Z_1, [Z_2, Z_3]]$. In the exterior E of the bottom-surface tubes, shallow caps give f_j and compressed rim tori give g_j . The unused upper caps kill label-comparison loops in $\pi_1(E)$: cancel the two inner cap sheets while holding the outer copy fixed. Individual embeddedness and product framings of the spheres also give the diagonal equations and $\tilde{\mu} = 0$.

[Lemma 2.4 · pp. 11–13](#)

Compress the bottom surfaces using hypothetical replacement caps. Their prescribed framings ensure the required locally flat compression and product collars. Regularize a map to the plumbing tree away from the actual disc images, obtaining connected walls and vertex regions. In the parity double-cover quotient, exactly m of the $2m$ edges joining A and B are active. One original coordinate direction on each boundary torus dies and the other survives; together with Mayer–Vietoris restriction isomorphisms, this gives complementary internal tangent spaces.

[Proposition 2.2; Lemmas 2.5–2.6 · pp. 7, 13–15](#)

2. Correct vertex graphs with their residual ideals fixed

Each edge's based connection slice has a central potential f_i and a square-zero boundary variation $f_i + t_i b_i^{\text{form}}$. Its internal Jacobian ideal equals the full curvature ideal. Fix vertex residual lists r_P and $I_P = (r_P)$ for $P = A, B$. Stokes and duality give

$$S_P(H_P) \in I_P^2, \quad (\partial_h S_P)(H_P) = M_P r_P,$$

where M_P has a unit full-column minor. Corrections by I_P -multiples make the values exactly zero while retaining the residual quotients and markings. Apply Theorem 5.1 separately to the edges, recenter the background, and only then pass to $K = \mathbb{C}((\sigma))$.

Lemmas 6.3–6.4 · pp. 41–43

3. Transport triple laws on the nonreduced quotients

Nine adjoint matrices span $\text{End}(\mathfrak{sl}_2 \otimes K)$. Independently conjugated Brunnian laws therefore detect every coefficient belonging to three distinct active edges. Higher terms lie in the maximal ideal times the relevant triple ideal. Nakayama gives $t_i t_j t_k \in I_A$ on side A and products of three distinct edge holonomy deviations in I_B on side B .

The holonomy logarithm itself is not claimed to be algebraic. Using algebraic matrix lifts G_i , replace the perturbations by

$$b_i = -\epsilon_i \text{tr} \left(T_i \left((G_i - 1) - \frac{1}{2} (G_i - 1)^2 \right) \right).$$

The error is at least cubic in the deviations. The relation $G_i(H_A) - 1 \in (I_A, t_i)$ and $t_i^2 = 0$ control the resulting value and gradient errors. A finite correction increasing powers of the full parameter ideal restores the zero graph. One does not impose $(t_1, \dots, t_m)^2 = 0$: mixed squarefree monomials remain. Proposition 6.1 now supplies every hypothesis of Theorem 3.1. Its contradiction proves that the fixed disc problem in E has no solution.

Proposition 6.1 · p. 37; Lemmas 6.5–6.7; Corollary 6.8 · pp. 44–47

3. Proof of Theorem 5.1 — Algebraization preserving critical-scheme markings

Proof target : Theorem 5.1 · p. 30

Theorem 5.1 — Algebraic edge potential with markings

Let Y be a compact connected oriented smooth three-manifold with connected torus boundary. Choose based coordinate circles α, β so that the image of $H^1(Y) \rightarrow H^1(\partial Y)$ evaluates on α and vanishes on β . For the central potential f and boundary complex C of Proposition 4.5, put $N = \dim(H^1(C) \otimes \mathfrak{sl}_2)$, $R = \mathbb{C}[[h_1, \dots, h_N]]$, and $J = \text{Jac}(f)$. There are an invertible formal coordinate change $x = \phi(h)$ and an algebraic power series $F \in (x)^3$ with $f = F \circ \phi$. For any fixed finite collection of based loops, every matrix entry of their holonomies is the pullback of an algebraic power series in the x coordinates on $\text{Spf}(R/J)$. This identification holds on the entire possibly nonreduced critical scheme, in particular for the based β -holonomy.

Theorem 5.1 · p. 30

1 A filling with sphere boundary

$Z = Y \cup_{T^2} (\text{punctured solid torus}), \quad \partial Z = S^2$
 $\mathcal{X} = \text{Loc}_{SL_2}(Z)^{\text{fr}}$

Transgression, relative orientation, Lagrangian intersection, and a minimal chart

[PTVV13, Theorem 2.5; Corollary 2.6(4), pp. 33, 37]

[Cal, Definition 2.6; Claim 2.7; Theorem 2.9, pp. 11–12]

[PTVV13, Theorem 2.9, p. 39]

[BBJ, Example 5.15; Theorem 5.18(i), pp. 34, 36]

2 Algebraic Darboux chart

$\mathcal{X} : (-1)\text{-shifted}, \quad F \in (x)^3$

Based holonomy matrix entries admit algebraic lifts from the critical quotient.

