

OpenAI Math Digest

CATALOGUE 287

All nonabelian free group factors are isomorphic

Catalogue guide & one proof overview

Independent reading guide · English edition

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Contents

The catalogue guide is followed by the article. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

All nonabelian free group factors are isomorphic

This single-paper catalogue follows the construction of small perturbations of free Haar generators and a generating limit, then the amplification step yielding $L(\mathbb{F}_2) \cong L(\mathbb{F}_3)$. It also covers the consequences for countably infinite rank, fundamental groups, and the dependence of free entropy dimensions on generators.

1. Paper in this catalogue

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 287 · 2026-10-15 checked source snapshot

#

01

Manuscript and overview

An isomorphism of the free group factors →

Short name: An isomorphism of the free group factors

September 23, 2026 · 23 pages

Original paper (GitHub) ↗

Principal claim

Consecutive ranks $n \geq 3$, ranks two and three, and all interpolated parameters $1 < r \leq \infty$ are isomorphic. Fundamental groups are $\mathbb{R}_{>0}$; four free entropy dimensions depend on generators.

Theorems 1.1–1.2 · pp. 1–2; Corollaries 6.1, 7.1 · pp. 21–22 ↗

Role and available overview

Small cocycle → tracial flow → generating limit with adaptive error budgets; followed by amplification, dichotomy, and transport of tuples. Bilingual account and self-check prepared.

Read the proof overview →

2. External inputs

The passage to rank two uses Dykema's amplification formula; inclusion of the infinite parameter uses the classical dichotomy. For entropy dimensions, semicircular tuples and algebraic group-algebra generators are transported separately. The [article's inputs and applications](#) distinguish statement-and-application checks from complete proof verification.

3. Sources and checking scope

The official overview title is All nonabelian free group factors are isomorphic; CONTENTS uses Isomorphism of the free group factors. Entry 287 belongs to Operator algebras and appears on overview print/PDF p.31. The source commit is 1581d4cbdd69a0faf34b2089614fe2a814316a6a; the included manuscript is by OpenAI, dated September 23, 2026, and has 23 pages.

OpenAI Research Catalog · Catalogue 287, Operator algebras · p. 31

The author of this account read the central passages of §§2–7 and performed a self-check, not an independent review. Complete proofs of external inputs and further dependencies were not verified. Rădulescu's dichotomy was checked through Goldbring–Pi's explicit restatement; the Brannan et al. input was checked in author preprint v2 §5. See the [source reading guide and unresolved scope](#).

287 / 01

[Read on the website ↗](#)

CATALOGUE 287 · PAPER 01 / 1 Overview

An isomorphism of the free group factors

Absorb an extra free generator using a small cocycle, then reach rank two by normalized amplification.

The manuscript constructs a freely generating Haar n -tuple inside $L(\mathbb{F}_{n+1})$ for $n \geq 3$. An iteration of small automorphisms establishes the limiting distribution and generation separately; interpolation enters afterwards through amplification. This account checks the manuscript's argument and direct inputs against source passages. It does not constitute independent referee certification.

Manuscript author: OpenAI · September 23, 2026 · 23 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Consecutive ranks

For every integer $n \geq 3$, there is a unital trace-preserving normal $*$ -isomorphism

$$L(\mathbb{F}_n) \cong L(\mathbb{F}_{n+1}).$$

The factors carry their canonical normalized traces.

[Theorem 1.1 · p. 1](#)

Theorem 1.2 — Ranks two and three

There exists a unital normal trace-preserving $*$ -isomorphism

$$\Phi : L(\mathbb{F}_2) \longrightarrow L(\mathbb{F}_3).$$

[Theorem 1.2 · p. 2](#)

Corollary 6.1 — The interpolated family and fundamental groups

For every $1 < r, s \leq \infty$, there is a unital normal trace-preserving $*$ -isomorphism $L(\mathbb{F}_r) \cong L(\mathbb{F}_s)$. Moreover,

$$\mathcal{F}(L(\mathbb{F}_r)) = \mathbb{R}_{>0} \quad (1 < r \leq \infty).$$

Finite noninteger parameters denote interpolated free group factors; ∞ denotes countably infinite rank. The parameter $r = 1$ is excluded.

