

OpenAI Math Digest

CATALOGUE 159

Erdős's reciprocal-sum conjecture and quasipolynomial Szemerédi bounds

Catalogue guide & one proof overview

Independent reading guide · English edition

Edition: 2026-10-15

Created by: Masataka Iwai · Osaka University

AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The catalogue guide is followed by the article. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

Erdős's reciprocal-sum conjecture and quasipolynomial Szemerédi bounds

This catalogue concerns density bounds for sets avoiding arithmetic progressions of a fixed length, strong enough to force convergence of their reciprocal sums. The manuscript constructs density increments on triangular cells retaining earlier polynomial constraints, then controls dimension and width losses to close a finite iteration.

1. Papers in official catalogue order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 159 · 2026-10-15 checked source snapshot

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01

Manuscript and overview

Quasipolynomial Bounds for Arithmetic Progressions →

Short name: Quasipolynomial Bounds for Arithmetic Progressions

September 23, 2026 · 198 pages

Original paper (GitHub) ↗

Principal claim

For fixed $k \geq 3$, bounds the largest nonconstant- k -term-progression-free subset by $C_k N \exp\{-c_k(\log N)^{\varepsilon_k}\}$. Every set of positive integers with divergent reciprocal sum contains progressions of every finite length, nonconstant for lengths at least two.

Theorem 1.1; Corollary 1.2 · p. 4 ↗

Role and available overview

Controls triangular-cell dimensions and widths from higher weights, iterates a fixed multiplicative density gain, and passes from the finite bound to reciprocal sums by dyadic summation.

Read the proof overview →

2. Sources and checking scope

[OpenAI26Catalog] denotes OpenAI's OpenAI Research Catalog. The official subject is **Combinatorics**, and the official catalogue number is 159. We use the OpenAI Research Catalog at commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a, with the catalogue entry on overview p. 18 and the manuscript dated September 23, 2026.

[OpenAI26Catalog, Catalogue 159, p. 18]

The article on Quasipolynomial Bounds for Arithmetic Progressions checks the principal hypotheses and quantifiers, triangular iteration, inversion of the threshold, summation for reciprocal sums, and statements and application points of the direct external inputs. It does not independently verify the entire analytic construction. Unchecked technical portions are identified in the article. Proof connections to other catalogues remain uninvestigated; this does not mean that no such dependencies exist.

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[Read on the website ↗](#)

CATALOGUE 159 · PAPER 01 / 1 Overview

Quasipolynomial Bounds for Arithmetic Progressions

Controlling triangular polynomial-cell width losses closes the density increment at a quasipolynomial scale.

OpenAI's September 23, 2026 manuscript establishes a density bound for each fixed progression length and deduces Erdős's reciprocal-sum conjecture for sets of positive integers. This article presents the authors' claims and proof route. It is not an independent verification of all 198 pages; the passages compared with the source and the remaining unchecked arguments are distinguished below.

Manuscript author: OpenAI · September 23, 2026 · 198 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Density bound and sufficient interval length

Fix an integer $k \geq 3$. Let $r_k(N)$ be the maximum cardinality of a subset of $[N] = \{1, \dots, N\}$ containing no nonconstant k -term arithmetic progression. Such a progression has the form $a, a + d, \dots, a + (k - 1)d$ with $d > 0$. There are $C_k, c_k, \varepsilon_k > 0$, depending only on k , such that for every $N \geq 2$,

$$r_k(N) \leq C_k N \exp\{-c_k (\log N)^{\varepsilon_k}\}.$$

Equivalently, there is $A_k \geq 1$ such that, for every $0 < \alpha \leq 1$, a subset of $[N]$ of density at least α contains a nonconstant k -term progression whenever

$$\log N \geq A_k (2 + \log(1/\alpha))^{A_k}.$$

All logarithms are natural. “Quasipolynomial” describes the sufficient length $N = \exp(O_k((2 + \log(1/\alpha))^{4k}))$ as a function of $1/\alpha$. The cardinality bound $r_k(N)$, the density $r_k(N)/N$, and a sufficient N are distinct quantities. Constants and exponents depend on the fixed k ; neither a uniform estimate in k nor an optimal exponent is asserted.