Transport the full relative trace, beyond the tangent pairing

Proposition 5.2; Lemma 5.3; Appendix A · pp. 32, 35, 48–52

3 Compare full closed-form classes

$f' = F \circ x, \quad J = \text{Jac}(f), \quad f' - f \in J^2$

Include the relative integration homotopy and use the completed Cartan calculation.

Use the formal flow of a vector field with coefficients in J

Lemma 5.4; proof of Theorem 5.1 · pp. 35–36

4 Remove the error while fixing markings

$\Phi \equiv \text{id} \pmod{J}, \quad (f + e) \circ \Phi = f, \quad f = F \circ \phi$

Holonomy is retained on the entire critical scheme.

Compare the shifted symplectic chart with the Chern–Simons potential

Open figure ↗ TeX source ↓

1. Obtain an algebraic critical chart from a punctured filling

Glue a solid torus minus an open ball to Y , taking α as the meridian, to obtain $\partial Z = S^2$.

Transgress the trace-pairing 2-shifted form on BG over the sphere to obtain a 0-shifted form.

Relative orientation and Poincaré–Lefschetz duality make $\mathrm{Loc}_G(Z) \rightarrow \mathrm{Loc}_G(S^2)$ Lagrangian. Its derived intersection with the based-sphere Lagrangian gives a (-1) -shifted form on $\mathrm{Loc}_G(Z)^{\mathrm{fr}}$.

Section 5.1 checks finite CW type, duality with perfect coefficients, and finite presentation. A minimal Darboux chart then supplies $F \in (x)^3$. Regular based holonomies lift from its critical quotient to the smooth chart.

[PTVV13, Theorem 2.5; Corollary 2.6(4), pp. 33, 37]

[Cal, Definition 2.6; Claim 2.7; Theorem 2.9, pp. 11–12]

[PTVV13, Theorem 2.9, p. 39]

[BBJ, Example 5.15; Theorem 5.18(i), pp. 34, 36]

2. Compare full relative closed forms to obtain a squared error

An isomorphism of critical schemes or agreement of tangent pairings alone does not identify the prescribed f . Proposition 5.2 and Appendix A carry the formal group nerve's trace cocycle $\frac{1}{2}\kappa(\delta a, \delta a)$ through derived coefficients, source restrictions, and relative integration, comparing the full closed-form class and the boundary isotropic homotopy. Rather than simply reinterpreting Getzler's ordinary nilpotent comparison on derived coefficients, Appendix A constructs that extension separately.

Consequently the Hamiltonian forms for the transferred Darboux potential f' and for f are homotopic as full closed forms. In the completed Cartan calculation, j contractions land in J^j and the remainder tends to arbitrarily high order. After normalizing constants, this gives $f - f' \in J^2$.

[Getzler09, Theorem 5.4, print p. 295 / PDF p. 27]

[Getzler09, Corollary 5.11, print p. 299 / PDF p. 31]

Proposition 5.2; Lemma 5.3; Appendix A · pp. 32, 35, 48–52

3. Use a flow that is the identity on the critical quotient

Write $e = f' - f \in J^2$ as $e = \sum v_i \partial_i f$ with $v_i \in J$. Since $f \in (h)^3$, the matrix D in $\nabla e = D\nabla f$ has entries in the maximal ideal. The formal flow of $X_t = -(1 + tD)^{-T}v$ for $f_t = f + te$ removes the error. Its coefficients lie in J , so the flow is the identity on R/J and preserves every previously identified holonomy marking. This proves both the identity of functions and marking preservation on the full critical scheme.

Lemma 5.4; proof of Theorem 5.1 · pp. 35–36

4. Proof of Theorem 3.1 — A finite tensor obstruction to the two triple laws

Proof target : Theorem 3.1 · p. 16

Theorem 3.1 — Finite tensor obstruction

Let K have characteristic zero, let $h = (h_1, \dots, h_n)$ be coordinate blocks of total dimension N , and let $I \subset \{1, \dots, n\}$ be an active set with $|I| = m \geq 5$. Take algebraic power series $F_i(h_i)$ for $1 \leq i \leq n$ and $b_i(h_i)$ for $i \in I$, each depending only on its own block, and set

$$T = K[t_i : i \in I] / (t_i^2 : i \in I), \quad S_0 = \sum_i F_i, \quad S_t = S_0 + \sum_{i \in I} t_i b_i.$$

For a coordinates x and $N - a$ coordinates y , there cannot exist $H_L \in (x, t)T[[x]]^N$ and $H_R \in (y)K[[y]]^N$ with injective complementary central tangents, $S_t(H_L) = S_0(H_R) = 0$, and, for every three distinct $i, j, k \in I$,

$$t_i t_j t_k \in ((\partial_{h_\nu} S_t)(H_L))_\nu, \quad (b_i b_j b_k)(H_R) \in ((\partial_{h_\nu} S_0)(H_R))_\nu.$$

Differentiate before substitution with external parameters fixed. These are actual gradient ideals, without passing to radicals.