Corollary 6.1 · p. 21

Corollary 7.1 — Dependence of free entropy dimensions on generators

Let $M = L(\mathbb{F}_2)$ with its canonical trace. For every integer $n \geq 2$, there are a self-adjoint n -tuple $X^{(n)}$ and a self-adjoint $2n$ -tuple $Y^{(n)}$ such that

$$\begin{aligned} W^*(X^{(n)}) &= W^*(Y^{(n)}) = M, \\ \delta(X^{(n)}) &= \delta_0(X^{(n)}) = n, \quad \delta^*(Y^{(n)}) = \delta^*(Y^{(n)}) = n. \end{aligned}$$

Consequently none of the four dimensions is invariant under replacing a finite self-adjoint W^* -generating tuple, and each is unbounded on generating tuples of this single factor. The two tuple families need not coincide.

Corollary 7.1 · pp. 21–22

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [An isomorphism of the free group factors](#). External inputs use the following keys.

- [Dyk94](#) Ken Dykema — Interpolated free group factors: Normalized amplification formula, used in Theorem 1.2.. Pacific Journal of Mathematics 163 (1994), 123–135; publisher PDF with cover.
- [GP25](#) Isaac Goldbring and Jennifer Pi — On the first-order free group factor alternative: Explicit restatement of the classical isomorphism dichotomy including the infinite parameter, used in Corollary 6.1.. arXiv:2305.08168v2, 2 January 2025.
- [V90](#) Dan Voiculescu — Circular and semicircular systems and free product factors: Semicircular realization of free group factors, used in Corollary 7.1.. Progress in Mathematics 92 (1990), 45–60; author-hosted two-page scan.
- [V01](#) Dan Voiculescu — Free entropy: Microstates dimension $\delta = n$ for a free semicircular family.. arXiv:math/0103168v1, 26 March 2001.

- [BEHY23](#) Michael Brannan, Floris Elzinga, Samuel J. Harris, and Makoto Yamashita — Crossed product equivalence of quantum automorphism groups of finite dimensional C_* -algebras: The opening of §5 states $\delta_0=n$ for free semicircular tuples; checked in author preprint v2 rather than the journal layout.. arXiv:2202.04714v2, 21 February 2023; §5 corresponding to the cited published article.
- [MS05](#) Igor Mineyev and Dimitri Shlyakhtenko — Non-microstates free entropy dimension for groups: Computes δ^* and δ from L^2 -Betti numbers for self-adjoint algebraic generators of a group algebra.. Geometric and Functional Analysis 15 (2005), 476–490; published PDF.
- [CS03](#) Alain Connes and Dimitri Shlyakhtenko — L^2 -homology for von Neumann algebras: Free-group L^2 -Betti numbers $\beta_0=0$ and $\beta_1=n-1$.. arXiv:math/0309343v1, 20 September 2003.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — Small perturbations and a generating limit

Proof target : [Theorem 1.1](#) · [p. 1](#)

Proof route. Start with a freely generating Haar tuple $(A_1, \dots, A_n, C)$ in $M = L(\mathbb{F}_{n+1})$. A flow driven by conjugates of $S = \arg C$ turns a small group cocycle into a small automorphism. A word in an almost unchanged n -tuple then approximates the old generator C . Finally, an adaptive budget protects each approximation against subsequent perturbations. The diagram covers the entire route for this theorem; no interpolation is used here.