[Theorem 1.1 · p. 4](#)

Corollary 1.2 — Divergent reciprocal sums

Let $\mathbb{N} = \{1, 2, \dots\}$. For every set $A \subseteq \mathbb{N}$, if $\sum_{a \in A} 1/a = \infty$, then for every finite length $\ell \geq 1$ there are $a, d \in \mathbb{N}$ such that $a, a + d, \dots, a + (\ell - 1)d \in A$. For lengths at least two, $d > 0$ makes the progression nonconstant.

[Corollary 1.2 · p. 4](#)

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Quasipolynomial Bounds for Arithmetic Progressions](#). External inputs use the following keys.

- [LSS24](#) James Leng, Ashwin Sah, Mehtaab Sawhney — Quasipolynomial bounds on the inverse theorem for the Gowers $U^{s+1}[N]$ -norm: Interval Gowers inverse theorem and converse correlation bound; distinct from this manuscript’s box and product-cyclic extensions. v3, 10 August 2024.
- [SS14](#) Tomasz Schoen, Olof Sisask — Roth’s theorem for four variables and additive structures in sums of sparse sets: Almost-periodicity retaining the input Bohr radius multiplicatively, used through Lemma C.2 in degree-one shift comparison. v1, 11 August 2014.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1 · p. 4](#)

Proof route. Start with a progression-free set and iterate Theorem 2.1 finitely many times. The state is a two-box certificate on a cell retaining earlier polynomial constraints, rather than just a density on a subinterval. The next section explains one increment; first we identify the output dependencies that make iteration possible.

1 Start on an unconstrained interval

$$f = 1_A, \quad a_0 = \alpha/2, \quad p = C(2 + \log(1/\alpha))$$

$$T = \lceil \log(2/\alpha) / \log(1 + \eta_k) \rceil + 1 = O_k(p)$$

Fix dimensions first; width increments use only higher widths
Propositions 10.2–10.3; §§10.2–10.3 · pp. 77–80

2 Close forecasts from higher weights

$$d_i(t) \leq d_*(p), \quad Q_i(t) \leq V_i = \text{poly}_k(p)$$

$$V_i = Q_i(0) + TP_i(p, d_*, V_{>i}), \quad D = D_k$$

Increment or impossibility; initial $\log N$ pays all costs
Theorem 2.1; §2.2 · pp. 9–10 Proposition 10.4 · p. 81

3 Sustain the side lengths and stop finitely

$$\sum \log \text{slice costs} + \log \text{side cutoff} \leq \text{poly}_k(p)$$

$$a_T \geq (\alpha/2)(1 + \eta_k)^T \geq 1, \quad fB^- \leq B^+$$

Certificate contradiction gives the bound; slack $\alpha N/2$ handles endpoints
Proof of Theorem 1.1 · pp. 81–82

4 Invert the threshold

$$r_k(N)/N \leq C_k e^{-c_k(\log N)^{\varepsilon_k}}$$

$$\log N \geq A_k(2 + \log(1/\alpha))^{A_k} \implies k\text{-AP}$$

Fix density, dimension, and width forecasts; close the finite iteration and invert the threshold

Open figure ↗ TeX source ↓

1. Fix the iteration horizon first

For a progression-free set of density $\alpha > 0$, start with $f = 1_A$, no cell constraints, and $a_0 = \alpha/2$. Fix one resource parameter $p = C(2 + \log(1/\alpha))$ and reserve

$$T = \left\lceil \frac{\log(2/\alpha)}{\log(1 + \eta_k)} \right\rceil + 1 = O_k(p).$$

Each increment multiplies the target by at least $1 + \eta_k$ and adds at most $d_0 = (2 + p)^{C_k}$ fresh absolute coordinates. Discounted intermediate targets remain above a fixed positive multiple of a_0 , so the same p ensures $a \geq e^{-p}$ throughout. Temporary higher-dimensional path boxes do not replace the root spatial dimension in the next round: the root remains an integer-affine slice of the original interval.