Theorem 3.1 · p. 16

1 Separate algebraic functions and zero graphs

$$S_t = \sum_i F_i + \sum_{i \in I} t_i b_i, \quad t_i^2 = 0, \quad |I| \geq 5$$

$$S_t(H_L) = S_0(H_R) = 0, \quad T_0 H_L \oplus T_0 H_R = K^N$$

Approximate the final equations; specialize a finite coefficient ring

[\[Artin69, Theorem 1.10, print p. 26 / PDF p. 5\]](#)

[Lemma 3.2; Remark 3.3 · pp. 17–18](#)

2 Nonzero pairing in a finite Frobenius complex

$$\Omega_p = \Omega_{k[z]/(z_v^p)}^\bullet, \quad D = dS_0 \wedge, \quad \int U_0 \wedge V \neq 0$$

Use the target triple law and a kernel-preserving chain projection

[Lemmas 3.7–3.8 · pp. 21–22](#)

3 At least three suitable active factors

$$[V'] \in \bigotimes_i H_i, \quad B_i v_i = \delta_i v_i = 0 \text{ in at least } m - 2 \geq 3 \text{ slots}$$

$$\int U_0 \wedge V' = \int U_0 \wedge V$$

Pair the actual source-triple boundary with the lifts in the same complex

[§3.4; proof of Theorem 3.1 · pp. 22–23](#)

4 Every tensor pairing vanishes

$$(D_i + t_i db_i \wedge)(a_i + t_i w_i) = 0$$

$$t_i t_j t_k P(t) = 0 \implies P(0) = 0$$

Summing over tensors contradicts the nonzero pairing.

Nonzero pairing of transverse graphs versus three-parameter lifts

[Open figure ↗](#) [TeX source ↓](#)

1. Approximate and specialize only the final equations

Fix F_i, b_i and expand graph coordinates and membership witnesses in the squarefree parameter basis. Apply Artin approximation to separate finite polynomial systems on the two sides, retaining first jets. No equations differentiate unknown graph functions, and no nested approximation is needed. Descend finite étale presentations and inverse units to one finitely generated coefficient ring and specialize to large positive characteristic. If $K = \mathbb{C}((\sigma))$, treat each coefficient appearing in these finite presentations as a single element; do not reduce Laurent series term by term.

[\[Artin69, Theorem 1.10, print p. 26 / PDF p. 5\]](#)

[Lemma 3.2; Remark 3.3 · pp. 17–18](#)

2. Use two homologies to obtain three suitable factors

In characteristic $p > 2$, let Ω_p be the forms over the coordinate Frobenius quotient $k[z]/(z_v^p)$ and put $D = dS_0 \wedge$. Normal cycles U_0, V of the transverse graphs have nonzero pairing. Block separation gives $H(\Omega_p, D) = \bigotimes_i H_i$. Let δ_i be induced by ordinary exterior differentiation and B_i be multiplication by b_i .

The target triple law puts $[V]$ in a sum of tensors with all but at most two active factors in $\ker B_i$. A chain projection for δ replaces it, without changing the pairing, by a sum with at least $m - 2 \geq 3$ factors in $\ker B_i \cap \ker \delta_i$. The actual representing forms need not be d -closed.

Lemmas 3.7–3.8 · pp. 21–22

3. Lift each tensor over three parameters and kill its constant pairing

On the three selected factors, $[\delta_i, B_i] = [db_i \wedge]$ acts trivially, so solve $D_i w_i = -db_i \wedge a_i$. Tensor the lifts $a_i + t_i w_i$ to obtain an actual cycle over $k[t_i, t_j, t_k]/(t_i^2, t_j^2, t_k^2)$. The product includes mixed terms.

The source triple law supplies an actual boundary identity in the same complex. Pairing gives $t_i t_j t_k P(t) = 0$, hence $P(0) = 0$ by the squarefree basis. Choose a suitable triple separately for each tensor and sum: the original nonzero pairing must vanish. The argument does not interchange homology with a nonflat coefficient specialization.

§3.4; proof of Theorem 3.1 · pp. 22–23

5. Which papers supply which steps

PTVV transgression and Lagrangian intersection, Calaque's relative orientation, and the BBJ Darboux chart supply the algebraic model in Theorem 5.1. The full relative trace comparison with the manuscript's potential, the squared-Jacobian error, and the flow fixing markings are internal steps. Artin approximation enters Theorem 3.1 only after this algebraization and the vertex corrections.

The boundary-only manuscript in this catalog asserts nonfilling of boundary images without prescribing output framings. The present theorem excludes replacements with prescribed boundary maps and induced framings, again without prescribing relative homotopy classes. Their configurations and algebraization methods differ; neither is treated here as a proof input for the other. This manuscript makes no separate counterexample claim for ordinary or simple surgery or five-dimensional s -cobordism.

