

OpenAI Math Digest

CATALOGUE 246

Cannon's conjecture

Catalogue guide & one proof overview

Independent reading guide · English edition

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Contents

The catalogue guide is followed by the article. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

Cannon's conjecture

This catalogue concerns realizing a hyperbolic group with sphere boundary by isometries of hyperbolic three-space. Its official subject is Group theory. The single manuscript claims an unconditional solution through bounded combinatorial modulus on the boundary. The exposition separates the author's claim from the editor's checking scope.

1. Paper in this catalogue

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 246 · 2026-10-15 checked source snapshot

#

01

Manuscript and overview

A Modulus Proof of Cannon's Conjecture →

Short name: Modulus proof of Cannon's conjecture

September 23, 2026 · 31 pages

Original paper (GitHub) ↗

Principal claim

Claims a proper cocompact isometric action on hyperbolic three-space with finite kernel for a hyperbolic group with sphere boundary; in the torsion-free case, the group is a closed hyperbolic three-manifold group.

Theorem 1.1 · p. 2; Corollary 8.1 · p. 29 ↗

Role and available overview

Extracts a function and measure from unbounded 2-modulus, excludes both growth regimes, then constructs the action by uniformization and an invariant conformal structure.

Read the proof overview →

2. Enter the argument

The [branching proof overview](#) shows where two contradictions to unbounded modulus join at uniformization. The [construction of the action on hyperbolic space](#) distinguishes topological homeomorphism, quasi-Möbius equivalence, and equivariant conjugacy, continuing to the finite-kernel conclusion for the original group.

Direct inputs from uniformization, quasiconformal mapping theory, and hyperbolic groups are collected with their checking scope in the article's [source-and-application table](#). Some cross-catalogue relations remain uninvestigated; this is not a declaration of no dependencies.

3. Sources and checking scope

The fixed official catalogue lists entry 246 as “Cannon’s conjecture”. Group theory is the eleventh subject in official display order, covering entries 246–259; entry 246 is on page 27 of [overview.pdf](#). There is one manuscript in official order, dated September 23, 2026, with 31 pages including references.

[OpenAI Research Catalog](#) · [Catalogue 246; Group theory](#) · p. 27

[Official inventory](#)

The article self-checks the main theorem, the proof connections of the principal intermediate propositions and direct inputs, and the torsion-free manifold consequence. It is not an independent verification of the entire proof. The parallel HK95 route, virtual-structure inputs to Corollary 8.2, and systematic cross-catalogue comparisons are outside this check. Details appear in the article’s [return-to-sources](#) guide.

A Modulus Proof of Cannon's Conjecture

Two contradictions bounding boundary modulus, followed by an action on hyperbolic space

The manuscript claims to resolve Cannon's conjecture by putting a visual metric on a hyperbolic group boundary homeomorphic to the two-sphere and proving boundedness of its combinatorial 2-moduli. A real-valued function and a finite measure extracted from hypothetical unboundedness form the core of the new argument. This article follows their construction, the exclusion of exponential and subexponential growth, and the realization of the group action after uniformization.

Manuscript author: OpenAI · September 23, 2026 · 31 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A geometric action with finite kernel

Let G be a hyperbolic group whose Gromov boundary $Z = \partial G$ is homeomorphic to S^2 . There exists a homomorphism

$$\rho : G \longrightarrow \text{Isom}(\mathbb{H}^3)$$

whose isometric action is proper and cocompact and whose kernel $\ker \rho$ is finite.

Properness means that $\{g \in G : \rho(g)K \cap K \neq \emptyset\}$ is finite for every compact $K \subset \mathbb{H}^3$. No torsion-free, faithful-boundary-action, or orientation-preserving hypothesis is imposed. The isometry group includes orientation-reversing isometries.

[Theorem 1.1 · p. 2](#)

Corollary 8.1 — A closed hyperbolic manifold in the torsion-free case

If, in addition, G is torsion-free, then it is the fundamental group of a closed hyperbolic three-manifold. The manifold may be nonorientable.

Corollary 8.1 · p. 29

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A Modulus Proof of Cannon’s Conjecture](#). External inputs use the following keys.

- [BK13](#) Marc Bourdon and Bruce Kleiner — Combinatorial modulus, the combinatorial Loewner property, and Coxeter groups: Modulus uniformization for an approximately self-similar metric sphere. *Groups, Geometry, and Dynamics* 7 (2013), 39–107; published version.
- [AIM09](#) Kari Astala, Tadeusz Iwaniec, and Gaven Martin — Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane: Metric-to-analytic quasiconformality, null sets, differentials, and conformal structures. Princeton University Press, 2009; publisher preview (70 PDF pages).
- [Tuk80](#) Pekka Tukia — On two-dimensional quasiconformal groups: Invariant measurable conformal structures from hyperbolic circumcenters. *Annales Academiæ Scientiarum Fennicæ, Series A I Mathematica* 5 (1980), 73–78; published version.
- [AB60](#) Lars Ahlfors and Lipman Bers — Riemann’s mapping theorem for variable metrics: Composition of weak differentials and the measurable Riemann mapping theorem. *Annals of Mathematics, Second Series* 72 (1960), 385–404; scanned published version with cover page.
- [BH99](#) Martin R. Bridson and André Haeffliger — Metric Spaces of Non-Positive Curvature: Quasi-geodesic cyclic orbits of infinite-order elements, used in Lemma 4.4. Springer, *Grundlehren der mathematischen Wissenschaften* 319, 1999; scanned book.
- [HK95](#) Juha Heinonen and Pekka Koskela — Definitions of quasiconformality: Parallel metric characterization cited by Proposition 2.6; external statement unchecked here. *Inventiones Mathematicæ* 120 (1995), 61–79; bibliography verified, original statement not inspected.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

