

OpenAI Math Digest

CATALOGUE 087

The Mahler conjectures and symplectic width

Catalogue synthesis & 3 proof overviews

Independent reading guide · English edition

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AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 3 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

The Mahler conjectures and symplectic width

Catalogue 087 contains three manuscripts on the symmetric Mahler inequality and Hanner equality classification, the Santaló-centered inequality and simplex rigidity for general convex bodies, and the Gromov width of symmetric polar products. This guide describes the authors' claims in fixed source versions and the routes to their proofs. The three papers do not form a sequential chain of assumptions.

1. Papers and guides in official order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 087 · 2026-10-15 checked source snapshot

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01

Manuscript and overview

The symmetric Mahler conjecture and its equality cases →

Short name: Symmetric Mahler equality cases

2026-09-22 · 26 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Every origin-symmetric body has volume product $\geq 4^n/n!$, with equality exactly for linear Hanner bodies.

[Theorem 1.1 · p. 2](#) ↗

Role and available overview

Convert holomorphic mass to a real simplex integral. Vanishing equality deficit gives a common feasibility witness, metric medians and Hanner classification.

[Read the proof overview](#) →

#

02

[Manuscript and overview](#)

The Mahler Conjecture for General Convex Bodies →

Short name: General Mahler conjecture

2026-09-22 · 72 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Every convex body has Santaló-centered volume product $\geq (n+1)^{n+1}/(n!)^2$, with equality exactly for simplices.

[Theorem 1.1 · p. 1](#) ↗

Role and available overview

Positive-dual cone Laplace integrals, simultaneous normalization of Gaussian projections, noncommutative layer defects and strict interval estimates.

[Read the proof overview](#) →

#

03

Manuscript and overview

Symplectic Balls in Symmetric Polar Products →

Short name: Symplectic balls in polar products

2026-09-22 · 19 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For $n \geq 2$, the symmetric polar product has Gromov width 4. Balls of every capacity $0 < c < 4$ embed, yielding the symmetric volume bound.

[Theorem 1.1 · p. 1](#) ↗

Role and available overview

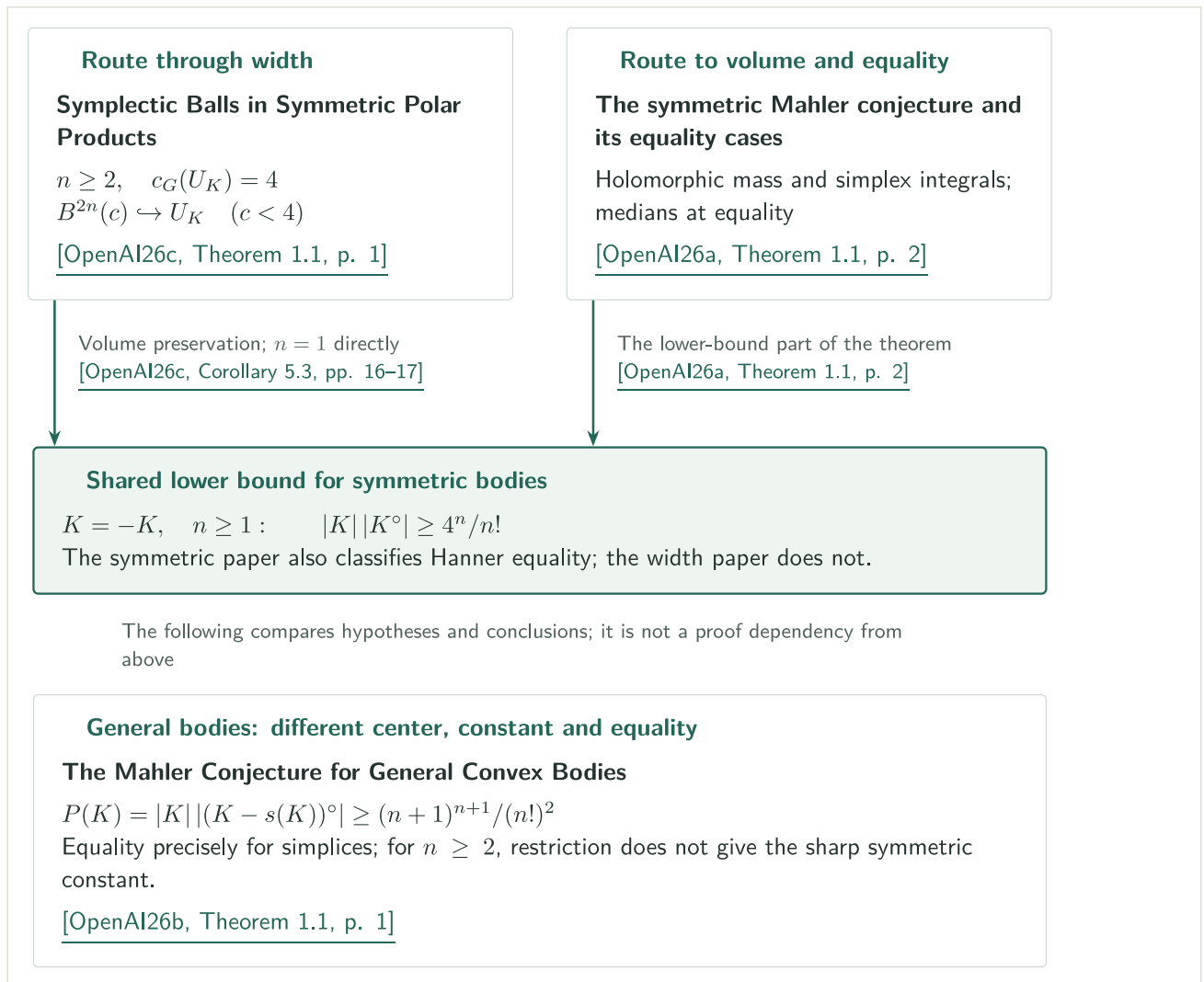
Combine a ball from a holomorphic zero with uniform slice control; finite polar approximation handles arbitrary symmetric bodies, while a supporting cylinder gives the upper bound.

[Read the proof overview](#) →

2. Where conclusions meet and where hypotheses differ

The diagram's citation keys all refer to OpenAI manuscripts dated September 22, 2026:

[OpenAI26a], The symmetric Mahler conjecture and its equality cases; [OpenAI26b], The Mahler Conjecture for General Convex Bodies; and [OpenAI26c], Symplectic Balls in Symmetric Polar Products. Citations in the diagram open the original sources.



Two symmetric routes meet at the same volume bound; the general case uses a different center and constant

[Open figure ↗](#) [TeX source ↓](#)

The symmetric paper turns mass at a holomorphic zero into a real simplex integral. When missing volume vanishes, a common feasibility witness yields metric medians. This supplies both the lower bound and the direction asserting that equality forces a linear Hanner body. See the [symmetric theorem](#) and [both directions of equality](#).