Remark 7.1 · p. 48

6. Return to the sources

- Lemma 2.4 · pp. 11–13 — Read exterior group-ring labels and framings
- Theorem 5.1 · p. 30 — Check algebraization and the markings it preserves
- Proposition 5.2; Lemma 5.3; Appendix A · pp. 32, 35, 48–52 — Track the scope of the full relative closed-form comparison
- Proposition 6.1 · p. 37; Lemmas 6.5–6.7; Corollary 6.8 · pp. 44–47 — Read how the two vertices supply triple memberships
- Theorem 3.1 · p. 16 — Check every hypothesis of the final obstruction
- Remark 7.1 · p. 48 — Check the output conditions and the stated surgery scope

The reading covered the main theorem; framings, group-ring labels, and cuts in §2; approximation, the two homologies, and lifts in §3; full gradient ideals and algebraization in §§4–5; Appendix A's relative trace comparison; residual corrections and triple laws in §6; and the conclusion in §7. The statements and application conditions were checked for PTVV (numbered arXiv v4), Calaque, BBJ, Artin Theorem 1.10, and Getzler Theorem 5.4 and Corollary 5.11. Dupont's original text and other Getzler inputs were not rechecked here. This does not independently verify every cochain transfer, sign in the derived comparison, or local plumbing operation; complete proofs of external results are also outside the checking scope.

305 / 03

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CATALOGUE 305 · PAPER 03 / 3 Overview

A PD4 group without an aspherical manifold model

Transporting a marked filling obstruction to closed manifold nonrealization through reflection and stabilized sphere removal

The manuscript constructs an oriented integral Poincaré duality group with a finite four-dimensional classifying space and asserts that it is not the fundamental group of any closed aspherical topological four-manifold. Its starting point is a Poincaré pair of free-group graph type whose prescribed boundary map admits no manifold filling.

Manuscript author: OpenAI · September 24, 2026 · 32 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A PD4 group with finite BG and no manifold model

There exist a finitely presented group G and a finite, oriented, four-dimensional aspherical Poincaré complex Q with $\pi_1 Q = G$ such that

$$H^i(G; \mathbb{Z}G) = 0 \quad (i \neq 4), \quad H^4(G; \mathbb{Z}G) \cong \mathbb{Z},$$

with trivial right G -action on the last group, but no closed aspherical topological four-manifold has fundamental group G . In particular, the trivial left $\mathbb{Z}G$ -module $\mathbb{Z}$ has a finite resolution by finitely generated free modules.

Theorem 1.1 · pp. 1–2

This is an existence assertion in dimension four, not a counterexample in every dimension. The manuscript's conclusion is distinct from the editorial checking scope stated below.

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to A PD4 group without an aspherical manifold model. External inputs use the following keys.