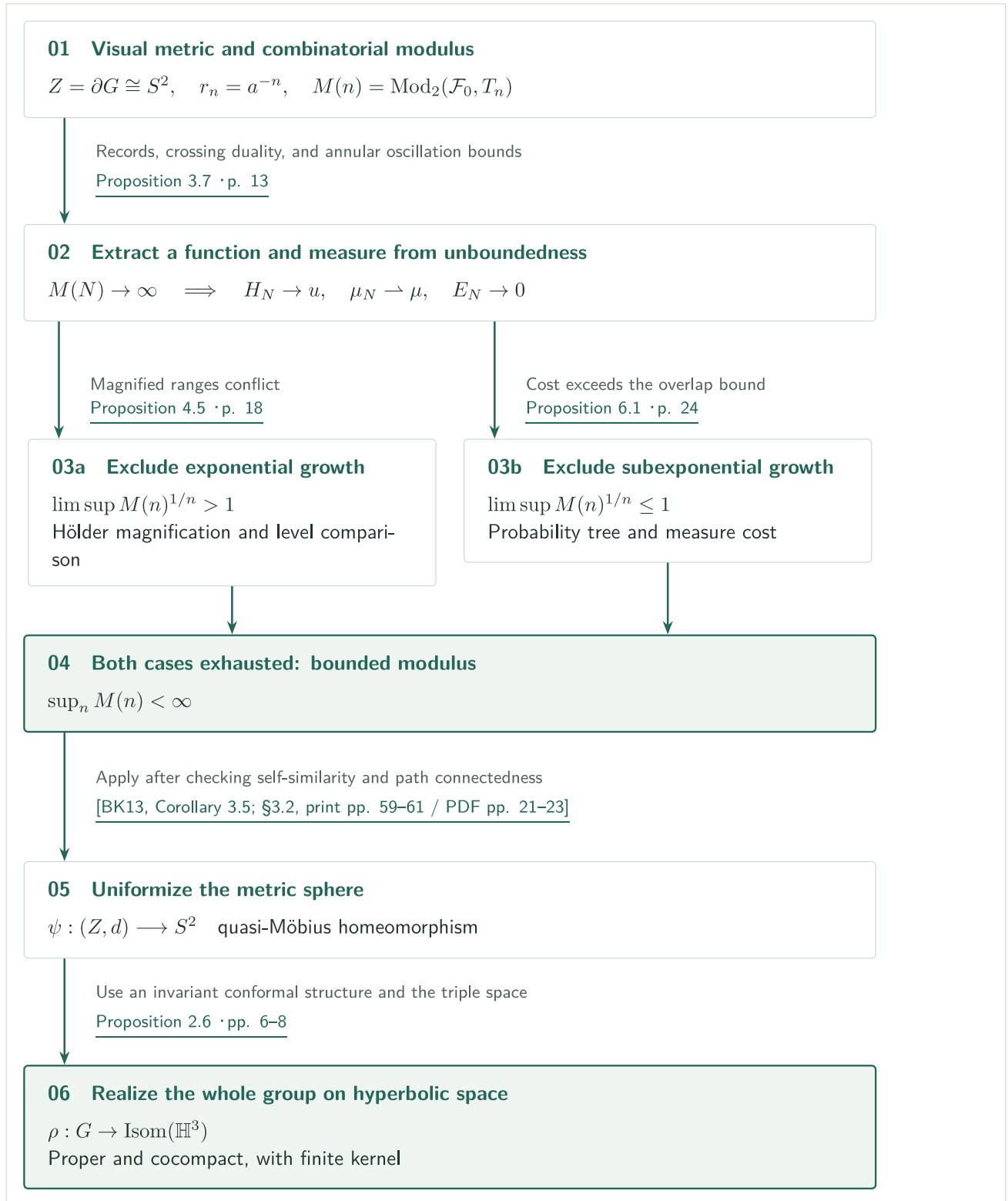
Proof target : [Theorem 1.1](#) · p. 2

Fix the hypotheses and full conclusion of the main theorem. In the Cayley graph for a finite symmetric generating set, take basepoint e and a visual metric $d(y, z) \asymp a^{-(y|z)_e}$ for some $a > 1$. All balls, diameters, and oscillations below refer to d . Write B for open balls and $\overline{B}$ for closed balls. Set $r_n = a^{-n}$, choose a maximal r_n -separated set T_n , and use the open cells $B(q, 2r_n)$.