The width paper combines a ball from holomorphic vanishing with a uniform horizontal integral estimate, embedding balls of every capacity $c < 4$ into the real $2n$ -dimensional domain $U_K = \text{int } K \times \text{int } K^\circ$. Volume preservation yields the same symmetric lower bound $4^n/n!$, but not the Hanner classification. The papers describe a shared lens construction as background; the width proof does not use the symmetric paper's volume theorem or equality classification. See the [width construction and independent upper bound](#), and the [volume consequence](#).

[\[OpenAI26c, Background and prior constructions, p. 2\]](#)

The general paper studies $P(K) = |K| |(K - s(K))^\circ|$. It passes to Laplace integrals on positive-dual cones, then uses simultaneous normalization of Gaussian projections, noncommutative layer defects, and strict spectral interval estimates to obtain simplex rigidity. For $n \geq 2$, its constant $(n+1)^{n+1}/(n!)^2$ is smaller than the symmetric constant $4^n/n!$. Restricting the general theorem to symmetric bodies therefore does not give the sharp symmetric result or the Hanner classification. See the [general lower bound](#) and [simplex equality classification](#).

[OpenAI26b, Introduction, comparison with symmetric case, p. 1]

[OpenAI26a, §1.1, comparison with general case, p. 3]

3. Consequences branching after the main theorems

The geometric-to-functional passage uses the Fradelizi–Meyer implication through auxiliary convex bodies of increasing dimension, rather than a reformulation in one fixed dimension. For even proper lower-semicontinuous convex potentials, the integral product is at least 4^n ; for general potentials without evenness, it is at least e^n . Both require $0 < \int e^{-\varphi} < \infty$ and allow an infinite conjugate integral. The geometric equality classifications do not automatically classify all functional equality cases.

[OpenAI26a, Corollaries 1.2–1.3 and proofs, pp. 4–5]

[OpenAI26b, §8; Corollaries 1.2 and 8.1, pp. 40–41]

Starting point

The all-dimensional symmetric lower bound

Consequence and conditions

Even functional product $\geq 4^n$; for symmetric full-dimensional log-concave probability measures, essential continuity additionally gives entropy–transport

Guide

[Functional · transport](#)

Starting point

The all-dimensional general lower bound

Consequence and conditions

General functional product $\geq e^n$; full-dimensional log-concave probability measures with essential continuity give entropy–transport

Guide

[Functional · transport](#)

Starting point

Balls of every capacity $c < 4$ in the width paper

Consequence and conditions

For $n \geq 3$, strict volume and pairwise-capacity conditions on finitely many positive R_i give disjoint neighborhood embeddings of closed balls

Guide

[Sufficient packing criterion](#)

The last row uses Theorem 1.1 of catalogue 343's Symplectic Ball Packings in Higher Dimensions, OpenAI, September 23, 2026 ([OpenAI26p]). Its application is Corollary 5.5 of the width paper, not the main width theorem or the Mahler inequality. The inequalities $\sum_i R_i^n < 4^n$ and $R_i + R_j < 4$ ($i < j$) allow one $c < 4$; a packing into that ball is composed with the width embedding. The resulting conditions for the polar product are sufficient, not claimed necessary and sufficient.

[OpenAI26p, Theorem 1.1, p. 2] [OpenAI26c, Corollary 5.5, p. 17]

4. Sources, versions, and checking scope

The official heading is The Mahler conjectures and symplectic width. The subject is the fourth field in the official table of contents, Convex and metric geometry (087–101), following print/PDF p.10 of overview.pdf. The alternate CONTENTS heading, The Mahler conjectures, functional inequalities and polar-product symplectic width, is retained in the inventory.

OpenAI Research Catalog · Catalogue 087, Convex and metric geometry · p. 10

All three manuscripts are dated September 22, 2026, at fixed official commit 1581d4c. The bilingual writing and content checks use explicitly selected `gpt-6-astra`, comparing the manuscripts' hypotheses and conclusions with selected direct inputs and their uses. Each article states its scope. Complete recomputation of the general paper's rational certificates, interpolation and tail bounds, full external proofs, and recursive verification of the bibliography remain incomplete. These checks are not expert peer review or independent verification of complete proofs.

The symmetric Mahler conjecture and its equality cases

From a conformal boundary law to simplex volumes, and from equality to the three-ball property

OpenAI's September 22, 2026 manuscript claims the symmetric Mahler inequality in every dimension, with equality precisely for linear Hanner bodies. Its central device is an integral identity for feasible simplices, obtained from a planar conformal map and the mass of an isolated holomorphic zero. For equality, adjoining one dimension turns vanishing integrated deficit into an optimal representation at each point and direction, and then into metric medians for the norm with the polar body as unit ball.

Manuscript author: OpenAI · 2026-09-22 · 26 pages, including references

[Read the manuscript on GitHub ↗](#)

This article explains the manuscript's argument and its direct inputs. The editorial checking scope and the unverified scope, including complete proofs of external theorems, are distinguished at the end.

1. Main results

Write $|K|$ for Lebesgue volume in the dimension of K , $K^\circ = \{y : \langle x, y \rangle \leq 1 \ (x \in K)\}$ for polarity at the **fixed origin**, and $P(K) = |K||K^\circ|$. Hanner polytopes are obtained from centered nondegenerate intervals by finitely many operations $H_1 \oplus_\infty H_2 = H_1 \times H_2$ and $H_1 \oplus_1 H_2 = \text{conv}((H_1 \times \{0\}) \cup (\{0\} \times H_2))$. A linear Hanner body is an invertible linear image of such a polytope; translations are not included.

Theorem 1.1 — The symmetric Mahler inequality and equality classification

For every integer $n \geq 1$ and every origin-symmetric compact convex body $K = -K \subset \mathbb{R}^n$ with nonempty interior,

$$P(K) \geq \frac{4^n}{n!}.$$

Equality holds if and only if K is a linear Hanner body.

Theorem 1.1 · p. 2

Corollary 1.2 — The even functional Mahler inequality

Let $n \geq 1$ and let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be even, proper, lower-semicontinuous and convex, with $0 < \int_{\mathbb{R}^n} e^{-\varphi} < \infty$. For its Fenchel–Legendre transform $\varphi^*(y) = \sup_x (\langle x, y \rangle - \varphi(x))$,

$$\left(\int_{\mathbb{R}^n} e^{-\varphi(x)} dx \right) \left(\int_{\mathbb{R}^n} e^{-\varphi^*(y)} dy \right) \geq 4^n.$$

The second integral may be $+\infty$. The constant is sharp: for $\varphi(x) = \sum_i |x_i|$, both integrals equal 2^n . The corollary does not classify all equality potentials.

