

OpenAI Math Digest

CATALOGUE 197

Nonsofic groups and group-ring counterexamples

Catalogue synthesis & 4 proof overviews

Independent reading guide · English edition

Edition: 2026-10-18

Created by: Masataka Iwai · Osaka University

AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 4 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

00 · Catalogue synthesis	3
01 · A Torsion-Free Group Algebra That Is Not Directly Finite	12
02 · A Counterexample to Kaplansky's Direct-Finiteness Conjecture in Characteristic Two	23
03 · A Counterexample to the Group-Ring Determinant Conjecture	35
04 · A Counterexample to Kaplansky's Direct-Finiteness Conjecture in Odd Characteristic	44

Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

Catalogue 197 · Synthesis

[Read on the website ↗](#)

CATALOGUE 197

Nonsofic groups and group-ring counterexamples

The three scalar direct-finiteness counterexamples differ in coefficient field, torsion, and construction. The finite-field characteristic-two example also supplies an input for a counterexample to the integer group-ring determinant lower bound. The guide separates scalar elements, group-ring matrices, nonsoficity, and cellular-automaton consequences.

Papers in official catalogue order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 197 · 2026-10-18 checked source snapshot

#

01

Manuscript and overview

A Torsion-Free Group Algebra That Is Not Directly Finite →

Short name: Torsion-free direct finiteness

October 4, 2026 · 31 pages

Original paper (GitHub) ↗

Principal claim

A finitely presented torsion-free G and $a, b, c \in \mathbb{F}_2[G]$ satisfy $ab=1$, $ac=0$, $c \neq 0$, hence $ba \neq 1$. G has a finite two-dimensional classifying complex.

Theorem 1.1 · p. 1 ↗

Role and available overview

An independent construction combining simultaneous-walk parity with cone complexes and arrangement exclusion, separately securing a nonzero coefficient and torsion-freeness.

Read the proof overview →

#

02

[Manuscript and overview](#)

A Counterexample to Kaplansky's Direct-Finiteness Conjecture in Characteristic Two →

Short name: Direct finiteness in characteristic two

September 23, 2026 · 28 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

A scalar pair $ab=1 \neq ba$ over a finite characteristic-two field K and a finitely presented group with odd-prime torsion; also an injective nonsurjective CA on all configurations.

[Theorem 1.1 · p. 1](#); [Corollary 7.3 · p. 26](#) ↗[Role and available overview](#)

Punctured finite-geometry pairings, embeddings, character packing, and scalarization; supplies the characteristic-two input used by the determinant paper.

[Read the proof overview](#) →

#

03

[Manuscript and overview](#)

A Counterexample to the Group-Ring Determinant Conjecture →

Short name: Group-ring determinant

September 23, 2026 · 9 pages

[Original paper \(GitHub\)](#) ↗[Principal claim](#)

An integral group-ring matrix over finitely generated G_D is invertible over $\mathbb{Q}[G_D]$, has Fuglede–Kadison determinant strictly between 0 and 1, and has a finite negative logarithmic integral.

[Theorem 1.1 · p. 2](#) ↗[Role and available overview](#)

Turns paper 2’s scalar defect into a binary factorization/real-trace gap, lifts modulo 2^k , and calculates an integral-matrix determinant.

[Read the proof overview](#) →

#

04

Manuscript and overview

A Counterexample to Kaplansky's Direct-Finiteness Conjecture in Odd Characteristic →

Short name: Direct finiteness in odd characteristic

September 26, 2026 · 25 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For one specified odd prime p , a field K of order p^4 and a finitely generated group containing torsion admit a scalar pair $ab=1 \neq ba$ and an injective nonsurjective CA on all configurations.

Theorem 1.1 · p. 1; Proposition 8.1 · p. 23 ↗

Role and available overview

High-degree modular idempotents, a common central sector, directed triangular compression, and finite-factor scalarization; does not depend on the characteristic-two paper.

[Read the proof overview](#) →

How the constructions and results connect

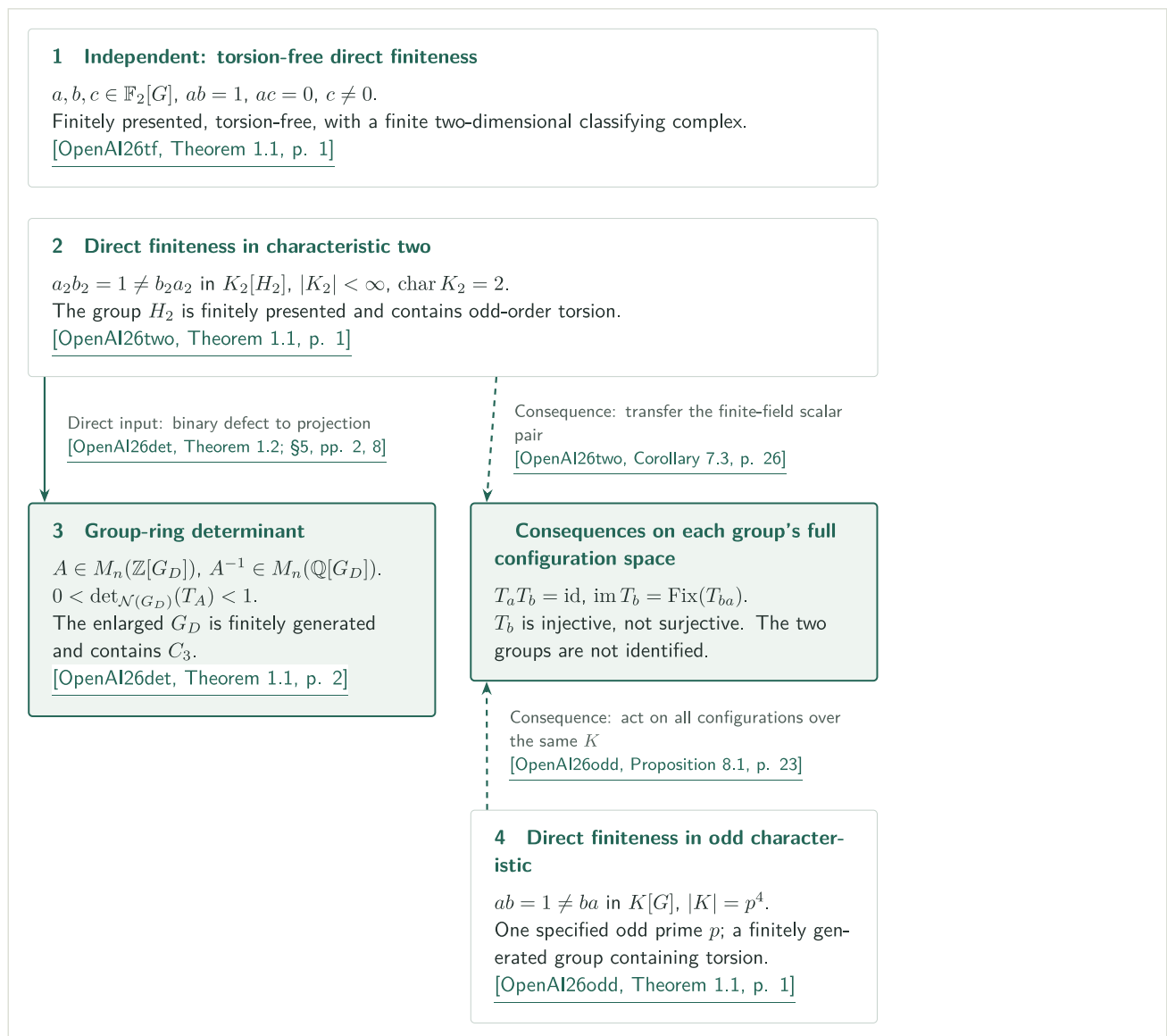
The solid arrow records an actual inter-paper proof input. Dashed arrows are cellular-automaton consequences of completed scalar pairs, not inputs for constructing another group-algebra counterexample. The torsion-free construction at the top is independent.

References on the diagram

- **OpenAI26tf** — OpenAI, A Torsion-Free Group Algebra That Is Not Directly Finite, October 4, 2026. The independent prime-field $\mathbb{F}_2$, torsion-free construction. [[OpenAI26tf](#), Theorem 1.1, p. 1]
- **OpenAI26two** — OpenAI, A Counterexample to Kaplansky's Direct-Finiteness Conjecture in Characteristic Two, September 23, 2026. Supplies the scalar pair used by the determinant paper and its own CA consequence. [[OpenAI26two](#), Theorem 1.1, p. 1]

- **OpenAI26det** — OpenAI, A Counterexample to the Group-Ring Determinant Conjecture, September 23, 2026. Converts a binary defect into an integral group-ring matrix. [OpenAI26det, Theorem 1.1, p. 2]
- **OpenAI26odd** — OpenAI, A Counterexample to Kaplansky's Direct-Finiteness Conjecture in Odd Characteristic, September 26, 2026. An independent construction for one specified odd prime and its CA consequence. [OpenAI26odd, Theorem 1.1, p. 1]

All four versions are fixed to the same official commit. Citations give original result numbers and pages, with printed and PDF pagination agreeing.



Independent torsion-free construction, the characteristic-two determinant input, and cellular-automaton consequences of two scalar constructions

[Open figure ↗](#) [TeX source ↓](#)

The scalar pair from [Direct finiteness in characteristic two](#) is restricted to a binary matrix pair in [Group-ring determinant](#). A projection over the enlarged group $G_D = (\bigoplus_{H_2} C_3) \rtimes H_2$ then yields an integral matrix with determinant strictly between zero and one. The input and output groups differ, and the latter contains order-three torsion. Since the starting field need not be $\mathbb{F}_2$ itself, the coefficient-restriction dimension d must be retained.

[Direct finiteness in odd characteristic](#) does not replace this binary input. Transferring its scalar pair over the specified p to T_c , using the same right-neighbor convention as the characteristic-two paper, gives $T_a T_b = \text{id}$ and $\text{im } T_b = \text{Fix}(T_{ba})$ in both cases. A nonzero coefficient of the reverse defect detects nonsurjectivity on the full configuration space.

Kind

Direct input

Supplying result $\rightarrow$ use

Characteristic two, Theorem 1.1 $\rightarrow$ Determinant, Theorem 1.2 and §5

Content and hypotheses

A scalar pair over a finite characteristic-two field and a finitely generated group. Finite presentation of the input is weakened to finite generation here.

Kind

Internal consequence

Supplying result $\rightarrow$ use

Characteristic two, Theorem 1.1 $\rightarrow$ Proposition 7.2 and Corollary 7.3

Content and hypotheses

A finite field, finite support, and $ab = 1 \neq ba$ give an injective nonsurjective CA and its fixed-point image description.

Kind

Internal consequence

Supplying result $\rightarrow$ use

Odd characteristic, Theorem 1.1 $\rightarrow$ Proposition 8.1

Content and hypotheses

The same algebraic conditions are used over the unchanged specified field of order p^4 .

[OpenAI26two, Theorem 1.1, p. 1]

[OpenAI26det, Theorem 1.2; §5, pp. 2, 8]

[OpenAI26two, Corollary 7.3, p. 26]

[OpenAI26odd, Proposition 8.1, p. 23]

Nonsoficity is not a construction principle in the reverse direction. The external stable-finiteness theorem **ES04** (Gábor Elek and Endre Szabó, Sofic groups and direct finiteness, arXiv:math/0305440v2, 19 April 2004) implies that each of the three scalar constructions has a nonsofic group. For the determinant output group, applying the bound in **ES05** (the same authors, Hyperlinearity, essentially free actions and L^2 -invariants. The sofic property, arXiv:math/0408400v2, 7 October 2014) to A^*A gives a contradiction. These external theorems explain consequences of the completed counterexamples.