For the family $\mathcal{F}_0$ of continuous curves of diameter at least d_0 , define

$$M(n) = \min_{\omega \geq 0} \left\{ \sum_{q \in T_n} \omega(q)^2 : \sum_{B(q, 2r_n) \cap \pi \neq \emptyset} \omega(q) \geq 1 \quad (\pi \in \mathcal{F}_0) \right\}.$$

Fix $d_0 > 0$ small enough both for expanded scale- R curves to belong to $\mathcal{F}_0$ and for the uniformization criterion. **Proof route.** Assuming that $M(n)$ is unbounded produces a common scalar limit. The two possible growth regimes then lead to contradictions. Metric uniformization is invoked only after boundedness has been obtained.



The two unbounded-growth regimes are excluded and the argument joins at uniformization and the isometric action

[Open figure ↗](#) [TeX source ↓](#)

1. Separate the topological input from the metric conclusion

The topological sphere supplies separation and crossing continua in coordinate rectangles. Quasi-Möbius equivalence of the visual metric to the round sphere is not inferred from this topological hypothesis alone. Choose a vertex v at depth k on a ray to x , and put $h = v^{-1}$, $R = r_k$, and $D_y = \max\{R, d(x, y)\}$. Lemma 2.1 gives

$$d(hy, hz) \asymp \frac{R d(y, z)}{D_y D_z}.$$

On a small ball this is a uniform bi-Lipschitz comparison after rescaling distances by R^{-1} , supplying approximate self-similarity. The packing bound $\# \lesssim (t/s)^Q$ in Lemma 2.3 gives doubling; Lemma 2.4 supplies quantitatively controlled paths.

Lemmas 2.1–2.4 · pp. 4–6

2. Turn boundedness into the uniformization input

With the scalar limit of Proposition 3.7 as their common input, Proposition 4.5 excludes exponential growth and Proposition 6.1 excludes unbounded subexponential growth. Thus $\sup_n M(n) < \infty$. The external criterion used in Theorem 2.5 requires an arcwise connected, approximately self-similar metric sphere and uniform bounds for the 2-moduli of large curves in its graph approximations. Choosing $2^{-m}/a < r_n \leq 2^{-m}$ makes the cells above a fixed approximation for any $\kappa > 2a$. No Coxeter-group hypothesis is required. The resulting $\psi : (Z, d) \rightarrow S^2$ is a quasi-Möbius homeomorphism; equivariance with a Möbius action has not yet been established. Proposition 2.6 constructs a further conjugacy and realizes the whole group.

§2.3; Theorem 2.5 · p. 6

[BK13, Corollary 3.5; §3.2, print pp. 59–61 / PDF pp. 21–23]

3. Proposition 3.7 — From unbounded modulus to a scalar limit

Proof target : Proposition 3.7 · p. 13

Proposition 3.7 — The scalar limit

Assume that $M(n)$ is unbounded. If $\limsup M(n)^{1/n} > 1$, choose

$1 < \lambda < \limsup M(n)^{1/n}$; otherwise take $\lambda = 1$. Record indices for $M(N)/\lambda^N$ satisfy $M(N) \rightarrow \infty$ and $M(N - \ell) \leq \lambda^{-\ell} M(N)$. In the coordinate rectangles $Q \subset \widehat{Q}$ selected in §3, let $U = \text{int } \widehat{Q}$. The construction gives continuous H_N with band values 0, 1 and weights ρ_N ; put $\mu_N = M(N) \sum_q \rho_N(q)^2 \delta_q$.

Along a subsequence, $H_N \rightarrow u$ locally uniformly on U and $\mu_N \rightarrow \mu$, where $u : U \rightarrow [0, 1]$ is continuous and nonconstant and μ is finite. The function is respectively 0, 1 on the interiors of the two bands; it is locally Hölder if $\lambda > 1$. For each compact $K \subset U$, all sufficiently small $s > 0$, and uniformly for $x \in K$,

$$\left(\text{osc}_{\overline{B}(x,s)} u \right)^2 \leq C \mu(\overline{B}(x, C_\mu s)).$$

For each $t \in (0, 1)$ there is a continuum $C_t \subset Q$ of diameter $\geq d_1$, with $u = t$ on $C_t \cap U$, meeting every compact path in U whose endpoint values strictly straddle t . The fixed-transform energy bounds of Lemma 3.5 remain valid along this subsequence. No extension of u outside U is asserted.

Proposition 3.7 · p. 13

01 Fix records and a crossing rectangle

$$M(N - \ell) \leq \lambda^{-\ell} M(N), \quad \text{Mod}_2(\text{horizontal crossings}) \gtrsim M(N)$$

Least-norm incidence vector; min–max envelopes

[Lemmas 3.1–3.4 · pp. 8–10](#)

02 Dualize crossings and regularize band values

$$\sum \rho_N^2 \lesssim M(N)^{-1}, \quad H_N|_{\text{top}} = 0, \quad H_N|_{\text{bottom}} = 1$$

Level continua block straddling paths

Expand level crossings and use modulus at the smaller scale

[Lemma 3.6 · p. 12](#)