1 Choose finite real coefficients

$D_{x_1} = h, D_{x_j} = 0 \ (j \geq 2), \ \|h\|_{\ell^2}, \|D_w - \delta_e\|_{\ell^2} < \eta$
 Free-prefix Catalan limit $\longrightarrow$ cutoff inverse at low spectrum

Formal trace identity and analyticity; descend relations and extend normally
Lemmas 3.2–3.4 · pp. 7–10
Section 3.3 · pp. 10–11

2 Automorphisms preserving moments

$\beta_t(S) = S, \ \beta_t \in \text{Aut}(M)$
 $\|\beta_1(A_j) - A_j\| \leq 3\pi\|h_j\|_{\ell^2}, \ \|\beta_1(A_w) - CA_w\| \leq 3\pi\|D_w - \delta_e\|_{\ell^2}$

Combine the small cocycle with a free-basis change
Lemma 4.1; Sections 4.1–4.3 · pp. 11–16
Proposition 5.1 · pp. 16–17

3 Absorb the old completion

$\gamma(A_j) = A_j, \ \gamma(C) = CA_w, \ \alpha = \gamma^{-1}\beta_1$
 $\max_j \|\alpha(A_j) - A_j\| < \varepsilon, \ \|\alpha(A_w) - C\| < \varepsilon$

Current $F_{j,k}$ uses the old $C^{(k-1)}$; word length affects the future budget
Section 5.2; equations (5.3)–(5.9) · pp. 17–19

4 Choose the future budget after the word

$F_{j,k} \longrightarrow \varepsilon_k \longrightarrow w_k \longrightarrow G_{j,k}(X) = F_{j,k}(X, X_{w_k})$
 $L_k \longrightarrow r_k \leq \min\{r_{k-1}/2, 2^{-k}/(1 + L_k)\}$

Tail control, conditional expectation, and spatial identification of a Haar tuple
Section 5.2; equations (5.3)–(5.9) · pp. 17–19
Lemma 2.2 · p. 4

5 Secure Haar distribution and generation separately

$A_j^{(k)} \xrightarrow{\|\cdot\|} A_j^{(\infty)}, \ \|G_{j,k}(A^{(\infty)}) - y_j\|_2 \leq 3 \cdot 2^{-k}$
 $W^*(A^{(\infty)}) = M, \ L(\mathbb{F}_n) \cong L(\mathbb{F}_{n+1})$

From small coefficients to a limit with both Haar distribution and generation

[Open figure ↗](#) [TeX source ↓](#)

1. Descend the formal flow to normal automorphisms

Put $\Gamma = \mathbb{F}_n$, $S_g = A_g S A_g^*$, and $s(h) = \sum_g h(g) S_g$. Bounded Borel functional calculus gives S with $e^{iS} = C$, $\tau(S) = 0$, and $\|S\| \leq \pi$. For finitely supported real h_j , define

$$D_{x_j} = h_j, \quad D_{gb} = D_g + \lambda(g) D_b,$$

and solve the polynomial ODE in $B = C^*(A_1, \dots, A_n, S)$:

$$A'_j(t) = i \sum_g h_j(g) A_g(t) S A_g(t)^* A_j(t).$$

Faithfulness of the finite trace keeps the solution unitary. Its bounded derivative and completeness yield global solutions in both time directions.

Freeness at time t is not assumed. On the formal algebra of unitaries and one self-adjoint symbol, the manuscript proves $\tau \circ \text{ev}_0 \circ \partial = 0$ and applies this to every iterated derivative. A nonidentity group label forces zero trace. Identity-label monomials reduce to polynomials in the free conjugate components S_g ; the first variation conjugates one component by a unitary in the other components. Bounds on polynomial degree and coefficient sums establish real analyticity of actual solution moments. Their positive-order derivatives vanish initially, so the moments remain constant on the entire real line.

Lemmas 3.2–3.4 · pp. 7–10

Define $\varphi_t(\text{ev}_0 p) = \text{ev}_t p$. Applying moment preservation to p^*p and using faithfulness preserves the evaluation kernel, making the map well-defined. The formula

$$\|b\| = \lim_{k \rightarrow \infty} \tau((b^*b)^k)^{1/(2k)}$$

then gives isometry and injectivity. Solving the same ODE backwards inside

$B_t = C^*(A_1(t), \dots, A_n(t), S)$ proves $B_t = B$, hence surjectivity. The tracial isometry extends to a unitary on $L^2(M, \tau)$; conjugating the left representation gives a normal automorphism β_t .