Proposition 10.4 · p. 81

2. Triangular recurrences control the loss of length

The weight depth $D = D_k$ is fixed. The increase in the dimension at weight i is bounded by d_0 and a polynomial in strictly higher-weight dimensions. Within a preparation round, set $H_D = d_D$ and $H_i = d_i + H_{i+1}^2$; then $d_i^{\text{out}} \leq d_i + d_0 + H_{i+1}^2$, with $H_{D+1} = 0$. Descending induction bounds dimensions and cut counts across all T rounds by one polynomial forecast $d_*(p)$, fixed before numerical precisions.

For widths w_i , write $Q_i = \log(2/w_i)$. The decisive estimate is

$$Q_i(t+1) - Q_i(t) \leq P_i(p, d_*, Q_{i+1}(t), \dots, Q_D(t)).$$

Neither the current Q_i nor lower-weight widths enter the right-hand side. Define downward envelopes

$$V_D = Q_D(0) + TP_D(p, d_*), \quad V_i = Q_i(0) + TP_i(p, d_*, V_{i+1}, \dots, V_D).$$

Then $Q_i(t) \leq V_i = \text{poly}_k(p)$. Fixed D means fixed polynomial-composition depth. The sum of logarithmic root-slice losses, remaining logarithmic side requirements, and endpoint-rounding costs is also $\text{poly}_k(p)$. Actual length losses occur at the exponential of this scale; a polynomial loss in the side lengths themselves is not asserted. An initial $\log N$ exceeding the total sustains every round.

Propositions 10.2–10.3; §§10.2–10.3 · pp. 77–80

3. The stopping contradiction and equivalence of the two forms

An earlier impossibility alternative already contradicts the initial assumptions. Otherwise $a_T \geq (\alpha/2)(1 + \eta_k)^T \geq 1$. The certificate $\mathbb{E}fB^- > a_T \mathbb{E}B^+$ contradicts the pointwise inequality $fB^- \leq B^+$. Thus a progression-free set cannot satisfy $\log N \geq C_k(2 + \log(1/\alpha))^{A_k}$.

Substituting $\alpha = r_k(N)/N > 0$ and inverting gives

$$\alpha < e^2 \exp\{-C_k^{-1/A_k}(\log N)^{1/A_k}\},$$

which yields the density bound after renaming constants. Conversely, put $\bar{C}_k = \max(1, C_k)$. The bound implies $r_k(N) \leq \alpha N/2 < \alpha N$ whenever

$$\log N \geq \left(\frac{\log(2\bar{C}_k/\alpha)}{c_k} \right)^{1/\varepsilon_k}.$$

This slack covers equality in “density at least α .” Enlarging A_k gives the stated threshold throughout $0 < \alpha \leq 1$. The density bound is immediate when $\alpha = 0$.

Proof of Theorem 1.1 · pp. 81–82

3. Theorem 2.1 — One increment retaining the constraints

Proof target : Theorem 2.1; §2.2 · pp. 9–10

Theorem 2.1 — Triangular increment

Fix $k \geq 3$ and $s = k - 2$. Give spatial variables u weight one and integer blocks $b_h \in \mathbb{Z}^{d_h}$ weight h , with $C_h(u, b_{<h})$ of weighted degree at most h . Centers l_h are bounded and $0 < w_h \leq 1/32$. For sufficiently small $\gamma > 0$, let B^- and B^+ indicate the existence of an integer tuple satisfying, respectively, for all h ,

$$\|b_h - C_h(u, b_{<h}) - l_h\|_\infty \leq w_h, \quad \|b_h - C_h(u, b_{<h}) - l_h\|_\infty \leq (1 + \gamma)w_h.$$

Narrow widths and triangularity make the tuple unique.