03 Measure oscillation on annuli

$$|H_N(x) - H_N(y)|^2 \lesssim \lambda^{-\ell} \mu_N(A_\ell), \quad \mu_N = M(N) \sum \rho_N(q)^2 \delta_q$$

Sum disjoint annuli; closed-set weak limits; fix transformations first

[Proposition 3.7 · p. 13](#)

04 Keep three inputs at the limit

$$H_N \rightarrow u, \quad \mu_N \rightarrow \mu; \quad (\text{osc}_{\overline{B}(x,s)} u)^2 \lesssim \mu(\overline{B}(x, C_\mu s))$$

$$C_t \cap U \subset \{u = t\}, \quad \text{diam } C_t \geq d_1; \quad E_N^{(J)}(H_N \circ g^{-1}; K) \rightarrow 0$$

Dual crossing weights control oscillation while retaining a function, a measure, and large level continua

[Open figure ↗](#) [TeX source ↓](#)

1. Obtain both small energy and large level sets

Localization to finitely many coordinate rectangles selects a fixed rectangle whose horizontal crossing modulus is bounded below by a constant times $M(N)$. The least-norm point in the convex hull of the horizontal cell-incidence vectors supplies admissible weights ρ_N for the transverse crossings. This finite-dimensional duality yields $\sum \rho_N^2 \lesssim M(N)^{-1}$. The manuscript proves this step internally.

Interpolate a path-cost function on a coordinate triangulation, then apply a min–max envelope from the upper band and a max–min envelope from the lower band. These operations produce H_N without increasing path oscillations and allow strict sublevel and superlevel sets to be connected to their respective bands. Jordan separation then makes each transverse level continuum $C_{N,t}$ block every path straddling t .

[Lemmas 3.1–3.4 · pp. 8–10](#)

2. Distinguish estimates passed to the limit from those retained on approximants

Lemma 3.6 expands an annulus of radius $R = r_\ell$ and applies modulus at scale $N - \ell$. It bounds the squared endpoint-value difference by $M(N - \ell)$ times the sum of ρ_N^2 in the annulus. Inserting the record inequality and summing over disjoint annuli gives oscillation $O(k^{-1/2})$ at scale r_k . If $\lambda > 1$, one annulus gives $O(\lambda^{-k/2})$. These are the equicontinuity and additional Hölder estimates needed for local uniform convergence. The closed-ball measure inequality uses the closed-set upper bound for weak convergence.

Lemma 3.6 · p. 12

For each fixed $g \in G$, compact $K \subset gU$, and fixed $J \geq 2$, retain the local energy formed from closed-ball oscillations:

$$E_N^{(J)}(H_N \circ g^{-1}; K) = \sum_{\substack{q \in T_N \\ \overline{B}(q, 2r_N) \cap K \neq \emptyset}} \left(\text{osc}_{\overline{B}(q, Jr_N)}(H_N \circ g^{-1}) \right)^2 \longrightarrow 0.$$

The bi-Lipschitz constant of an individual group element need not be uniform over the group. Subsequent comparisons fix the transformations or the pair being compared before sending $N \rightarrow \infty$. The Hausdorff subsequence for a level continuum may depend on t ; no simultaneous selection for every t is required.

Lemma 3.5 · p. 11

4. Proposition 4.5 — Excluding exponential growth

Proof target : Proposition 4.5 · p. 18

Proposition 4.5 — Contradiction for exponential records

In the unbounded-modulus and record-index setup of Proposition 3.7, the case $\lambda > 1$ cannot occur. This is a conclusion within the contradiction setup assuming unbounded modulus.

Proposition 4.5 · p. 18

01 Magnify while preserving Hölder variation

$$\lambda > 1, \quad H_N \rightarrow u \quad \implies \quad W : Z \setminus \{b\} \rightarrow \mathbb{R} \text{ nonconstant}$$

Weighted oscillation records and diagonal selection

Lemmas 4.1–4.3 · pp. 14–17

02 Retain small energy on approximants

$$h_j, \sigma_j \text{ fixed first,} \quad f_j = \frac{H_{N_j} \circ h_j^{-1} - u(x_j)}{\sigma_j}, \quad E_{N_j}(f_j) \rightarrow 0$$

Bound level-continuum variation by the two energies

Lemmas 3.8–3.9 · p. 14

03 Compare transforms with different punctures

$$p_s \neq p_t \quad \implies \quad W_t(x) = W_t(p_s) \quad (x \in \mathcal{N}(W_s) \setminus \{p_t\})$$

Remagnify away from stabilizer directions; choose distinct punctures

[BH99, III.Γ, Corollary 3.10(1), print p. 462 / PDF p. 485]

Lemma 4.4 · pp. 17–18 Proposition 4.5 · p. 18

04 Separated ranges conflict with an equal value

$$W_j(D_1) \cap W_j(D_2) = \emptyset, \quad W_j(x_1) = W_j(p_i) = W_j(x_2) \\ x_\nu \in \mathcal{N}(W_i) \cap D_\nu \quad \implies \quad \perp$$

Magnified Hölder limits force the same value in two regions with separated ranges

Open figure ↗ TeX source ↓

1. Equip the magnified limit with comparable approximants

Expand a variation region of the Hölder function u and normalize its values by positive affine maps. Weighted oscillation records in Lemma 4.1 produce a nonconstant locally Hölder limit W on the punctured sphere $Z \setminus \{b\}$. Magnifying u alone would not automatically preserve the vanishing-energy estimates for fixed transformations. Lemma 4.2 first fixes the expansion h_j and normalization σ_j , then chooses N_j sufficiently large so that

$$f_j(z) = \frac{H_{N_j}(h_j^{-1}z) - u(x_j)}{\sigma_j}$$

also has vanishing local energy along the diagonal sequence.

