

OpenAI Math Digest

CATALOGUE 285

Counterexamples to Baum–Connes and Kadison–Kaplansky

Catalogue synthesis & 3 proof overviews

Independent reading guide · English edition

Edition: 2026-10-15

Created by: Masataka Iwai · Osaka University

AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 3 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

00 · Catalogue synthesis	3
01 · A torsion-free counterexample to reduced Baum–Connes injectivity	9
02 · A torsion-free counterexample to the Kadison–Kaplansky projection conjecture	22
03 · An irrational-trace counterexample to reduced Baum–Connes	34

Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

Counterexamples to Baum–Connes and Kadison–Kaplansky

Official catalogue 285. These three manuscripts claim counterexamples to injectivity and surjectivity of coefficient-free reduced Baum–Connes assembly, and to the projection conjecture for reduced group C^* -algebras of torsion-free groups. Each article distinguishes the manuscript's claims from the reductions and external inputs checked editorially.

Papers in official catalogue order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 285 · 2026-10-15 checked source snapshot

#

01

Manuscript and overview

A torsion-free counterexample to reduced Baum–Connes injectivity →

Short name: A torsion-free counterexample to reduced Baum–Connes injectivity

September 23, 2026 · 85 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Claims an infinite-order kernel class for coefficient-free reduced assembly of a finitely generated torsion-free group, hence failure of rational injectivity.

[Theorem 1.1 · p. 1](#) ↗

Role and available overview

Combines cohomological detection of a torus with vanishing of its reduced Bott class, retaining both in a finitely generated witness.

[Read the proof overview →](#)

#

02

[Manuscript and overview](#)

A torsion-free counterexample to the Kadison–Kaplansky projection conjecture

[→](#)

Short name: A torsion-free counterexample to the Kadison–Kaplansky projection conjecture

September 23, 2026 · 76 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Claims a projection with $0 < \tau(p) < 1/2$ in the scalar reduced group C^* -algebra of a finitely generated torsion-free group, contradicting the projection conjecture.

[Theorem 1.1 · p. 2](#) ↗

Role and available overview

Constructs a matrix projection of small positive trace in a torsion-free group, then compresses it to a scalar projection in a free product.

[Read the proof overview](#) →

#

03

Manuscript and overview

An irrational-trace counterexample to reduced Baum–Connes →

Short name: An irrational-trace counterexample to reduced Baum–Connes

September 23, 2026 · 23 pages

[Original paper \(GitHub\) ↗](#)

Principal claim

Claims an irrational-trace projection for a finitely generated group, obstructing surjectivity of coefficient-free reduced assembly; no torsion-free condition is imposed.

[Theorem 1.1 · p. 2 ↗](#)

Role and available overview

Code relations give exact local probabilities; a global spectral gap places the projection in the C^* -algebra. Its trace is compared with the rational trace of the assembly image.

[Read the proof overview →](#)**Separate the kernel from the image of assembly**

The injectivity article concerns the kernel of

$$\mu_r : K_0(BG_{\text{inj}}) \longrightarrow K_0(C_r^*(G_{\text{inj}}))$$

for a torsion-free group G_{inj} . Degree-two cohomology detects the Dirac class h of an embedded torus and keeps it of infinite order, while a separate construction kills its Bott image in the reduced group C^* -algebra. Here $K_0(BG)$ is compactly supported K-homology, defined by the direct limit over finite subcomplexes. The manuscript explains that maximal assembly detects the same h rationally, distinguishing its conclusion from counterexamples to maximal assembly or the classical Novikov conjecture.[OpenAI26a, Theorem 1.1, p. 1] [OpenAI26a, Section 1.2, p. 2]

The [projection](#) article first constructs a matrix projection of small positive trace for a torsion-free group Γ_{proj} , then compresses it to a scalar projection in the free product $G_{\text{proj}} = \Gamma_{\text{proj}} * F_N$. Its conclusion is $p \in C_r^*(G_{\text{proj}})$ with $0 < \tau(p) < 1/2$, extending beyond the existence of a matrix projection.[OpenAI26b, Theorem 1.1, p. 2]

The irrational-trace article concerns the image of

$$\mu_0^{G_{\text{sur}}} : K_0^{G_{\text{sur}}}(\underline{EG}_{\text{sur}}) \longrightarrow K_0(C_r^*(G_{\text{sur}}))$$

for a finitely generated group G_{sur} , with no torsion-free assumption. The domain is equivariant K-homology with G -compact supports on the classifying space for proper actions. Exact local distributions from codes and a global spectral gap produce an actual projection $b_0 \in C_r^*(G_{\text{sur}})$. Its irrational trace is compared with the rational trace of the assembly image. The manuscript also deduces failure of rationalized surjectivity.[OpenAI26c, Theorems 1.1–1.2, p. 2] [OpenAI26c, Proposition 6.1 and proof of Theorem 1.1, pp. 21–22]

Compare the two trace obstructions and their construction methods

Section 11 of the projection manuscript also records failure of trivial-coefficient reduced assembly surjectivity for Γ_{proj} , using the matrix projection and the integer-valued trace of assembly for torsion-free groups. This does not depend on the separate irrational-trace construction. A noninteger trace alone does not exclude membership in the rationalized image. The irrational-trace manuscript instead uses rational-valued traces on assembly for arbitrary countable groups, so both the group hypotheses and the strength of the obstruction differ. [OpenAI26b, Section 11, p. 73] [OpenAI26c, Proposition 6.1 and proof of Theorem 1.1, pp. 21–22]

There is an **explicit methodological reference from the projection manuscript to the injectivity manuscript**. Section 2 (p.7) of the latter adapts the padded-stream construction and recovery scheme from Sections 4 and 6 of the former, establishing the exponents, cutoffs and gated corrections needed for its own interfaces. Section 7.4 (p.64) also follows the cone-chain and prime-order obstruction method of Section 7.6 of the projection manuscript, reproving the exact-chain interfaces for its current sheets. The projection theorem is not used as an existence input; extension of the integral detector and Bott realization are separate arguments in the injectivity manuscript.[OpenAI26a, Section 2, p. 7] [OpenAI26a, Section 7.4, p. 64] [OpenAI26b, Section 7.6, pp. 36–38]

This directed relation is recorded as adaptation and reproof of a method, distinguished from direct application of a main theorem. General similarities involving expanders or graphical constructions do not by themselves justify further dependency arrows. The individual articles give the external inputs, their applications and the diagrams of the proof reductions.

Sources and checking scope

All citation keys refer to manuscripts by OpenAI dated September 23, 2026. OpenAI26a denotes A torsion-free counterexample to reduced Baum–Connes injectivity (85 pages); OpenAI26b denotes A torsion-free counterexample to the Kadison–Kaplansky projection conjecture (76 pages); OpenAI26c denotes An irrational-trace counterexample to reduced Baum–Connes (23 pages).

The order follows the [official contents](#) and page 31 of the [official overview](#), under Operator algebras (285–303), at fixed commit `1581d4cbdd69a0faf34b2089614fe2a814316a6a`. This guide checks the objects, hypotheses and conclusions of the claims and searches for direct inter-manuscript citations. It does not independently verify all three proofs. Detailed reading locations and unresolved points are recorded at the end of each article.

A torsion-free counterexample to reduced Baum–Connes injectivity

Manuscript author: OpenAI · September 23, 2026 · 85 pages, including references

[Read the manuscript on GitHub ↗](#)

An integrally detected torus and Bott cancellation in the reduced group C algebra*

The manuscript claims an infinite-order kernel class for coefficient-free reduced Baum–Connes assembly of a finitely generated torsion-free group. An integral cohomology class detects a torus in a Heisenberg subgroup, while localized Bott representatives move through a tower of heights to cancel its analytic image. The final step retains both properties in the same finitely generated subgroup.

1. Main results

Theorem 1.1 — Failure of rational injectivity for coefficient-free reduced assembly

There exist a finitely generated torsion-free discrete group G_{inj} , an injective homomorphism $i : \mathbb{Z}^2 \hookrightarrow G_{\text{inj}}$, and an integral class $c \in H^2(BG_{\text{inj}}; \mathbb{Z})$ such that

$$\langle (Bi)^*c, [T^2] \rangle = 1, \quad i_*(\beta) = 0 \quad \text{in } K_0(C_r^*(G_{\text{inj}})).$$

Here $\beta \in K_0(C_r^*(\mathbb{Z}^2)) \cong K^0(T^2)$ is a rank-zero Bott generator, and i_* is induced by the inclusion of reduced group C* algebras. If $[D_{T^2}]$ is the spin Dirac class of the oriented torus, then

$$h = (Bi)_*[D_{T^2}] \in K_0(BG_{\text{inj}})$$

has infinite order and $\mu_r^{G_{\text{inj}}}(h) = 0$. The map in question is

$$\mu_r^{G_{\text{inj}}} : K_*(BG_{\text{inj}}) \longrightarrow K_*(C_r^*(G_{\text{inj}})),$$

with coefficient algebra $\mathbb{C}$ and trivial action. All K-theory is complex; the domain is compactly supported K-homology, defined as the colimit over finite subcomplexes. Thus $h \otimes 1 \neq 0$ but $(\mu_r^{G_{\text{inj}}} \otimes \mathbb{Q})(h \otimes 1) = 0$: injectivity fails even after rationalization. Finite presentation is not part of the conclusion.

Theorem 1.1 · p. 1

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A torsion-free counterexample to reduced Baum–Connes injectivity](#). External inputs use the following keys.