Point-norm continuity is asserted on B , not on all of M . Since B includes the bounded logarithm $S = \arg C$, it is not identified here with the canonical reduced group C^* -algebra.

Only **after** constructing the automorphism does the proof use the creation–annihilation–diagonal estimate $\|s(h)\| \leq 3\pi\|h\|_{\ell^2}$. Differentiating $e^{-itS}A_w(t)$ in the correct order gives

$$\|\beta_1(A_j) - A_j\| \leq 3\pi\|h_j\|_{\ell^2}, \quad \|\beta_1(A_w) - CA_w\| \leq 3\pi\|D_w - \delta_e\|_{\ell^2}.$$

Thus coefficient estimates control operator-norm displacements.

Lemmas 2.3–2.5 · pp. 4–6

Section 3.3 · pp. 10–11

2. Construct a small cocycle by cutoff inversion

The condition $n \geq 3$ enters through the separate choices $a = x_1$ and $b_j = x_2^j x_3 x_2^{-j}$. The prefixes $p_j = (ab_1) \cdots (ab_j)$ form a free family. With $w_m = p_m a$ and $T_m = 1 + \sum_{j=1}^m \lambda(p_j)$, the coefficients $h_1 = h$, $h_j = 0$ ($j \geq 2$) give $D_{w_m} = T_m h$.

A Catalan moment count using noncrossing pairings of free words shows that the spectral measures of Q_m/m , where $Q_m = T_m T_m^*$, converge weakly to

$$d\nu(x) = \frac{1}{2\pi} \sqrt{\frac{4-x}{x}} 1_{(0,4)}(x) dx.$$

The required property is $\nu(\{0\}) = 0$, not invertibility of every Q_m . First choose small $d > 0$, then large m , and put

$$E_m = 1_{(dm, \infty)}(Q_m), \quad R_m = Q_m^{-1} E_m, \quad h^{(0)} = T_m^* R_m \delta_e.$$

Here R_m denotes bounded functional calculus with the low spectrum removed. The identities $T_m h^{(0)} = E_m \delta_e$, $\|h^{(0)}\|_{\ell^2}^2 \leq (dm)^{-1}$, and $\|T_m h^{(0)} - \delta_e\|_{\ell^2}^2 = \tau_\Gamma(1 - E_m)$ make both required quantities small. Commutation with coefficientwise real conjugation makes the vector real. After m is fixed, approximate it by a finitely supported real vector; $\|T_m\| \leq m + 1$ controls the final error.

Lemma 4.1; Sections 4.1–4.3 · pp. 11–16

3. Approximate the old completion and respect the order of choices

The flow alone approximates CA_w . The free-basis transformation $\gamma(A_j) = A_j$, $\gamma(C) = CA_w$ corrects this: setting $\alpha = \gamma^{-1} \circ \beta_1$ gives

$$\max_j \|\alpha(A_j) - A_j\| < \varepsilon, \quad \|\alpha(A_w) - C\| < \varepsilon.$$

The new completion is $C' = \alpha(C) = CA_w^*$, whereas the approximation target is the **old** C . The tuple $(\alpha(A), C')$ remains freely generating Haar, so the step can be repeated.

Proposition 5.1 · pp. 16–17

Fix a countable dense family (y_j) in $L^2(M)$. At stage k , choose word polynomials $F_{j,k}$ in the current tuple $(A^{(k-1)}, C^{(k-1)})$ approximating $y_1, \dots, y_k$ within 2^{-k} . From their common Lipschitz bound H_k , choose

$$0 < \varepsilon_k < \min\{r_{k-1}/2, 2^{-k}/(1 + H_k)\}.$$

Only **after** the perturbation returns a word w_k , define $G_{j,k}(X) = F_{j,k}(X, X_{w_k})$, obtain a common Lipschitz bound L_k , and choose

$$0 < r_k \leq \min\{r_{k-1}/2, 2^{-k}/(1 + L_k)\}.$$

In the first estimate, X_{w_k} occupies a single unitary slot of $F_{j,k}$; its as-yet unknown word length is not needed to choose ε_k . The longer word only reduces the future budget r_k .