[ES04, Corollary 4.7, p. 8]

[ES05, Theorem 5, p. 18]

Where to enter the proofs

- **Prime field and torsion-freeness together:** read parity, finite-arrangement exclusion, and cone surgery in [Torsion-free direct finiteness](#). The product identities, the nonzero certificate c , and the classifying-complex properties require separate connections.
- **From matrices to scalars:** in the [characteristic-two article](#), follow character packing and detection of the defect in an embedded subalgebra; in the [odd-characteristic article](#), follow the common sector, triangular compression, and finite Heisenberg matrix units. The coefficient hypotheses and local mechanisms differ.
- **Real trace versus reduction modulo two:** in the [determinant article](#), compare the projection's real trace with the binary factorization's column count, then separate integrality from rational invertibility in the next section.

Sources and checking scope

The official field is **Algebra**, the number is **197**, and the overview title is **Nonsofic groups and group-ring counterexamples** (PDF p. 21). The CONTENTS.md heading is instead **A torsion-free group algebra that is not directly finite**. The table preserves the four-paper CONTENTS order; internal paper IDs are distinct from the catalogue number. The reference commit is `1581d4cbdd69a0faf34b2089614fe2a814316a6a`, and the overview title-page date is October 6, 2026.

OpenAI Research Catalog · Catalogue 197 / Algebra · p. 21

The main results, core proof passages, and recorded input statements and applications were compared with the official sources. Both ends of the characteristic-two determinant input and the CA image calculation were checked. Unrecorded citation relationships are not treated as absent dependencies. Individual articles identify probabilistic estimates, topological cases, and other details not independently recalculated. Characteristic-two Corollary 7.4 and the external type- F_∞ embedding are outside this guide's checked claims. Finite prescriptions have not been executed; recursive verification of all external proofs, expert peer review, and formal verification have not been performed.

A Torsion-Free Group Algebra That Is Not Directly Finite

Simultaneous-step parity produces a one-sided inverse; exclusion of spherical arrangements supplies torsion-freeness.

The manuscript constructs a group by attaching cones on finite labelled graphs and uses characteristic-two cancellation to produce a scalar one-sided inverse. Vertex types enforce the algebraic parity conditions, while random matchings, planar separation and cone surgeries supply nonvanishing and a finite classifying space.

Manuscript author: OpenAI · October 4, 2026 · 31 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Failure of direct finiteness for a torsion-free group

There exist a finitely presented torsion-free group G and $a, b, c \in \mathbb{F}_2[G]$ satisfying

$$ab = 1, \quad ac = 0, \quad c \neq 0.$$

Consequently $ba \neq 1$. Moreover, G admits a finite two-dimensional classifying complex. This asserts failure in the group algebra itself over the prime field, rather than only in a matrix ring.

[Theorem 1.1 · p. 1](#)

The proof selects a sufficiently large finite matching with the required properties. It gives a finite presentation and finite scalar sums once that matching is selected, but does not list its edges. Non-soficity follows from this conclusion and Elek–Szabó; it is not an input used to obtain the one-sided inverse.

[Section 7 · p. 30](#) [\[ES04, Corollary 4.7, p. 8\]](#)

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A Torsion-Free Group Algebra That Is Not Directly Finite](#). External inputs use the following keys.

- [LT79](#) Richard J. Lipton and Robert Endre Tarjan — A Separator Theorem for Planar Graphs: Planar separation in Lemma 5.5; required statement checked in the publisher abstract, external PDF text and pagination unverified. SIAM Journal on Applied Mathematics 36(2), 177–189 (1979); publisher abstract checked 2026-10-18.
- [Hat02](#) Allen Hatcher — Algebraic Topology: Hurewicz and Whitehead for contractibility in Section 6. 2002 book; author-hosted PDF checked 2026-10-18.
- [ES04](#) Gábor Elek and Endre Szabó — Sofic groups and direct finiteness: Used only to infer non-soficity as a consequence of the main theorem. arXiv:math/0305440v2; PDF header 19 April 2004.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1](#) · [p. 1](#)

Proof route. A projective plane with $q = 128$ and seven additional Fano-plane labels prescribe the parity. Matchings conditioned on large girth exclude reduced spherical arrangements, making the cone complex aspherical and protecting the roots. Summing vertex labels in the root components then gives the algebraic witnesses.

1 Prescribe vertex types

$q = 128$, projective plane + 7 Fano labels
 $|S_x \cap S_{x'}| \equiv 1$ except (x_A, x_A) ; $|S_x \cap S_y| \equiv 1$

Preserve parity and probabilistically exclude arrangements

Proposition 3.1 · pp. 5–6

Theorem 5.2 · p. 19

2 Select a finite matching

$\text{girth } \Gamma \geq L$, no reduced spherical arrangement

Cone surgeries and finite dimension two

Proposition 6.1 · pp. 24–25

3 Recover topology from cones

$X = F \cup \bigcup K_\Lambda$, $\tilde{X} \simeq *$;
 root protection, torsion-free, finitely presented

Simultaneous-step parity and the root coefficient

Proposition 2.1 · pp. 3–4

Section 7 · p. 30

4 Scalar one-sided inverse and nonzero annihilator

$a, b, c \in \mathbb{F}_2[G]$:
 $ab = 1, ac = 0, c \neq 0 \implies ba \neq 1$

From prescribed types to topology and scalar witnesses

[Open figure ↗](#) [TeX source ↓](#)

1. Separate the properties required of the graphs

Attach the abstract cone on each connected component Λ to a rose F by its label map, setting $X = F \cup_{\Lambda \rightarrow F} \bigcup_\Lambda K_\Lambda$ and $G = \pi_1(X)$. The outgoing label sets S_x are prescribed before choosing edges, so their parities survive changes of matching. Girth, diameter and absence of arrangements instead come from the matching choice. Theorem 5.2 supplies a finite outcome satisfying these conditions simultaneously; Proposition 6.1 converts it into a classifying complex with protected roots.

[Proposition 3.1 · pp. 5–6](#) [Theorem 5.2 · p. 19](#) [Proposition 6.1 · pp. 24–25](#)

2. Connect every conclusion

Proposition 2.1 gives $ab = 1$, $ac = 0$ and $c \neq 0$. If $ba = 1$, then $c = (ba)c = b(ac) = 0$, a contradiction. Finiteness and dimension two come from the cone construction; asphericity and torsion-freeness come from Proposition 6.1. The group-ring identities and the topological conclusions therefore merge distinct inputs.

Proposition 2.1 · pp. 3–4 Section 7 · p. 30

3. Proof of Proposition 2.1 — Keep one even-degree exception

Proof target : Proposition 2.1 · pp. 3–4

Proposition 2.1 — Parity criterion

Let finite loopless graphs Γ_A, Γ_B immerse in the same rose, with roots x_A, x_B . Write A', B' for the vertex sets of the root components and S_x for outgoing labels. Assume that $|S_x \cap S_{x'}|$ is odd on $A' \times A'$ except at (x_A, x_A) , that $|S_{x_A}|$ is even, and that $|S_x \cap S_y|$ is odd on $A' \times B'$. Assume also that every path starting at either root and ending at a different vertex has nontrivial label in G . If g_x, h_y are the labels of root-to-vertex paths, then

$$a = \sum_{x \in A'} g_x, \quad b = \sum_{x \in A'} g_x^{-1}, \quad c = \sum_{y \in B'} h_y^{-1}$$

satisfy $ab = 1$, $ac = 0$, $c \neq 0$, and hence $ba \neq 1$.

Proposition 2.1 · pp. 3–4

1 Keep each product summand constant

$$(g_x t)(g_{x'} t)^{-1} = g_x g_{x'}^{-1}$$

$$\deg(x, x') = |S_x \cap S_{x'}|$$

Each component has an even number of odd-degree vertices
Proposition 2.1 · pp. 3–4

2 Count the degree exceptions

$A' \times A'$: one even-degree vertex (x_A, x_A)
 $A' \times B'$: all degrees odd

Only the exceptional component is odd; root protection isolates 1
Proposition 2.1 · pp. 3–4

3 Apply cancellation in characteristic two

$$ab = 1, \quad ac = 0; \quad [1]c = 1 \implies c \neq 0$$

Associativity and nonvanishing
Theorem 1.1 · p. 1

4 Exclude the reverse identity

$$ba = 1 \implies c = (ba)c = b(ac) = 0, \quad \text{a contradiction}$$

Degree parity of simultaneous-step graphs determines both products

[Open figure ↗](#) [TeX source ↓](#)

1. One exception in $A' \times A'$ gives $ab = 1$

Join pairs of vertices by simultaneous steps with a common label t . A product summand is constant on each connected component, since

$$(g_x t)(g_{x'} t)^{-1} = g_x g_{x'}^{-1}.$$

The degree at (x, x') is $|S_x \cap S_{x'}|$. A component avoiding the exceptional vertex has only odd-degree vertices, hence even cardinality, so its contribution cancels in characteristic two. The component containing (x_A, x_A) has exactly one even-degree vertex, hence odd cardinality; its contribution is 1.

[Proposition 2.1 · pp. 3–4](#)

2. Cancellation on $A' \times B'$ and nonvanishing of c

All degrees in the mixed simultaneous-step graph are odd, so every component has even cardinality and $ac = 0$. On the other hand, $h_{x_B} = 1$, while root protection says that no other h_y is the identity. The identity coefficient in c is therefore exactly 1. Pairwise distinctness of all vertex labels is not required; protection of x_B alone suffices for this coefficient calculation.

Proposition 2.1 · pp. 3–4

4. Proof of Theorem 5.2 — From bounded paths to all arrangements

Proof target : Theorem 5.2 · p. 19

Theorem 5.2 — Absence of reduced spherical arrangements

Set $v = 16513$, $n = vm$, and $L = \lfloor c_0 \log n \rfloor$, with $c_0 > 0$ fixed as in Section 3.3. Choose independent uniform matchings between the prescribed inverse-label slot sets and condition on girth at least L .

The probability that Γ admits a reduced spherical arrangement of Definition 5.1 tends to zero as $m \equiv 1 \pmod{4}$ tends to infinity. Thus every sufficiently large admissible m admits a matching satisfying the girth bound, the component-diameter bound of Proposition 3.4, and absence of such arrangements simultaneously. A paired pair of occurrences carries inverse labels and traverses distinct underlying graph edges. At most one boundary may be a nonclosed path starting at a root, and at least one ordinary closed boundary is required.

Theorem 5.2 · p. 19

1 Bound paths through multiplicity stages

$$\sum_j (V_j - E_j - k_j) \leq 0, \quad \log \prod_j B_j \leq -\delta H/2 + a_0 b_0 + o(L)$$

Word-weight decay and first-moment bounds

[Lemma 4.3; Section 4.3–4.4 · pp. 12–18](#)

[Theorem 4.1 · pp. 10–11](#)

2 Fix the loss tolerance first

$$\exists \varepsilon > 0 \forall K_0, C, I : \text{probability of a bounded system} = o(1)$$

Control closure and separator losses in total

[Lemmas 5.3–5.5; Section 5.4 · pp. 19–24](#)

3 Extract from an arrangement of any size

Split, close by short paths, recursively separate

$$\varepsilon, D_0, U, \eta, K_0, C, I \text{ fixed before } n \rightarrow \infty$$

Discard bad clusters and retain positive total length

[Theorem 5.2 · p. 19](#)

[Lemmas 5.3–5.5; Section 5.4 · pp. 19–24](#)

4 A remaining cluster contradicts the bound

$$L \leq H_Z \leq CL, \quad b_Z \leq \varepsilon H_Z, \quad I_Z \leq I$$

Hence no reduced spherical arrangement

Bounded-path estimates and planar extraction exclude all arrangements

[Open figure ↗](#) [TeX source ↓](#)

1. Do not count repeated traversals as independent prescriptions

Theorem 4.1 fixes $\varepsilon > 0$ before the arbitrary fixed bounds K_0, C, I . It rules out, with probability tending to one, systems with at most K_0 paths, total length $L \leq H \leq CL$, at most I comparison pairs, and at most $b_0 \leq \varepsilon H$ unpaired positions. The proof organizes an image graph by edge multiplicities: $\Delta_j = \{e : m_e \geq j\}$. Fixing the root once compensates for the interval endpoints, giving $\sum_j (V_j - E_j - k_j) \leq 0$ for the net vertex/edge exponent.