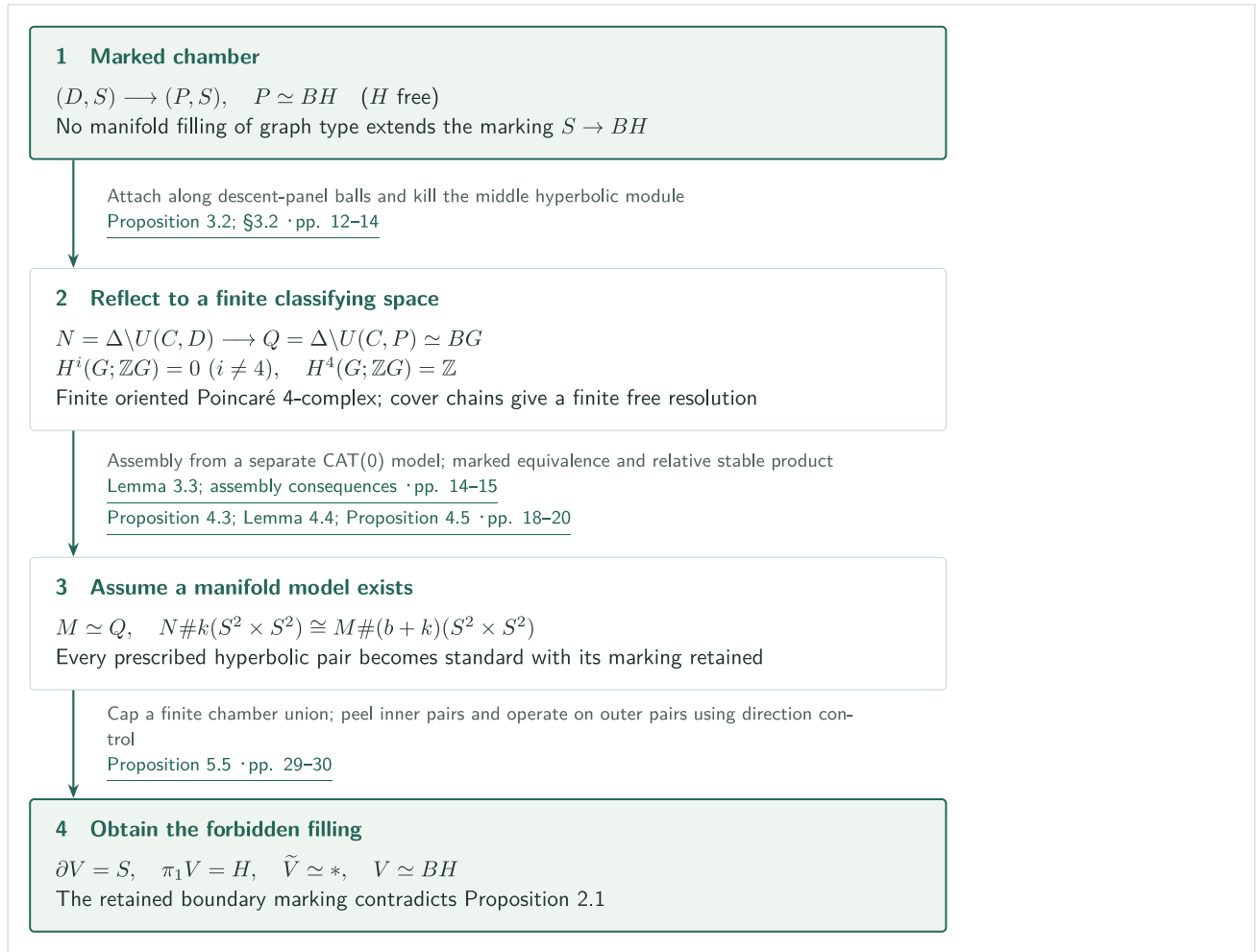
- [OpenAI26b](#) OpenAI — A marked tensor obstruction to four-dimensional disk embedding: Marked geometry, algebraic potentials, and the finite tensor obstruction. September 24, 2026.
- [KL22](#) Daniel Kasprowski and Markus Land — Topological 4-manifolds with 4-dimensional fundamental group: Normal-invariant removal and s-cobordism. arXiv:2007.03399v3.
- [PRT25](#) Mark Powell, Arunima Ray, and Peter Teichner — The 4-dimensional disc embedding theorem and dual spheres: Towers and dual spheres after obtaining null loops. Selecta Mathematica 31 (2025), article 80; published PDF.
- [FQ13](#) Michael H. Freedman and Frank Quinn — Topology of 4-manifolds: Two-phase separation of small capped gropes. January 2013 reformatted, unrevised edition.
- [Qui83](#) Frank Quinn — The stable topology of 4-manifolds: Relative stable product; original text not checked. Topology and its Applications 15 (1983), 71–77; full text not acquired.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — Finite BG and a reverse reduction from a hypothetical manifold

Proof target : [Theorem 1.1](#) · pp. 1–2

Proof route. Reflect the [marked chamber](#) to construct $Q \simeq BG$. A hypothetical manifold model M gives a stable product preserving all prescribed hyperbolic pairs and hence a sphere problem in finitely many chambers. [Direction control](#) and [sphere surgery](#) remove its middle homology and produce the forbidden boundary filling.



Finite classifying space and manifold nonrealization

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1. Reflection supplies finiteness, asphericity, and integral duality separately

A finite flag triangulation of $S = \partial D$ defines a right-angled Coxeter group C . Taking the finite chamber quotient by $\Delta = \ker(C \rightarrow (\mathbb{Z}/2)^V)$ gives a closed oriented four-manifold $N = \Delta \backslash U(C, D)$ and a finite complex $Q = \Delta \backslash U(C, P)$. In the reflected development using the universal cover of P , each new chamber attaches along a ball of descent panels. The resulting universal cover is contractible.

Meanwhile, Q is obtained from N by attaching three-cells along its entire prescribed middle sphere basis. Nonsingularity of the hyperbolic form lets the cap-product map descend to the quotient by the middle homology summand and its cohomological annihilator. This gives four-dimensional Poincaré duality. The orientation-preserving action gives the trivial right action on $H^4(G; \mathbb{Z}G)$. The cellular chains of the finite $K(G, 1)$ supply the finite free resolution, and its finiteness also supplies finite presentability.

Proposition 3.2; §3.2 · pp. 12–14

2. Stabilization must retain the specified sphere marking

The dualizing module makes the hypothetical $M \simeq Q$ orientable. For $L = M \#^b(S^2 \times S^2)$, the modules $\pi_2 N$ and $\pi_2 L$ carry the same prescribed hyperbolic form. In the common second Postnikov stage $EG \times_G K(A, 2)$, integral cup-product tests identify the fundamental classes and yield a homotopy equivalence inducing this **particular** module isomorphism.

Separately, a CAT(0) model over a support tree supplies Farrell–Jones assembly. It does not directly provide four-dimensional disk embedding. Kasprowski–Land's normal-invariant calculation gives an s -cobordism; Quinn's relative stable product then makes all sphere pairs, including the added pairs, geometrically standard. A stable homeomorphism of the ends alone would not preserve the chamber labels needed for the subsequent removals.