Section 5.2; equations (5.3)–(5.9) · pp. 17–19

4. Close the distribution and generation arguments separately

The bound $\sum_{\ell > k} \varepsilon_\ell \leq r_k$ makes each $A_j^{(k)}$ converge in operator norm to a unitary $A_j^{(\infty)}$. Norm continuity of each fixed word and of the trace preserves the zero trace of every nonidentity word. Generation requires a separate estimate:

$$\|G_{j,k}(A^{(\infty)}) - y_j\|_2 \leq L_k r_k + 2 \cdot 2^{-k} \leq 3 \cdot 2^{-k} \quad (j \leq k).$$

For $N = W^*(A^{(\infty)})$, the trace-preserving conditional expectation E_N extends to the orthogonal projection in L^2 . It fixes this dense family, hence $N = M$. Neither uniform operator-norm bounds for the approximating polynomials nor convergence of $C^{(k)}$ is required.

Finally $V\delta_g = A_g^{(\infty)}$ defines a unitary $\ell^2(\mathbb{F}_n) \rightarrow L^2(M)$ intertwining the regular representation with left multiplication. The spatial identification of double commutants gives $L(\mathbb{F}_n) \rightarrow M$. Surjectivity follows from $N = M$, trace preservation from $V\delta_e = 1$, and normality and injectivity from the spatial realization. The proof does not take a limit of the finite-stage automorphisms on the entire algebra.

Section 5.2; equations (5.3)–(5.9) · pp. 17–19

Lemma 2.2 · p. 4

3. Proof of Theorem 1.2 — Match the amplification parameters

Proof target : Theorem 1.2 · p. 2

Write $N_s = L(\mathbb{F}_s)$. In $Q^t = p(Q \otimes \overline{B(\ell^2)})p$, the projection has $(\tau_Q \otimes \text{Tr})(p) = t$ and the corner trace is $t^{-1}(\tau_Q \otimes \text{Tr})$; Tr assigns value one to rank-one projections. This is the normalization behind the exponent t^{-2} .

1 Use ranks covered by the construction

$$N_3 \cong N_4 \cong N_5, \quad \Theta : N_3 \longrightarrow N_5$$

Compose the isomorphisms for $n = 3, 4$ and restrict $\Theta \otimes \text{id}$ to corners

[Theorem 1.1 · p. 1](#)

[Section 6; equation \(6.1\) · p. 20](#)

2 Amplify to corners of the same trace

$$\Theta^{\sqrt{2}} : N_3^{\sqrt{2}} \longrightarrow N_5^{\sqrt{2}}$$

$$\tau(p) = \sqrt{2}, \quad \tau_{Q^{\sqrt{2}}} = \frac{1}{\sqrt{2}}(\tau_Q \otimes \text{Tr})$$

Apply the normalized formula $N_s^t \cong N_{1+(s-1)t^{-2}}$

[\[Dyk94, Theorem 2.4, print p. 128 / PDF p. 7\]](#)

3 Identify the corners with ranks two and three

$$N_3^{\sqrt{2}} \cong N_{1+2/2} = N_2, \quad N_5^{\sqrt{2}} \cong N_{1+4/2} = N_3$$

$$\Phi : N_2 \xrightarrow{\cong} N_3 \quad \text{unital, normal, trace-preserving}$$

Amplify the isomorphism between ranks three and five by $\sqrt{2}$

[Open figure ↗](#) [TeX source ↓](#)

1. Why the proof passes from rank three to rank five

Apply Theorem 1.1 at $n = 3, 4$ to obtain $\Theta : N_3 \rightarrow N_5$. The external formula

$$N_s^t \cong N_{1+(s-1)t^{-2}} \quad (s > 1, t > 0)$$

with $t = \sqrt{2}$ gives $N_3^{\sqrt{2}} \cong N_2$ and $N_5^{\sqrt{2}} \cong N_3$. Compose these identifications with $\Theta^{\sqrt{2}}$ to obtain Φ . The map $\Theta \otimes \text{id}$ sends a projection to one with the same finite trace and restricts to the normalized corners. Each map is unital, normal, trace-preserving, and bijective, and their composition retains these properties. The small-cocycle construction is not applied directly at $n = 2$.