An approximately compatible grid cuts the chains into long blocks so that square-summability of the word weights $P(W)$ can be used. For stage first-moment bounds B_j , realization probability is at most $\min_j B_j$, and the manuscript proves

$$\log \prod_j B_j \leq -\frac{\delta}{2} H + a_0 b_0 + o(L).$$

It does not multiply stage probabilities under an independence assumption. A sufficiently small ε and the sum over $\exp(o(L))$ patterns yield the bounded-path exclusion.

Theorem 4.1 · pp. 10–11 Lemma 4.3; Section 4.3–4.4 · pp. 12–18

2. Retain one bounded cluster from any large arrangement

Split long boundaries into pieces of length $O(UL)$ and close each piece by adding at most D_0L edges using the diameter estimate. Added occurrences remain unpaired. A planar Euler count makes the number of comparison intervals linear in the number of pieces. Recursive separation in Lemma 5.5 then deletes few pieces and leaves components with at most K_0 pieces. Choosing parameters in the order $\varepsilon, D_0, U, \eta, K_0, C, I, n$ permits application of the preceding estimate for one fixed triple (K_0, C, I) .

Small loss is established in total, not separately for every cluster. Discard clusters with excessive unpaired proportion, excessive comparison count, and the unique possible exceptional cluster of length below L . Positive total length remains. A remaining cluster contradicts Theorem 4.1. The required constant-bearing separator statement was checked against LT79's publisher abstract; its external PDF text and pagination remain unverified.

Lemmas 5.3–5.5; Section 5.4 · pp. 19–24

5. Proof of Proposition 6.1 — Remove same-edge pairings by surgery

Proof target : Proposition 6.1 · pp. 24–25

Proposition 6.1 — Cone criterion

Let a finite graph $\Gamma = \Gamma_A \sqcup \Gamma_B$ immerse in the rose F , with roots x_A, x_B . For the cone complex X and group G of (2.1), absence of reduced spherical arrangements of Definition 5.1 implies $\pi_2(X) = 0$ and nontriviality of every root-to-different-vertex path label in G .

Moreover, X is a finite aspherical complex of dimension at most two, and G is finitely presented and torsion-free.

Proposition 6.1 · pp. 24–25

1 Represent a failure by a shortest cone picture

$\pi_2(X) \neq 0$ or root protection fails

Use the same abstract-cone lift for both caps
[Section 6.2; Equations \(6.1\)–\(6.3\) · pp. 26–29](#)

2 Remove same-edge pairs by band surgery

Preserve root endpoints and an essential sphere

$$[\tilde{f}] = [\tilde{f}_1] + [\tilde{f}_2] \text{ in } H_2(\tilde{X})$$

Shortening forbids every same-edge pair
[Proposition 6.1 · pp. 24–25](#)
[Section 6.2; Equations \(6.1\)–\(6.3\) · pp. 26–29](#)

3 Minimality produces a forbidden arrangement

Each pair traverses distinct underlying edges

No arrangement $\implies \pi_2(X) = 0$ + root protection

Dimension at most two, Hurewicz–Whitehead, free resolution
[Section 6.3 · pp. 29–30](#)

4 Contractible universal cover and torsion-freeness

$$\tilde{X} \simeq *;$$

$$C_\ell \leq G \implies H^k(C_\ell; \mathbb{F}_\ell) = 0 \ (k > 2)$$

Contradicts nonvanishing from the periodic resolution

Convert arrangement exclusion into root protection and a finite classifying space

[Open figure ↗](#) [TeX source ↓](#)

1. A shortest cone picture would give a reduced arrangement

Assume either $\pi_2(X) \neq 0$ or a trivial label on a path from a root r to a different vertex z . Choose a cone picture minimizing total boundary length. Preimages of interior points of rose edges give disjoint arcs pairing inverse boundary letters. If one arc pairs traversals of the same underlying graph edge, a band surgery shortens the picture.

In the rooted case, retaining the side containing the marked break preserves the original endpoints r, z . When one inner cone disk splits in the sphere case, fill both caps using the same lift of the abstract cone. In the universal cover this gives $[\tilde{f}] = [\tilde{f}_1] + [\tilde{f}_2]$ in H_2 . Hurewicz

ensures that at least one sphere remains essential. No embedding of the cone image in $\mathbf{X}$ is assumed. Minimality excludes all same-edge pairings, producing the prohibited reduced arrangement.

Section 6.2; Equations (6.1)–(6.3) · pp. 26–29 [Hat02, Theorem 4.32, print p. 366 / PDF p. 375]

2. Recover the classifying complex and torsion-freeness

The universal cover is simply connected, has vanishing π_2 , and has dimension at most two. It is therefore acyclic and, by Hurewicz and Whitehead, contractible. The finite CW structure gives a finite presentation of G . If G contained a prime-order subgroup C_ℓ , restricting the cellular resolution to $\mathbb{Z}C_\ell$ would give a free resolution of length at most two. The cyclic group's periodic resolution instead gives $H^k(C_\ell; \mathbb{F}_\ell) \cong \mathbb{F}_\ell$ for all $k \geq 0$, a contradiction. This periodic calculation is reproduced in the manuscript. Thus the finite two-dimensional classifying complex, finite presentation and torsion-freeness all have explicit proof routes.

Section 6.3 · pp. 29–30 [Hat02, Theorem 4.5, print p. 346 / PDF p. 355]

6. External inputs and their roles

Input

LT79 planar separation

Use

Lemma 5.5; Theorem 5.2

Role and checking scope

Recursively separate the planar graph after deleting loops and parallel edges. Required statement and use checked; external PDF text and pagination unverified.

Input

Hat02 Hurewicz and Whitehead

Use

Sections 6.1–6.3

Role and checking scope

$\pi_2 \simeq H_2$ and contractibility from acyclicity for simply connected CW complexes. Statements and application hypotheses checked.

Input

ES04, Corollary 4.7

Use

Non-soficity in the Introduction

Role and checking scope

Contraposition of stable finiteness over any division ring for sofic groups. A consequence, not an input to the construction.

The separate characteristic-two counterexample is not used as an input to this main theorem. The zero-divisors manuscript is credited for the parity and arrangement-exclusion method, but the estimates and topology required here are reproduced. A complete comparison with that manuscript has not been performed.

7. Return to the sources

- [Proposition 2.1](#) · pp. 3–4 — Simultaneous-step parity and the identity coefficient.
- [Theorem 4.1](#) · pp. 10–11 — The quantifier order and repeated graph edges.
- [Lemmas 5.3–5.5; Section 5.4](#) · pp. 19–24 — Extract a bounded cluster from an arbitrarily large arrangement.
- [Section 6.2; Equations \(6.1\)–\(6.3\)](#) · pp. 26–29 — Cone surgeries preserving root endpoints and an essential sphere.
- [Section 7](#) · p. 30 — Assemble the selected finite matching into the theorem.

The reading covered the main theorem, parity calculation, multiplicity-stage estimates and parameter quantifiers, planar extraction, cone surgeries and final assembly. Hurewicz, Whitehead and Elek–Szabó were checked at the level of statements and use; LT79 was checked through its publisher abstract. All probability constants in Section 3, every grid estimate in Lemma 4.4, and proofs of external inputs have not been independently reverified. This is a guide to the manuscript's argument, not a certification of its entire proof.

A Counterexample to Kaplansky's Direct-Finiteness Conjecture in Characteristic Two

Pairings on a punctured finite affine space become a scalar counterexample through character packing and finite-group matrix units.

The construction separates a one-sided inverse from the certificate of a nonzero reverse defect. Vanishing sums on lines force the forward matrix product; specialization inside an embedded subgroup algebra detects the reverse defect. Only after adding a finite group factor does the construction yield two scalar group-algebra elements.

Manuscript author: OpenAI · September 23, 2026 · 28 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A scalar counterexample over a finite field of characteristic two

There exist a finite field K of characteristic two, a finitely presented group G containing an element of odd prime order, and $a_{\text{out}}, b_{\text{out}} \in K[G]$ with

$$a_{\text{out}}b_{\text{out}} = 1, \quad b_{\text{out}}a_{\text{out}} \neq 1.$$

The field, group and finite sums are specified by a terminating prescription using finite sets and finite-field arithmetic. The theorem does not assert $K = \mathbb{F}_2$; the group actually contains odd-prime torsion and is not torsion-free.

[Theorem 1.1 · p. 1](#)

Termination is proved, but the manuscript has not executed the search. The identities and nonvanishing defect have algebraic certificates that do not require a solution of the group word problem. By the contrapositive of Elek–Szabó's theorem, this group is also nonsofic. No reverse implication from non-soficity alone to a group-ring counterexample is used.

Section 6; Equation (6.2) · pp. 22–24 [ES04, Corollary 4.7, p. 8]

Corollary 7.3 — An injective nonsurjective cellular automaton

For the same $K, G, a_{\text{out}}, b_{\text{out}}$ as in Theorem 1.1, define for a finitely supported $c = \sum_u c_u u$

$$(T_c x)(g) = \sum_u c_u x(gu) \quad (x \in K^G).$$

The map $T_{b_{\text{out}}} : K^G \rightarrow K^G$ is injective but not surjective, and its image is the fixed set of $T_{b_{\text{out}} a_{\text{out}}}$. Thus G is not surjunctive. Configurations are not restricted to finite support.

Corollary 7.3 · p. 26

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A Counterexample to Kaplansky's Direct-Finiteness Conjecture in Characteristic Two](#). External inputs use the following keys.

- [Xyl11](#) Triantafyllos Xylouris — On the least prime in an arithmetic progression and estimates for the zeros of Dirichlet L-functions: Uniform least-prime bound in Section 5 and termination of the prime search in Section 6. Acta Arithmetica 150(1), 65–91 (2011), publisher PDF.
- [ES04](#) Gábor Elek and Endre Szabó — Sofic groups and direct finiteness: Non-soficity as a consequence of the main theorem. arXiv:math/0305440v2; PDF header 19 April 2004.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1](#) · p. 1

Proof route. Construct full affine lines on the complement V of holes O , with exact degree $D = \ell + 1$ and incidence girth at least **12**. Interpolate the hole indicators by low-degree polynomials to obtain matrix pairings. Theorem 3.1 converts these finite data into scalar witnesses, and Section 6 turns existence into a terminating finite search.

1 Construct full lines and holes

$$V = Q^n \setminus O, \quad \deg(v) = \ell + 1, \quad \text{girth} \geq 12$$

Low-degree full-line sums and hole-size bounds

Theorem 5.1 · p. 15

Proposition 2.3 · pp. 5–6

2 Interpolate at holes to obtain pairings

$$\sum_{v \in f} x_v x_v^T = 0, \quad \sum_{v \in V} x_v x_v^T = -I_m$$

$$0 < m < t \leq m\ell^2$$

Supply all hypotheses of the incidence criterion

Theorem 3.1 · p. 6

3 Embed, pack, and transfer to scalars

$$a_{\text{out}} b_{\text{out}} = 1, \quad b_{\text{out}} a_{\text{out}} - 1 \neq 0$$

Finitely presented group retaining odd-prime torsion

Success for all sufficiently large h ; no word oracle

Section 6; Equation (6.2) · pp. 22–24

4 Specify all choices by a terminating finite search

$$K = \mathbb{F}_{2^{h(\ell-1)}}, \quad G = L \times F$$

Finite expressions specified; the search has not been executed

From finite geometry to scalar witnesses and a terminating prescription

[Open figure ↗](#) [TeX source ↓](#)

1. Supply geometry, algebra and size separately

Enumerate a field $Q = \{a_0, \dots, a_{q-1}\}$ of order $q = 2^h$ and define the index simplex $S_d = \{(a_{k_1}, \dots, a_{k_n}) : k_i \geq 0, \sum_i k_i \leq d\}$. Set $A = Q^n$, take $O \subset S_{(q-2)/2}$, and write $m = |O|$. Interpolate the indicator values of each hole o by f_o , and set $x_v = (f_o(v))_{o \in O}$. Every entry of $x_v x_v^T$ has polynomial degree at most $q - 2$, so its sum vanishes both on full lines and on A . At holes the values are coordinate vectors; restricting to $V = A \setminus O$ yields

$$\sum_{v \in f} x_v x_v^T = 0, \quad \sum_{v \in V} x_v x_v^T = -I_m.$$

The two bounds on the number of holes imply $0 < m < t = |V| \leq m\ell^2$. Incidence girth protects local subgroup embeddings, the upper bound enables character packing, and $m < t$ detects the reverse defect.