- [EH20](#) Heath Emerson; Dan Hudson — Baum–Connes and the Fourier–Mukai transform: Assembly of the torus Dirac class; the full-torus case of Theorem 1.3. [arXiv:2001.06124v2](#), 27 July 2020.
- [KP20](#) Jens Kaad; Valerio Proietti — Index theory on the Miščenko bundle: Comparison of the Miščenko index with reduced assembly, applied to the compact torus cycle. [arXiv:1807.05757v2](#), 5 March 2020.
- [BvE16](#) Paul F. Baum; Erik van Erp — K-homology and Fredholm operators I: Dirac Operators: Twisted Dirac index on a closed even-dimensional spin-c manifold, detecting the degree of a line bundle on the torus. [arXiv:1604.03502v1](#), 12 April 2016.
- [HS07](#) Bernhard Hanke; Thomas Schick — The strong Novikov conjecture for low degree cohomology: Comparison and consequence only: degree-two detection makes the maximal assembly image rationally nonzero. [arXiv:0705.2578v1](#), 17 May 2007.
- [OpenAI26b](#) OpenAI — A torsion-free counterexample to the Kadison–Kaplansky projection conjecture: Methodological predecessor; cone chains and torsion exclusion are reproved here, rather than imported as a main-theorem input. September 23, 2026.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1 · p. 1](#)

Proof route. Construct the [integral detector](#) and [Bott vanishing](#) for the same accuracy sequence of groups Γ_ν . Choose one sufficiently late coordinate and pass to a subgroup containing a finite witness for the K-theory equality. The index pairing and naturality of the Miščenko index then identify topological nonvanishing and zero reduced assembly.

1 Two outputs of one construction sequence

$H \hookrightarrow \Gamma_\nu$, Γ_ν is torsion-free.

$$\langle c_{\Gamma_\nu}|_{T^2}, [T^2] \rangle = -1, \quad \beta \mapsto 0 \in K_0(C_r^*\Gamma_\nu) \quad (\nu \gg 0).$$

Choose detector and Bott vanishing; retain finite polynomial supports

[Proposition 7.11 · p. 69](#) [Proposition 8.8 · p. 80](#)

[Lemmas 8.7, 8.9 · pp. 80–81](#)

2 Recover a witness in a finitely generated subgroup

$H \leq G_{\text{inj}} \leq \Gamma_\nu$, $i : \mathbb{Z}^2 \hookrightarrow G_{\text{inj}}$.

$$c = -c_{\Gamma_\nu}|_{BG_{\text{inj}}}, \quad \langle (Bi)^*c, [T^2] \rangle = 1, \quad i_*(\beta) = 0.$$

Twisted Dirac index and naturality of the pairing

[\[BvE16, Theorem 1, p. 2\]](#) [§9.2 · pp. 82–83](#)

3 Detect nonvanishing by an integer index

$h = (Bi)_*[D_{T^2}]$, $c_1(L) = c$.

$$\langle [L] - [1], h \rangle = \langle (Bi)^*c, [T^2] \rangle = 1 \implies h \otimes 1 \neq 0.$$

Torus Bott image and extension of Miščenko–bundle coefficients

[\[EH20, Theorem 1.3, p. 2\]](#)

[\[KP20, Corollary 1.4; \(1.6\), pp. 5–7\]](#)

4 The same class has zero reduced assembly

$$\mu_r^{G_{\text{inj}}}(h) = i_*\mu_r^{\mathbb{Z}^2}([D_{T^2}]) = \pm i_*(\beta) = 0.$$

$$\mu_r^{G_{\text{inj}}} \otimes \mathbb{Q} : K_0(BG_{\text{inj}}) \otimes \mathbb{Q} \longrightarrow K_0(C_r^*G_{\text{inj}}) \otimes \mathbb{Q}$$

is not injective. The coefficient algebra is $\mathbb{C}$ with trivial action.

Combining the integral detector and Bott cancellation in a finitely generated group

[Open figure ↗](#) [TeX source ↓](#)

1. Retain both constructions in a finitely generated subgroup

Embed the integral Heisenberg group $H = \langle x, y, z \mid [x, y] = z, [x, z] = [y, z] = 1 \rangle$ in Γ_ν .

Proposition 7.11 supplies torsion-freeness of Γ_ν and a class c_{Γ_ν} evaluating to -1 on the ordered torus (z, x) . Proposition 8.8 asserts integral vanishing of its Bott image for all sufficiently large ν .

Lemma 8.9 approximates the finite matrix entries of a stabilization projection and a witnessing partial isometry by group polynomials. Their finite supports, together with generators of H , generate G_{inj} . The reduced inclusion of a subgroup is isometric, so the small-defect bounds remain valid in this subgroup. Spectral cutting and polar normalization recover an exact witness there. Torsion-freeness and the embedded torus pass to the subgroup; setting $c = -c_{\Gamma_\nu}|_{BG_{\text{inj}}}$ changes the evaluation to 1 . No limit group of the accuracy sequence is used.

Proposition 7.11 · p. 69 Proposition 8.8 · p. 80 Lemmas 8.7, 8.9 · pp. 80–81

2. Detect the topological class by an integer index

Choose a line bundle L with $c_1(L) = c$. Naturality of the index pairing on the compact torus cycle gives

$$\langle [L] - [1], h \rangle = \text{index}(D_{T^2} \otimes (Bi)^* L) - \text{index}(D_{T^2}) = \langle (Bi)^* c, [T^2] \rangle = 1.$$

The tangent bundle is trivial, so the characteristic class in the index formula is **1**, and the untwisted Dirac index is zero. An integer-valued homomorphism taking h to **1** proves infinite order. Compactness of BG_{inj} is not required.

[BvE16, Theorem 1, p. 2] §9.2 · pp. 82–83

3. Identify the reduced assembly image of the same class

In the full-torus case of Emerson–Hudson, the dual subtorus is a point, giving $\mu_r^{\mathbb{Z}^2}([D_{T^2}]) = \pm\beta$. The sign depends on orientation conventions and does not affect vanishing. Put $A = C_r^*(\mathbb{Z}^2)$, $B = C_r^*(G_{\text{inj}})$ and let $j : A \rightarrow B$ be inclusion. The Miščenko bundle becomes $E_A \otimes_j B$. The manuscript checks on this torus cycle that extension of coefficients commutes with the index Kasparov product. The comparison theorem therefore gives

$$\mu_r^{G_{\text{inj}}}(h) = j_* \mu_r^{\mathbb{Z}^2}([D_{T^2}]) = \pm j_*(\beta) = 0$$

This concerns injectivity with trivial coefficients, a different assertion from known failures of surjectivity with nontrivial coefficients or the existence of a nontrivial projection in the scalar algebra.

[EH20, Theorem 1.3, p. 2] [KP20, Corollary 1.4; (1.6), pp. 5–7] §9.2 · pp. 82–83

3. Proposition 7.11 — Torsion exclusion and the integral detector

Proof target : Proposition 7.11 · p. 69

Proposition 7.11 — Detecting the embedded Heisenberg torus

The graphical group Γ constructed in §§2–6 is torsion-free. The map $i_\Gamma : \mathbb{Z}^2 \rightarrow \Gamma$ sending the ordered basis to (z, x) is injective, and there is a class $c_\Gamma \in H^2(B\Gamma; \mathbb{Z})$ such that

$$\langle (Bi_\Gamma)^* c_\Gamma, [T^2] \rangle = -1$$

This assertion concerns the group Γ with the sheet and cone-chain structure below, rather than an arbitrary group containing the Heisenberg group.

Proposition 7.11 · p. 69

1 Embed the cover and decompose sheet cycles

$$\tilde{X} \hookrightarrow \text{Cay}(\Gamma, S), \quad H \hookrightarrow \Gamma, \quad Q \in \Gamma/H.$$

$$\bigoplus_Q Z_1(Q; R) \cong Z_1(\text{Cay}(\Gamma, S); R).$$

Turn finite-support cycle coordinates into cone boundaries

[Proposition 6.4 · p. 57](#) [Lemma 7.6; \(7.12\) · p. 65](#)

2 Exact cone chains

$$0 \rightarrow C_2(Z) \rightarrow C_1(Z) \rightarrow C_0(Z) \rightarrow \mathbb{Z} \rightarrow 0.$$

$$C_1, C_2 \text{ are free } \mathbb{Z}\Gamma\text{-modules, } C_0 \cong \mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} \mathbb{Z}.$$

Periodic-resolution contradiction / length-one resolution and restriction

[Lemma 7.8 · p. 66](#) [Lemma 7.10 · pp. 68–69](#)

3 Exclude torsion and extend degree two

$$C_p \leq \Gamma \implies \text{Tor}_n^{\mathbb{F}_p[C_p]}(\mathbb{F}_p, \mathbb{F}_p) = 0 \ (n > 2) \text{ is impossible.}$$

$$K = \ker(C_0 \rightarrow \mathbb{Z}), \quad \text{Ext}_{\mathbb{Z}\Gamma}^2(K, \mathbb{Z}) = 0$$

$$\implies H^2(B\Gamma; \mathbb{Z}) \rightarrow H^2(BH; \mathbb{Z}).$$

Extend the Heisenberg cocycle and preserve its bar-cycle evaluation

[Lemma 7.9 · pp. 67–68](#) [Proposition 7.11 · p. 69](#)

4 An integral torus detector

$$\omega(g, g') = \alpha\gamma' + \binom{\alpha}{2}\beta', \quad \omega(z, x) - \omega(x, z) = -1.$$

$$\Gamma \text{ is torsion-free, } i_\Gamma : \mathbb{Z}^2 \hookrightarrow \Gamma, \quad \langle (Bi_\Gamma)^* c_\Gamma, [T^2] \rangle = -1.$$

The cone-chain structure supplies torsion-freeness and the integral detector

[Open figure ↗](#) [TeX source ↓](#)

1. Embed the cover and control finite-support chains

Unique recovery of long words and exclusion of short circuits let the manuscript prove, by plane diagrams, that a null-labeled source path has equal endpoints. The voltage cover $\tilde{X}$ and its deck group H consequently embed in the Cayley graph; translated sheets are indexed by Γ/H . The direct decomposition into sheet cycle coordinates makes the chain complex of the two-dimensional complex Z , obtained by coning each sheet, exact over any commutative coefficient ring.

$$0 \longrightarrow C_2(Z) \longrightarrow C_1(Z) \longrightarrow C_0(Z) \longrightarrow \mathbb{Z} \longrightarrow 0, \quad C_0(Z) \cong \mathbb{Z}\Gamma \oplus \mathbb{Z}\Gamma \otimes_{\mathbb{Z}H} \mathbb{Z}.$$

The modules C_1, C_2 are free over $\mathbb{Z}\Gamma$. All chains and cycle decompositions have finite support; no independence of infinite sums is assumed.