Lemma 3.3; assembly consequences · pp. 14–15

[KL22, Lemma 2.3; Theorem 3.1; Lemma 3.3, pp. 5–7]

Proposition 4.3; Lemma 4.4; Proposition 4.5 · pp. 18–20

3. The hypothetical model produces the prohibited filling

Remove the near-central standard pairs in the periodically lifted system and cap the noncentral boundary factors by simply connected complementary fillings. Direction control solves the remaining sphere problem and produces V with $\partial V = S$, $\pi_1 V = H$, and contractible $\tilde{V}$, retaining the original map $S \rightarrow BH$. This contradicts Proposition 2.1(3). The preservation of that marking is what turns the construction into a nonrealization argument.

Proposition 5.5 · pp. 29–30

3. Proof of Proposition 2.1 — Excluding fillings with the prescribed boundary marking

Proof target : Proposition 2.1 · p. 4

Proposition 2.1 — The marked chamber

There exist a compact connected oriented smooth four-manifold D with connected boundary S , a finitely generated free group H , and a map $c : D \rightarrow BH$ inducing an isomorphism on fundamental groups. The space D has the homotopy type of a finite graph with $2k$ two-spheres attached and has a specified hyperbolic $\mathbb{Z}H$ -basis of $\pi_2 D$, represented by framed immersed spheres with vanishing Wall self-intersection obstructions. Attaching three-cells along the entire basis gives $P \simeq BH$, and (P, S) is an oriented Poincaré pair of formal dimension four. There is no compact oriented topological four-manifold D' with identified boundary S and a homotopy equivalence $D' \rightarrow BH$ restricting to a map homotopic to $c|_S$.

Proposition 2.1 · p. 4

1 Geometry with recorded conjugating paths

Marked grope bodies, exterior disks f_j , and dual spheres g_j
 $\lambda(f_i, g_j) = \delta_{ij}, \quad \lambda(g_i, g_j) = 0$

Homotopy pushout retaining tube sides and the intersection-form calculation
[\[OpenAI26b, Lemmas 2.3–2.4, pp. 9–12\]](#)
[Lemma 2.2; §§2.2–2.3 · pp. 6–8](#)

2 Attach compression bodies and kill the basis by cells

$D = E \cup \coprod_{F_j \times S^1} \coprod (V_j \times S^1), \quad P \simeq BH$
 Prescribed hyperbolic basis of $\pi_2 D$; Poincaré duality for (P, S)

The marking retains inverse-surgery cocores and every boundary law
[§2.4; Lemma 2.4 · pp. 8–10](#)

3 Cut a hypothetical topological filling

$D' \simeq BH \implies W \implies A, B, Y_i$
 Punctured smoothing; a form algebra computing filled cohomology

All conjugated laws produce Vertex data excluded by the finite tensor theorem
[\[OpenAI26b, Proposition 6.1, p. 37\]](#)
[\[OpenAI26b, Theorem 3.1, p. 16\]](#)

4 Impossible tensor input

$T = K[t_i]/(t_i^2), \quad S_t(H_L) = S_0(H_R) = 0$
 $t_i t_j t_k \in (\partial S_t)(H_L), \quad (b_i b_j b_k)(H_R) \in (\partial S_0)(H_R)$
 Complementary tangents, algebraic potentials, $m \geq 5$; actual ideals, not radicals

The marked chamber and the tensor obstruction

Open figure ↗ TeX source ↓

1. The companion supplies more than its nonembedding theorem

The inputs are finitely many grope bodies with independent conjugating paths and algebraic dual spheres in their exterior. Remove tubes of the lower punctured tori F_j , obtaining E , and attach $V_j \times S^1$, where V_j compresses F_j to a disk. The homotopy-pushout identification includes the actual tube sides; it adds one free generator and two two-spheres for each F_j . The dual-sphere intersection calculation supplies a hyperbolic basis. Killing the whole basis with three-cells gives $P \simeq BH$. No embedded sphere surgery is used at this stage.

[OpenAI26b, Lemmas 2.3–2.4, pp. 9–12]

Lemma 2.2; §§2.2–2.3 · pp. 6–8

2. A topological filling produces cut data

Given D' , attach the two-handles undoing the boundary surgery to obtain W with the original graph type and boundary map. Their cocores kill the required boundary words by embedded disks. Quinn's punctured smoothing permits regular levels of a tree-valued map away from the disks and punctures, giving connected vertex pieces and cuts with torus boundary.

The topological vertices cannot simply be treated as smooth. Lemma 2.4 uses positive-degree forms vanishing near the punctures and degree-zero functions separately constant near each puncture. This algebra computes the cohomology of the **filled space** and preserves coefficientwise Stokes and based-holonomy calculations. This additional step transfers the companion's smooth vertex argument to arbitrary topological fillings.