[\[Dyk94, Theorem 2.4, print p. 128 / PDF p. 7\]](#)

[Section 6; equation \(6.1\) · p. 20](#)

4. Proof of Corollary 6.1 — The infinite parameter and fundamental groups

Proof target : [Corollary 6.1 · p. 21](#)

1 Rule out the nonisomorphic alternative

$$N_2 \cong N_3$$

Dichotomy for $1 < r \leq \infty$: all isomorphic, or pairwise nonisomorphic

An isomorphism of distinct finite ranks excludes the second alternative

[Theorem 1.2 · p. 2](#)

[\[GP25, Theorem 1.1, p. 2\]](#)

2 Isomorphism including countably infinite rank

$$N_r \cong N_s \quad (1 < r, s \leq \infty)$$

Use amplification for finite parameters and $N_\infty \cong N_2$ for infinity

[\[Dyk94, Theorem 2.4, print p. 128 / PDF p. 7\]](#)

[Corollary 6.1 · p. 21](#)

3 Absorb every positive amplification

$$N_r^t \cong N_{1+(r-1)t^{-2}} \cong N_r \quad (1 < r < \infty)$$

$$N_\infty^t \cong N_2^t \cong N_2 \cong N_\infty$$

$$\mathcal{F}(N_r) = \mathbb{R}_{>0} \quad (1 < r \leq \infty)$$

Use the full-parameter dichotomy and amplification to obtain the consequences

[Open figure ↗](#) [TeX source ↓](#)

1. The infinite rank enters through the dichotomy

The classical dichotomy for the entire range $1 < r \leq \infty$ says that all factors are isomorphic or that distinct parameters always give nonisomorphic factors. Theorem 1.2 excludes the latter. The full-range statement was checked in Goldbring–Pi Theorem 1.1, which explicitly restates Rădulescu Corollary 4.7. The argument does not simply take an inductive limit of finite-rank isomorphisms.

[\[GP25, Theorem 1.1, p. 2\]](#)

2. Absorb every positive amplification

For finite $r > 1$ and $t > 0$, the parameter satisfies $1 + (r - 1)t^{-2} > 1$. Amplification and the preceding conclusion give $N_r^t \cong N_r$. For $r = \infty$, amplify $N_\infty \cong N_2$ to obtain