Corollary 1.2 · pp. 4–5

Corollary 1.3 — The symmetric entropy–transport inequality

Let $n \geq 1$. For $i = 1, 2$, let $\eta_i(dx) = e^{-V_i(x)} dx$ be full-dimensional origin-symmetric log-concave probability measures, with proper lower-semicontinuous convex potentials V_i . Assume their densities are essentially continuous: $e^{-V_i} = 0$ at $\mathcal{H}^{n-1}$ -almost every point of $\partial \text{supp } \eta_i$. Then the moment measures $\nu_i = (\nabla V_i)_\# \eta_i$ have finite first moments, the entropies $H(\eta_i) = \int \log(d\eta_i/dx) d\eta_i = - \int V_i d\eta_i$ are finite, and

$$H(\eta_1) + H(\eta_2) \leq -n \log(4e^2) + \mathcal{T}(\nu_1, \nu_2).$$

Here, for the class $\mathcal{F}$ of proper lower-semicontinuous convex functions,

$$\mathcal{T}(\mu_1, \mu_2) = \inf_{f \in \mathcal{F}} \left\{ \int f d\mu_1 + \int f^* d\mu_2 \right\}$$

is allowed to be $+\infty$. The sign convention for H is the negative of differential Shannon entropy.

Corollary 1.3 · p. 5

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [The symmetric Mahler conjecture and its equality cases](#). External inputs use the following keys.

- [HL81](#) Allan B. Hansen; Åsvald Lima — The structure of finite dimensional Banach spaces with the 3.2. intersection property: Classify finite-dimensional real Banach spaces with the three-ball property by ℓ^1 and ℓ^∞ sums. *Acta Mathematica* 146 (1981), 1–23; published PDF.
- [FM08](#) Matthieu Fradelizi; Mathieu Meyer — Some functional inverse Santaló inequalities: Pass from the symmetric geometric inequality in all dimensions to the even functional inequality. *Advances in Mathematics* 218 (2008), 1430–1452; author's personal copy with cover.
- [CEK15](#) Dario Cordero-Erausquin; Bo'az Klartag — Moment measures: Supply finite first moments of moment measures and finite entropy. *Journal of Functional Analysis* 268 (2015), 3834–3866; published PDF.
- [FGZ21](#) Matthieu Fradelizi; Nathael Gozlan; Simon Zugmeyer — Transport proofs of some functional inverse Santaló inequalities: Use the equivalence of even inverse Santaló and symmetric entropy–transport inequalities. [arXiv:2109.00871v3](#), 9 September 2021.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1](#) · [p. 2](#)

Proof route. The argument joins the lower bound, classification under equality, and attainment by Hanner bodies. The next section explains the real simplex identity underlying the bound; the following two sections explain the common witness and median constructions underlying classification. The general nonsymmetric Mahler manuscript is not an input.

1 Fix the origin and approximate by strip bodies

$$K = -K \subset \mathbb{R}^n, \quad n \geq 1, \quad Q = K^\circ$$

The uniform boundary coordinate gives the identity; polar approximation passes to the limit.

[Proposition 6.1 · pp. 15–16](#) [Corollary 6.2 · pp. 16–17](#)

2 Convert holomorphic mass to a simplex integral

$$1 \leq S = \frac{d!}{4^d} \int_A |\Sigma_X| dX \leq \frac{d!}{4^d} P(A) \implies P(K) \geq 4^n/n!$$

The sum-volume formula preserves equality; product neighborhoods and compactness apply.

[§7.1, \(36\)–\(38\) · pp. 18–19](#) [Proposition 7.1 · pp. 18–20](#)

3 Assume equality and lift by one dimension

$$P(K) = 4^n/n!, \quad B = K \oplus_1 [-1, 1], \quad B^\circ = Q \times [-1, 1]$$

Vanishing deficit supplies a common witness for each X, ξ .

Endpoint scaling and separation give medians; apply real finite-dimensional classification.

[Proposition 8.1 · pp. 21–22](#) [Lemma 8.2 · p. 23](#)

[\[HL81, Corollary 7.4, p. 15\]](#)

4 From endpoint costs to medians, three balls and classification

$$(\mathbb{R}^n, p_Q) : \text{every triple has a median} \implies Q = TH$$

$$K = Q^\circ = T^{-T} H^\circ : \text{a linear Hanner body}$$

Product 4 for intervals, polar closure and linear invariance give attainment by all Hanner bodies.

[Lemmas 2.1–2.2 · p. 6](#)

5 The converse follows by recursion from intervals

$$P(A \oplus_s A') = \binom{k+l}{k}^{-1} P(A)P(A'), \quad s \in \{1, \infty\}$$

$$P(K) = 4^n/n! \iff K \text{ is a linear Hanner body}$$

Lower bound, equality classification and converse in Theorem 1.1

[Open figure ↗](#) [TeX source ↓](#)

1. From integrated simplex volume to arbitrary convex bodies

Start with a finite strip body $A = \{X : |b_i \cdot X| \leq 1\}$. Using the conformal lens width λ , Proposition 6.1 constructs a union of feasible simplices $\Sigma_X \subset A^\circ$ such that

$$1 \leq S = \frac{d!}{4^d} \int_A |\Sigma_X| dX \leq \frac{d!}{4^d} |A| |A^\circ|$$

The first inequality comes from holomorphic mass, the identity from the uniform boundary law, and the last inequality from containment in the polar. For an arbitrary K , approximate $Q = K^\circ$ by inner symmetric polytopes Q_j . The inclusions $(1 - \delta_j)Q \subset Q_j \subset Q$ give $K \subset Q_j^\circ \subset (1 - \delta_j)^{-1}K$ by polarity. Taking the limits of both volumes proves the bound, without smoothness or general-position assumptions.

Proposition 6.1 · pp. 15–16 Corollary 6.2 · pp. 16–17

2. From equality to the three-ball property of the polar

If $P(K) = 4^n/n!$, then $B = K \oplus_1 [-1, 1]$ also attains equality in dimension $d = n + 1$. The integrated missing volume for approximations to $B^\circ = Q \times [-1, 1]$ tends to zero. Proposition 7.1 thereby obtains, at every $X \in \text{int } B$ and for every $\xi \neq 0$, **one** feasibility witness that compares with all competing representations. Proposition 8.1 lets X approach the apex $(0, 1)$ and uses the cost comparison to construct a median for each triple in $(\mathbb{R}^n, p_Q)$.

Lemma 8.2 converts medians to the three-ball property for arbitrary radii. For pairwise-intersecting balls with centers x_i and radii r_i , their distances d_i to a median m satisfy $d_i + d_j \leq r_i + r_j$. If m is outside one ball, moving it toward that center by the excess distance places it in all three balls. Radius zero is included. The normed space is finite-dimensional, real and complete, so Hansen–Lima's classification applies and gives $Q = TH$. Polarity yields $K = T^{-T}H^\circ$; closure of the Hanner class under polarity finishes the necessary classification.

Proposition 7.1 · pp. 18–20 Proposition 8.1 · pp. 21–22 Lemma 8.2 · p. 23 [HL81, Corollary 7.4, p. 15]
Theorem 8.3; Proposition 8.4 · p. 23

3. Every linear Hanner body attains equality

The converse does not require the classification theorem. An interval has volume product 4, and for bodies A, A' of dimensions k, l and $s \in \{1, \infty\}$,

$$P(A \oplus_s A') = \binom{k+l}{k}^{-1} P(A)P(A')$$

The recursive construction thus preserves $4^{k+l}/(k+l)!$. Polarity exchanges the ℓ_1 and ℓ_∞ sums, and the two determinant factors of an invertible linear map cancel in the volume product. This proves attainment for the whole equality family, including the base dimension $n = 1$.