§2.4; Lemma 2.4 · pp. 8–10

3. Mixed nilpotent terms and actual gradient ideals carry the contradiction

Choose $m \geq 5$ and $p \geq 9$ in the construction. Mayer–Vietoris in the parity cover makes the degree-two difference-of-restrictions map an isomorphism; the degree-one map is an isomorphism after setting extra-loop evaluations to zero. The surviving peripheral directions determine $m \geq 5$ active edges. All independently conjugated boundary laws enter the companion's Vertex data construction, yielding

$$T = K[t_i : i \in I]/(t_i^2), \quad S_0 = \sum F_i, \quad S_t = S_0 + \sum_{i \in I} t_i b_i$$

and formal graphs H_L, H_R with complementary tangent spaces. They satisfy

$S_t(H_L) = S_0(H_R) = 0$ and, for every distinct active triple,

$$t_i t_j t_k \in ((\partial_{h_\nu} S_t)(H_L))_\nu, \quad (b_i b_j b_k)(H_R) \in ((\partial_{h_\nu} S_0)(H_R))_\nu.$$

K has characteristic zero (it may be $\mathbb{C}((\sigma))$), and each $F_i(h_i), b_i(h_i)$ depends only on the separate coordinate block for edge i . This separated algebraicity, retention of mixed square-free terms in T , and membership in actual gradient ideals rather than their radicals are exactly the hypotheses needed by the Finite tensor obstruction. These data therefore cannot exist. The argument extracts constructions, potentials, algebraization, and the tensor theorem from the marked paper, rather than applying its main nonembedding theorem directly.

[OpenAI26b, Proposition 6.1, p. 37]

[OpenAI26b, Theorem 3.1, p. 16]

4. Proof of Proposition 5.5 — Sphere surgery after direction control

Proof target : Proposition 5.5 · pp. 29–30

Proposition 5.5 — Removing the remaining hyperbolic planes

The compact manifold $Z = Z_{R,L}$ constructed in §5 has $\partial Z = S$, $\pi_1 Z = H$, and universal-cover reduced homology concentrated in degree two, freely generated by the prescribed hyperbolic pairs with chamber labels $R < |v| \leq L$. Interior surgeries on this Z produce a compact oriented topological four-manifold V with boundary S , fundamental group H , and contractible universal cover. Its boundary map $S \rightarrow BH$ is the original central chamber marking.

Proposition 5.5 · pp. 29–30

1 Hyperbolic planes remaining outside

$\partial Z = S$, $\pi_1 Z = H$, $H_2(\tilde{Z}) = \bigoplus_{R < |v| \leq L}$ marked hyperbolic planes
 Central boundary marking retained; other reduced homology vanishes

Use the original pair marking and chamber group-ring pairings
[§5.1, Equation \(20\) · pp. 21–23](#)
[Lemma 5.2 · pp. 23–26](#)

2 Uniform propagation during preparation

Geometric duals, height-eight Whitney gropes, protected Clifford duals
 Finite templates give a propagation bound independent of R

Small radial directions far out and identification of the central-factor map
[Lemmas 5.3–5.4 · pp. 26–28](#)
[\[FQ13, Lemma 2.10.1; §2.10.2, print pp. 39–40 / PDF pp. 53–54\]](#)

3 Put each whole loop system in one patch

Separate finitely many collections, then upper components inside their neighborhoods
 Entire direction image in a contractible patch $\Rightarrow$ cap loops are null

Enter the null-loop stage after the use of goodness; no later diameter control is required
[\[PRT25, Lemma 4.2, proof, pp. 17–18\]](#)
[Proposition 5.5 · pp. 29–30](#)

4 From towers to sphere surgery

$\pi_1(M') \cong \pi_1 Z$; $H_i(V; \mathbb{Z}H) = 0$ ($i > 0$)
 Duals preserve the group; the acyclic universal cover is contractible
 The equivalence $V \simeq BH$ extends the original boundary marking

Direction control and a marked aspherical filling

[Open figure ↗](#) [TeX source ↓](#)

1. The removed-pair marking identifies the surviving homology

Capping the noncentral boundary factors of a finite chamber union changes the free product of chamber groups to its projection onto the central factor H . Replace the standard $S^2 \times S^2$ neighborhoods labelled by $|v| \leq R$ with four-balls. Including their based homotopies to the original chamber pairs in the finite region ensures that exactly the near-central labelled summands are deleted. This gives the surviving hyperbolic basis in Equation (20).

[§5.1, Equation \(20\) · pp. 21–23](#)

2. Keep the entire loop system in one direction patch

Lemma 5.2 prepares the outer sphere system with geometric duals and high Whitney gropes, with propagation independent of R . Radial directions of bounded-diameter sets far out in the Davis complex become uniformly small. Lemma 5.3 identifies the homomorphism induced by the direction map followed by $S \rightarrow BH$ with the central-factor map. Consequently a loop system whose **entire image lies in one contractible patch** is nullhomotopic in Z .