Proposition 2.3 · pp. 5–6 Theorem 3.1 · p. 6

2. Turn existence into a terminating prescription

For $h = 2, 3, \dots$, choose the least prime $\ell \equiv -1 \pmod{q^{100}}$, the field $K = \mathbb{F}_{2^{h(\ell-1)}}$ defined by the first irreducible polynomial in the specified order, its subfield Q , and the first primitive ℓ th root. For each h , enumerate the finitely many pairs $(O, \mathcal{F})$ and test size, full-line containment, regularity and girth. Theorem 5.1 guarantees success for all sufficiently large h and every field enumeration, so the outer search terminates too.

Finite-field linear algebra, finite local multiplication tables, prescribed conjugation relations and finite-group matrix units then determine the group and two finite expressions. There is no need to normalize those expressions by deciding equality of group words. This is a theoretical terminating prescription, not a claim of practical complexity or already computed output.

Section 6; Equation (6.2) · pp. 22–24 [Xyl11, Theorem 1.1, print p. 66 / PDF p. 2]

3. Proof of Theorem 3.1 — Rectangular identity, reverse defect, scalar transfer

Proof target : Theorem 3.1 · p. 6

Theorem 3.1 — Incidence criterion

Let ℓ be an odd prime, let K be a finite field of characteristic two containing a primitive ℓ th root ζ , and set $\kappa = \ell^2$. Let $|V| = t$ and let $\mathcal{F}$ be a finite family of nonempty subsets such that every $v \in V$ belongs to exactly $\ell + 1$ members and the bipartite incidence graph has girth at least 12. Suppose an integer m and columns $x_v \in K^m$ satisfy

$$0 < m < t \leq m\kappa, \quad \sum_{v \in V} x_v x_v^T = -I_m, \quad \sum_{v \in f} x_v x_v^T = 0 \quad (f \in \mathcal{F}).$$

Then the constructions below give a finitely presented group G containing an element of order ℓ and finite sums $a_{\text{out}}, b_{\text{out}} \in K[G]$ with $a_{\text{out}} b_{\text{out}} = 1$ and $b_{\text{out}} a_{\text{out}} \neq 1$.

Theorem 3.1 · p. 6

1 Embed local groups and take averages

$$K_v \hookrightarrow N, \quad P_v = \ell^{-2} \sum_{u \in K_v} u$$

$$UP = U, \quad PT = T, \quad UT = I_m$$

Use $t \leq m\ell^2$ and orthogonal idempotents for square factors

[Lemma 3.2 · pp. 7–9](#)

[Sections 4.2–4.3; Equations \(4.5\)–\(4.13\) · pp. 10–12](#)

2 Pack characters into slots

$$H = N \times C_\ell \hookrightarrow L, \quad J_0 C = eP$$

$$a_{\text{mat}} = UJ_0 + (1 - e)I_m, \quad b_{\text{mat}} = CT + (1 - e)I_m$$

Rank defect from $m < t$; specialize only on $K[H]$

[Section 4.4; Equations \(4.14\)–\(4.17\) · p. 12](#)

3 Detect the defect inside the subgroup algebra

$$\Delta = b_{\text{mat}} a_{\text{mat}} - I_m; \quad J_0 \Delta C = e(TU - P)$$

$$\epsilon : K[H] \rightarrow K, \quad \epsilon(e(TU - P)) = X^T X - I_t \neq 0$$

Independent matrix units; the complement annihilates the image

[Section 4.5; Equations \(4.18\)–\(4.25\) · pp. 13–14](#)

[Section 4.6 · p. 14](#)

4 Embed into a finite-group corner and complete the identity

$$\Phi : \text{Mat}_m(K[L]) \hookrightarrow K[L \times F], \quad \Phi(I_m) = f_{\text{blk}}$$

$$a_{\text{out}} = 1 - f_{\text{blk}} + \Phi(a_{\text{mat}})$$

$$b_{\text{out}} = 1 - f_{\text{blk}} + \Phi(b_{\text{mat}})$$

$$a_{\text{out}} b_{\text{out}} = 1, \quad b_{\text{out}} a_{\text{out}} - 1 = \Phi(\Delta) \neq 0$$

Build and detect a matrix defect, then embed it into a scalar group algebra

[Open figure ↗](#) [TeX source ↓](#)

1. Local embeddings justify averages and the rectangular identity

Put $K_v^0 = (\mathbb{F}_\ell^2, +)$ at each point and $L_f^0 = (\mathbb{F}_\ell, +)$ at each member. Glue incident L_f^0 into distinct one-dimensional subspaces of K_v^0 to form N . Lemma 3.2 protects these local groups by reducing a hypothetical collapse to a weighted planar graph with one exceptional vertex. After minimization, black degrees are at least three, ordinary white degrees at least two, and all but possibly one face have length at least twelve, contradicting Euler's formula. Merely writing a presentation would not supply this embedding guarantee.

Use averages $P_v = \kappa^{-1} \sum_{u \in K_v} u$ in the embedded local groups and put $P = \text{diag}(P_v)$, $U_{-,v} = -\kappa x_v P_v$, $T_{v,-} = P_v x_v^T$. The denominator is invertible because ℓ is odd. Partitioning $K_v \setminus \{1\}$ by incident one-dimensional subgroups and using the two pairings gives

$$UP = U, \quad PT = T, \quad UT = I_m.$$

The factors are still rectangular and do not yet constitute a direct-finiteness counterexample.

Lemma 3.2 · pp. 7–9 Section 4.1; Equation (4.4) · p. 10

2. After character packing, return to the subgroup algebra

Set $H = N \times \langle z \rangle$, $|z| = \ell$, and adjoin stable letters implementing coordinate isomorphisms and character twists to form L . The manuscript directly proves $H \hookrightarrow L$ by extending the regular permutation action of H . With the central idempotent $e = \ell^{-1} \sum_r \zeta^{-r} z^r$, orthogonal character idempotents pack the $t \leq m\kappa$ coordinates into m slots, yielding $J_0 C = eP$. Then

$$a_{\text{mat}} = UJ_0 + (1 - e)I_m, \quad b_{\text{mat}} = CT + (1 - e)I_m$$

satisfy $a_{\text{mat}} b_{\text{mat}} = I_m$.

The reverse defect $\Delta = b_{\text{mat}} a_{\text{mat}} - I_m$ is detected through

$$J_0 \Delta C = e(TU - P) \in \text{Mat}_t(K[H]).$$

Apply $\epsilon : K[H] \rightarrow K$, $nz^r \mapsto \zeta^r$. The result is $X^T X - I_t$, where $X = (x_v)$. Since $m < t$, the kernel of X is nonzero, so this matrix is nonzero. The embedding $K[H] \hookrightarrow K[L]$ preserves that fact. No extension of ϵ to all of $K[L]$ is asserted.

Sections 4.2–4.3; Equations (4.5)–(4.13) · pp. 10–12 Section 4.4; Equations (4.14)–(4.17) · p. 12

3. Complete the identity in a finite-group matrix corner

Character idempotents of $F = (\mathbb{Z}/\ell)^m \rtimes S_m$ give linearly independent elements p_{ij} with $p_{ij} p_{kr} = \delta_{jk} p_{ir}$. For $G = L \times F$, the map

$$\Phi : \text{Mat}_m(K[L]) \hookrightarrow K[G], \quad \Phi(Z) = \sum_{i,j} Z_{ij} p_{ij}$$

is injective and multiplicative, but $\Phi(I_m) = f_{\text{blk}}$ is the corner identity. Set

$$a_{\text{out}} = 1 - f_{\text{blk}} + \Phi(a_{\text{mat}}), \quad b_{\text{out}} = 1 - f_{\text{blk}} + \Phi(b_{\text{mat}}).$$

The complement annihilates the image on both sides, so the forward product is 1 and the reverse defect is $\Phi(\Delta) \neq 0$. The construction inverts only the odd number $|(\mathbb{Z}/\ell)^m|$, never $|S_m|$.

Local zero-sum relators of length at most three give a finite presentation for N . Finitely many stable letters and relations, followed by the finite factor F , give a finite presentation for G . The embeddings $K_v \leq N \leq H \leq L \leq G$ retain elements of order ℓ . Finite presentation is not inferred merely from finite generation of the support subgroup.

Section 4.5; Equations (4.18)–(4.25) · pp. 13–14 Section 4.6 · p. 14

4. Proof of Theorem 5.1 — Trades preserve regularity around the holes

Proof target : Theorem 5.1 · p. 15

Theorem 5.1 — Full lines with exact degree and large incidence girth

Let $q = 2^h$ and let ℓ be the least prime congruent to $-1 \pmod{q^{100}}$. Set $D = \ell + 1$, $s = \lfloor \log_q D \rfloor$, $b = s + 20$, $n = 10b$ and $N_{\text{pt}} = q^n$. For any enumerated field Q of order q , put $A = Q^n$ and $S = S_{(q-2)/2}$.

For every sufficiently large h and every such Q and enumeration, there are $O \subset S$ and a set $\mathcal{F}$ of full affine Q -lines contained in $V = A \setminus O$ such that

$$\frac{2N_{\text{pt}}}{D^2} \leq |O| \leq \frac{7N_{\text{pt}}}{D^2},$$

every point of V lies on exactly D lines, and the incidence graph has girth at least 12.

Theorem 5.1 · p. 15

1 Obtain a uniform dimension bound

$$\ell \equiv -1 \pmod{q^{100}}, \quad \ell \leq Cq^{100L}$$

$$s = \lfloor \log_q(\ell + 1) \rfloor, \quad n = 10(s + 20) = O(1)$$

Uniform least-prime bound, Vandermonde spaces, deletion of holes
[\[Xyl11, Theorem 1.1, print p. 66 / PDF p. 2\]](#)
[Lemmas 5.2–5.4 · pp. 15–18](#)

2 Choose compatible holes and direction spaces

$$2q^n/D^2 \leq |O| \leq 7q^n/D^2$$

Exclude translate collisions and replacement rectangles

Classify polygons of lengths three to five and use the local lemma
[Lemmas 5.5–5.7 · pp. 19–21](#)

3 Exclude short polygons with random slabs

Each event uses at most four variables

$$\max_{P_{j,C}} \sum_{A \ni P_{j,C}} \Pr(A) = O(q^{-7}) < 1/16$$

At each non-hole point, one incidence is lost and gained, or neither
[Lemmas 5.8–5.9; proof of Theorem 5.1 · pp. 21–22](#)

4 Trade fans for parallel lines while preserving degree

$$q \text{ fan lines} \mapsto q - 1 \text{ parallel lines}$$

Remove holes; retain degree D and girth ≥ 12

Uniform prime bound, compatible holes, random slabs and exact trades

[Open figure ↗](#) [TeX source ↓](#)

1. Choose uniform parameters and compatible holes

Apply Xylouris to the coprime residue $q^{100} - 1$ and modulus q^{100} to obtain $\ell \leq Cq^{100L}$ for absolute constants $C > 0, L$; Xylouris gives $L = 5.18$. Thus s , and consequently n , have bounds independent of h , making the asymptotic estimates uniform. Partition the D directions into bundles of size q and use a Vandermonde construction of spaces W_r, E_j such that any at most ten have direct sum.