Lemmas 2.1–2.2 · p. 6

3. Proposition 6.1 — From holomorphic mass to the real simplex identity

Proof target : Proposition 6.1 · pp. 15–16

Proposition 6.1 — The feasible-simplex identity

Let $d \geq 1$ and let $b_1, \dots, b_m$ be nonzero, pairwise nonproportional real rows spanning $(\mathbb{R}^d)^*$. Put $A = \{X : |b_i X| \leq 1\}$, $C = A^\circ$, and use the lens width λ from Lemma 3.1. For $X \in \text{int } A$, let $L_X = \{Y : |b_i Y| \leq \lambda(b_i X) \forall i\}$. For a basis I and signs ϵ_i , the pair is feasible if $Y_{I,\epsilon} = B_I^{-1}(\epsilon_i \lambda(b_i X))_{i \in I}$ belongs to L_X . Let Σ_X be the union of the corresponding simplices $\text{conv}(0, \epsilon_i b_i : i \in I)$.

For independent uniform boundary angles form $Z_I = B_I^{-1}(F(e^{i\theta_i}))_{i \in I}$. Let P_I be the probability that every row coordinate lies in the closed lens $\overline{D}$, and $S = \sum_I P_I$. Then

$$1 \leq S = \frac{d!}{4^d} \int_A |\Sigma_X| dX \leq \frac{d!}{4^d} |A| |A^\circ|,$$

$$0 \leq \int_A (|A^\circ| - |\Sigma_X|) dX \leq |A| |A^\circ| - \frac{4^d}{d!}.$$

Proposition 6.1 · pp. 15–16

1 Parametrize the lens boundary by a uniform coordinate

$F(e^{i\Theta}) \stackrel{\text{law}}{=} \epsilon\lambda(T) + iT, \quad T \sim \text{Unif}[-1, 1]$
 ϵ is an independent fair sign; $g = F^{-1}$.

Verify compact sublevels inside the lens and use the Stokes mass bound.

[Lemma 3.1 · pp. 7–8](#) [Lemma 4.1 · pp. 10–12](#)

2 Apply isolated-zero mass to the high-power map

$f_k = (g(b_j z)^k)_j, \quad \sum_I M_{I,k} \geq (\pi k)^d / d!$
 Spanning controls the zero and the nonvanishing leading term.

Cancel k^d and apply upper Fatou on a fixed finite measure space.

[Lemma 5.1 · pp. 12–14](#)

3 Rescale radii to obtain boundary probabilities

$u_i = \rho_i^{1/k} e^{i\theta_i}, \quad \limsup_{k \rightarrow \infty} M_{I,k} / k^d \leq \frac{\pi^d}{d!} P_I$
 $S = \sum_I P_I \geq 1$

Extra boundary equalities are null; unique maximizers make simplex interiors disjoint.

[Lemma 5.2 · p. 14](#) [Proposition 6.1 · pp. 15–16](#)

4 Remove boundary ties and sum simplex volumes

$t_I = B_I X, \quad d! |\Sigma_X| = \sum_{I, \epsilon} |\det B_I| \mathbf{1}_{\text{feasible}} \quad a.e.$
 $S = \frac{d!}{4^d} \int_A |\Sigma_X| dX, \quad \Sigma_X \subset C = A^\circ$

Insert containment and $S \geq 1$ into the exact identity.

[Proposition 6.1 · pp. 15–16](#)

5 Obtain the bound and the deficit for equality

$P(A) \geq 4^d / d!, \quad 0 \leq \int_A (|C| - |\Sigma_X|) dX \leq P(A) - 4^d / d!$

Conversion of holomorphic mass into boundary probabilities and real simplex volumes

[Open figure ↗](#) [TeX source ↓](#)

1. High powers at an isolated zero produce boundary probabilities

The conformal map of Lemma 3.1 is $F(z) = \frac{8}{\pi^2} \sum_{j \geq 0} (-1)^j z^{2j+1} / (2j+1)^2$. Its boundary law is $\epsilon\lambda(T) + iT$, where T is uniform on $[-1, 1]$ and ϵ is an independent fair sign. Set $g = F^{-1}$ and $f_k = (g(b_j z)^k)_j$ on $\Omega = \{z : b_j z \in D \forall j\}$. Spanning ensures the zero is unique and the degree- k homogeneous leading term is nonzero away from the origin. A sublevel $|f_k|^2 \leq s < 1$ confines every row coordinate to a fixed compact subset of the lens, also verifying the compact-sublevel hypothesis of Lemma 4.1.

With $d^c = \frac{i}{4}(\bar{\partial} - \partial)$, Lemma 4.1 bounds the integrated sum of squared Jacobian minors below by $(\pi k)^d/d!$. Its proof compares sublevel mass by Stokes using the positivity of $dd^c \log |f_k|^2$, and calculates the homogeneous leading term's sphere flux as $(\pi k)^d$. Invariance under common phase rotation makes the correction to radial homogeneous flux vanish; this is the normalization step.

For each basis I , change variables to $u_i = g(b_i z)$, then set $u_i = \rho_i^{1/k} e^{i\theta_i}$. The factor k^d cancels. On the fixed region $\sum_i \rho_i^2 < 1$, the inferred coordinates approach the boundary samples. Upper Fatou on a finite measure space yields $\limsup M_{I,k}/k^d \leq (\pi^d/d!)P_I$. Summing over finitely many bases gives $S \geq 1$. Additional boundary equalities do not have to be excluded at this stage.

[Lemma 3.1 · pp. 7–8](#) [Lemma 4.1 · pp. 10–12](#) [Lemma 5.1 · pp. 12–14](#)

2. Avoid double counting and retain the deficit

For each fixed sign choice the boundary law has measure $4^{-d} dt_I$. With $t_I = B_I X$ and $D_I = |\det B_I|$, it gives $P_I = 4^{-d} D_I \int_A \sum_{\epsilon} \mathbf{1}_{(I,\epsilon) \text{ feasible}} dX$. By strict curvature of the lens boundary and nonproportionality of the rows, Lemma 5.2 makes the angles at which an extra row also lies on the boundary a null set. Thus, for almost every X , different feasible pairs give different vertices of L_X .

A vector in the interior of a feasible simplex defines a linear functional uniquely maximized at the corresponding vertex. Hence distinct simplices have disjoint interiors, allowing their volumes $D_I/d!$ to be summed. This proves the central identity. Containment $\Sigma_X \subset C$ and $S \geq 1$ then give

$$\int_A (|C| - |\Sigma_X|) dX = |A||C| - \frac{4^d}{d!} S$$

and its nonnegative upper bound. The output for the equality argument is this **integrated deficit**; it does not assert $\Sigma_X = C$ at each X .