Proposition 6.4 · p. 57 Lemmas 6.3, 7.3; Proposition 7.4 · pp. 55–57, 62–64 Lemma 7.6; (7.12) · p. 65

2. Rule out prime-order subgroups by resolution length

The group H is torsion-free, cone-vertex stabilizers are conjugates of H , and other cell stabilizers are trivial. If $C_p \leq \Gamma$, it acts freely on cells, and the chains over $\mathbb{F}_p$ give a length-two free resolution of the trivial $\mathbb{F}_p[C_p]$ module. But tensoring the periodic resolution over $\mathbb{F}_p[C_p] \cong \mathbb{F}_p[t]/(t^p)$ with the trivial module leaves homology $\mathbb{F}_p$ in every degree. Resolution comparison gives a contradiction. Hence there is no subgroup C_p and Γ is torsion-free.

Lemma 7.8 · p. 66

3. Extend an explicit cocycle to the ambient group

Use Heisenberg coordinates with the following product.

$$\begin{aligned} (\alpha, \beta, \gamma)(\alpha', \beta', \gamma') \\ = (\alpha + \alpha', \beta + \beta', \gamma + \gamma' + \alpha\beta'). \end{aligned}$$

The integral 2-cocycle

$$\omega(g, g') = \alpha\gamma' + \binom{\alpha}{2}\beta'$$

evaluates to -1 on the bar cycle $[z|x] - [x|z]$. The module $K = \ker(C_0(Z) \rightarrow \mathbb{Z})$ has a length-one free resolution, so $\text{Ext}_{\mathbb{Z}\Gamma}^2(K, \mathbb{Z}) = 0$. This makes $H^2(B\Gamma; \mathbb{Z}) \rightarrow H^2(BH; \mathbb{Z})$ surjective. The manuscript also compares chain maps from the induced resolution and the bar resolution, identifying this map with actual subgroup restriction. An extension of $[\omega]$ therefore retains the required evaluation.

Lemma 7.9 · pp. 67–68 Lemma 7.10 · pp. 68–69

4. Proposition 8.8 — Bott vanishing in the reduced algebra

Proof target : Proposition 8.8 · p. 80

Proposition 8.8 — Integral vanishing at sufficiently late coordinates

Consider the compatible sequence of constructions Γ_ν , with parameters chosen in the order of §9.1. For every sufficiently large ν , the image of the rank-zero Bott generator for $\langle z, x \rangle \cong \mathbb{Z}^2$ is zero in

$$K_0(C_r^*(\Gamma_\nu))$$

No rationalization is needed. Height $i \geq 0$ runs inside one tower, whereas accuracy ν indexes different constructions. Matrix sizes remain fixed along the accuracy sequence.

Proposition 8.8 · p. 80

1 Approximate localized Bott families by finite Cayley formulas

$$a_i = p_i - e, \quad \|b_f - UDI f^{\text{mod}} I^* DU^*\| < \eta.$$

Full regular norm, uniform over all heights; fixed matrix size.

Uniform mesh approximation via sampling, constant extraction and forest bounds

[Lemmas 8.1–8.4 · pp. 72–76](#) [Lemma 8.5; \(8.17\)–\(8.18\) · pp. 76–77](#)

2 Obtain an exact homomorphism and paths in the quotient

$$Q_r = \prod_{\nu} C_r^*(\Gamma_{\nu}) / \bigoplus_{\nu} C_r^*(\Gamma_{\nu}), \quad \Theta : \mathcal{B} \rightarrow Q_r.$$

The algebra $\mathcal{B}$ is generated by bounded model families and two continuous paths.

Rotate and rescale disjoint adjacent pairs; use finite additivity

[\(8.19\)–\(8.20\) · pp. 77–78](#)

3 Move even to odd to positive-even heights

$$b_I = [e + A_I] - [e] \in K_0(\mathcal{B}),$$

$$b_{\text{ev}} = b_{\text{od}} = b_{\text{ev},+}, \quad b_{\text{ev}} = b_0 + b_{\text{ev},+} \implies b_0 = 0.$$

A fixed-size projection path using isometry only on the support

[Lemma 8.6; \(8.23\)–\(8.25\) · pp. 79–80](#)

4 Identify the root with the physical torus Bott class

$$v^* v a^{\text{phys}} = a^{\text{phys}}, \quad \Theta(A_0) = v a^{\text{phys}} v^*.$$

$$[e + a^{\text{phys}}] - [e] = \Theta_*(b_0) = 0 \quad \text{in } K_0(Q_r).$$

Lift the quotient witness; correct it by spectral cutting and polar normalization

[Lemmas 8.7, 8.9 · pp. 80–81](#) [Proposition 8.8 · p. 80](#)

5 Integral vanishing at sufficiently late coordinates

$$\beta \mapsto 0 \quad \text{in } K_0(C_r^*(\Gamma_{\nu})) \quad (\nu \gg 0).$$

Distinguish height i from accuracy ν and keep the witness size finite and fixed.

From finite Cayley formulas to quotient Bott cancellation and coordinate witnesses

[Open figure ↗](#) [TeX source ↓](#)

1. Realize finite Cayley formulas in full regular norm

Treat the localized differences $a_i = p_i - e$, adjacent-height rotations and rescalings as finite families of profiles. Proposition 5.2 imposes short-circuit exclusion and full regular operator-norm approximation on the same voltage choice. Small spectral-carrier traces and bounds on excluded coordinates control the loss when restricting to disjoint sheet domains. Products of incidences extract normalized constant modes, while a fixed gap $\rho < 1$ suppresses their complements. The remaining Cayley edges form a forest, whose operator-norm error is $O(\sqrt{|S_{\text{wt}}|}/q)$.

Lemma 8.4 obtains finite matrix group-algebra elements with arbitrarily small error. It does not assume physical height-selection projections or membership of Borel carriers in the reduced C^* algebra. The order of choices is essential: first profiles, meshes and weight nets, then the degree parameter, and finally the initial height.

Proposition 5.2; §5.8 · pp. 41–42, 49–50 Lemmas 8.1–8.4 · pp. 72–76 §9.1 · pp. 81–82

2. Remove the initial term using two homotopies and finite additivity

Fix regular representations of $A_\nu = C_r^*(\Gamma_\nu)$ and put $Q_r = \prod_\nu A_\nu / \bigoplus_\nu A_\nu$, where products are bounded sequences and direct sums are norm-null sequences. Let $\mathcal{B}$ be the C^* algebra generated by four bounded model families and two paths. Compression by almost-isometries gives an exact $*$ -homomorphism Θ in the quotient. Uniform approximation on finite meshes proves $\Theta(\mathcal{B}) \subset Q_r$.

Denote the even, odd, positive-even and initial-height differences by $A_{\text{ev}}, A_{\text{od}}, A_{\text{ev},+}, A_0$. Using external unitization, set $b_I = [e + A_I] - [e] \in K_0(\mathcal{B})$. Rotations on disjoint adjacent pairs, followed by rescaling, give

$$b_{\text{ev}} = b_{\text{od}} = b_{\text{ev},+}, \quad b_{\text{ev}} = b_0 + b_{\text{ev},+}, \quad b_0 = 0.$$

The second equality comes from a finite-size projection homotopy using orthogonal supports. The all-height families have already been constructed as bounded operators; the displayed argument does not sum infinitely many K-theory classes.

Lemma 8.5; (8.17)–(8.18) · pp. 76–77 (8.19)–(8.20) · pp. 77–78

3. Identify the physical Bott class and recover equality in one group

The initial profile and the subgroup Bott representative a^{phys} are related by a bounded v satisfying $v^*va^{\text{phys}} = a^{\text{phys}}$ and $\Theta(A_0) = va^{\text{phys}}v^*$. Supported transport in Lemma 8.6 constructs a fixed-size projection path from precisely these relations. Hence the physical Bott class vanishes in $K_0(Q_r)$.

Lemma 8.7 lifts a stabilization projection and a partial isometry from the quotient to bounded sequences, then uses spectral cutting and polar normalization at sufficiently late coordinates. This recovers equality in $K_0(A_\nu)$ with a fixed finite witness size. The unitary equivalence from right to left regular representations sends each group element to that same group element, yielding Proposition 8.8.

Lemma 8.6; (8.23)–(8.25) · pp. 79–80 Lemmas 8.7, 8.9 · pp. 80–81

5. Which papers supply which steps

Source

[BvE16, Theorem 1, p. 2]

Used at

§9.2, index detection above

Content and checking scope

The closed spin torus and pulled-back line bundle meet the index theorem's hypotheses. Statement and use checked.

Source

[EH20, Theorem 1.3, p. 2]

Used at

§9.2, assembly image

Content and checking scope

The dual of the full torus is a point. Identification with the rank-zero Bott class, up to sign, checked.

Source

[KP20, Corollary 1.4; (1.6), pp. 5–7]

Used at

§9.2, subgroup naturality

Content and checking scope

Apply the comparison to a countable-group principal bundle with compact torus base. The manuscript computes coefficient extension.

Source

[OpenAI26b, §7.6, pp. 36–38]

Used at

§7.4, cone chains

Content and checking scope

Methodological reference. Exactness and torsion exclusion for the present sheets are reproved here.