The level comparison of Lemma 3.8 says that if w oscillates by at least b_0 on continua where $f = t$, then $b_0|I|$ is at most the sum of the products of their cell oscillations. Cauchy–Schwarz contradicts this when both energies vanish. Applying this comparison in a common compact domain, Lemma 4.3 obtains, for transforms W_s, W_t with distinct punctures p_s, p_t ,

$$W_t(x) = W_t(p_s) \quad (x \in \mathcal{N}(W_s) \setminus \{p_t\}),$$

where $\mathcal{N}(f)$ is the set of points at which f is not locally constant.

Lemmas 4.1–4.3 · pp. 14–17

Lemmas 3.8–3.9 · p. 14

2. Move the punctures and contradict range separation

Lemma 4.4 bounds the accumulation directions of $\text{Stab}_G(b)$ by two. The argument compares a linear count of short stabilizer elements with the quadratic number of products that two different axis directions would produce. The input from BH99, Corollary 3.10(1), states that the cyclic orbit of an infinite-order element in a hyperbolic group is a quasi-geodesic. The manuscript uses this to treat the element as loxodromic with two boundary endpoints. The statement and use were compared.

[BH99, III.Г, Corollary 3.10(1), print p. 462 / PDF p. 485]

Lemma 4.4 · pp. 17–18

Magnify W again in a variation region avoiding this finite set and b . Its punctures can then be chosen pairwise distinct. Choose two small disks in the new limit, each containing variation and with disjoint range intervals. For two sufficiently late magnified functions, the preceding comparison forces one to have the same value at non-locally-constant points of the other in both disks. Its two range intervals are disjoint, giving the contradiction.

Proposition 4.5 · p. 18

5. Proposition 6.1 — Excluding subexponential growth

Proof target : Proposition 6.1 · p. 24

Proposition 6.1 — Unbounded subexponential modulus is impossible

In the boundary, metric, and cell setup of the main theorem, $M(n)$ cannot be unbounded while satisfying $\limsup_{n \rightarrow \infty} M(n)^{1/n} \leq 1$.

Proposition 6.1 · p. 24

01 Subdivide value intervals at a uniform depth

$$\limsup M(n)^{1/n} \leq 1, \quad J_y \subset J, \quad t_y = |J_y|/|J|$$

$$\sum_y \min(a^{-\gamma D} t_y, a^{\tau D} t_y^2) \geq 1$$

Cancel interval length before fixing D ; square bound and Jensen
[Lemma 5.2 · pp. 21–23](#) [\(5.20\)–\(5.22\)](#); [Lemma 5.3 · pp. 23–24](#)

02 Flow probability and retain good nodes

$$P_i \leq a^{-\gamma D n} \Delta_i / \Delta_0, \quad \sum_i \Delta_i^2 \leq E_0$$

Good mass is at least $1/2$ at infinitely many generations

Control near pairs by inverse prefixes and conditional probabilities
[Lemmas 6.2–6.3 · pp. 25–26](#)

03 Select many far puncture pairs at one depth

$$k = k_0 + Dn, \quad R = a^{-k}, \quad \ell = D \lfloor \gamma n / Q \rfloor$$

$$NP_* \gtrsim k^{-1}, \quad \#\{\{i, j\} : d_{ij} > a^{-\ell}\} \gtrsim N^2$$

Fix a pair for level comparison; control the enlarged ball
[Lemma 6.4 · pp. 26–27](#)

04 Assign an oscillation cost to one endpoint

$$D_{ij} = \overline{B}(h_i^{-1} p_j, C_\mu s_{ij}), \quad s_{ij} \asymp R^2 / d_{ij}^2$$

$$\mu(D_{ij}) \gtrsim \kappa^2 \Delta_i^2, \quad h_i D_{ij} \subset \overline{B}(p_j, C_3 R)$$

Count enlarged charged balls directly; no measure invariance
[\(6.9\)–\(6.14\); §7 · pp. 28–29](#)

05 Cost and overlap cannot coexist

$$ck^{-2} a^{2\gamma D n} \leq \sum \mu(D_{ij}) \leq C a^{Q\ell} \mu(Z)$$

$$Q\ell \leq \gamma D n \implies a^{\gamma D n} \leq C k^2 \perp$$

A probability tree of value intervals produces many far puncture pairs, whose measure costs exceed their allowed overlap

[Open figure ↗](#) [TeX source ↓](#)

1. Choose one subdivision depth even for short value intervals

The function u need not be Hölder in this regime. Section 5 instead builds a tree of centers x_i , radii R_i , and nonempty open value intervals $J_i \subset (0, 1)$ realized nearby. Write $\Delta_i = |J_i|$. Child intervals are nested inside parent intervals, and radii decrease by a^{-D} for one fixed D . At a “good” node, two fixed target balls after expansion each contain variation of at least $\kappa\Delta_i$, while their **whole-ball range intervals** are separated by at least $\Delta_i/4$. Balls need not be connected.