Choose holes with small independent probability and delete collisions. The retained holes meet each translate of W_r at most once, have bounded occupancy in relevant cosets, and avoid four simultaneous witnesses for an exceptional rectangle of replacement lines. These three conditions control compatibility of the later trades and the event counts.

[Xyl11, Theorem 1.1, print p. 66 / PDF p. 2] Lemmas 5.2–5.4 · pp. 15–18

2. Exclude short slab polygons, then exchange full lines

For each coset of $U_j = d_j \oplus E_j$, independently choose a two-dimensional space $P_{j,C} \subset E_j$ and partition into three-dimensional slabs. Directness of the direction spaces restricts candidate triangles, quadrilaterals and pentagons to a few types. Each bad event depends on at most four variables, and the sum of event probabilities at any variable is $O(q^{-7})$. The weighted local lemma proved in the manuscript simultaneously eliminates collisions of holes within slabs and the short polygons.

Partition each slab into full lines. At every hole o and bundle r , remove the fan of q lines through o and insert $q - 1$ parallel lines in $o + W_r$. Every non-hole point either loses and gains exactly one incidence or changes neither; the hole loses all its incidences. Thus the degree on V stays exactly D . A short incidence cycle would yield either an excluded generic polygon or a replacement rectangle already forbidden by the hole choice.

Lemmas 5.5–5.7 · pp. 19–21 Lemmas 5.8–5.9; proof of Theorem 5.1 · pp. 21–22

5. Proof of Corollary 7.3 — Right-neighbor action and local equations for the image

Proof target : Corollary 7.3 · p. 26

Proof route. In the right-neighbor convention of the main results, $T_c T_d = T_{cd}$. The left inverse gives injectivity, and the reverse product acts as a projection identifying the image.

1 Use the right-neighbor action

$$(T_c x)(g) = \sum_u c_u x(gu), \quad T_c T_d = T_{cd}$$

Configurations x may have infinite support

Compose finite coefficient sums

[Lemma 7.1 · p. 25](#)

[Proposition 7.2 · p. 26](#)

2 Identify the left inverse and the image

$$ab = 1 \implies T_a T_b = \text{id}$$

$$\text{im } T_b = \{y : T_{ba} y = y\}$$

Scalar reverse defect and finite alphabet K

[Corollary 7.3 · p. 26](#)

3 Use a nonzero defect coefficient to find a missing configuration

$$(ba - 1)_h \neq 0 \implies (T_{ba} \delta_h - \delta_h)(1_G) \neq 0$$

$\delta_h \notin \text{im } T_b$; G is not surjective

From scalar witnesses to an injective nonsurjective map on all configurations

[Open figure ↗](#) [TeX source ↓](#)

1. The left inverse acts on every configuration

The coefficient supports are finite, so the double sum remains finite even for infinitely supported $x \in K^G$: $T_c(T_d x)(g) = \sum_{u,v} c_u d_v x(guv) = T_{cd} x(g)$. The order is cd , not dc . With $a = a_{\text{out}}$ and $b = b_{\text{out}}$, we have $T_a T_b = T_{ab} = \text{id}$, hence T_b is injective. The finite field supplies a finite alphabet, and the finite support of b supplies finite memory.

[Lemma 7.1 · p. 25](#)

2. A defect coefficient selects a missing point indicator

If $y = T_b x$, then $T_{ba} y = y$; conversely, this equality gives $y = T_b(T_a y)$. Hence

$$\text{im } T_b = \{y : T_{ba} y = y\}.$$

Choose a coefficient $(ba - 1)_h \neq 0$. The point indicator δ_h satisfies

$$(T_{ba} \delta_h - \delta_h)(1_G) = (ba - 1)_h \neq 0,$$

so it is outside the image. The same image is described by the finite local equations

$\sum_u (ba - 1)_u y(gu) = 0$. This proves nonsurjectivity and non-surjunctivity. It does not claim that a particular nonzero coefficient has actually been computed by executing the search.

[Proposition 7.2 · p. 26](#) [Corollary 7.3 · p. 26](#)

6. External inputs and the catalogue connection

Input or connection

Xyl11, Theorem 1.1

Use

Uniform bound on s in Section 5; prime search in Section 6

Checking scope

Coprime modulus and residue, statement with effective constant, and application checked.

External proof not verified.

Input or connection

ES04, Corollary 4.7

Use

Non-soficity in the Introduction

Checking scope

Stable finiteness for sofic groups over any division ring and the contrapositive checked. A consequence, not an input to the construction.

Input or connection

Theorem 1.1 here $\rightarrow$ determinant counterexample

Use

Characteristic-two input for the group-ring determinant construction

Checking scope

Scalar conclusion checked here; the consuming step belongs to the determinant article and catalogue connection record.

The HNN-type embedding, finite-factor matrix-to-scalar transfer, interpolation and local lemma are proved within this manuscript. References to earlier constructions and coding papers are not promoted to unchecked direct-dependency arrows.

7. Return to the sources

- Proposition 2.3 · pp. 5–6 — Obtain both pairings by interpolation at the holes.
- Theorem 3.1 · p. 6 — Separate incidence conditions and the size inequalities.
- Section 4.4; Equations (4.14)–(4.17) · p. 12 — Return to the subgroup algebra to detect the reverse defect.
- Section 4.5; Equations (4.18)–(4.25) · pp. 13–14 — Use matrix units and the complementary identity to obtain scalars.
- Theorem 5.1 · p. 15 — Trades preserving degree and exclusion of short cycles.
- Section 6; Equation (6.2) · pp. 22–24 — Termination of finite search and specification of finite expressions.
- Proposition 7.2 · p. 26 — The image on all configurations and a missing point indicator.

The reading covered the pairings, local embeddings, packing, reverse defect and scalar transfer in Sections 2–4; the parameters and geometric construction in Section 5; the terminating prescription in Section 6; and the automaton and image proof in Sections 7.1–7.2. All hole/slab probability constants have not been independently recomputed, external theorems' full proofs have not been verified, and the search has not been executed. Corollary 7.4 in Section 7.3 and the separate type- F_∞ embedding manuscript are outside this article's checking scope. This guide describes the author's existence and termination claims rather than certifying the entire proof.

A Counterexample to the Group-Ring Determinant Conjecture

From a binary one-sided inverse defect to excess real trace and a negative integral-matrix logarithmic determinant

The manuscript takes a scalar group-algebra counterexample in characteristic two as input and constructs a counterexample to the integer group-ring Determinant Conjecture. Its central device is a rational orthogonal projection whose real trace exceeds the number of columns through which its reduction modulo two factors; that discrepancy is converted into a Fuglede–Kadison determinant.

Manuscript author: OpenAI · September 23, 2026 · 9 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A rationally invertible integral group-ring counterexample

There exist a finitely generated discrete group G_D , an integer $n \geq 1$, and $A \in M_n(\mathbb{Z}[G_D])$ invertible over $\mathbb{Q}[G_D]$ such that its left regular operator satisfies

$$0 < \det_{\mathcal{N}(G_D)}(T_A) < 1.$$

For the probability spectral measure μ_A of $T_A^*T_A$ and the unnormalized trace $\tau_n(I_n) = n$, the convention is

$$\log \det_{\mathcal{N}(G_D)}(T_A) = \frac{1}{2} \tau_n \log(T_A^*T_A) = \frac{n}{2} \int_{(0,\infty)} \log t \, d\mu_A(t).$$

This integral is finite and strictly negative. The output is a **matrix** over the integer group ring; invertibility over $\mathbb{Z}[G_D]$ is not asserted. The constructed group contains elements of order three. Neither finite presentation nor torsion-freeness is part of the conclusion.

Theorem 1.1 · p. 2

Equations (1.1)–(1.2) · p. 1

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A Counterexample to the Group-Ring Determinant Conjecture](#). External inputs use the following keys.

- [OpenAI26two](#) OpenAI — A Counterexample to Kaplansky’s Direct-Finiteness Conjecture in Characteristic Two: The sole external construction input: a scalar one-sided inverse pair over a finite field of characteristic two. September 23, 2026; fixed commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a.
- [ES05](#) Gábor Elek; Endre Szabó — Hyperlinearity, essentially free actions and L^2 -invariants. The sofic property: The determinant lower bound used only for the subsequent nonsocicity consequence. arXiv:math/0408400v2, 2014-10-07, 20 pages.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — The complete construction

Proof target : [Theorem 1.1 · p. 2](#)

Proof route. Start with a finite field K_2 of characteristic two and a scalar pair $a_2 b_2 = 1 \neq b_2 a_2$. Put $d = [K_2 : \mathbb{F}_2]$, pass to binary matrices, and apply the [projection construction](#) followed by the [integral-matrix conversion](#). The starting existence theorem supplies a finite prescription with proved termination, not an executed numerical instance. It does not assert $K_2 = \mathbb{F}_2$.

1 Scalar input in characteristic two
 $a_2 b_2 = 1 \neq b_2 a_2$ in $K_2[H_2]$, $d = [K_2 : \mathbb{F}_2]$.

Restrict coefficients and separate one coefficient using cyclic coordinates
[\[OpenAI26two, Theorem 1.1, p. 1\]](#); [Lemma 2.1 · p. 3](#); [Proposition 2.2 · pp. 3–4](#)

2 Binary defect to excess real trace
 $G_D = (\bigoplus_{H_2} C_3) \rtimes H_2$, $P \in M_{d+1}(\mathbb{Z}[1/3][G_D])$.
 $P = P^* = P^2$, $\tau(P) = d + \theta$, $\bar{P} = uv$ (through d columns).

Lift modulo q and cancel denominators
[Lemmas 3.1–3.2 · pp. 5–6](#)

3 Integral matrix and rational inverse
 $q = 2^k \equiv 1 \pmod{3^{1+|S|}}$, $q > 1$.
 $A \in M_{2d+1}(\mathbb{Z}[G_D])$, $A^{-1} \in M_{2d+1}(\mathbb{Q}[G_D])$.

Remove shears and compute the trace difference
[Proposition 4.3 · p. 7](#); [§5; Equations \(5.1\)–\(5.2\) · p. 8](#)

4 All conclusions
 $\det_{\mathcal{N}(G_D)}(T_A) = q^{-\theta} \in (0, 1)$, $\theta = \frac{1}{3}(\frac{2}{3})^{|S|} > 0$.
 $\int \log t \, d\mu_A = -\frac{2\theta \log q}{2d+1} > -\infty$.
A generating set of H_2 together with z_1 generates G_D .

Binary defect, excess projection trace, and the determinant of an integral matrix

[Open figure ↗](#) [TeX source ↓](#)

1. From the scalar input to an integral matrix over a finitely generated group

Applying the regular representation $K_2 \hookrightarrow M_d(\mathbb{F}_2)$ coefficientwise gives $X, Y \in M_d(\mathbb{F}_2[H_2])$ and $C = I_d - YX \neq 0$. The identity $XY = I_d$ implies $C^2 = C$ and $XC = CY = 0$. Proposition 2.2 constructs

$$G_D = \left(\bigoplus_{h \in H_2} C_3 \right) \rtimes H_2, \quad P \in M_{d+1}(\mathbb{Z}[1/3][G_D]), \quad \tau_{d+1}(P) = d + \theta,$$

with $\bar{P}$ factoring through d columns. A finite generating set of H_2 , together with the identity-coordinate generator z_1 , generates G_D . Proposition 4.3 with $r = d$, $m = d + 1$ supplies an integral matrix of size $n = 2d + 1$ and its finite-support rational inverse.

[\[OpenAI26two, Theorem 1.1, p. 1\]](#)

[Lemma 2.1 · p. 3](#)

[§5; Equations \(5.1\)–\(5.2\) · p. 8](#)

2. From excess trace to strict bounds and a finite logarithmic integral

A finite set $S \subset H_2 \setminus \{1\}$ determines

$$\theta = \frac{1}{3} \left(\frac{2}{3} \right)^{|S|} > 0, \quad D = 3^{1+|S|}.$$

Choose $q = 2^k > 1$ with $q \equiv 1 \pmod{D}$. Then

$$\det_{\mathcal{N}(G_D)}(T_A) = q^{-\theta} \in (0, 1), \quad \int_{(0, \infty)} \log t \, d\mu_A(t) = -\frac{2\theta \log q}{2d+1} < 0.$$

The regular representation of the rational group-ring inverse is bounded, so the positive spectrum lies in a compact interval away from zero. This separately establishes finiteness of the logarithmic integral: the example violates the lower bound one, not determinant class.