§2; §7.4 · pp. 7, 64

For comparison with maximal assembly, detection by the degree-two class c meets Hanke–Schick's hypothesis, so the maximal assembly image of the same h is rationally nonzero. That image dies under the regular quotient $C_{\max}^*(G_{\text{inj}}) \rightarrow C_r^*(G_{\text{inj}})$. The manuscript does not claim a counterexample to maximal rational injectivity or the classical Novikov conjecture.

[HS07, Theorem 1.2, pp. 1–2] §§1.1–1.2 · pp. 1–3

The projection-counterexample manuscript in this catalogue supplies adapted stream/recovery and cone-chain methods, but its projection-existence theorem is not used as an input producing this kernel class. No direct theorem dependence on the irrational-trace manuscript was found in the proof route and bibliography inspected here.

6. Return to the sources

- Theorem 1.1 · p. 1 — Check the group, coefficients and the two conclusions.
- Proposition 5.2; §5.8 · pp. 41–42, 49–50 — The simultaneous geometric and full-regular-norm requirements.
- Proposition 7.11 · p. 69 — Combine torsion exclusion and the integral detector.
- Lemma 8.5; (8.17)–(8.18) · pp. 76–77 — Pass from bounded operator families to exact quotient homotopies.
- Lemmas 8.7, 8.9 · pp. 80–81 — Recover equality in one group and then a finitely generated subgroup.
- §9.2 · pp. 82–83 — Compare the nonzero index pairing with zero assembly.

The reading covered the main statement, the simultaneous-selection interface in §5.8, §§6.3–6.4, cycle/cone arguments and integral detection in §7, finite formulas, quotient algebras, Bott cancellation and finite witnesses in §8, and the conclusion in §9. Statements and application hypotheses were checked in fixed versions of the three external inputs and the maximal-assembly comparison.

The complete sequential probability, recovery and pruning estimates in §§2–4, and the Fourier envelopes, finite regular-norm tests and centered moment calculations in §§5.4–5.7, have not been independently audited. These are important unresolved parts on which the later analytic realization depends. This is not a certification of the complete 85-page proof, an independent verification of all external proofs, expert peer review or formal verification.

A torsion-free counterexample to the Kadison–Kaplansky projection conjecture

A separated spectral band of small trace, followed by scalar compression in a free product

The manuscript claims a nontrivial projection in the reduced group C^* -algebra of a torsion-free group. Its main construction retains a spectral gap for a finite matrix in the regular representation while making the **unnormalized** trace of the upper projection small. A separate free-product estimate then converts the matrix projection into a scalar one.

Manuscript author: OpenAI · September 23, 2026 · 76 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A nontrivial scalar projection in the reduced group C^* -algebra

There exist a finitely generated torsion-free discrete group G_{proj} and a projection $e = e^* = e^2 \in C_r^*(G_{\text{proj}})$ such that the canonical trace satisfies

$$0 < \tau_{G_{\text{proj}}}(e) < \frac{1}{2}.$$

In particular, $e \neq 0, 1$. The target is the scalar reduced group C^* -algebra; this is not a statement about the group ring, the maximal group C^* -algebra, or merely a matrix algebra. Finite presentation is not asserted.

[Theorem 1.1 · p. 2](#)

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A torsion-free counterexample to the Kadison–Kaplansky projection conjecture](#). External inputs use the following keys.

- [MT10](#) Robin A. Moser and Gábor Tardos — A constructive proof of the general Lovász Local Lemma: Asymmetric local lemma for selecting expanding finite blocks. arXiv:0903.0544v3, 20 May 2009; published 2010.
- [BGW16](#) Paul Baum, Erik Guentner and Rufus Willett — Exactness and the Kadison–Kaplansky Conjecture: Consequence after construction: integral trace on the trivial-coefficient assembly image. Author-hosted 40-page manuscript, retrieved 2026-10-15; published 2016.
- [OpenAI26a](#) OpenAI — A torsion-free counterexample to reduced Baum–Connes injectivity: Comparison: failure of assembly injectivity; not a direct input. September 23, 2026.
- [OpenAI26c](#) OpenAI — An irrational-trace counterexample to reduced Baum–Connes: Comparison: irrational-trace failure of surjectivity; not a direct input. September 23, 2026.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — Connecting the matrix construction to a free product

Proof target : [Theorem 1.1](#) · p. 2

Proof route. A voltage cover of a tower labeled by a finite alphabet embeds in a Cayley graph, giving a torsion-free group Γ_{proj} and a finite matrix T . The local model's upper spectral band passes to T , while equivariant restrictions to all copies bound the trace of its projection. Passing to $\Gamma_{\text{proj}} * F_N$ then compresses the matrix projection using a single row. Section 3 explains the matrix construction; Section 4 explains the final compression.

1 Finite alphabet and voltage cover

$$Y \leftarrow P, \quad \Gamma_{\text{proj}} = \langle \mathcal{S} \mid \text{closed-walk labels in } P \rangle$$

Long-word recovery, girth, and a nilpotent torsion-free deck group F

Copy geometry, a separated band, and an equivariant trace bound

[Theorem 7.1 · pp. 26–27](#); [Theorem 10.11 · p. 72](#)

2 Combine geometry and analysis for the matrix construction

$$T = T^* \in M_N(\mathbb{C}[\Gamma_{\text{proj}}]), \quad p_0 \in M_N(C_r^*(\Gamma_{\text{proj}}))$$

$$0 < a = \sum_{j=1}^N \tau_{\Gamma}((p_0)_{jj}) < \frac{1}{2}$$

Γ_{proj} is finitely generated and torsion-free

Use the unnormalized trace $a < 1/2$ in the normal-form estimate

[Proposition 2.2 · pp. 4–6](#)

3 Free product and a row bounded below

$$G_{\text{proj}} = \Gamma_{\text{proj}} * F_N, \quad U = (\lambda(t_1), \dots, \lambda(t_N))$$

$$p_0 U^* U p_0 \geq (\sqrt{1-a} - \sqrt{a})^2 p_0$$

Corner inverse, partial isometry, and preservation of trace

[Section 11 · p. 73](#)

4 Scalar projection and all conclusions

$$e = (U p_0)(p_0 U^* U p_0)^{-1} (U p_0)^* \in C_r^*(G_{\text{proj}})$$

$$e = e^* = e^2, \quad 0 < \tau_{G_{\text{proj}}}(e) = a < \frac{1}{2}, \quad e \neq 0, 1$$

The free product also preserves finite generation and torsion-freeness

Geometry, analysis, and the free product leading to the main theorem

[Open figure ↗](#) [TeX source ↓](#)

1. Establishing the group hypotheses separately from the analytic output

The voltage group F is a nilpotent torsion-free group with integer coordinates. It is a central product of two integer affine groups, identifying their centers with opposite signs. The identified subgroup is primitive in the central $\mathbb{Z}^2$, leaving an integer central coordinate. Positive powers multiply the first nonzero graded coordinate by that positive integer, excluding torsion. This alone does not imply that the presented group Γ_{proj} is torsion-free. Theorem 7.1 uses long-word recovery and a girth condition on the cover to obtain embedded copies, a single finite ball containing each copy's earlier intersections, and a forest of remaining edges. Proposition 7.14 uses the embedding of copy stabilizers in F and the acyclicity of the cone complex. A subgroup

of order p would give the trivial module over $\mathbb{F}_p[C_p]$ a free resolution of length two, contradicting the periodic resolution with nonzero homology in arbitrarily high degrees. Finite generation follows from finiteness of the final alphabet.

[Lemma 8.3; equation \(8.5\) · pp. 40–42](#) [Theorem 7.1 · pp. 26–27](#) [Proposition 7.14 · pp. 36–38](#)

2. Obtaining an actual projection before passing to the free product

Theorem 10.11 supplies an actual projection $p_0 \in M_N(C_r^*(\Gamma_{\text{proj}}))$, not just an abstract K_0 class. The matrix trace is

$$\tau_{\Gamma,N}(p_0) = \sum_{j=1}^N \tau_{\Gamma}((p_0)_{jj}),$$

without division by N . Its value lies in $(0, 1/2)$, so Proposition 2.2 applies. The free product $G_{\text{proj}} = \Gamma_{\text{proj}} * F_N$ preserves finite generation and torsion-freeness, and its constructed scalar projection has the same trace.

[Theorem 10.11 · p. 72](#) [Section 11 · p. 73](#)

3. Proof of Theorem 3.1 — Preserving the gap and controlling the trace

Proof target : [Theorem 3.1 · p. 6](#)

Theorem 3.1 — A matrix projection of small trace

There exist a finitely generated torsion-free discrete group Γ_{proj} , a finite integer $N \geq 1$, and a self-adjoint $T \in M_N(\mathbb{C}[\Gamma_{\text{proj}}])$ whose spectrum in the regular representation has a nonempty separated subset. Its spectral projection p_0 satisfies

$$p_0 \in M_N(C_r^*(\Gamma_{\text{proj}})), \quad 0 < \tau_{\Gamma_{\text{proj}},N}(p_0) < \frac{1}{2}.$$

Separated means that the distance to the remainder of the spectrum is positive.