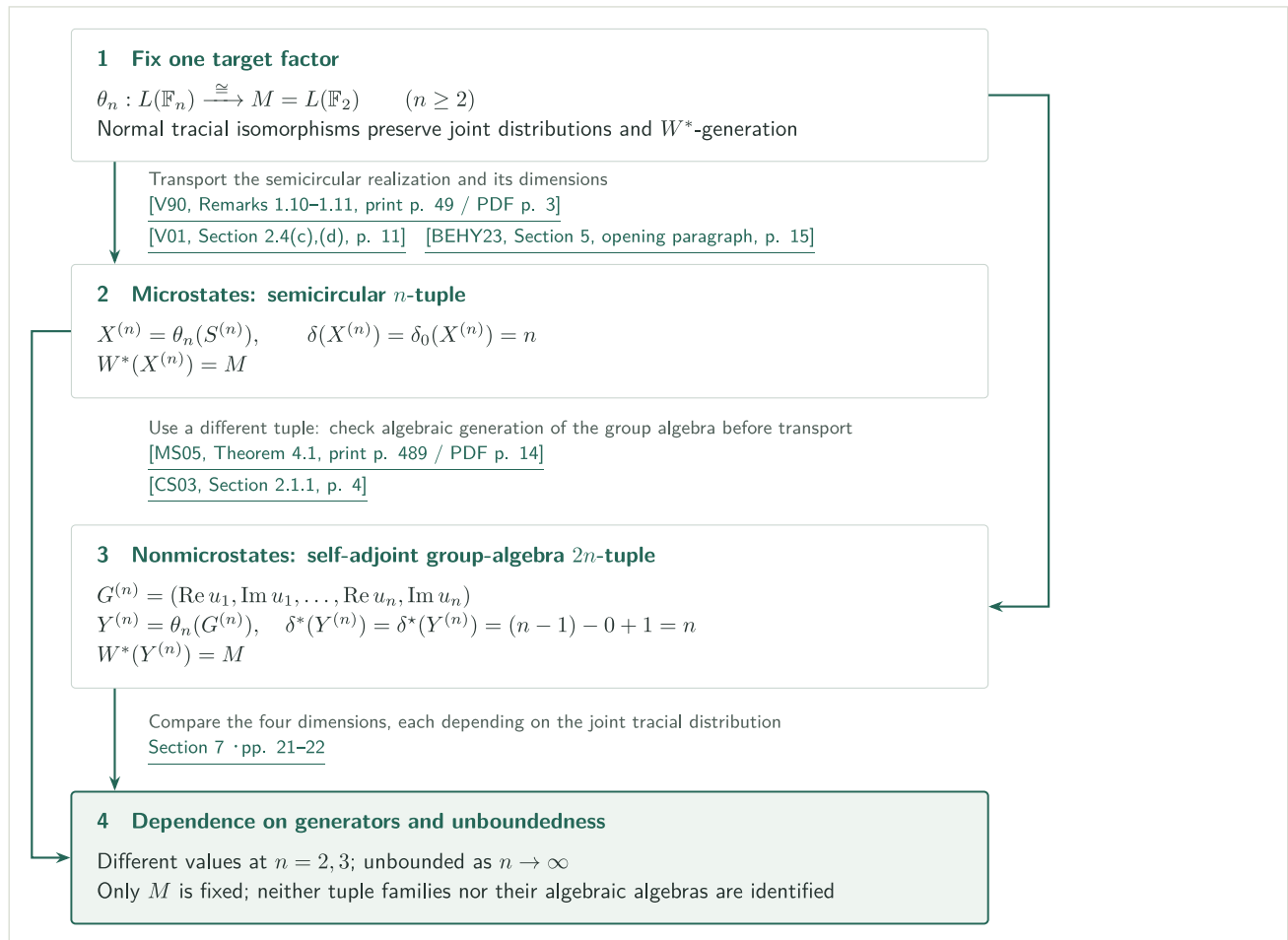
$N_\infty^t \cong N_2^t \cong N_2 \cong N_\infty$. Thus every fundamental group is $\mathbb{R}_{>0}$. No formal substitution of infinity into the finite-parameter formula is needed.

[Corollary 6.1 · p. 21](#)

5. Proof of Corollary 7.1 — Transport two different tuple families

Proof target : Corollary 7.1 · pp. 21–22

Theorems 1.1–1.2 give $\theta_n : L(\mathbb{F}_n) \rightarrow M = L(\mathbb{F}_2)$, with the identity for $n = 2$. Trace preservation gives equality of the joint $*$ -distributions of a self-adjoint tuple Z and $\theta_n(Z)$. Normality and surjectivity give $W^*(\theta_n(Z)) = \theta_n(W^*(Z))$.



Transport semicircular tuples and algebraic group-algebra generators to one factor

Open figure ↗ TeX source ↓

1. Compute microstates dimensions on semicircular tuples

A free variance-one semicircular n -tuple $S^{(n)}$ generates $L(\mathbb{F}_n)$. The semicircular realization was checked in Voiculescu's account. Its semicircular normalization can be rescaled to variance one without changing the generated algebra. The one-variable formula for atomless laws and free additivity give $\delta(S^{(n)}) = n$; the opening of the external §5 states $\delta_0(S^{(n)}) = n$. Then $X^{(n)} = \theta_n(S^{(n)})$ retains generation and both values.