There are fixed $D = D_k$, $\eta_k > 0$, and polynomials depending only on k with the following property. Suppose $p \geq 2$, $a \geq e^{-p}$, and $0 \leq f \leq 1$ on a positive-dimensional integer box has support containing no nonconstant integer-vector k -term progression and satisfies $\mathbb{E}fB^- > a\mathbb{E}B^+$. If each logarithmic root side exceeds the specified polynomial cutoff in p , the dimension data, and all $Q_h = \log(2/w_h)$, then either the hypotheses are impossible or a root slice carries a new certificate at target $(1 + \eta_k)a$.

Old determining data are retained through exact integer reparametrizations. At most $d_0 = (2 + p)^{C_k}$ fresh absolute slots are added, together with downward preparation copies. Dimensions and widths satisfy

$$d'_i - d_i \leq d_0 + P_i(p, d_{i+1}, \dots, d_D), \quad Q'_i - Q_i \leq P'_i(p, \text{all dimension forecasts}, Q_{i+1}, \dots, Q_D).$$

The logarithmic root-slice cost is polynomial in the same data including all Q_i . Exponents and gain are independent of the number of rounds; gaps and discounts can be chosen in advance from dimension and cut-count forecasts.

Theorem 2.1; §2.2 · pp. 9–10

In the diagram, λ denotes the target after the terminal absolute increment.

1 Two-box certificate with old constraints

$\mathbb{E}fB^- > a\mathbb{E}B^+$, $Q_i = \log(2/w_i)$
 Retain integer slots b_i and polynomials $C_i(u, b_{<i})$

Transform old integer arguments exactly; stopped paths have zero mass
[Proposition 4.1; Lemma 4.3 · pp. 25–26](#) [Proposition 6.2 · pp. 42–44](#)

2 Prepare and descend to terminal boxes

$\mathbb{E}Y \geq (1 - \delta)a$, $\Pr(Y \geq (1 - 2\delta)a) \geq \delta a$
 Rank cuts and injective integer-affine paths

Positive counting and relative lifting at old rank 0
[Lemma 2.2 · pp. 11–12](#) [Proof of Theorem 1.15 · pp. 194–195](#)

3 Obtain fresh slots by the absolute increment

$\mathbb{E}(f - \lambda)B > 0$, $d_{\text{fresh}} \leq (2 + p)^{C_k}$
 Fresh rank is independent of terminal dimension

Passive/active return, then solve modular and real constraints
[Proposition 7.8; Corollary 7.9; Theorem 8.2 · pp. 57–62](#) [Proposition 9.1; §9 · pp. 69–70](#)

4 Return both payoffs together

$j > s : g^U(u)$, $j \leq s : \pi^0 g = P_Y$ exact
 Restore old determining equations and integer arguments exactly

Compress to simultaneous boxes and retain a fixed net gain
[Lemma 10.1 · pp. 76–77](#) [Propositions 10.2–10.3; §§10.2–10.3 · pp. 77–80](#)

5 Next certificate with triangular losses

$a' = (1 + \eta_k)a$, $Q'_i - Q_i \leq P_i(p, d_*, Q_{>i})$
 Root logarithmic slice cost is polynomial in all width logs

Preparation, constrained sampling, the absolute increment, and exact restoration

[Open figure ↗](#) [TeX source ↓](#)

1. Do not charge the inverse of a small certificate surplus

A preparation rank cut uses a relation at a failing degree, lowers the rational value-space dimension, and, when needed, copies a higher block down one weight. Every higher determining polynomial is transformed exactly on its integer arguments. Since coefficients are unrestricted, continuity in an approximate integer slot cannot replace this operation.