[Theorem 3.1 · p. 6](#)

1 A fiber projection small at both endpoints

$$p(t) = p_t \otimes p_{t-\delta}[V_t], \quad 0 < t < \delta$$

$$0 < s_p(t) \leq CNt^{n_f}(\delta - t)^{n_f}, \quad N = q^{n_f}$$

Combine expansion with windows on two adjacent layers

[Proposition 6.3 · pp. 24–25](#); [Lemma 9.1 · pp. 50–51](#)

2 Use constant vectors to make an upper band

$$O(t)E_*(t) = E_*(t), \quad O(t)|_{1-E_*(t)} \leq 1 - \chi$$

$$\text{tr}_{\text{id}} E_*(t) = s_p(t), \quad |J(t)| \leq 2$$

Finite tests and simultaneous sampling; localize the weighted trace

[Theorem 9.2 · pp. 52–63](#); [Propositions 10.3–10.4 · pp. 65–67](#)

3 Simultaneous girth and full regular-norm approximation

$$\|T_P - \Pi O \Pi\| \leq \eta, \quad |\mathcal{S}| = o(dD)$$

$$S = \sum_i 2^{C \cdot H_i} \sum_{v \text{ in layer } i} e_v = o(1)$$

One-ball overlaps and small perturbations; continuous functional calculus

[Theorem 7.1 · pp. 26–27](#); [Proposition 10.7 · pp. 68–69](#)

4 Trim domains and control the remaining forest

$$\|1_{B(Q)} E_Q\| \leq \sqrt{S}, \quad \|T - T_D\| = o(1)$$

$$0 \neq p_0 = 1_{[1-\chi/4, \infty)}(T) \in M_N(C_r^*(\Gamma_{\text{proj}}))$$

Count all incidences separately from the nonequivariant domain assignment

[Lemmas 10.8–10.10 · pp. 70–72](#)

5 Compare traces through an equivariant map to all copies

$$\mathcal{A}\xi = (E_Q(\xi|_Q))_Q, \quad K = \ker \mathcal{A}$$

$$p_0 \mathcal{H} \hookrightarrow K^\perp, \quad 0 < \tau_{\Gamma, N}(p_0) \leq S_0 \leq S < \frac{1}{2}$$

From a local model of small trace to a matrix projection in the reduced group C*-algebra

[Open figure ↗](#) [TeX source ↓](#)

1. Combining expansion with trace decay at both endpoints

The tower $\mathbf{Y}^0$ before pruning in Proposition 6.3 preserves normalized constants on each bipartite incidence and contracts their orthogonal complements by at least a uniform gap χ . This constant-preservation property is not assumed unchanged for the pruned graph $\mathbf{Y}$; the subsequent comparison uses the compression $\Pi O \Pi$. The vertex fraction β_ℓ lost in the pruning needed for label recovery tends uniformly to zero. To select expanding finite blocks, the proof

estimates the dependency neighborhoods and probabilities of bad-cut events and applies the asymmetric Lovász local lemma. It does not choose a different final parameter for every layer of the infinite tower.

Proposition 6.3 · pp. 24–25 Proposition 5.4 · pp. 18–20 [MT10, Theorem 1.1, p. 1]

On the analytic side, for the central Fourier variable t of F , the construction combines a twisted-convolution projection p_t with $p_{t-\delta}$ shifted by a finite magnetic representation. Here $n_f = m(m-1)/2$ counts the strictly upper triangular coordinates used in the construction. With $\delta = q^{-1}$ and $N = q^{n_f}$, for $0 < t < \delta$ one has

$$p(t) = p_t \otimes p_{t-\delta}[V_t], \quad 0 < s_p(t) = N \operatorname{tr}_t(p_t) \operatorname{tr}_{t-\delta}(p_{t-\delta}) \leq CNt^{n_f}(\delta - t)^{n_f}.$$

Absorbing the finite magnetic representation contributes the factor N to the trace. Decay at both endpoints must survive this factor. Windows satisfying $\sum_i h_i(t)^2 = 1$ occupy at most two adjacent layers; their supporting layers escape toward the two ends of the tower as t approaches the endpoints. A symmetric combination of normalized constant vectors gives the model $O(t)$ the eigenvalue $+1$. Expansion bounds all remaining positive spectrum by $1 - \chi$.

Proposition 8.7 · pp. 45–46 Lemma 8.9; equations (8.20)–(8.21) · pp. 47–48 Lemma 9.1 · pp. 50–51

2. Enforcing girth and regular-norm approximation with the same voltages

Theorem 9.2 simultaneously chooses edge voltages $z_e \in F$ and finitely many matrix weights W_e so that the cover has the required girth and

$$\|T_P - \Pi O \Pi\| \leq \eta, \quad |\mathcal{S}| = o(dD).$$

Here Π retains the surviving vertices and $\mathcal{S}$ is the finite alphabet augmented by weight tags.

Lemma 9.3 averages compressions to coordinate boxes in the nilpotent group, transferring finite matrix tests back to the **full regular operator norm**. Centered-moment estimates and bounds on the probability that a short closed walk has trivial holonomy make the total failure probability over all layers less than one. The claimed gap is not inferred merely from selected finite-dimensional representations.

Theorem 9.2 · pp. 52–63

3. Preserving the spectral gap with a nonequivariant domain assignment

Compression and a small perturbation of the model produce the cover's upper projection E_P . Let e_v be its identity diagonal summed over matrix indices at a base vertex, and let $w_i = 2^{C \cdot H_i}$ be the geometric ball-size weight. Endpoint decay and decay of the resolvent across layers yield

$$S := \sum_i w_i \sum_{v \text{ in layer } i} e_v \leq 96\mu_\ell \longrightarrow 0.$$

There is no extra layer-cardinality or matrix-dimension factor in this estimate. Removing earlier intersections from a copy Q produces disjoint domains D_Q and loses at most $\sqrt{S}$ in the norm of its upper projection. The remaining edges form a forest, so their weighted adjacency has norm at most $4C_\eta\sqrt{|\mathcal{S}|/(dD)} = o(1)$. These two small errors give, with $\alpha = \chi/1000$,

$$\text{spec}(T) \subset (-\infty, 1 - \chi + 4\alpha] \cup [1 - 3\alpha, 1 + 3\alpha].$$

The upper part is nonempty, and $1 - \chi/4$ lies in the gap. Continuous functional calculus therefore places $p_0 = 1_{[1-\chi/4, \infty)}(T)$ inside the reduced C-algebra*. Existence of a spectral projection in the von Neumann algebra alone would not establish this membership.

Proposition 8.11 · pp. 49–50 Propositions 10.3–10.4 · pp. 65–67 Proposition 10.7 · pp. 68–69

4. Bounding the trace through an equivariant map to all copies

The assignment of D_Q need not be equivariant, so the preceding ordinary partial-isometry comparison does not by itself compare canonical traces. Set $\mathcal{H} = \ell^2(\Gamma_{\text{proj}}) \otimes \mathbb{C}^N$. Instead use all indexed copies and define $\mathcal{A}\xi = (E_Q(\xi|_Q))_Q$. The equivariant incidence bijection $(Q, x) \leftrightarrow (x, v) \in \Gamma_{\text{proj}} \times V(Y)$ identifies the target upper space's identity trace with $S_0 = \sum_v e_v \leq S$.

The overlap-loss bound makes $\mathcal{A}$ bounded. On $K = \ker \mathcal{A}$, one has $\langle T\xi, \xi \rangle \leq (1 - \chi + 3\alpha)\|\xi\|^2$, whereas the lower bound on the range of p_0 is $1 - 3\alpha$. Hence $P_{K^\perp} : p_0\mathcal{H} \rightarrow K^\perp$ is injective. Equivariant polar decomposition gives

$$0 < \tau_{\Gamma, N}(p_0) \leq \text{Tr}_\Gamma(P_{K^\perp}) \leq S_0 \leq S < \frac{1}{2}.$$

Strict positivity follows from $p_0 \neq 0$ and faithfulness of the trace. Geometry, magnetic representation, and approximation accuracy are fixed in order, followed by one sufficiently large finite ℓ . The final step takes neither a limit group nor a limit projection.

Lemmas 10.8–10.10 · pp. 70–72 Theorem 10.11 · p. 72

4. Proof of Proposition 2.2 — From normal forms to compression inside the C^* -algebra

Proof target : Proposition 2.2 · pp. 4–6

Proposition 2.2 — Scalar compression after a free product

Let Λ be a discrete group and let $p \in M_N(C_r^*(\Lambda))$ be a projection satisfying $0 < a := \tau_{\Lambda, N}(p) < 1/2$. Set $K = \Lambda * F_N$, let $t_1, \dots, t_N$ freely generate F_N , and put $U = (\lambda_K(t_1), \dots, \lambda_K(t_N))$ and $A = pU^*Up$. Then A is invertible in the corner $pM_N(C_r^*(K))p$, and

$$e = (Up)A^{-1}(Up)^* \in C_r^*(K)$$

is a projection with $\tau_K(e) = a$. If Λ is finitely generated and torsion-free, so is K .

Proposition 2.2 · pp. 4–6

1 Projection with small unnormalized trace

$$p \in M_N(C_r^*(\Lambda)), \quad a = \tau_{\Lambda, N}(p) \in (0, \tfrac{1}{2})$$

$$K = \Lambda * F_N, \quad U = (\lambda_K(t_1), \dots, \lambda_K(t_N))$$

Subgroup inclusion preserves trace; diagonal traces control bad inputs

[Lemma 2.1 · p. 4](#); [equations \(2.3\)–\(2.7\) · p. 5](#)

2 Separate good and bad inputs by normal form

$$\xi = p\xi : \quad \|U\xi_g\| \geq \sqrt{1-a}\|\xi\|, \quad \|U\xi_b\| \leq \sqrt{a}\|\xi\|$$

$$\|U\xi\| \geq c_a\|\xi\|, \quad c_a = \sqrt{1-a} - \sqrt{a} > 0$$

The positive lower bound permits functional calculus in the corner

[equation \(2.2\) and proof · pp. 5–6](#)

3 Invert inside the C*-corner

$$A = pU^*Up, \quad c_a^2 p \leq A \leq Np$$

$$A^{-1/2} \in pM_N(C_r^*(K))p, \quad V = UpA^{-1/2}, \quad V^*V = p$$

Traciality and cyclically reduced free-product normal forms

[Proposition 2.2 · pp. 4–6](#)

4 Scalar projection, trace, and group conditions

$$e = VV^* = (Up)A^{-1}(Up)^* \in C_r^*(K), \quad \tau_K(e) = a$$

$$0 < a < \tfrac{1}{2} \implies e \neq 0, 1$$

If Λ is finitely generated and torsion-free, so is K

Normal-form lower bounds give a corner inverse and a scalar projection

[Open figure ↗](#) [TeX source ↓](#)

1. Controlling cancelling inputs by the trace

Subgroup inclusion preserves the reduced norm and trace. In coordinate j of $\xi = p\xi$, call an input bad if its normal form starts with t_j^{-1} and good otherwise. Good outputs have distinct initial letters t_j , hence disjoint supports. Each bad input position belongs to a unique bad coordinate. Cauchy–Schwarz using the projection's diagonal coefficients gives

$$\|U\xi_g\| \geq \sqrt{1-a}\|\xi\|, \quad \|U\xi_b\| \leq \sqrt{a}\|\xi\|.$$

Thus $\|U\xi\| \geq c_a\|\xi\|$ with $c_a = \sqrt{1-a} - \sqrt{a} > 0$. A normalized matrix trace cannot replace a in this estimate.