[\[V90, Remarks 1.10–1.11, print p. 49 / PDF p. 3\]](#)

[V01, Section 2.4(c),(d), p. 11]

[BEHY23, Section 5, opening paragraph, p. 15]

2. Compute nonmicrostates dimensions on group-algebra generators

For the canonical group unitaries u_j , put $a_j = (u_j + u_j^*)/2$, $b_j = (u_j - u_j^*)/(2i)$, and $G^{(n)} = (a_1, b_1, \dots, a_n, b_n)$. Since $u_j = a_j + ib_j$, this tuple generates $\mathbb{C}\mathbb{F}_n$ algebraically. It therefore meets Mineyev–Shlyakhtenko's hypotheses, yielding

$$\delta^*(G^{(n)}) = \delta^*(G^{(n)}) = \beta_1^{(2)}(\mathbb{F}_n) - \beta_0^{(2)}(\mathbb{F}_n) + 1 = n.$$

Perform this computation before transport and set $Y^{(n)} = \theta_n(G^{(n)})$. The group-algebra theorem is not applied directly to an arbitrary W^* -generating tuple.

[MS05, Theorem 4.1, print p. 489 / PDF p. 14]

[CS03, Section 2.1.1, p. 4]

3. Noninvariance and unboundedness

All four dimensions depend only on the joint tracial distribution, so transport preserves the relevant values. Comparing $n = 2$ and $n = 3$ gives different values on generators of the same factor; letting n increase gives unboundedness. The algebraic $*$ -algebras generated by the transported tuples are not fixed as n varies. There is consequently no conflict with algebraic-generator invariance in the group-algebra theorem.

Section 7 · pp. 21–22

6. External inputs and their uses

Input

Dykema's amplification formula

Used at

Theorem 1.2; Corollary 6.1

Hypotheses and checking scope

Finite real parameter $s > 1$ and $t > 0$. Theorem 2.4 and the normalization on manuscript p.20 were compared.

Input

Rădulescu's dichotomy as restated by Goldbring–Pi

Used at

Corollary 6.1

Hypotheses and checking scope

The range $1 < r \leq \infty$ was checked; the original Rădulescu proof was not read.

Input

Semicircular realization and the δ, δ_0 computations

Used at

The $X^{(n)}$ part of Corollary 7.1

Hypotheses and checking scope

Statements and values for free semicircular tuples checked in external sources; their complete proofs were not reverified.

Input

Mineyev–Shlyakhtenko and free-group Betti numbers

Used at

The $Y^{(n)}$ part of Corollary 7.1

Hypotheses and checking scope

Self-adjoint algebraic generators of the group algebra of a finitely generated group; checked before transport.

The free-sum estimate, flow, and Catalan calculation in §§2–5 are proved in the manuscript. References to Fox calculus, cyclomorphy, free monotone transport, and Ricard–Xu estimates provide methodological background; they are not recorded here as external theorems substituted into the proof. As the manuscript explicitly states, it also uses no result from Shlyakhtenko's Fuchsian-group-factor preprint.

7. Return to the sources

- Lemmas 3.2–3.4 · pp. 7–10 — From a formal derivation to actual moment invariance.

- Section 3.3 · pp. 10–11 — Distinguish relation descent, surjectivity, and normal extension.
- Lemma 4.1; Sections 4.1–4.3 · pp. 11–16 — Read the free prefixes, spectral limit, and cutoff inverse.
- Section 5.2; equations (5.3)–(5.9) · pp. 17–19 — Order of choices after finding a word, and generation in the limit.
- Section 6; equation (6.1) · p. 20 — Normalization and the $\sqrt{2}$ amplification to rank two.
- Corollary 6.1 · p. 21 — Dichotomy for the infinite parameter and fundamental groups.
- Section 7 · pp. 21–22 — Two different tuple families and transport of distributions.

The indicated passages in §§2–7 were read and self-checked for every stated conclusion, definition and extension of maps, rank restrictions, approximation topology, order of error choices, amplification normalization, and the two generator families. Checking an input's statement and application is distinct from verifying its entire proof.

Unresolved scope: direct reading of Rădulescu's original Corollary 4.7 (an explicit restatement is used here); correspondence with the journal pagination of Brannan–Elzinga–Harris–Yamashita (checked in author preprint v2 §5); and the proofs of external inputs and their further dependencies. No independent expert review or formal verification was performed.