[Lemma 5.2 · p. 14](#) [Proposition 6.1 · pp. 15–16](#)

4. Proposition 7.1 — From vanishing deficit to one common witness

Proof target : [Proposition 7.1 · pp. 18–20](#)

Proposition 7.1 — A representation minimizing the lens cost

Let $K = -K \subset \mathbb{R}^n$, $n \geq 1$, be a convex body with $P(K) = 4^n/n!$. Put $Q = K^\circ$, $d = n + 1$, $B = K \oplus_1 [-1, 1]$ and $C = Q \times [-1, 1]$. For every $X \in \text{int } B$ and $0 \neq \xi = (q, a) \in \mathbb{R}^{n+1}$, set $T = p_C(\xi) = \max\{p_Q(q), |a|\}$. There are $1 \leq r \leq d$ rows $c_i \in Q \times \{-1, 1\}$, weights $\alpha_i \geq 0$ and $Y \in \mathbb{R}^d$ satisfying

$$\xi = \sum_i \alpha_i c_i, \quad \sum_i \alpha_i = T,$$

$$|h \cdot Y| \leq \lambda(h \cdot X) \quad (h \in C), \quad c_i \cdot Y = \lambda(c_i \cdot X) \quad (1 \leq i \leq r)$$

In particular, for every finite representation $\xi = \sum_j \beta_j h_j$ with $\beta_j \geq 0$ and $h_j \in C$,

$$\sum_i \alpha_i \lambda(c_i \cdot X) \leq \sum_j \beta_j \lambda(h_j \cdot X).$$

The choice may depend on X, ξ , but one choice works for all competing representations.

Proposition 7.1 · pp. 18–20

1 Lift while preserving equality and use inner polytopes

$$B = K \oplus_1 [-1, 1], \quad C = Q \times [-1, 1], \quad P(B) = 4^d/d! \\ C_j = Q_j \times [-1, 1], \quad B_j = C_j^\circ, \quad D_j = \int_{B_j} (|C_j| - |\Sigma_{j,X}|) dX \rightarrow 0$$

Missing V_k for every $Z \in U_k$ would force too large an integrated deficit.

[§7.1, \(36\)–\(38\) · pp. 18–19](#) [§7.2, \(39\)–\(41\) · pp. 19–20](#)

2 Capture each prescribed point and direction

$$X \in \text{int } B, \quad e \in \partial C, \quad D_{j_k} < |U_k| |V_k| \\ X_k \rightarrow X, \quad e_k \rightarrow e, \quad e_k \in \Sigma_{j_k, X_k}$$

Concavity extends vertex constraints to their hull; limiting rows need not stay independent.

[§7.2, \(39\)–\(41\) · pp. 19–20](#)

3 Bound a common witness and allow degenerate limits

$$Y_k \in \lambda(0)B_{j_k}, \quad c_{ik} \rightarrow c_i, \quad Y_k \rightarrow Y, \quad \theta_{ik} \rightarrow \theta_i \\ |h \cdot Y| \leq \lambda(h \cdot X) \quad (h \in C), \quad c_i \cdot Y = \lambda(c_i \cdot X)$$

Use $1 = p_C(e) \leq \sum \theta_i \leq 1$; the same Y works against every competitor.

[Proposition 7.1 · pp. 18–20](#)

4 Fix the total weight by the boundary gauge and compare all representations

$$e = \xi/T, \quad \sum_i \theta_i = 1, \quad \alpha_i = T\theta_i \\ \sum_i \alpha_i \lambda(c_i \cdot X) = \xi \cdot Y \leq \sum_j \beta_j \lambda(h_j \cdot X)$$

Integral deficit, product neighborhoods and a common witness allowing degenerate limits

[Open figure ↗](#) [TeX source ↓](#)

1. Product neighborhoods capture every point and direction

We have $|B||C| = \frac{4}{n+1} P(K) = 4^d/d!$. Choose symmetric inner polytopes $t_j Q \subset Q_j \subset Q$ with $t_j \rightarrow 1$, and put $C_j = Q_j \times [-1, 1]$, $B_j = C_j^\circ$. Proposition 6.1 gives

$$D_j = \int_{B_j} (|C_j| - |\Sigma_{j,X}|) dX \rightarrow 0.$$

For prescribed $X \in \text{int } B$ and $e \in \partial C$, choose small positive-volume balls $U_k \subset \text{int } B$ and $V_k \subset \text{int } C$ close to X and e . Choose $\overline{V_k} \subset \text{int } C$; then $t_j C \subset C_j$ and $t_j \rightarrow 1$ let us take j_k large enough to ensure both $V_k \subset C_{j_k}$ and $D_{j_k} < |U_k| |V_k|$. If every $\Sigma_{j_k, Z}$ with $Z \in U_k$ missed V_k , the deficit would be at least that product. Therefore there are $X_k \rightarrow X$, $e_k \rightarrow e$ with $e_k \in \Sigma_{j_k, X_k}$. This uses product neighborhoods, not a claim of pointwise convergence of the deficit at the prescribed point.

[§7.1, \(36\)–\(38\) · pp. 18–19](#) [§7.2, \(39\)–\(41\) · pp. 19–20](#)

2. Preserve the witness even when bases become singular

A simplex containing e_k supplies d rows c_{ik} , weights $\theta_{ik} \geq 0$ with $\sum_i \theta_{ik} \leq 1$, and a common witness Y_k . Concavity of λ extends feasibility from vertices to all of C_{j_k} . Since $\lambda \leq \lambda(0)$, we have $Y_k \in \lambda(0)B_{j_k}$; the polar sandwich makes these witnesses uniformly bounded. No uniform bound on the inverse basis matrices is needed, and the limiting rows may be dependent.

Take a simultaneous convergent subsequence of Y_k, c_{ik}, θ_{ik} . For any $h \in C$, use $t_{j_k} h \in C_{j_k}$ to pass to all feasibility inequalities with the same limit Y . Since $p_C(e) = 1$, we get $1 \leq \sum_i \theta_i \leq 1$, so no mass remains at the simplex vertex 0. Setting $e = \xi/T$ and $\alpha_i = T\theta_i$ gives the desired representation. Finally,

$$\sum_i \alpha_i \lambda(c_i \cdot X) = \xi \cdot Y = \sum_j \beta_j h_j \cdot Y \leq \sum_j \beta_j \lambda(h_j \cdot X)$$

proves optimality against every competitor using that common witness.

§7.2, (39)–(41) · pp. 19–20 Proposition 7.1 · pp. 18–20

5. Proposition 8.1 — From apex cost comparison to medians

Proof target : Proposition 8.1 · pp. 21–22

Proposition 8.1 — Metric medians of an equality body

Let $n \geq 1$ and let $K = -K \subset \mathbb{R}^n$ be a convex body with $P(K) = 4^n/n!$. Put $Q = K^\circ$.