[Lemma 2.1 · p. 4](#) [equations \(2.3\)–\(2.7\) · p. 5](#)

2. Constructing the actual projection using the corner inverse

The inequalities $c_a^2 p \leq A \leq Np$ imply that A^{-1} and $A^{-1/2}$ belong to the same C^* -corner. The row $V = UpA^{-1/2} \in M_{1,N}(C_r^*(K))$ satisfies $V^*V = p$, and $e = VV^*$ is a projection in the scalar algebra. Traciality gives

$$\tau_K(e) = \sum_j \tau_K(V_j V_j^*) = \sum_j \tau_K(V_j^* V_j) = a.$$

The inequalities $0 < a < 1/2$ exclude both $e = 0$ and $e = 1$. This construction uses functional calculus in the corner and a finite row; it does not infer the existence of a projection merely from comparison of K_0 classes.

[equation \(2.2\) and proof](#) · pp. 5–6

3. Preserving the group conditions and completing the theorem

Take $\Lambda = \Gamma_{\text{proj}}$ and $p = p_0$. Adjoining finitely many free generators preserves finite generation. Every nontrivial free-product element is conjugate either into a factor or to a cyclically reduced word of length at least two; positive powers of the latter have increasing length. As both factors are torsion-free, the free product is torsion-free. This supplies the group conditions and all projection conclusions of Theorem 1.1.

[Proposition 2.2](#) · pp. 4–6 [Section 11](#) · p. 73

5. Which papers supply which steps

Source and result

MT10, Theorem 1.1

Used at

Proposition 5.4, pp.18–20

Role and required hypotheses

Simultaneous avoidance from dependency neighborhoods and asymmetric probability bounds for finitely many bad-cut events

Checking scope

Statement and application compared; not every probability count independently recomputed

Source and result

BGW16, Proposition 10.3

Used at

§11, p.73

Role and required hypotheses

Integral trace on the trivial-coefficient reduced assembly image for a countable torsion-free group

Checking scope

Statement and application hypotheses compared; proof of the index theorem outside scope

[MT10, Theorem 1.1, p. 1] [BGW16, Proposition 10.3, p. 36]

The latter is a consequence after the projection construction, not an input to it. The nonintegral trace of $[p_0]$ excludes it from the image of

$$\mu_{\Gamma,0}^r : K_0(B\Gamma) \longrightarrow K_0(C_r^*(\Gamma)).$$

The coefficient algebra is $\mathbb{C}$ with trivial action, $\Gamma = \Gamma_{\text{proj}}$ is torsion-free, and the failure is surjectivity. This argument does not construct an assembly kernel class.

Section 11 · p. 73 [BGW16, Proposition 10.3, p. 36]

Within the catalog, A torsion-free counterexample to reduced Baum–Connes injectivity asserts a different construction with a kernel class of infinite order. An irrational-trace counterexample to reduced Baum–Connes constructs an irrational-trace scalar projection without asserting torsion-freeness. The small nonintegral trace here must not be identified with an irrational trace. Within the compared main statements and introductions, and this manuscript's construction and bibliography, neither of those two papers was identified as a direct input to the present proof.

[OpenAI26a, Theorem 1.1, p. 1] [OpenAI26c, Theorem 1.1, p. 2]

6. Return to the sources

- Proposition 2.2 · pp. 4–6 — The separate matrix-to-scalar reduction.
- Theorem 7.1 · pp. 26–27 — Hypotheses and geometric outputs for embedding the cover.
- Lemma 8.9; equations (8.20)–(8.21) · pp. 47–48 — The fiber trace small at both endpoints, including the factor N .
- Theorem 9.2 · pp. 52–63 — Simultaneous girth and regular-norm approximation.
- Lemmas 10.8–10.10 · pp. 70–72 — The equivariant trace bound using all indexed copies.

- [Section 11 · p. 73](#) — Completion and the assembly consequence.

The editorial reading covered the compression in §2, incidence and torsion exclusion in §§7.5–7.6, twisted projections and endpoint trace bounds in §8, the model and simultaneous-realization reductions in §9, and the gap, trace, and completion arguments in §§10–11. The external statements and uses of the local lemma and assembly-image trace integrality were compared with their sources.

Not independently verified are all labeling tables and probability counts in §§4–6, all diagram transformations and the single-ball overlap estimate in §§7.2–7.4, the kernel identification in Lemma 8.5, and every template and rounding error in §§9.4–9.6. The article explains Theorems 7.1 and 9.2 as inputs from these internal constructions; it does not certify the complete main theorem. The separate manuscript cited for absence of group-ring idempotents was also outside this comparison. Complete external proofs and their further dependencies were not recursively verified.

An irrational-trace counterexample to reduced Baum–Connes

Exact local distributions from codes, and a spectral gap on the entire graph

The manuscript claims to construct an irrational-trace projection in the reduced group C^* -algebra of a semidirect product, using a finite-field lamp module modulo code relations on graphs. The decisive step is a uniform spectral gap that places the kernel projection in the reduced group C^* -algebra itself. Comparing its trace with the trace restriction on assembly then obstructs surjectivity of the coefficient-free reduced Baum–Connes map.

Manuscript author: OpenAI · September 23, 2026 · 23 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Irrational trace and failure of coefficient-free surjectivity

There exist a finitely generated discrete group G_{sur} and a projection $b_0 \in C_r^*(G_{\text{sur}})$ such that, for the canonical trace normalized by $\tau_{G_{\text{sur}}}(1) = 1$,

$$\tau_{G_{\text{sur}}}(b_0) \notin \mathbb{Q}, \quad [b_0] \notin \text{im}\left(\mu_0^{G_{\text{sur}}} : K_0^{G_{\text{sur}}}(\underline{E}G_{\text{sur}}) \longrightarrow K_0(C_r^*(G_{\text{sur}}))\right).$$

The manuscript therefore asserts that the coefficient-free reduced Baum–Connes conjecture for countable discrete groups is false. K-theory is integral complex K-theory, and the coefficient algebra is $\mathbb{C}$ with trivial action. The space $\underline{E}G$ classifies proper actions: fixed-point sets are contractible for finite subgroups and empty for infinite subgroups. No finite-dimensional or cocompact model is assumed. The domain is equivariant K-homology with

G -compact supports, defined as the direct limit of $KK_0^G(C_0(Z), \mathbb{C})$ over closed invariant G -compact subspaces Z of a proper G -CW model. No torsion-free condition is imposed on G_{sur} .

Theorem 1.1 · p. 2

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [An irrational-trace counterexample to reduced Baum–Connes](#). External inputs use the following keys.