A node that is not good has many separated child candidates retaining a definite fraction of the value interval. At every node, the capacity estimate of Lemma 5.2 gives

$$\sum_y \min\{a^{-\gamma D} t_y, a^{\tau D} t_y^2\} \geq 1, \quad t_y = |J_y|/|J|,$$

where $\tau = 1/20$, $\beta = \tau/[4(Q+1)]$, and $\gamma = \beta/2$. The proof integrates level values $t \in J$ over many radial bins at the intermediate scale $a^{-\lceil \beta D \rceil} R$. After removing candidates with large ranges, it transfers modulus to the remaining small-range candidates and applies Cauchy–Schwarz. The factor $|J|$ cancels before D is chosen, so the same D works even as intervals shorten deep in the tree.

Lemma 5.2 · pp. 21–23

2. Retain substantial probability on good nodes

Choose child probabilities p_y with $p_y \leq a^{-\gamma D} t_y$ at every node, $p_y \leq a^{-D/2} t_y^2$ at nodes that are not good, and $p_y \leq a^{\tau D} t_y^2$ at good nodes. For a generation- n node i of probability P_i with m_i ancestors that are not good,

$$P_i \leq a^{-\gamma D n} \frac{\Delta_i}{\Delta_0}, \quad \Delta_i^2 \geq \Delta_0^2 P_i a^{D((1/2+\tau)m_i - \tau n)}.$$

Separation of centers and the measure estimate in Proposition 3.7 give $\sum_i \Delta_i^2 \leq E_0 < \infty$. Jensen’s inequality bounds the mean proportion of bad ancestors by $1/11 + o(1)$. Consequently infinitely many generations have good-node mass at least $1/2$. Conditional probabilities for descendants m generations below an ancestor A also satisfy $P_i/P_A \leq a^{-\gamma D m}$.

(5.20)–(5.22); Lemma 5.3 · pp. 23–24

3. Compare pairwise oscillation costs with measure multiplicity

At such a generation set $k = k_0 + Dn$ and $R = a^{-k}$. Select one of finitely many target-ball patterns and one of $O(k)$ probability classes, obtaining N nodes with $P_* \leq P_i \leq 2P_*$ and $NP_* \gtrsim k^{-1}$. For the punctures $p_i = h_i z_0$ of their expansions h_i , inverse-word prefixes at

common depth bound multiplicity at radius $O(R)$. Combining prefixes with the ancestor conditional-probability bound shows that, for $\ell = D\lfloor \gamma n/Q \rfloor$, the mass within $a^{-\ell}$ of a puncture is $O(a^{-\gamma D\lfloor \gamma n/Q \rfloor})$. Thus at least cN^2 pairs have $d_{ij} = d(p_i, p_j) > a^{-\ell} \gg R$.

Lemmas 6.2–6.3 · pp. 25–26

Fix a far pair before passing to the limit in the approximation index. The level comparison in Lemma 6.4 forces one endpoint, say i , to have u -oscillation at least $\kappa\Delta_i/16$ on $h_i^{-1}\overline{B}(p_j, A_h R)$. If both endpoints qualify, either assignment is allowed.

Lemma 6.4 · pp. 26–27

With $z_{ij} = h_i^{-1}p_j$ and $s_{ij} \asymp R^2/d_{ij}^2$, the closed ball $D_{ij} = \overline{B}(z_{ij}, C_\mu s_{ij})$ pays for this oscillation: $\mu(D_{ij}) \gtrsim \kappa^2 \Delta_i^2$. Crucially, the **enlarged closed ball** still satisfies $h_i D_{ij} \subset \overline{B}(p_j, C_3 R)$. Indeed, $C_\mu s_{ij} / \max(R, d(x_i, z_{ij})) \lesssim R/d_{ij} \rightarrow 0$, keeping the expansion denominators comparable throughout the enlarged ball. Combining bounded multiplicity in j for fixed i with packing for $D_{ij} \subset \overline{B}(x_i, C_4 R a^\ell)$ gives

$$ck^{-2}a^{2\gamma Dn} \leq \sum_{\text{assigned far pairs}} \mu(D_{ij}) \leq Ca^{Q\ell}\mu(Z).$$

Since $Q\ell \leq \gamma Dn$, this yields $a^{\gamma Dn} \leq Ck^2$, contradicting $k = k_0 + Dn$. The argument counts the actual enlarged charged balls; it does not assume that μ is invariant under the group.

(6.9)–(6.14); §7 · pp. 28–29

6. Proposition 2.6 — From the uniformized boundary to the whole group action

Proof target : Proposition 2.6 · pp. 6–8

Proposition 2.6 — The bridge to hyperbolic space

Let G, Z, d be as in the [main theorem setup](#). If (Z, d) is quasi-Möbius homeomorphic to the round sphere, then G admits a proper cocompact isometric action on $\mathbb{H}^3$ with finite kernel.