Every triple x_0, x_1, x_2 in $(\mathbb{R}^n, p_Q)$ admits a point m such that, for all $0 \leq i < j \leq 2$,

$$p_Q(x_i - x_j) = p_Q(x_i - m) + p_Q(m - x_j)$$

Uniqueness of the median is not asserted.

Proposition 8.1 · pp. 21–22

1 Convert three distances into two sign weights

$$\|\cdot\| = p_Q, \quad p_1 = \|x\|, \quad p_2 = \|y\|, \quad p = \|x - y\|$$

$$s = p_1 + p_2, \quad a = p_1 - p_2, \quad A_{\pm} = (p \pm a)/2, \quad c = (s - p)/2$$

Proposition 7.1 gives total weight p ; endpoint scaling handles varying representations uniformly.

[Proposition 7.1 · pp. 18–20](#) [Lemma 3.2 · pp. 8–9](#)

2 Approach the apex and normalize the optimal cost

$$X_{\delta} = (\delta u, 1 - \delta), \quad u \in \text{int } K, \quad \xi = (x - y, a)$$

$$\lambda(1 - \delta t)/\lambda(1 - \delta) \rightarrow t$$

uniformly on compact positive intervals

Use the cost of the competitor $p_1(x/p_1, 1) + p_2(-y/p_2, -1)$ as an upper bound.

[§8, \(42\)–\(46\) · pp. 21–22](#)

3 Compare with a two-term representation to bound a support function

$$D_0 = \{r_+ - r_- : r_+ + r_- = x - y, \|r_{\pm}\| \leq A_{\pm}\}$$

$$\langle x + y, u \rangle \leq h_{D_0}(u) + s - p$$

Homogenize the support bound to every direction; the upper bounds sum to the original distances.

[Proposition 8.1 · pp. 21–22](#)

4 Use separation and triangle inequalities to obtain a median

$$x + y \in D_0 + (s - p)Q, \quad m = x - r_+ = y + r_-$$

$$\|m\| \leq c, \quad \|x - m\| \leq A_+, \quad \|y - m\| \leq A_-$$

Distances are additive for all three pairs, including $c = 0$ or $A_{\pm} = 0$.

Endpoint asymptotics, support-function comparison and a metric median

[Open figure ↗](#) [TeX source ↓](#)

1. Compare the two sign costs at the apex

Translate the triple to $0, x, y$. Write $\|\cdot\| = p_Q$, $p_1 = \|x\|$, $p_2 = \|y\|$, $p = \|x - y\|$, $s = p_1 + p_2$, $a = p_1 - p_2$, $A_{\pm} = (p \pm a)/2$ and $c = (s - p)/2$. Repeated points are immediate, so consider distinct points. Fix $u \in \text{int } K$ and apply Proposition 7.1 at $X_{\delta} = (\delta u, 1 - \delta) \in \text{int } B$ to $\xi = (x - y, a)$. Since $p \geq |a|$, the total weight is p and the two sign weights are $A_{\pm}$.

The aggregates $r_{\pm} = \sum_{\sigma_i = \pm 1} \alpha_i b_i$ satisfy $r_+ + r_- = x - y$ and $\|r_{\pm}\| \leq A_{\pm}$. Their possible differences form the compact convex set

$$D_0 = \{r_+ - r_- : r_+ + r_- = x - y, \|r_{\pm}\| \leq A_{\pm}\}$$

Lemma 3.2 gives $\lambda(1 - \eta) = \frac{2}{\pi} \eta \log(1/\eta) + O(\eta)$, hence $\lambda(1 - \delta t)/\lambda(1 - \delta) \rightarrow t$ uniformly on compact positive intervals. Here $t = 1 - \sigma \langle b, u \rangle$ lies in $[1 - p_K(u), 1 + p_K(u)] \subset (0, \infty)$. This uniformity gives the lower bound $p - h_{D_0}(u)$ for the limiting normalized optimal cost, even though the chosen representation varies with δ .

Compare with $(x - y, a) = p_1(x/p_1, 1) + p_2(-y/p_2, -1)$. Its normalized cost tends to $s - \langle x + y, u \rangle$, so optimality yields

$$\langle x + y, u \rangle \leq h_{D_0}(u) + s - p \quad (u \in \text{int } K)$$

No uniform selection of representations across u is needed.

Lemma 3.2 · pp. 8–9 §8, (42)–(46) · pp. 21–22

2. Convex separation forces additive distances

Extend the inequality to K by continuity and use $p_K = h_Q$ to homogenize it. For every direction v we obtain $h_{\{x+y\}}(v) \leq h_{D_0}(v) + (s - p)h_Q(v)$. Convex separation gives $x + y \in D_0 + (s - p)Q$. Choose the corresponding $r_{\pm}$ and let $m = x - r_+ = y + r_-$. Then

$$\|m\| \leq c, \quad \|x - m\| \leq A_+, \quad \|y - m\| \leq A_-.$$

Because $c + A_+ = p_1$, $c + A_- = p_2$ and $A_+ + A_- = p$, all three triangle inequalities become equalities. The same argument permits $c = 0$ or $A_{\pm} = 0$. This is the median used in the main theorem's classification.

Proposition 8.1 · pp. 21–22

6. Corollary 1.2 — From geometry in all dimensions to even functions

Proof target : Corollary 1.2 · pp. 4–5

1 Use the symmetric geometric inequality in every dimension

$P(K) \geq 4^N/N!$ for every $N \geq 1$, $f = e^{-\varphi}$ is even

The implication for even log-concave functions requires the all-dimensional hypothesis.

[FM08, Proposition 1(a), print p. 1433 / PDF p. 5]

[FM08, Proof of Proposition 1(a), print pp. 1434–1435 / PDF pp. 6–7]

2 Increase the auxiliary dimension to obtain a functional product

$N = n + m$, $m \rightarrow \infty$, $\mathcal{P}(f) = (\int f) \inf_z \int f^z \geq 4^n$

The product at $z = 0$ dominates the translation infimum; $\varphi = \|x\|_1$ proves sharpness.

Corollary 1.2 · pp. 4–5

3 Identify the polar function at zero with the Legendre transform

$f^0 = e^{-\varphi^*}$, $(\int e^{-\varphi}) (\int e^{-\varphi^*}) \geq \mathcal{P}(f) \geq 4^n$
The second integral is allowed to be infinite.

The auxiliary-body limit is not an equality-classification argument for functions.

[FM08, Remark 2, print p. 1437 / PDF p. 9]

Scope of equality classification

The Hanner classification of bodies does not classify all equality potentials.

The all-dimensional geometric inequality yields the even functional constant

Open figure ↗ TeX source ↓

1. Return from translation minimization to the origin

Set $f = e^{-\varphi}$. Fradelizi–Meyer's product is $\mathcal{P}(f) = (\int f) \inf_z \int f^z$, where $f^z(y) = \inf_{f(x)>0} e^{-\langle x-z, y-z \rangle} / f(x)$. Their Proposition 1(a) assumes the symmetric geometric Mahler inequality in every dimension and concludes $\mathcal{P}(f) \geq 4^n$ for even log-concave f . Theorem 1.1 supplies this assumption. Since $f^0 = e^{-\varphi^*}$, the product at the origin is at least $\mathcal{P}(f)$. If the second integral is infinite, the required bound is immediate.