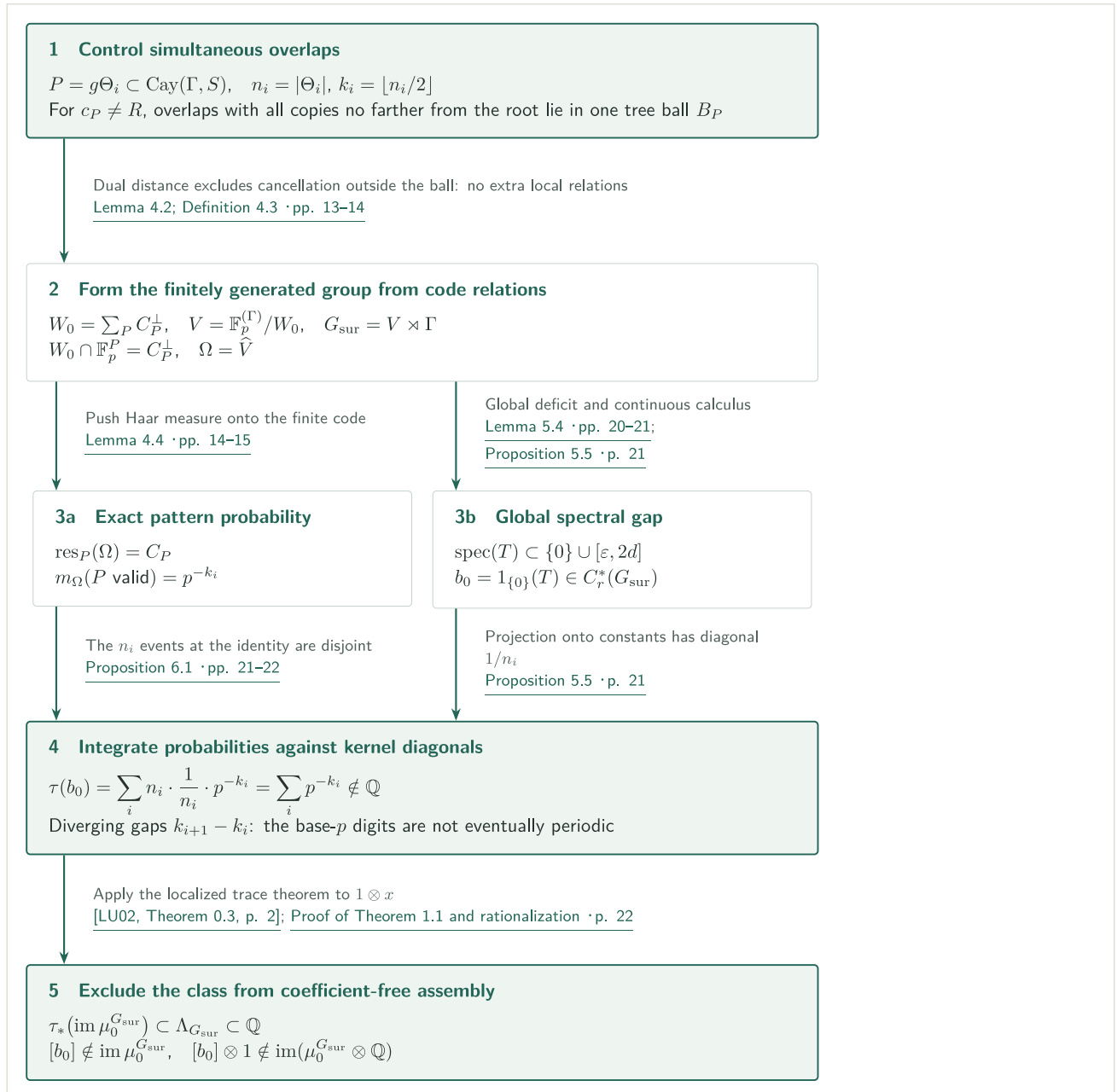
- [LU02](#) Wolfgang Lück — The relation between the Baum-Connes Conjecture and the Trace Conjecture: Trace restriction on assembly; use the localized image without assuming surjectivity. 2002 author-hosted typeset PDF, numbered 1–30 (not final journal pagination 123–152).
- [Osa20](#) Damian Osajda — Small cancellation labellings of some infinite graphs and applications: Finite labelling of expanders and isometric embeddings into a Cayley graph. *Acta Mathematica* 225 (2020), 159–191.
- [BH99](#) Martin R. Bridson, André Haefliger — *Metric Spaces of Non-Positive Curvature: Uniform hyperbolicity from a coarse linear filling bound*. Springer, 1999; scanned book, print/PDF offset +23.
- [LPS88](#) Alexander Lubotzky, Ralph Phillips, Peter Sarnak — Ramanujan graphs: Fixed-degree expanders with growing girth. *Combinatorica* 8 (1988), 261–277; published scan with visible folios.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — Joining local distributions and spectral estimates through the trace

Proof target : [Theorem 1.1 · p. 2](#)

Proof route. Codes on isometrically embedded expander copies define a finitely generated group $G_{\text{sur}} = V \rtimes \Gamma$. Dual distance gives exact probabilities for local patterns. Primal distance and the geometry of overlapping copies give a uniform gap. The resulting projection is measured using the local probabilities, and its trace is compared with Lück's restriction on assembly. The next section explains the global gap argument.



Local codes and a uniform gap combine to give an irrational-trace projection

[Open figure ↗](#) [TeX source ↓](#)

1. From simultaneous overlap control to the group and its local distributions

Choose expanders Θ_i of fixed degree d , with a common Laplacian gap $\rho > 0$ and growing girth σ_i . Put $n_i = |\Theta_i|$ and $k_i = \lfloor n_i/2 \rfloor$, passing to a subsequence so that k_i strictly increases and $k_{i+1} - k_i \rightarrow \infty$. The LPS family supplies the bounded degree, growing girth and bounded diameter/girth ratio required by Osajda's labelling theorem. Its finite alphabet S defines a graphical presentation of a group Γ in whose Cayley graph X the Θ_i embed isometrically.

[LPS88, Theorems 3.4 and 4.1, print pp. 267, 269 / PDF pp. 7, 9]

[Osa20, Theorem 2.7, print p. 173 / PDF p. 15] [Osa20, Lemma 3.1; Theorem 3.2, print p. 176 / PDF p. 18]
Proposition 3.1 · pp. 9–10

Cone off every translated copy $P = g\Theta_i$ by adding a center c_P . The manuscript bounds the number of arcs in compressed diagrams, obtains a uniform coarse linear filling inequality, and invokes Bridson–Haeffliger to obtain a hyperbolicity constant independent of the labels and λ . Only then is λ chosen small. For a root R and a copy with $c_P \neq R$, the intersections of P with **all** other copies whose centers are no farther from R (including equal distances) consequently lie in one intrinsic induced tree ball B_P . Its radius is at most $\sigma(P)/16$ and it has fewer than $\eta|P|$ vertices, where $\eta = 1/100$. Pairwise small intersections alone do not supply this simultaneous containment.

[BH99, III.H.2.1; III.H.2.9, print pp. 414, 419 / PDF pp. 437, 442] Proposition 2.3; Lemma 2.4 · pp. 7–9

Fix one sufficiently large prime $p > 2$ and encode each allowed set of d outgoing labels by a nonzero element of $\mathbb{F}_p$. Write w_P for the prescribed word on a copy. Lemma 4.1 constructs a code $w_P \in C_P \subset \mathbb{F}_p^P$ of dimension k_i , with every nonzero word in either C_P or $C_P^\perp$ supported on more than $\eta|P|$ vertices. Existence follows by choosing a random k_i -dimensional subspace containing w_P : the number of low-weight words is less than $p^{n_i/4}$, and a union bound makes the probability of a low-weight primal or dual word less than one. Impose all translated dual relations:

$$W_0 = \sum_P C_P^\perp \subset \mathbb{F}_p^{(\Gamma)}, \quad V = \mathbb{F}_p^{(\Gamma)} / W_0, \quad G_{\text{sur}} = V \rtimes \Gamma.$$

The subspace W_0 is Γ -invariant, and V is generated by the Γ -orbit of $v_* = [e_1]$. Thus the lifts of S , together with $(v_*, 1)$, generate G_{sur} .

Lemma 4.1 · pp. 12–13 Lemma 4.2; Definition 4.3 · pp. 13–14

The key equality is $W_0 \cap \mathbb{F}_p^P = C_P^\perp$. If a finite sum supported on P used a relation from another copy, choose a participating copy Q farthest from the root c_P . All its intersections with the other supports lie in B_Q , whereas the dual word has larger support. A coordinate outside this ball cannot cancel, a contradiction. Dualizing to $\Omega = \text{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p)$ shows that restriction $\omega \mapsto X_\omega|_P$ has image exactly C_P , and pushes Haar measure to its uniform distribution. The prescribed pattern therefore has probability p^{-k_i} .

Lemma 4.2; Definition 4.3 · pp. 13–14 Lemma 4.4 · pp. 14–15

The uniform gap for the operator T in the next section gives $b_0 = 1_{\{0\}}(T) \in C_r^*(G_{\text{sur}})$. This is an actual projection in the ordinary reduced algebra, not merely a class in a matrix algebra or a projection available only in the von Neumann algebra.

Proposition 5.5 · p. 21

2. Canceling copy counts against diagonal entries

Call a copy valid when it has the prescribed pattern. Valid copies are disjoint isolated components of the enabled graph, and $\pi_\omega(b_0)$ projects onto constants on each of them and vanishes on the remainder. Its diagonal entry on a type i copy is $1/n_i$. Exactly n_i type i copies contain the identity: their translations are $g = \varphi_i(v)^{-1}$ for the embedded vertices v . The events that these copies are valid are mutually exclusive, including across types. Hence

$$\tau_{G_{\text{sur}}}(b_0) = \int_{\Omega} \langle \pi_\omega(b_0) \delta_1, \delta_1 \rangle dm_\Omega = \sum_i n_i \frac{1}{n_i} p^{-k_i} = \sum_i p^{-k_i}.$$

No independence between copies is needed. Since $p > 2$, the base- p expansion has digit one exactly at the positions k_i . The infinitely many ones have diverging gaps, so this expansion is not eventually periodic. The trace is therefore irrational and lies strictly between zero and one.

Section 5.1, (17)–(21) · pp. 15–16 Proposition 6.1 · pp. 21–22

3. Excluding the class from proper-action assembly

For any countable discrete group G , set $\Lambda_G = \mathbb{Z}[1/|F| : F \leq G \text{ finite}] \subset \mathbb{Q}$. Lück's Theorem 0.3 identifies the image of localized assembly followed by the canonical trace with Λ_G . Evaluating at $1 \otimes x$ for an integral class x yields

$$\tau_{G*}(\mu_0^G(x)) \in \Lambda_G \subset \mathbb{Q}.$$

Neither surjectivity of assembly nor injectivity of localization is required. This is the manuscript's Theorem 1.2. Lück's Section 4 uses the direct limit of equivariant KK-groups over cocompact subcomplexes of a proper space, and Section 5 uses its analytic index. The manuscript employs this proper-action assembly.

[LU02, Theorem 0.3, p. 2] [LU02, Sections 4–5, pp. 23, 25] Theorem 1.2 · p. 2

The irrational trace of $[b_0]$ excludes it from the image. The identification $C_r^*(V \rtimes \Gamma) \cong C(\Omega) \rtimes_r \Gamma$ is used to analyze the operator; the obstructed map is the coefficient-free assembly map for G_{sur} itself. A finite rational linear combination of assembly classes still has rational trace, so $[b_0] \otimes 1$ also lies outside the rationalized assembly image, as observed in the unnumbered consequence on manuscript p.22.