Constrained paths along integer-affine maps carry the prepared cell to terminal boxes. Stopped paths retain zero mass; survivors are not renormalized. Proposition 6.2 gives a terminal mean Y with $\mathbb{E}Y \geq (1 - \delta)a$, and at least a δa fraction of paths have $Y \geq (1 - 2\delta)a$. This avoids paying the inverse of the original strict surplus $\mathbb{E}fB^- - a\mathbb{E}B^+$. Injective integer-affine pullback and removal of points preserve progression-freeness.

Proposition 4.1; Lemma 4.3 · pp. 25–26 Proposition 6.2 · pp. 42–44

2. Obtain a fresh-coordinate count independent of dimension

Lemma 2.2 supplies a fixed multiplicative gain and a degree- s polynomial patch with at most $(2 + p)^{C_k}$ fresh coordinates on a terminal box of density $a \geq e^{-p}$. The fresh count is independent of its dimension n , although logarithmic side requirements, slice costs, Lipschitz bounds, and inverse surplus may depend on $p + n$.

Positive counting in fixed dimension alone does not give this distinction. The manuscript obtains Proposition D.3's absolute increment from Theorem C.1's shift comparison, then converts the niltest into a patch by Lemma D.5. The relative lifting theorem, Theorem I.15, bounds the output rank by $d + d_0$ for an old patch of rank d . Applying it to the constant old patch $\mathbf{1}$, with $d = 0$, gives the dimension-independent d_0 . We follow the statements and their combination in Lemma 2.2; the complete proofs in Appendices C–I have not been independently verified.

Lemma 2.2 · pp. 11–12 Proposition D.3 · pp. 118–119 Proof of Theorem I.15 · pp. 194–195

3. Keep integer boxes and finite groups distinct

For the interval-to-box inverse-theorem extension, embed $Q = \prod_i \{0, \dots, T_i - 1\}$ in an interval by a mixed-radix affine map with bases $4T_i$, and extend by zero. Two-face additive identities are reflected without carries, giving equality of unnormalized cube sums. Group embeddings retain the cube-count normalization ratio. To pass from a product of cyclic groups back to boxes, divide each coordinate into at most four intervals of diameter less than one-third of the period, eliminating wraparound. Factors of $\exp(O_s(n))$ enter the logarithmic resource budget.

Positive counting for the absolute increment uses a separate passage. A box with distinct prime side lengths P_j is identified by the Chinese remainder theorem with $\mathbb{Z}/(\prod_j P_j)\mathbb{Z}$; $P_j > k - 1$ ensures invertibility of the small slope differences. Proposition D.3 inserts a smooth nonnegative weight supported on $x_j/P_j \in (1/4, 1/3)$ and $d_j/P_j \in (1/(20k), 1/(10k))$. On this support every progression lifts to the integer box without wraparound and has $d \neq 0$. Positive group counting contradicts the zero count forced by integer progression-freeness. Arbitrary cyclic progressions are not read unconditionally as integer progressions.

Theorem A.7; Lemma A.9 · pp. 90–93 Proposition D.3 · pp. 118–119

4. Return the positive score and close the width dependencies

The return transports the positive and negative payoffs together, retaining positivity of their difference. At passive layers $j > s$, Proposition 7.8 and Corollary 7.9 give an ordinary ambient orbit and a retained modulus with small logarithmic cost. At active layers $j \leq s$, Theorem 8.2 retains the determining projection as an exact identity. Auxiliary subdivisions in short variables or moduli could feed a current-width cost back into the scoring slice; instead these variables remain in labeled polynomial families.

Proposition 9.1 solves the inactive integer equations, real constraints, and congruence conditions in turn, restoring the old determining equations exactly. Selected affine planes solve the active layer's large full-modulus congruences identically, avoiding subdivision at that modulus. Rather than control the returned polynomial's coefficients, extraction localizes its normalized domain, bounding $\log(w_j/r'_j)$ by higher-layer data independent of the current width. Finally Lemma 10.1 uses coverage averaging to compress to simultaneous residual boxes, with no new slots. Section 10.3 orders dimension forecasts, gaps, descending numerical forecasts, and the final accuracy, rank, and side choices; total discounts remain below the absolute gain, leaving $\eta_k > 0$.