The external implication uses auxiliary bodies in dimension $n + m$ and lets $m \rightarrow \infty$; the geometric inequality in dimension n alone is insufficient. For $\varphi = \|x\|_1$, the conjugate is zero on the cube and $+\infty$ outside, proving sharpness. This limiting argument does not classify equality for general functions.

[FM08, Proposition 1(a), print p. 1433 / PDF p. 5]

[FM08, Proof of Proposition 1(a), print pp. 1434–1435 / PDF pp. 6–7]

[FM08, Remark 2, print p. 1437 / PDF p. 9] Corollary 1.2 · pp. 4–5

7. Corollary 1.3 — Establish integrability before applying entropy–transport equivalence

Proof target : Corollary 1.3 · p. 5

1 Check symmetry, full dimensionality and essential continuity

$$\eta_i = e^{-V_i} dx, \quad \eta_i(\mathbb{R}^n) = 1, \quad \nu_i = (\nabla V_i)_\# \eta_i \\ e^{-V_i} = 0 \quad \mathcal{H}^{n-1}\text{-a.e. on } \partial \text{supp } \eta_i$$

Use gradient integrability and the potential estimate in the proof of Proposition 7.

[CEK15, Lemma 4, print p. 3840 / PDF p. 7]

[CEK15, Proof of Proposition 7, (14)–(15), print p. 3844 / PDF p. 11]

2 Obtain finite first moments and finite entropy

$$\int |y| d\nu_i = \int |\nabla V_i| e^{-V_i} dx = \int |\nabla e^{-V_i}| dx < \infty \\ V_i \in L^1(\eta_i), \quad H(\eta_i) = - \int V_i d\eta_i \in \mathbb{R}$$

Corollary 1.2 supplies the left side; match the density assumptions and entropy sign.

Corollary 1.2 · pp. 4–5 [FGZ21, Theorem 1.3, p. 3]

3 Apply the equivalence at the even functional constant 4

$$\text{IS}_{n,s}(4) \iff \text{ET}_{n,s}(4)$$

Insert $c = 4$ in the equivalence.

[FGZ21, Definitions 1.1–1.2, pp. 2–3] Corollary 1.3 · p. 5

4 Obtain the symmetric entropy–transport inequality

$$H(\eta_1) + H(\eta_2) \leq -n \log(4e^2) + \mathcal{T}(\nu_1, \nu_2) \\ \mathcal{T} = +\infty \text{ is permitted.}$$

Finite first moments and finite entropy allow application of the equivalence theorem

Open figure ↗ TeX source ↓

1. Establish the domains of the cost and entropy

For $\rho_i = e^{-V_i}$, the identity $|\nabla \rho_i| = |\nabla V_i| e^{-V_i}$ holds almost everywhere in the support's interior. Cordero-Erausquin–Klartag's Lemma 4 gives $\int |\nabla \rho_i| < \infty$, so the pushforward definition implies $\int |y| d\nu_i(y) < \infty$. The proof of their Proposition 7 also gives $V_i \in L^1(\eta_i)$, making $H(\eta_i) = - \int V_i d\eta_i$ finite. The **statement** of the external Proposition 7 concerns integrability of V_i^* against the moment measure; the integrability of V_i used here is obtained from equations (14)–(15) in its proof.

[CEK15, Lemma 4, print p. 3840 / PDF p. 7]

[CEK15, Proof of Proposition 7, (14)–(15), print p. 3844 / PDF p. 11]

2. Insert the constant 4 in the symmetric equivalence

Fradelizi–Gozlan–Zugmeyer Theorem 1.3 states, with credit to Gozlan, the equivalence of $\mathbf{IS}_{n,s}(c)$ and $\mathbf{ET}_{n,s}(c)$. Their Definitions 1.1–1.2 match the manuscript's essential continuity, entropy sign and cost $\mathcal{T}$. Apply Corollary 1.2 with $c = 4$ to obtain $-n \log(4e^2)$. Origin symmetry places the potentials and moment measures in the even/symmetric setting, while full dimensionality supplies the setting of log-concave probabilities with densities. The conclusion remains meaningful if $\mathcal{T} = +\infty$.

[FGZ21, Definitions 1.1–1.2, pp. 2–3] [FGZ21, Theorem 1.3, p. 3] Corollary 1.3 · p. 5

8. Which papers supply which steps

Source and result

[HL81, Corollary 7.4, p. 15]

Used at

Theorem 8.3 / Proposition 8.4, p. 23; classification

Role and required hypotheses

The 3.2 intersection property of a finite-dimensional **real** Banach space gives iterated ℓ_1/ℓ_∞ sums

Checking scope

Statement on p. 15, closed-ball/arbitrary-radius convention in §1, and application checked; complete classification proof unverified

Source and result

[FM08, Proposition 1(a), print p. 1433 / PDF p. 5]

Used at

Corollary 1.2, pp. 4–5; functional inequality

Role and required hypotheses

Transfer the symmetric geometric inequality in **all dimensions** to even functional products

Checking scope

Print p. 1433 / PDF p. 5 and application checked; auxiliary-body setup read; complete limit argument unverified

Source and result

[CEK15, Lemma 4, print p. 3840 / PDF p. 7]; [CEK15, Proof of Proposition 7, (14)–(15), print p. 3844 / PDF p. 11]

Used at

Corollary 1.3, p. 5; finiteness

Role and required hypotheses

Integrability of log-concave gradients and of the convex potential

Checking scope

Statements and relevant proof paragraph on print pp. 3840, 3844 / PDF pp. 7, 11 checked

Source and result

[FGZ21, Theorem 1.3, p. 3]

Used at

Corollary 1.3, p. 5; equivalence

Role and required hypotheses

Symmetric inverse Santaló gives entropy–transport for essentially continuous densities

Checking scope

Definitions and statement in cited v3, pp. 2–3 checked; Gozlan's original proof not recursively verified

Gross's conformal construction and Demailly's generalized Lelong numbers are cited as background in the manuscript, but the required lens properties and mass bound are proved internally; they are not counted here as external direct inputs. Section 1.1 explicitly states independence from the general nonsymmetric Mahler paper. The nonsymmetric constant $(n+1)^{n+1}/(n!)^2$ alone does not yield the symmetric constant $4^n/n!$.

§1.1 · p. 3

9. Return to the sources

- Lemma 4.1 · pp. 10–12 — Check the isolated-zero hypotheses and mass normalization.
- Proposition 6.1 · pp. 15–16 — Read the exact conversion from probability to real simplex volume.