Proof of Theorem 1.1 and rationalization · p. 22

3. Proof of Proposition 5.5 — From local deficits to a global gap

Proof target : Proposition 5.5 · p. 21

Proposition 5.5 — A kernel projection in the reduced group C*-algebra

Use the graphs, codes and group selected in Proposition 3.1 and Section 4, and put $\Omega = \widehat{V}$. Let $f_s \in C(\Omega)$ be the clopen-set projection recording that the coordinate $\omega(v_*)$ allows the outgoing label s , and let u_s be the canonical unitary for $s \in \Gamma$. Define

$$T = dI - \sum_{s \in S^{\pm 1}} f_s u_s f_{s^{-1}} \in C(\Omega) \rtimes_r \Gamma \cong C_r^*(G_{\text{sur}}).$$

Let $\mathcal{G}_\omega$ retain the Cayley edges allowed at both endpoints, and let U_ω be the complement of the valid copies. Then $\pi_\omega(T) = dI - \text{Adj}(\mathcal{G}_\omega)$. There is a projection $b_0 \in C_r^*(G_{\text{sur}})$ such that, for every ω , $\pi_\omega(b_0)$ is the orthogonal projection onto $\ker \pi_\omega(T)$. It projects onto constants on each valid copy and is zero on $\ell^2(U_\omega)$. Its diagonal entry at the identity is $1/n_i$ if the unique valid copy containing that vertex has type i , and zero if there is no such copy.

Section 5.1, (17)–(21) · pp. 15–16 Proposition 5.5 · p. 21

In the diagram, A is the graph of all enabled edges on a finite set $Y \subset U_\omega$, and L_j is the subspace of $Z_1(A; \mathbb{F}_2)$ whose previously eliminated copy coordinates vanish.

1 Missing edges on invalid copies

$$X_\omega|_P - w_P \in C_P \setminus \{0\}, \quad |\text{supp}(X_\omega|_P - w_P)| > \eta|P|$$

$$d|B| - 2e_{\omega,P}(B) \geq c|B| \quad (B \subset P)$$

Extra edges created by cancellation lie inside the overlap ball B_P
[Lemma 3.2 · p. 11](#); [Lemmas 5.2–5.3 · pp. 16–19](#)

2 Eliminate cycle coordinates from farthest copies

$$Z_1(X; \mathbb{F}_2) = \bigoplus_P Z_1(P; \mathbb{F}_2)$$

$$\text{pr}_P(L_j) \subset Z_1(D_P; \mathbb{F}_2), \quad D_P = B_P \cup (A \cap P)$$

Girth bounds extra attachment cycles; elimination avoids double counting
[Lemmas 5.2–5.3 · pp. 16–19](#)

3 Charge rank to disjoint exterior vertex sets

$$b_1(D_P) \leq (d/2 - 1 - c/4)|(Y \cap P) \setminus B_P|$$

$$d|Y| - 2|E(A)| \geq (c/2)|Y| \quad (Y \subset U_\omega \text{ finite})$$

Level sets and Cauchy–Schwarz, retaining $q(x) = d - \deg_\omega(x)$
[Lemma 5.4 · pp. 20–21](#)

4 Obtain a uniform gap on copies and the remainder

$$\pi_\omega(T)|_{\ell^2(U_\omega)} \geq c^2/(8d), \quad \varepsilon = \min\{\rho, c^2/(8d)\} > 0$$

$$\ker \pi_\omega(T) = \bigoplus_{P \text{ valid}} \mathbb{C}1_P, \quad \text{spec}(\pi_\omega(T)) \subset \{0\} \cup [\varepsilon, 2d]$$

Transfer the common gap to the faithful sum and use continuous calculus
[Section 5.1, \(17\)–\(21\) · pp. 15–16](#); [Proposition 5.5 · p. 21](#)

5 Construct the kernel projection in the reduced algebra

$$\bigoplus_\omega \pi_\omega \text{ faithful}, \quad b_0 = \chi(T) \in C_r^*(G_{\text{sur}})$$

$$\chi(0) = 1, \quad \chi|_{[\varepsilon, 2d]} = 0, \quad b_0^2 = b_0 = b_0^*$$

Eliminating cycle coordinates gives a uniform gap and continuous functional calculus

[Open figure ↗](#) [TeX source ↓](#)

1. Counting local deficits without overlap

On an invalid copy, the support of $X_\omega|_P - w_P \in C_P \setminus \{0\}$ exceeds $\eta|P|$. Every discrepancy removes at least one prescribed edge incidence. Combining these missing incidences with the expander boundary estimate gives $d|B| - 2e_{\omega,P}(B) \geq c|B|$ for every $B \subset P$, where $c = (\eta/2) \min\{\kappa, 1\} > 0$ and κ is the edge-expansion constant. Overlaps prevent simply summing this inequality over copies.

Lemmas 5.2–5.3 · pp. 16–19

The finite-support cycle space decomposes as $Z_1(X; \mathbb{F}_2) = \bigoplus_P Z_1(P; \mathbb{F}_2)$. For the enabled graph A on a finite set $Y \subset U_\omega$, eliminate its cycle coordinates in decreasing order of distance from a root. Coordinate projections are not restrictions of edge chains, but any extra edge arising through cancellation with remaining copies lies in B_P . The projected coordinate therefore belongs to the cycle space of $D_P = B_P \cup (A \cap P)$.

Lemma 3.2 · p. 11 Lemmas 5.2–5.3 · pp. 16–19

If a component C outside B_P has l_C attachment edges to the ball, its contribution is $b_1(C) + \max(0, l_C - 1)$. The girth and radius bounds separate distinct attachment endpoints by more than $\sigma(P)/2$, bounding the extra rank by $4|C|/\sigma(P)$. The local deficit and $\sigma(P) > 20/c$ give

$$b_1(D_P) \leq (d/2 - 1 - c/4)|(Y \cap P) \setminus B_P|.$$

The exterior sets $(Y \cap P) \setminus B_P$ are pairwise disjoint in the elimination order. Their charges can therefore be summed. Euler's formula then gives the global deficit

$$d|Y| - 2|E(A)| \geq (c/2)|Y|.$$

This charging to exterior vertices is the central step from individual expander estimates to the whole graph.

Lemmas 5.2–5.3 · pp. 16–19

2. A uniform lower bound on the remaining components

On U_ω , retain the vertex potential $q(x) = d - \deg_\omega(x) \geq 0$. Integrating the finite-set deficit over level sets of $|\xi|^2$ and applying Cauchy–Schwarz jointly to the edge and potential terms gives

$$\langle \pi_\omega(T)\xi, \xi \rangle = \sum_{\{x,y\} \in E} |\xi(x) - \xi(y)|^2 + \sum_x q(x) |\xi(x)|^2 \geq \frac{c^2}{8d} \|\xi\|^2.$$

Each valid copy has gap ρ off its constants, and no enabled edge joins different valid copies or the remainder. Thus, uniformly in ω ,

$$\text{spec}(\pi_\omega(T)) \subset \{0\} \cup [\varepsilon, 2d], \quad \varepsilon = \min\{\rho, c^2/(8d)\} > 0, \quad \ker \pi_\omega(T) = \bigoplus_{P \text{ valid}} \mathbb{C}1_P.$$

Lemma 5.4 · pp. 20–21

3. Returning through a faithful representation to continuous functional calculus

The direct sum $\bigoplus_{\omega} \pi_{\omega}$ of the regular representations induced from evaluations is faithful on the reduced crossed product. Uniformity of the gap prevents nonzero spectrum from approaching zero in this direct sum. Consequently T has the same gap in $C_r^*(G_{\text{sur}})$. Choose a continuous function with $\chi(0) = 1$ and $\chi|_{[\epsilon, 2d]} = 0$. Then $b_0 = \chi(T)$ is a projection in that C*-algebra. Functional calculus commutes with every π_{ω} , yielding the asserted kernel and diagonal descriptions.

Section 5.1, (17)–(21) · pp. 15–16 Proposition 5.5 · p. 21

4. Which papers supply which steps

LPS supplies the expander family in Proposition 3.1; Osajda supplies its finite labelling and isometric embeddings; Bridson–Haefliger supplies uniform hyperbolicity in Proposition 2.3; and Lück supplies Theorem 1.2 and the final trace comparison. The two code distances, exact local relations and global cycle-rank estimate are arguments within this manuscript.

Inspection of its introduction, construction, final proof and bibliography did not identify direct use of the other two manuscripts in this catalogue. Shared use of expanders or projections alone is not recorded as a proof dependency.

5. Return to the sources

- Proposition 2.3; Lemma 2.4 · pp. 7–9 — Contain simultaneous overlaps in one tree ball.
- Lemma 4.2; Definition 4.3 · pp. 13–14 — Exact local relations and finite generation.
- Lemmas 5.2–5.3 · pp. 16–19 — Eliminate cycle coordinates and charge ranks to disjoint exterior sets.
- Lemma 5.4 · pp. 20–21 — Obtain a uniform bound including the remaining graph.
- Proposition 6.1 · pp. 21–22 — Compute the trace from copy counts, probabilities and diagonal entries.
- Proof of Theorem 1.1 and rationalization · p. 22 — Coefficient-free assembly and failure of rationalized surjectivity.

Editorial checking covered the cited reductions and calculations in Sections 2–6, the main theorem's domain, codomain and conclusions, and the statements and applications of the four direct external inputs. Lück's author-hosted typeset PDF is numbered 1–30, distinguished here from the final journal pagination 123–152. In the published LPS scan, Theorems 3.4 and 4.1 and the visible printed folios 267 and 269 were checked on PDF pages 7 and 9.

We have not independently reconstructed every singular plane-diagram surgery in Lemma 2.2 or the disk-filling conversion in Proposition 2.3. Complete proofs of external inputs, their recursive references, and the Davis–Lück comparison papers remain unchecked. This account does not constitute expert refereeing or independent verification of the entire counterexample.