Proposition 2.6 · pp. 6–8

01 Conjugate the boundary action onto the sphere

$$\psi : (Z, d) \rightarrow S^2, \quad \mathcal{G} = \psi G \psi^{-1}$$

Infinitesimal distortion criterion; hyperbolic circumcenters

[AIM09, Theorems 2.5.3–2.5.4, print pp. 26–27 / PDF pp. 45–46]

[Tuk80, Theorem and invariant-structure construction, print pp. 73–76 /

PDF pp. 1–4]

02 Build an invariant structure from uniform quasiconformality

$$\sup_{f \in \mathcal{G}} K(f) < \infty, \quad q(x) = \text{circumcenter}\{f^* q_{\text{round}}(x)\}$$

Measurable Riemann mapping and composition, including reversals

[AB60, Theorem 6, print p. 396 / PDF p. 13]

03 Integrate the measurable structure and extend isometrically

$$F^* q_{\text{round}} = q, \quad F \mathcal{G} F^{-1} \subset \text{Möb}^\pm(S^2)$$

$$\rho : G \rightarrow \text{Isom}(\mathbb{H}^3)$$

$F\psi$ conjugates the boundary actions

Boundary expansion supplies properness and cocompactness on triples

Proposition 2.6 · pp. 6–8

04 Transfer the action from triples to hyperbolic space

$$T(S^2) \simeq I/H \longrightarrow I/K_o \simeq \mathbb{H}^3, \quad |H| = 2$$

Equivariant proper surjection: proper, cocompact, finite kernel

Integrate an invariant conformal structure to construct an isometric representation, then transfer properness and cocompactness from triples

Open figure ↗ TeX source ↓

1. Convert the quasi-Möbius action into an equivariant isometric action

Under $\psi : Z \rightarrow S^2$, the conjugated group $\mathcal{G} = \psi G \psi^{-1}$ has one quasi-Möbius distortion function for all its elements. Letting three points coalesce in a four-point cross-ratio estimate bounds infinitesimal linear distortion uniformly. The metric and analytic characterizations in planar charts then make $\mathcal{G}$ uniformly quasiconformal. For a reversing map, apply the characterization after a conformal reflection.

[AIM09, Theorems 2.5.3–2.5.4, print pp. 26–27 / PDF pp. 45–46]

Countability and preservation of null sets provide an invariant conull set where all differentials and composition rules hold. Positive definite quadratic forms on each tangent space, modulo positive scalars, form a hyperbolic plane; pullback by any invertible linear map, of either orientation, is an isometry. The orbit of the round conformal structure under pullbacks is

bounded. Its unique circumcenter defines an invariant measurable conformal structure q . The manuscript applies Tukia's circumcenter construction to the whole group using this quadratic-form description.

[AIM09, Theorem 3.1.2, print p. 49 / PDF p. 68]

[AIM09, §2.3; Theorem 2.3.2, print pp. 18–20 / PDF pp. 37–39]

[Tuk80, Theorem and invariant-structure construction, print pp. 73–76 / PDF pp. 1–4]

The measurable Riemann mapping theorem gives a quasiconformal homeomorphism F with $F^*q_{\text{round}} = q$. The differential composition rule and invariance make $F\mathcal{G}F^{-1}$ Möbius or anti-Möbius. Unique hyperbolic isometric extensions define the homomorphism ρ . The composite $F\psi$ now conjugates the original boundary action to the action of ρ at infinity; this is the map carrying equivariance.

[AB60, Lemma 10; Theorem 5, print pp. 393–395 / PDF pp. 10–12]

[AB60, Theorem 6, print p. 396 / PDF p. 13]

2. Transfer properness, cocompactness, and finite kernel through triples

The boundary expansion estimates make the action on ordered distinct triples $T(Z)$ proper and cocompact. A long group element brings two points of a uniformly separated triple close, giving properness. Expansion at the scale of its closest pair separates all three points uniformly, giving cocompactness. Conjugation by $F\psi$ preserves these properties.

Put $I = \mathbf{Isom}(\mathbb{H}^3)$. The stabilizer H of a standard ordered ideal triple consists of the identity and reflection in the hyperbolic plane through that triple. Choose an H -fixed point o and its compact stabilizer K_o . Then

$$T(S^2) \simeq I/H \longrightarrow I/K_o \simeq \mathbb{H}^3$$

is an equivariant proper surjection. Properness and cocompactness transfer from triples to hyperbolic space through this map. The kernel lies in a point stabilizer and is therefore finite. The faithful action belongs to $G/\ker \rho$; the conclusion for the original G is an action with finite kernel. No restriction to a finite-index orientation-preserving subgroup, or replacement by a quasi-action, is used as the final conclusion.

Proposition 2.6 · pp. 6–8

7. Corollary 8.1 — From a free action to a closed manifold

Proof target : Corollary 8.1 · p. 29

Use the additional hypothesis and conclusion of Corollary 8.1.