Proposition 7.8; Corollary 7.9; Theorem 8.2 · pp. 57–62 Proposition 9.1; §9 · pp. 69–70
 Lemma 10.1 · pp. 76–77 Propositions 10.2–10.3; §§10.2–10.3 · pp. 77–80

4. Corollary 1.2 — From finite bounds to reciprocal sums

Proof target : Corollary 1.2 · p. 4

1 Fix one length and argue by contradiction

$$k \geq 3, \quad A \subseteq \mathbb{N}, \quad \sum_{a \in A} a^{-1} = \infty$$

A is assumed to contain no nonconstant k -term AP

Translate to an interval of length 2^m ; each reciprocal is at most 2^{-m}
[Theorem 1.1 · p. 4](#) [Proposition 11.1; Corollary 11.2 · p. 82](#)

2 Apply the finite bound on dyadic intervals

$$\sum_{a \in A \cap [2^m, 2^{m+1})} \frac{1}{a} \leq 2^{-m} r_k(2^m) \\ \leq C_k e^{-c_k(m \log 2)^{\varepsilon_k}} \quad (m \geq 1)$$

$m = 0$ contributes at most one; the remaining upper bounds are summable
[Corollary 1.2 · p. 4](#)

3 Summability contradicts divergence

$$m^\varepsilon / \log m \rightarrow \infty \implies e^{-dm^\varepsilon} \leq m^{-2} \text{ eventually} \\ \sum_{a \in A} a^{-1} < \infty$$

Constants may depend on each fixed k ; no uniform threshold is needed
[Corollary 1.2 · p. 4](#)

4 Quantify over every finite length

$\forall \ell \geq 1 \exists a, d \in \mathbb{N} : \quad a + jd \in A \quad (0 \leq j < \ell)$
 $k \geq 3$ was arbitrary; infinitude supplies lengths 1 and 2

Sum the finite bound over dyadic intervals and contradict divergence

[Open figure ↗](#) [TeX source ↓](#)

1. Fix one length and sum

Fix $k \geq 3$ and suppose $A \subseteq \mathbb{N}$ contains no nonconstant k -term progression. Each half-open dyadic interval $[2^m, 2^{m+1}) \cap \mathbb{Z}$ translates to an integer interval of length 2^m , so

$$\sum_{a \in A \cap [2^m, 2^{m+1})} \frac{1}{a} \leq 2^{-m} r_k(2^m) \leq C_k \exp\{-c_k(m \log 2)^{\varepsilon_k}\} \quad (m \geq 1).$$

For every $d, \varepsilon > 0$, $m^\varepsilon / \log m \rightarrow \infty$, hence $e^{-dm^\varepsilon} \leq m^{-2}$ for all sufficiently large m . Thus the right-hand side is summable. The interval $m = 0$ contributes at most one, and $\sum_{a \in A} 1/a < \infty$, contradicting divergence. The qualitative bound $r_k(N) = o(N)$ alone would not give this summability.

[Corollary 1.2 · p. 4](#) [Proposition 11.1; Corollary 11.2 · p. 82](#)

2. Recover every finite length

The contradiction applies to every fixed $k \geq 3$, so a divergent-sum set contains a nonconstant progression of each such length. Constants and progressions may differ with k ; no single uniform threshold is required. Divergence also makes the set infinite, supplying lengths one and two. This covers all quantifiers in Corollary 1.2. Section 11.1 extends the same dyadic partition to nonnegative weights w , but the subsequent weighted consequences are not all presented as main results here.