Proposition 4.3 · p. 7

§5; Equations (5.1)–(5.2) · p. 8

3. Proposition 2.2 — Coefficient separation and excess trace

Proof target : Proposition 2.2 · pp. 3–4

Proposition 2.2 — A projection with trace exceeding its binary factorization size

Let $d \geq 1$ and $X, Y \in M_d(\mathbb{F}_2[H_2])$ satisfy $XY = I_d \neq YX$. In the group G_D above there exist a finite set $S \subset H_2 \setminus \{1\}$ and an orthogonal projection $P \in M_{d+1}(\mathbb{Z}[1/3][G_D])$ such that $\tau_{d+1}(P) = d + \theta$, $\theta = \frac{1}{3} \left(\frac{2}{3} \right)^{|S|} > 0$, and $\bar{P} = uv$. Here u is $(d+1) \times d$ and v is $d \times (d+1)$ over $\mathbb{F}_2[G_D]$, and $D = 3^{1+|S|}$ clears all coefficients of P .

Proposition 2.2 · pp. 3–4

1 Translate a nonzero coefficient to the identity

$$C = I_d - YX \neq 0, C^2 = C, XC = CY = 0.$$

$$h_0^{-1}C_{ij} = 1 + \sum_{s \in S} s, S \subset H_2 \setminus \{1\}.$$

The factors p_s in $sp s^{-1}$ and $1 - p_s$ in p kill the product
[Proposition 2.2 · pp. 3–4](#)

2 Separate with commuting coordinate projections

$$p_h = (1 + z_h + z_h^2)/3, p = p_1 \prod_{s \in S} (1 - p_s).$$

$$p s p = 0 \ (s \in S), \tau(p) = \frac{1}{3} \left(\frac{2}{3}\right)^{|S|} = \theta > 0.$$

Use $XC = CY = 0$ and $C^2 = C$ to cancel the off-diagonal blocks
[Proposition 2.2 · pp. 3–4](#)

3 Separate real trace from binary column count

$$fCw = \bar{p}, P = \text{diag}(I_d, p),$$

$$u = \begin{pmatrix} X \\ fC \end{pmatrix}, \quad v = \begin{pmatrix} Y & Cw \end{pmatrix}, \quad uv = \bar{P}.$$

$$\tau(P) = d + \theta > d; D = 3^{1+|S|} \text{ clears denominators.}$$

Cyclic-coordinate averages separate a defect coefficient and create excess real trace

[Open figure ↗](#) [TeX source ↓](#)

1. Enlarge the group to kill the unwanted coefficients

Translate a coefficient equal to one in a nonzero entry C_{ij} to obtain $h_0^{-1}C_{ij} = 1 + \sum_{s \in S} s$. With coordinate averages $p_h = (1 + z_h + z_h^2)/3$, set

$$p = p_1 \prod_{s \in S} (1 - p_s).$$

These averages are commuting self-adjoint idempotents. Since $sp s^{-1}$ contains p_s while p contains $1 - p_s$, one has $p s p = 0$. The original and translated label sets need not be disjoint. Distinct original coordinates make the identity coefficient multiplicative, giving $\tau_1(p) = \theta > 0$.

[Proposition 2.2 · pp. 3–4](#)

2. Use the binary defect to save one column

Let f be the row with $\bar{p}h_0^{-1}$ in position i , and w the column with $\bar{p}$ in position j . Then $fCw = \bar{p}$. Define

$$P = \text{diag}(I_d, p), \quad u = \begin{pmatrix} X \\ fC \end{pmatrix}, \quad v = \begin{pmatrix} Y & Cw \end{pmatrix}.$$

The identities $XC = CY = 0$ and $C^2 = C$ give $uv = \text{diag}(I_d, \bar{p}) = \bar{P}$. The rational orthogonal projection P has the excess real trace; this is not a real-valued comparison of the binary projection's trace.

Proposition 2.2 · pp. 3–4

4. Proposition 4.3 — Integrality and shear invariance

Proof target : Proposition 4.3 · p. 7

Proposition 4.3 — Turning the trace gap into a determinant

Let G be discrete, $m, r \geq 1$, and $P = P^* = P^2 \in M_m(\mathbb{Q}[G])$. Suppose an odd $D > 0$ clears its denominators and $\bar{P} = uv$ factors into matrices of sizes $m \times r$ and $r \times m$ over $\mathbb{F}_2[G]$. For every $q = 2^k > 1$ with $q \equiv 1 \pmod{D}$, the matrix $A \in M_{r+m}(\mathbb{Z}[G])$ of Lemma 3.2 is invertible over $\mathbb{Q}[G]$ and satisfies

$$\tau_{r+m} \log(T_A^* T_A) = 2(r - \tau_m(P)) \log q, \quad \log \det_{\mathcal{N}(G)}(T_A) = (r - \tau_m(P)) \log q.$$

The positive-square spectral measure is supported in a compact subinterval of $(0, \infty)$.

Proposition 4.3 · p. 7

1 Lift the factorization in a corner ring

$$q = 2^k, e = P_q, W = e - eu_0v_0e, W^k = 0.$$

$$Q = e + W + \cdots + W^{k-1}, e = (Qeu_0)(v_0e).$$

Use $q \equiv 1 \pmod{D}$ and $UV \equiv P \pmod{q}$ for integrality
[Lemmas 3.1–3.2 · pp. 5–6](#)

2 Integrality and rational factorization

$$A = \begin{pmatrix} qI_r & V \\ U & q^{-1}\{UV - P - q(I_m - P)\} \end{pmatrix} = L_1 B R_1.$$

$$B = \text{diag}(qI_r, -q^{-1}P - (I_m - P)).$$

Use zero diagonal blocks and cyclicity of the trace
[Lemmas 4.1–4.2 · pp. 6–7](#)

3 Remove the shears along a path

$$M(x) = L_x B R_x, \tau(M' M^{-1}) = 0.$$

$$\frac{d}{dx} \tau \log(M^* M) = 2 \operatorname{Re} \tau(M' M^{-1}) = 0.$$

Split the orthogonal projection and use the bounded inverse
[Proposition 4.3 · p. 7](#)

4 Compute on the diagonal projection

$$\log(B^* B) = 2 \log q \operatorname{diag}(I_r, -P).$$

$$\log \det(T_A) = (r - \tau(P)) \log q.$$

$$A^{-1} \in M_{r+m}(\mathbb{Q}[G]) \Rightarrow \operatorname{supp} \mu_A \subseteq (0, \infty).$$

Lift modulo two to modulo q , then remove block shears to calculate the determinant

[Open figure ↗](#) [TeX source ↓](#)

1. A finite corner geometric series enables integrality

Put $R = (\mathbb{Z}/q\mathbb{Z})[G]$ and $e = P_q$. For arbitrary lifts u_0, v_0 , work in the corner ring $eM_m(R)e$ with $Z = eu_0v_0e$ and $W = e - Z$. Every coefficient of W is even, so $W^k = 0$ and $Q = e + W + \cdots + W^{k-1}$ is a corner inverse of Z . Thus $u_q = Qeu_0$, $v_q = v_0e$ give $e = u_qv_q$.

For integral lifts U, V set

$$A = \begin{pmatrix} qI_r & V \\ U & q^{-1}\{UV - P - q(I_m - P)\} \end{pmatrix}.$$

The matrix $P + q(I_m - P) = qI_m + (1 - q)P$ is integral because $D \mid q - 1$; its agreement with UV modulo q makes the lower-right block integral. Put $J = -q^{-1}P - (I_m - P)$ and $B = \text{diag}(qI_r, J)$. Then $A = L_1 B R_1$, where L_x, R_x are lower and upper block shears with off-diagonal blocks $xU/q, xV/q$. The identity $J^{-1} = -qP - (I_m - P)$ supplies an explicit rational inverse. No conditions such as $PU = U$ or $V = U^*$ are imposed on the lifts.

[Lemmas 3.1–3.2 · pp. 5–6](#)

2. Differentiate under the trace along a noncommutative path

For $M(x) = L_x B R_x$, Lemma 4.1 gives

$$\frac{d}{dx} \tau \log(M(x)^* M(x)) = 2 \operatorname{Re} \tau(M'(x) M(x)^{-1}).$$

The proof uses a uniformly convergent logarithm series and tracial cyclicity, without assuming that $M^* M$ commutes with its derivative. Each shear's logarithmic derivative has zero diagonal blocks; its trace, also after conjugation, is zero. Hence this quantity is constant along the path. Computing at $B^* B = \text{diag}(q^2 I_r, q^{-2} P + I_m - P)$ produces the difference $r - \tau_m(P)$. The bounded inverse gives $\|T_A^{-1}\|^{-2} I \leq T_A^* T_A \leq \|T_A\|^2 I$.

[Lemmas 4.1–4.2 · pp. 6–7](#)

[Proposition 4.3 · p. 7](#)

5. External inputs and consequences

The scalar characteristic-two pair is the external input treated in the [Characteristic Two](#) article; its construction is not reproved here. An odd-characteristic pair alone does not satisfy the binary premise. Coordinate projections and the tracial logarithmic derivative have earlier antecedents, but the required arguments are proved locally in this manuscript.

Nonsoficity is a subsequent consequence. If G_D were sofic, the ES05 lower bound would apply to the positive self-adjoint integral matrix $A^* A$, giving $\tau \log(T_A^* T_A) \geq 0$, contrary to the computed negative value. This theorem is not used to construct A .

[\[ES05, Theorem 5, p. 18\]](#)

[§5; Equations \(5.1\)–\(5.2\) · p. 8](#)

6. Return to the sources

- [Proposition 2.2 · pp. 3–4](#) — Separate one defect coefficient and create the gap between real trace and binary factorization

- Lemmas 3.1–3.2 · pp. 5–6 — Follow the corner inverse and the integrality calculation
- Lemmas 4.1–4.2 · pp. 6–7 — Use tracial cyclicity to handle noncommutative differentiation
- §5; Equations (5.1)–(5.2) · p. 8 — Check all conclusions and determinant normalization

We read §§2–5 and checked the connections between the main statement, projection, modular lift, integrality, trace calculation, and finite generation. We also compared the characteristic-two input statement with its use here, and the ES05 lower bound for positive integral matrices with the nonsolicity consequence. This does not include independent verification of all external proofs, numerical execution of the finite prescription, expert peer review, or formal verification.

A Counterexample to Kaplansky's Direct-Finiteness Conjecture in Odd Characteristic

High-degree modular representations, a common central sector, and triangular compression to scalar one-sided inverses

The manuscript constructs a direct-finiteness counterexample over a field of order p^4 for one specified odd prime p , using finite formulas and choices. Comparing a low-rank graph-fitting matrix with a high-regular-rank local idempotent is not sufficient by itself. The proof preserves finite-group embeddings, realizes the ranks as rectangular matrix identities over one nonzero ring, and then transfers the matrix defect injectively into a scalar group algebra.

Manuscript author: OpenAI · September 26, 2026 · 25 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A scalar counterexample in a specified odd characteristic

Put

$$m = \binom{1200}{600}, \quad p = \min\{h \in \mathbb{Z} : h > 1, h \mid (m!)^2 + 1\}.$$

Then p is an odd prime. There exist a specified field K of order p^4 , a specified finitely generated group G containing torsion, and specified finite sums $a, b \in K[G]$ such that

$$ab = 1, \quad ba \neq 1.$$

“Specified” means given by finite formulas and choices in ordered finite sets. The construction is not claimed to have been executed as a practical computation. It concerns this particular p , not every odd prime. Finite presentation is not stated as a conclusion of this theorem.