Lemma 5.4 first separates collections assigned to finitely many patches, then separates their individual upper components within each collection's neighborhood. The second phase need not bound each cap's gallery diameter uniformly. It preserves containment of the whole image in a single contractible patch, so even loops traversing many intersections are null.

Lemma 5.2 · pp. 23–26

[FQ13, Lemma 2.10.1; §2.10.2, print pp. 39–40 / PDF pp. 53–54]

Lemmas 5.3–5.4 · pp. 26–28

3. Replace precisely the goodness-dependent step and then construct towers

Geometric duals kill meridians, so inclusion of the complement M' of plumbed neighborhoods of the a_i into Z induces a fundamental-group isomorphism. Cap double-point loops are therefore nullhomotopic in M' as well. The manuscript then enters the proof of Powell–Ray–Teichner's Lemma 4.2 after its use of goodness. The protected Clifford capped surfaces must also remain geometric duals. The argument neither assumes nor proves goodness of the entire free group H .

From this point onward, null homotopies, tower caps, and final Whitney disks may range throughout Z . Surgery on the resulting framed embedded Lagrangian spheres preserves the fundamental group because of their dual spheres. Computing the surgery trace from both ends gives $H_i(V; \mathbb{Z}H) = 0$ for $i > 0$. The simply connected acyclic universal cover has CW homotopy type, so it is contractible and $V \simeq BH$. All surgeries are interior, preserving the boundary marking.

[PRT25, Lemma 4.2, proof, pp. 17–18]

§5.5, Equation (22) · pp. 28–29

Proposition 5.5 · pp. 29–30

5. Which papers supply which steps

Source and result

A marked tensor obstruction: Lemmas 2.3–2.4, Proposition 4.5, Theorem 5.1, Propositions 5.2 and 6.1, Theorem 3.1

Used at

Marked chamber

Role and required hypotheses

Marked bodies and exterior duals, markings on nonreduced critical schemes, algebraicity, gradient-ideal memberships for every active triple

Checking scope

Input statements and their uses in §§2.1 and 2.4 compared; the companion's full proofs were not independently verified

Source and result

Kasprowski–Land; Quinn

Used at

Marked stable realization

Role and required hypotheses

Almost-spin manifolds, Farrell–Jones, the specified normal invariant, and a relative product

Checking scope

The manuscript's hypotheses and uses were read; external-source checking is recorded below

Source and result

Freedman–Quinn; Powell–Ray–Teichner

Used at

Direction control and towers

Role and required hypotheses

Separation of small upper gropes, null double-point loops, and protected geometric duals

Checking scope

The manuscript's replacement null-loop step and subsequent input conditions were read; this is not verification of the full external proofs

The direct input within the catalogue consists of constructions and intermediate results from the marked tensor paper. No direct input from the boundary-only paper occurs in the citations and proof passages examined here.

6. Return to the sources

- Theorem 1.1 · pp. 1–2 — Hypotheses, quantifiers, and all conclusions
- Lemma 2.2; §§2.2–2.3 · pp. 6–8 — Construction retaining tube sides and duality
- §2.4; Lemma 2.4 · pp. 8–10 — Transfer from topological fillings to tensor inputs
- Proposition 3.2; §3.2 · pp. 12–14 — Finite BG and integral duality
- Proposition 4.3; Lemma 4.4; Proposition 4.5 · pp. 18–20 — Stabilization preserving the specified sphere marking
- Lemmas 5.3–5.4 · pp. 26–28 — Direction control of entire loop systems
- §5.5, Equation (22) · pp. 28–29 — Passage from null loops to towers
- Proposition 5.5 · pp. 29–30 — Surgery trace and contractibility

The passages read include Theorem 1.1, Proposition 2.1, reflection and duality in §3, preservation of markings in §4, and the filling, preparation, direction, tower, and surgery arguments in §5. Statements of direct inputs from the marked tensor paper were compared with their uses here. Particular checkpoints were the passage to topological vertices, integral markings, entire cap loop systems, and the final regular-coefficient surgery calculation.

External passages checked include Kasprowski–Land’s Lemmas 2.3 and 3.3 and Theorem 3.1, the statement and relevant proof steps of Powell–Ray–Teichner’s Lemma 4.2, and Freedman–Quinn’s Lemma 2.10.1 and §2.10.2. The original text of Quinn’s stable product theorem was not acquired; only its presentation and application in §4.2 were read. That external connection remains unchecked.

For the external $\text{CAT}(0)$, Farrell–Jones, and punctured-smoothing inputs, their presentation and use in this manuscript were read, but their statements in the cited original sources were not checked in this pass.

The complete external proofs concerning $\text{CAT}(0)$ geometry, assembly, smoothing, stable products, and disk embedding, as well as the marked paper's full algebraization and finite tensor proofs, were not independently verified. Nor does this constitute a complete independent verification of uniform finite-template control or every geometric operation from the marked stable product to the final sphere surgeries. This article explains the manuscript's proof architecture and input–application relations; it is not an editorial certification of a solution to Wall's problem.