01 Add torsion-freeness G is torsion-free, $G \curvearrowright \mathbb{H}^3$

Proper cocompact isometric action

Combine finite stabilizers from properness with torsion-freeness

[Corollary 8.1 · p. 29](#)**02 Finite point stabilizers become trivial**

$$|G_x| < \infty \implies G_x = \{1\} \quad (x \in \mathbb{H}^3)$$

Free action, cocompactness, and a simply connected universal cover

[Corollary 8.1 · p. 29](#)**03 Identify the fundamental group of a closed hyperbolic manifold**

$$M = \mathbb{H}^3/G, \quad \pi_1(M) \cong G$$

Orientability is not required

Torsion-freeness eliminates finite stabilizers and identifies the group with the fundamental group of a closed hyperbolic manifold

[Open figure ↗](#) [TeX source ↓](#)

1. Eliminate finite stabilizers

Point stabilizers in the proper action of Theorem 1.1 are finite. If G is torsion-free, they are trivial, so the action is faithful and free. Cocompactness makes $\mathbb{H}^3/G$ compact, the free isometric action makes it a hyperbolic manifold without boundary, and simple connectedness of $\mathbb{H}^3$ identifies its fundamental group with G . Orientation preservation was not assumed, so orientability is not concluded. Virtual fibering and the further conclusions of Corollary 8.2 require separate external inputs and are not among the principal results covered here.

[Corollary 8.1 · p. 29](#)

8. Which papers supply which steps

Source and result

[BK13, Corollary 3.5; §3.2, print pp. 59–61 / PDF pp. 21–23]

Used at

Theorem 2.5, overview

Input and hypotheses

Arcwise connected, approximately self-similar metric sphere; uniform 2-modulus bounds for a fixed approximation

Checking scope

Source hypotheses and their verification here compared; no recursive verification of the underlying Bonk–Kleiner theorem

Source and result

[AIM09, Theorems 2.5.3–2.5.4, print pp. 26–27 / PDF pp. 45–46]

Used at

Proposition 2.6, action construction

Input and hypotheses

Infinitesimal distortion in planar charts gives quasiconformality

Checking scope

Statements and use, including reflection, compared

Source and result

[AIM09, Theorem 3.1.2, print p. 49 / PDF p. 68]

Used at

Same step

Input and hypotheses

Null-set preservation and nondegenerate differentials for a countable group

Checking scope

Statement and use compared

Source and result

[Tuk80, Theorem and invariant-structure construction, print pp. 73–76 / PDF pp. 1–4]

Used at

Same step

Input and hypotheses

Circumcenter of a uniformly bounded orbit of conformal structures

Checking scope

Measurability and invariance construction compared; reversals also use the manuscript’s quadratic-form formulation

Source and result

[AB60, Theorem 6, print p. 396 / PDF p. 13]

Used at

Same step

Input and hypotheses

Integration of a bounded measurable Beltrami coefficient

Checking scope

Statement and application compared; the composition rule uses Lemma 10 and Theorem 5 of the same source

Source and result

[BH99, III. Γ, Corollary 3.10(1), print p. 462 / PDF p. 485]

Used at

Lemma 4.4, exponential case

Input and hypotheses

Infinite-order element of a hyperbolic group; quasi-geodesic cyclic orbit supplies two endpoints

Checking scope

Original statement and manuscript use compared

The BH99 input in Lemma 4.4 was checked as a direct dependency against its original statement and use. The parallel HK95 characterization cited in Proposition 2.6 was not checked separately; the table follows the inspected AIM09 route. Section 3’s finite convex duality was read in the

manuscript itself: another modulus-duality paper in the bibliography is not automatically treated as a direct proof input. Direct dependencies, generalizations, or alternative proofs in other catalogues were not systematically investigated; this does not mean that there are no such connections.

9. Return to the sources

- §2.3; Theorem 2.5 · p. 6 — Visual metric, cells, choice of d_0 , and uniformization hypotheses
- Proposition 3.7 · p. 13 — Limit domain, closed-ball measure estimate, and energy after fixed transformations
- Lemmas 4.1–4.3 · pp. 14–17 — Order of choosing normalizations and approximation indices
- Lemma 5.2 · pp. 21–23 — A subdivision depth uniform in the length of the value interval
- (6.9)–(6.14); §7 · pp. 28–29 — Images and multiplicity of enlarged charged balls; final contradiction
- Proposition 2.6 · pp. 6–8 — Bridge from quasi-Möbius conjugacy to an isometric action of the whole group

The editor read the statements, proof text of §§2–7, and Corollary 8.1 in the fixed September 23, 2026 version and self-checked the claim-to-proof connections presented here. Particular checks concerned closed-ball measure estimates, fixing transformations before choosing approximation indices, subdivision depth independent of interval length, and multiplicity of enlarged charged balls. External-input checks have the scope shown above. This is a single author’s content check, not independent peer review, independent verification of the entire proof, or an audit of formalization.

The manuscript and official catalogue claim an unconditional solution. This article presents that claim and its argument; it does not assert editorial certification that the conjecture has been resolved. The parallel HK95 route, external inputs to Corollary 8.2 and later material, and cross-catalogue relations retain the unchecked or out-of-scope status described above.