Theorem 1.1 · p. 1

Proposition 8.1 — One-sided inverses and the image on all configurations

Let k be a finite field, L a group, and $a, b \in k[L]$ finite sums with $ab = 1$. For $c = \sum_u c_u u$, define on all functions $x : L \rightarrow k$, without a finite-support restriction,

$$(T_c x)(g) = \sum_u c_u x(gu).$$

Then $T_b : k^L \rightarrow k^L$ is an injective cellular automaton, and

$$\text{im } T_b = \{y \in k^L : T_{ba} y = y\}.$$

For every h with $(ba - 1)_h \neq 0$, the point indicator δ_h is absent from its image. In particular, if $ba \neq 1$, then T_b is nonsurjective. Applying this to the scalar pair of Theorem 1.1 gives an injective, nonsurjective cellular automaton.

Proposition 8.1; Equation (8.1) · p. 23

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A Counterexample to Kaplansky’s Direct-Finiteness Conjecture in Odd Characteristic](#). External inputs use the following keys.

- [ES04](#) Gábor Elek; Endre Szabó — Sofic groups and direct finiteness: Stable finiteness for sofic groups, used only for nonsficity after the scalar construction. [arXiv:math/0305440v2](#), 2004-04-19, 9 pages.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — Connecting the construction

Proof target : [Theorem 1.1 · p. 1](#)

Proof route. The graph ratio $t/m < 1/100$ and local idempotent ratio $d/w > 1/96$ are converted into a split injection $S^{md} \rightarrow S^{wt}$ over $S = fK[G_1]$. Since $wt < md$, a nonzero kernel vector certifies a matrix defect, which is then scalarized using a finite group factor. The high-rank idempotent and central-sector realization are detailed under [Proposition 3.1](#) and [Proposition 5.1](#).

1 Specified field and two strict ratios

$p = \min\{h > 1 : h \mid (m!)^2 + 1\}$, $|K| = p^4$, $m = \binom{1200}{600}$.
 $F = \alpha\beta^T$, $t/m < 1/100$; $e^2 = e \in K[H]$, $d/w > 1/96$.

Directed annihilations, spherical gluing, and central-sector rank realization
[Corollary 3.3 · p. 8](#); [Lemma 4.1](#); [Proposition 4.2 · pp. 9–13](#); [Proposition 5.1 · p. 14](#)

2 Preserve finite groups and pass to one ring

$H_i \hookrightarrow G_0 \hookrightarrow G_1$, $S = fK[G_1]$, $f \neq 0$.
 $P_i P_j = 0$ (edges $i < j$ only), $Y_i J_i = fI_d$, $J_i Y_i = P_i I_w$.

Remove the strictly triangular part by a finite series and use a kernel vector
[Lemmas 6.1–6.2 · pp. 18–20](#)

3 Triangular compression and a nonzero defect

$D^\# C^\# = fI_{md}$, $r = wt < md$.
 $A_R B_R = I_r$, $(B_R A_R - I_r)v_* = -v_* \neq 0$.

Independent matrix units and the complementary identity preserve the defect
[Proposition 7.1](#); [Lemma 7.2](#); [Corollary 7.3 · pp. 20–22](#);
[Proof of Theorem 1.1](#); [finite prescriptions · pp. 22–23](#)

4 Scalarize using a finite group factor

$\Phi : M_n(K[G_1]) \hookrightarrow K[G_1 \times \Lambda]$, $g = \Phi(I_n)$.
 $a = 1 - g + \Phi(\tilde{A}_R)$, $b = 1 - g + \Phi(\tilde{B}_R)$: $ab = 1 \neq ba$.
The group $G = G_1 \times \Lambda$ is finitely generated and contains torsion.

Realize the two ratios over a common ring, compress triangularly, and encode matrices injectively as scalars

[Open figure ↗](#) [TeX source ↓](#)

1. Fix the coefficient field and low-rank graph factorization

If $p \leq m$, then p divides both $m!$ and $(m!)^2 + 1$, a contradiction; hence $p > m$. Also $(m!)^2 = -1$ in $\mathbb{F}_p$. Choose the least nonsquare $\eta \in \mathbb{F}_p^*$ and set

$$K_0 = \mathbb{F}_p[\xi]/(\xi^2 - \eta), \quad K = K_0[\nu]/(\nu^2 - \xi).$$

The norm $N_{K_0/\mathbb{F}_p}(\xi) = -\eta$ is nonsquare, so both extensions are quadratic and $|K| = p^4$. Fix $M = p^4 - 1$ and $\zeta \in K^*$ of order M .

The vertices of Γ are the 600-subsets I_i of $[1200]$, with adjacency defined by $|I_i \cap I_j| < 200$. Three pairwise adjacent vertices would have a union of at least 1203 elements, so the graph is triangle-free. Multilinearizing the degree-400 polynomial

$$Q_j(X) = \prod_{k=200}^{599} \frac{\sum_{x \in I_j} X_x - k}{600 - k}$$

gives $F_{ij} = \sum_{T \in \mathcal{T}} \alpha_i(T) \beta_j(T)$, where $\mathcal{T} = \{T \subset [1200] : |T| \leq 400\}$. Then $F_{ii} = 1$, $F_{ij} = 0$ for distinct nonneighbors, and $t = |\mathcal{T}|$ satisfies $t/m < 1/100$. The denominators are invertible because $p > m$.

Lemma 2.1; Proposition 2.2 · pp. 3–4

2. Glue the directed annihilations while retaining the finite groups

Put $P = (\mathbb{F}_p)^3$, $A_0 = (\mathbb{Z}/M\mathbb{Z})^P$, and $H = A_0 \rtimes P$. Let c be the identity-coordinate function, $u(s) = (0, s)$, and $v(s) = cu(s)c^{-1}$. Then $U = u(P)$ and $V = cUc^{-1}$ have $U \cap V = 1$. For $\ell = (p-1)/2$, the local idempotent satisfies

$$(u(s) - 1)^\ell e = 0, \quad e(v(s) - 1)^\ell = 0, \quad d/w > 1/96,$$

where $w = |H|$ and $d = \dim_K eK[H]$. Along each edge $i < j$, identify $v(1, \bar{j}, \bar{j}^2)$ in H_i with $u(1, \bar{i}, \bar{i}^2)$ in H_j to form G_0 . Calling the shared element y , the right annihilator in $K[\langle y \rangle] = K[X]/(X^p)$ gives $e_i = qX^{p-\ell}$; since $p - \ell = \ell + 1$, this implies $e_i e_j = 0$. No reverse ordered product is assumed.

Proposition 4.2 establishes $H_i \hookrightarrow G_0$. Nonvanishing of the required local words of length at most three follows from Vandermonde independence, using $p > m$, and $U \cap V = 1$. In Lemma 4.1, a hypothetical kernel yields a minimal spherical map with exactly one nonidentity face. Neutral edges, bridges, equal-type faces, and consecutive equal labels are reduced. Triangle-freeness and the short-word condition then force ordinary vertex degrees and face lengths to be at least four. The Euler sum is eight, whereas the exceptional face contributes at most three in the loop-free case; the remaining single-loop case contributes at most seven in total. Ordinary graph-of-groups normal form does not replace the injectivity proof for this first quotient.

Corollary 3.3 · p. 8

Lemma 4.1; Proposition 4.2 · pp. 9–13

3. Common-ring identities yield a certified matrix defect

Proposition 5.1 supplies $G_0 \hookrightarrow G_1$ and a nonzero central idempotent f . In $S = fK[G_1]$, whose identity is f , put $P_i = fe_i$. The identities $Y_i J_i = fI_d$, $J_i Y_i = P_i I_w$ hold simultaneously at every vertex.

For $C_0 = (\beta_j(T)P_j)_{T,j}$ and $D_0 = (\alpha_i(T)P_i)_{i,T}$, the product is $\mathcal{P} + N$, where $\mathcal{P} = \text{diag}(P_i)$ and N is strictly lower triangular. Thus $N^m = 0$ and $\mathcal{P}N = N = N\mathcal{P}$. A finite geometric sum corrects D_0C_0 to $\mathcal{P}$; taking w copies and composing with J_i, Y_i gives

$$D^\sharp C^\sharp = fI_{md}, \quad C^\sharp : S^{md} \rightarrow S^{wt}, \quad r = wt < md = N_0.$$

The inequalities here are ordinary integer inequalities. Let $\iota : S^r \rightarrow S^{N_0}$ and $\pi : S^{N_0} \rightarrow S^r$ include and project onto the first r coordinates. Set

$$A_R = \pi D^\sharp + (1 - f)I_r, \quad B_R = C^\sharp \iota + (1 - f)I_r.$$

Then $A_R B_R = I_r$. If u_* has last coordinate f and all others zero, $v_* = C^\sharp u_* \neq 0$, $D^\sharp v_* = u_*$, and $A_R v_* = 0$. Therefore $(B_R A_R - I_r)v_* = -v_* \neq 0$ certifies the reverse defect without deciding equality of group words.

Lemmas 6.1–6.2 · pp. 18–20

4. Preserve the defect under scalarization and finish the finite prescription

Choose the least $q \geq 0$ with $n = M^q \geq r$, and pad A_R, B_R to size n by identity matrices. The finite Heisenberg group $\Lambda = (\mathbb{Z}/M\mathbb{Z})^q \times (\mathbb{Z}/M\mathbb{Z})^q \times \mathbb{Z}/M\mathbb{Z}$ has multiplication

$$(x, y, k)(x', y', k') = (x + x', y + y', k + k' + y \cdot x').$$

The matrix units Ω_{uv} of Proposition 7.1 form a basis of a nonzero central sector $f'K[\Lambda] \simeq M_n(K)$. A shift/modulation representation sends them to standard matrix units, proving linear independence and hence injectivity of $\Phi : M_n(K[G_1]) \hookrightarrow K[G_1 \times \Lambda]$. Multiplication formulas alone would not establish injectivity.

Put $G = G_1 \times \Lambda$, $g = \Phi(I_n)$, $a = 1 - g + \Phi(\tilde{A}_R)$, and $b = 1 - g + \Phi(\tilde{B}_R)$. Cross terms with the complement vanish, giving $ab = 1$; injectivity gives $ba - 1 = \Phi(\tilde{B}_R \tilde{A}_R - I_n) \neq 0$. The coefficient field remains the same K throughout. Finitely many finite groups and the specified identifications generate G , while the embedded H_i retain torsion. Every choice of field element, character, basis, or finite inverse occurs in an ordered finite set.

Proposition 7.1; Lemma 7.2; Corollary 7.3 · pp. 20–22

Proof of Theorem 1.1; finite prescriptions · pp. 22–23

3. Proof of Proposition 3.1 — Constructing the large local idempotent

Proof target : Proposition 3.1; §3.1 · pp. 5–6

Proposition 3.1 — Prescribed two-sided annihilations and a regular-rank bound

Use p, K, M, P, A_0, H, u, v above, and put $\ell = (p-1)/2$, $w = |H|$. There is an idempotent $e \in K[H]$, specified by finite field operations, such that for every $s \in P$, $(u(s) - 1)^\ell e = 0$ and $e(v(s) - 1)^\ell = 0$. Its regular rank $d = \dim_K eK[H]$ satisfies $d/w > 1/96$.

Proposition 3.1; §3.1 · pp. 5–6

1 High-degree column space

$X_j^p = 0$, $D = 3(p-1)$, $\ell = (p-1)/2$.
 B : $\deg \geq D - \ell + 1$, $h = \binom{\ell+2}{3}$, $h/p^3 > 1/48$.

Degree gives annihilation; the grid bound retains more than half the characters
 Proposition 3.1; §3.1 · pp. 5–6; Lemma 3.2; §3.2 · pp. 6–7

2 Select nondegenerate weighted pairings

$N_\lambda = B^\top D_\lambda^{-1} B$.
 $\det(B^\top \text{diag}(\tau) B) \neq 0$; $|\mathcal{R}| > |A_0|/(2p^3)$.