Corollary 1.2 · p. 4

5. External inputs and their uses

The principal direct external inputs are the interval inverse theorem of Leng–Sah–Sawhney and the radius-sensitive almost-periodicity theorem of Schoen–Sisask. The first supplies the interval case of Theorem A.7; the second enters degree-one shift comparison through Lemma C.2. Nilsequence correlation alone does not finish the main theorem: positive counting, a dimension-independent fresh-coordinate count, and exact restoration of old constraints provide the intervening connections.

Theorem A.7; Lemma A.9 · pp. 90–93 Lemma C.2 · p. 101

LSS Theorem 1.2 supplies a nilsequence for a unit-bounded interval function with U^{s+1} norm at least δ , with dimension $\text{poly}_s(\log(1/\delta))$ and complexity and inverse correlation bounded by $\exp(\text{poly}_s(\log(1/\delta)))$. This manuscript sets $\delta = e^{-p}$, normalizes the correlator to have cap one, and compares coordinate metrics. The converse input, Lemma B.5, corresponds to Lemma A.9 here. The external interval results must be separated from this manuscript's own box and product-cyclic extensions.

[LSS24, Theorem 1.2, p. 1] [LSS24, Lemma B.5, p. 83]

SS Theorem 5.4 supplies almost-periods under regular-Bohr-set and small-sumset hypotheses, retaining the incoming radius ρ as a multiplicative factor in the output radius. Lemma C.2 applies it to two sets of relative density $\geq e^{-q}$ and obtains, for fixed error η , $\text{rank } D_* \leq D + C_\eta(1 + q)^4$ and $w(D_*) \geq w(S') \exp\{-C_\eta(1 + q + \log(2 + D))\}$. Dividing counting convolution by the product of the set cardinalities converts the error into a probability error. This relative width loss is the input for tracking radii through the degree-one iteration.

[SS14, Theorem 5.4, p. 14]

A reviewer other than the author compared the statements of these two sources and the applications above with the originals. The input theorems' proofs were not independently verified.

The source record distinguishes external editions, statement checks, and application checks. The principal internal connections appear in the stages above. The companion manuscript's coloring lower bound in §1.4 is a comparison, not an input to the density upper bound. Catalog 159 contains one paper; no within-catalog dependency on another paper is introduced.

6. Return to the sources

- [Theorem 1.1 · p. 4](#) — Quantifiers and the density-bound/threshold formulations
- [Theorem 2.1; §2.2 · pp. 9–10](#) — Two-box certificates and triangular loss bounds
- [Lemma 2.2 · pp. 11–12](#) — The connection supplying a dimension-independent fresh-slot count
- [Propositions 10.2–10.3; §§10.2–10.3 · pp. 77–80](#) — Structural choices first and descending width forecasts
- [Proposition 10.4 · p. 81](#) — Iteration horizon, side-length budget, and contradiction at target one
- [Proposition D.3 · pp. 118–119](#) — Chinese remaindering and the weight excluding wraparound
- [Corollary 1.2 · p. 4](#) — Dyadic summation and quantification over every finite length

The editorial reading covered the main statements on p.4; cell and increment setup on pp.8–14; preparation statements and the start of rank cutting on pp.24–26; forward mass comparison on pp.40–44; passive return on pp.57–60; active-return setup on pp.60–62; extraction setup and statement on pp.69–70; the iteration on pp.76–82; interval/box/group transitions on pp.90–93; almost-periodicity's application on p.101; positive counting to the absolute increment on pp.114–119; and the end of relative lifting on pp.194–195. We compared the inversion, dyadic summation, iteration forecasts, and roles of the cited internal results with the source.

The full constrained-sampling error estimates, active globalization proof, extraction estimates, and complete proofs in Appendices B–I remain unchecked. Accordingly, the analytic construction of Theorem 2.1 is presented as the authors' proof route, not as an independently verified chain in its entirety. Expert peer review and formal verification have not been performed. Editions and reading records are in [sources.json](#); timing and checks are in [status.md](#).