Use free-orbit matrix units and count the rank of all p^3 columns
 §3.3; Equations (3.6)–(3.7) · pp. 7–8

3 Place idempotents in Fourier blocks

$T_\lambda = B N_\lambda^{-1} B^\top D_\lambda^{-1}$, $T_\lambda^2 = T_\lambda$.
 $d = hp^3 |\mathcal{R}|$, $d/w > 1/96$.
 $(u(s) - 1)^\ell e = 0$, $e(v(s) - 1)^\ell = 0$ (every $s \in P$).

Realize the high-degree subspace as a group-algebra idempotent using weighted pairings and Fourier orbits

Open figure ↗ TeX source ↓

1. Augmentation degree annihilates columns and rows

For translations L_s on K^P , set $X_j = L_{\epsilon_j} - I$. Then $X_j^p = 0$ and the vectors $X_1^{n_1} X_2^{n_2} X_3^{n_3} e_0$, $0 \leq n_j < p$, form a basis. Put $D = 3(p-1)$ and let B consist of the basis columns of total degree at least $D - \ell + 1$. Then

$$(L_s - I)^\ell B = 0, \quad B^\top (L_s - I)^\ell = 0, \quad h = \text{rank } B = \binom{\ell+2}{3}, \quad h/p^3 > 1/48.$$

The first equality raises degree to at least $D + 1$; the second follows by applying the same argument to $-s$ and transposing. Transpose involves no scalar conjugation.

Proposition 3.1; §3.1 · pp. 5–6

2. Vary the weights to obtain enough nondegenerate pairings

For a character λ of A_0 , put $D_\lambda = \text{diag}((x\lambda)(c))$ and $N_\lambda = B^\top D_\lambda^{-1} B$. By Cauchy–Binet, $\det(B^\top \text{diag}(\tau_x) B)$ has squares of maximal minors as coefficients of distinct monomials. Full column rank makes it nonzero. The degree- h grid bound gives singular pairings a fraction at most $h/M < 1/4$. Characters with nontrivial stabilizer also occupy a fraction below $1/4$, so more than half have free orbits and invertible N_λ . Choosing one representative from each orbit containing such a character gives $|\mathcal{R}| > |A_0|/(2p^3)$. Admissibility need not be constant on an orbit.

Lemma 3.2; §3.2 · pp. 6–7

3. Convert representation rank to regular rank

Fourier matrix units identify a free-character-orbit block with $M_{p^3}(K)$. Place

$$T_\lambda = B N_\lambda^{-1} B^\top D_\lambda^{-1}$$

in each selected block and zero in the complement to define e . Since $T_\lambda B = B$, it is idempotent of rank h . Its left-multiplication image consists of matrices whose p^3 columns lie in $\text{im } B$, so its regular rank is hp^3 . Hence $d = hp^3|\mathcal{R}|$ and $d/w > h/(2p^3) > 1/96$. The weighted row factor annihilates the conjugate translations $v(s)$ on the prescribed side. Only the prime-to- p subgroup A_0 is diagonalized; semisimplicity of the entire modular group algebra is not assumed.

§3.3; Equations (3.6)–(3.7) · pp. 7–8

4. Proof of Proposition 5.1 — Realizing regular ranks over one ring

Proof target : Proposition 5.1 · p. 14

Proposition 5.1 — Rectangular matrices in a common central sector

Let K have characteristic p , let M be a positive integer prime to p , and suppose K contains ζ of order M . Put $H = (C_M)^{(\mathbb{F}_p)^3} \rtimes (\mathbb{F}_p)^3$ and $w = |H|$. Suppose a countable group G_0 contains specified copies $H_i \simeq H$, $1 \leq i \leq m$, and idempotents $e_i \in K[H_i]$ have the same positive integer rank $d = \dim_K e_i K[H_i]$.

There exist a group G_1 containing G_0 and a central element z of order M such that, with $f = M^{-1} \sum_{k=0}^{M-1} \zeta^{-k} z^k \neq 0$, $S = fK[G_1]$, and $P_i = f e_i$, specified matrices $J_i \in \text{Mat}_{w \times d}(S)$ and $Y_i \in \text{Mat}_{d \times w}(S)$ satisfy $Y_i J_i = f I_d$, $J_i Y_i = P_i I_w$. Their entries lie in the group algebra of specified finite subgroups of G_1 , multiplied by f , and are constructed by finite group operations and linear algebra over K .

Proposition 5.1 · p. 14

1 Attach finite groups and fix a central character

$L_i = (C_M)^{H_i} \rtimes H_i$, $z = \delta_i$, $|z| = M$.
 $f = M^{-1} \sum_k \zeta^{-k} z^k \neq 0$, $S = fK[G_1]$ (identity f).

The exponent sum one excludes prime-to- p stabilizers
[§§5.1–5.2; Lemma 5.2 · pp. 14–16](#)

2 Decompose the modular stabilizer blocks

$T_\theta \cap A_0 = 1$, $T_\theta \hookrightarrow (\mathbb{F}_p)^3$.
 $fK[L_i] \simeq \prod_\theta M_{s_\theta}(K[T_\theta])$.

Lift the augmented basis using a finite geometric series
[§5.3; Equations \(5.5\)–\(5.7\) · pp. 17–18](#)

3 A finite basis change over a nilpotent ideal

$\mathfrak{a}^{3(p-1)+1} = 0$, $Q^{-1}EQ = \text{diag}(I_{d_0}, 0)$.
 $d = d_0|T_\theta|$, $w = s_\theta|T_\theta|$, $wd_0 = s_\theta d$ (integer identities).

Pull blockwise coordinate pairings back into the same group algebra
[§5.3; Equations \(5.5\)–\(5.7\) · pp. 17–18](#)

4 Pair equally numerous coordinates

$J_i \in \text{Mat}_{w \times d}(S)$, $Y_i \in \text{Mat}_{d \times w}(S)$.
 $Y_i J_i = f I_d$, $J_i Y_i = P_i I_w$, $P_i = f e_i$.

Attach finite groups and use nilpotent modular augmentation ideals to realize exact ranks

[Open figure ↗](#) [TeX source ↓](#)

1. Align character sectors and control their stabilizers

Attach $L_i = (C_M)^{H_i} \rtimes H_i$ to $\langle z \rangle \times G_0$, identifying z with the constant function δ_i . Matching free actions orbit by orbit over $\langle z \rangle \times H_i$ proves that each factor embeds. Thus z retains its order and the identity coefficient M^{-1} of f is nonzero.

The stabilizer T_θ of a character with $\chi(z) = \zeta$ meets A_0 trivially. Otherwise an element of order $q > 1$ would force the exponent sum one to lie in $q(\mathbb{Z}/M\mathbb{Z})$, a contradiction. Hence $T_\theta \hookrightarrow (\mathbb{F}_p)^3$ is elementary abelian, possibly nontrivial and nonnormal. Explicit matrix units and dimension comparison give

$$fK[L_i] \simeq \prod_{\theta} M_{s_\theta}(K[T_\theta]), \quad s_\theta = [H_i : T_\theta].$$

§§5.1–5.2; Lemma 5.2 · pp. 14–16

2. Conjugate to a coordinate projection over a nilpotent ideal

In one block put $D_\theta = K[T_\theta]$ and $E = \rho_\theta(P_i)$. Its augmentation ideal satisfies $\mathfrak{a}^{3(p-1)+1} = 0$. Choose bases of the image and complementary image after augmentation, and form Q from the corresponding columns of E and $I - E$. Write $Q = Q_0(I + N)$ with entries of N in $\mathfrak{a}$; a finite geometric sum gives its inverse. Thus $Q^{-1}EQ = \text{diag}(I_{d_0}, 0)$.

Comparison with the original regular representation gives ordinary integer equalities $d = d_0|T_\theta|$, $w = s_\theta|T_\theta|$, and therefore $wd_0 = s_\theta d$. Pairing these equally numerous coordinates constructs the rectangular matrices blockwise, and pulling back all blocks produces J_i, Y_i . The proof does not jump from numerical K -linear ranks to a free-module isomorphism over S .

§5.3; Equations (5.5)–(5.7) · pp. 17–18

5. Proof of Proposition 8.1 — The defect persists on infinitely supported configurations

Proof target : Proposition 8.1; Equation (8.1) · p. 23

Proof route. With the convention in the [main statement](#), verify the composition order and then identify the image as a fixed-point set. No further characteristic or torsion assumption is needed.

1 From scalar elements to all configurations

$$ab = 1, (T_c x)(g) = \sum_u c_u x(gu), x \in k^L.$$

Expand the finite sums in the actual multiplication order
[Proposition 8.1; Equation \(8.1\) · p. 23](#)

2 Finite memory and a left inverse

$$T_c T_d = T_{cd}, T_a T_b = \text{id}.$$

The map T_b is an injective cellular automaton.

Characterize $y = T_b x$ and detect failure with a nonzero coefficient
[Proposition 8.1; Equation \(8.1\) · p. 23](#)

3 Fixed-point condition and missing configurations

$$\text{im } T_b = \{y : T_{ba} y = y\}.$$

$$(T_{ba} \delta_h - \delta_h)(1) = (ba - 1)_h \neq 0 \Rightarrow \delta_h \notin \text{im } T_b.$$

Even infinitely supported preimages are excluded.

Transfer multiplication to all configurations and identify a missing point indicator

[Open figure ↗](#) [TeX source ↓](#)

1. Composition order and injectivity

Expanding two finite sums gives $T_c T_d = T_{cd}$. Thus $T_a T_b = T_{ab} = \text{id}$, proving injectivity of T_b . The finite support of b supplies a memory set, so it is a cellular automaton.

2. Characterize the image and exclude point indicators

If $y = T_b x$, then $T_{ba} y = T_b T_a T_b x = y$. Conversely, $T_{ba} y = y$ implies $y = T_b(T_a y)$, so y lies in the image. Finally $(T_{ba} \delta_h - \delta_h)(1) = (ba - 1)_h \neq 0$ excludes δ_h . These computations hold on the full function space, so allowing infinitely supported preimages does not change the conclusion.

[Proposition 8.1; Equation \(8.1\) · p. 23](#)

6. External inputs, background, and consequences

The characteristic-two paper is a comparison with a different local construction, not an input to this odd-prime proof. The required graph fit, grid estimate, spherical gluing, central-sector realization, and finite Heisenberg encoding are proved within this manuscript. Its connections to earlier work are methodological antecedents rather than unproved imported steps.

Nonsoficity uses an external result only after the scalar pair has been completed. ES04 states that group algebras of sofic groups are stably finite over every division ring. The field K satisfies this coefficient hypothesis, so the constructed $ab = 1 \neq ba$ forces G to be nonsofic. In contrast, the cellular-automaton counterexample follows from the direct computation above without this external theorem.

[ES04, Corollary 4.7, p. 8]

7. Return to the sources

- Proposition 3.1; §3.1 · pp. 5–6 — Read the high-degree modular construction and distinguish regular rank
- Lemma 4.1; Proposition 4.2 · pp. 9–13 — Follow why the gluing quotient preserves the finite groups
- §5.3; Equations (5.5)–(5.7) · pp. 17–18 — Check the passage from integer ranks to identities over the same ring
- Lemmas 6.1–6.2 · pp. 18–20 — Read the directed-zero-product argument and the nonzero kernel certificate
- Proposition 7.1; Lemma 7.2; Corollary 7.3 · pp. 20–22 — Follow injective matrix units and complementary-identity scalarization
- Proposition 8.1; Equation (8.1) · p. 23 — Check multiplication order and missing points on all configurations

We read §§2–8 and compared the specified field, the two ratios, directed zero products, finite-group embeddings, common-sector integer rank identities, triangular compression, injective scalarization, and the image on all configurations. We also read the spherical-map reductions for loops, bridges, and equal-label smoothing and the Euler contradiction; this is not an independent formal verification of every topological case. The ES04 statement and its application were checked, without recursively verifying its proof or those of the background literature. The enormous finite prescription has not been executed, and expert peer review and formal verification have not been performed.