- Proposition 7.1 · pp. 18–20 — Follow the passage from integral deficit to each point and direction.
- Proposition 8.1 · pp. 21–22 — Read endpoint cost comparison and convex separation.
- Theorem 8.3; Proposition 8.4 · p. 23 — Read the final three-ball, classification and polarity steps.
- Corollary 1.2 · pp. 4–5 — Check the functional consequence requiring all dimensions.
- Corollary 1.3 · p. 5 — Check finiteness and application of the equivalence theorem.

The source is OpenAI's September 22, 2026 manuscript, a 26-page PDF at official commit `1581d4cbdd69a0faf34b2089614fe2a814316a6a`. Printed and PDF pages agree. This editorial pass newly read and compared the principal proof passages in §§2–8, the proofs of Corollaries 1.2–1.3, and the identified passages in four direct external sources. The specific checks concern mass normalization, simplex double counting, passage from integrated deficit to prescribed points, common-witness quantifiers under degenerate bases, uniformity of the endpoint estimate, and arbitrary-radius balls.

This is not an independent reproof of every calculation or measure-theoretic detail, or expert peer review. Complete proofs of the external classification theorem, the Fradelizi–Meyer limiting argument, and Gozlan's equivalence remain unverified. Use by other catalogue papers is reserved for the catalogue-level comparison of the receiving manuscript; a shared Mahler topic does not establish dependency. Source passages, file hashes and unverified scope are recorded in `sources.json`, and production and checks in `status.md`.

The Mahler Conjecture for General Convex Bodies

Dual-cone Laplace products, Gaussian projection fields, and simplex rigidity

OpenAI's manuscript dated 22 September 2026 claims the sharp Santaló-centered volume-product bound for general convex bodies, with a complete simplex equality classification. Its proof lifts a body to a cone in one higher dimension and closes an entropy estimate from Gaussian projection fields through interval inequalities for noncommuting matrices. This article explains that argument; it does not independently verify the entire proof, including its extensive scalar estimates.

Manuscript author: OpenAI · 2026-09-22 · 72 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — General Mahler inequality and equality

For every integer $n \geq 1$ and every convex body $K \subset \mathbb{R}^n$,

$$P(K) := |K| |(K - s(K))^\circ| \geq \frac{(n+1)^{n+1}}{(n!)^2},$$

with equality if and only if K is a simplex. A convex body is compact, convex and has nonempty interior. The Santaló point $s(K) \in \text{int } K$ uniquely minimizes $z \mapsto |(K - z)^\circ|$, where

$$(K - z)^\circ = \{y : \langle y, x - z \rangle \leq 1 \text{ for all } x \in K\}.$$

No symmetry or boundary regularity is assumed. Since $P(K)$ is the **minimum** over centers, the same lower bound holds for $|K| |(K - z)^\circ|$ at every interior z . Equality is not asserted at every center.

Theorem 1.1 · p. 1

Corollary 1.2 — Functional inequality

Let $n \geq 1$ and let $\varphi : \mathbb{R}^n \rightarrow \mathbb{R} \cup \{+\infty\}$ be proper, lower semicontinuous and convex, with $0 < \int e^{-\varphi} < \infty$. For the Fenchel–Legendre transform $\varphi^*(y) = \sup_x (\langle x, y \rangle - \varphi(x))$,

$$\left(\int_{\mathbb{R}^n} e^{-\varphi(x)} dx \right) \left(\int_{\mathbb{R}^n} e^{-\varphi^*(y)} dy \right) \geq e^n.$$

The second integral may be infinite. No centering, evenness or normalization is required. An extremizing example establishes sharpness; a complete functional equality classification is not a conclusion of this corollary.

Corollary 1.2 · p. 4

Corollary 8.1 — Entropy–transport inequality

Let $n \geq 1$ and let $\eta_i(dx) = e^{-V_i(x)}dx$, $i = 1, 2$, be full-dimensional log-concave probability measures with proper lower-semicontinuous convex potentials. Assume essentially continuous densities: $e^{-V_i} = 0$ at $\mathcal{H}^{n-1}$ -almost every point of $\partial \text{supp } \eta_i$. Then the moment measures $\nu_i = (\nabla V_i)_\# \eta_i$ have finite first moments,

$$H(\eta_i) := \int \log(d\eta_i/dx) d\eta_i = - \int V_i d\eta_i$$

is finite, and

$$H(\eta_1) + H(\eta_2) \leq -3n + \mathcal{T}(\nu_1, \nu_2), \quad \mathcal{T}(\mu_1, \mu_2) := \inf_f \left\{ \int f d\mu_1 + \int f^* d\mu_2 \right\}.$$

The infimum ranges over proper lower-semicontinuous convex f ; $\mathcal{T}$ may be infinite. Here H is the negative of differential Shannon entropy. Essential continuity is an additional assumption for this consequence.

Corollary 8.1 · p. 41

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [The Mahler Conjecture for General Convex Bodies](#). External inputs use the following keys.

- [Kla18](#) Bo'az Klartag — Isotropic constants and Mahler volumes: Cone/Laplace correspondence, recalculated with signs and factorials in Lemma 2.1. [arXiv:1710.08084v2](#), 1 March 2018.
- [Sol19](#) Valeriu Soltan — Moreau-Type Characterizations of Polar Cones: Moreau decomposition, reproved with positive duality in Lemma 3.1. [arXiv:1810.04206v1](#), 9 October 2018 (published 2019).
- [Cha13](#) Patrawut Chansangiam — Operator Monotone Functions: Characterizations and Integral Representations: Operator monotonicity of square roots keeps the covariance update in its spectral box. [arXiv:1305.2471v1](#), 11 May 2013.
- [HP02](#) Christian Houdré and Nicolas Privault — Concentration and deviation inequalities in infinite dimensions via covariance representations: Semigroup covariance representation, reproved by Hermite expansion in Lemma 5.1. Author manuscript for Bernoulli 8(6), 2002; manuscript pagination.
- [Nof17](#) Vanni Noferini — A formula for the Fréchet derivative of a generalized matrix function: Divided-difference differentiation of symmetric matrix functions gives interval averages between eigenvalues. Author accepted manuscript [gendk_SIMAX_rev2c](#); SIAM J. Matrix Anal. Appl. 38(2), 2017.
- [FM08](#) Matthieu Fradelizi and Mathieu Meyer — Some functional inverse Santaló inequalities: All-dimensional geometric Mahler implies the functional lower bound for general log-concave functions. *Advances in Mathematics* 218 (2008), 1430–1452; author personal copy with cover sheet.
- [FGZ21](#) Matthieu Fradelizi, Nathael Gozlan, and Simon Zugmeyer — Transport proofs of some functional inverse Santaló inequalities: Gozlan's equivalence of functional inverse Santaló and entropy–transport inequalities. [arXiv:2109.00871v3](#), 9 September 2021.
- [CEK15](#) Dario Cordero-Erausquin; Bo'az Klartag — Moment measures: Editorial supplement for finite moment-measure first moments and entropy; not an explicit citation in manuscript §8.2. *Journal of Functional Analysis* 268 (2015), 3834–3866; published PDF.

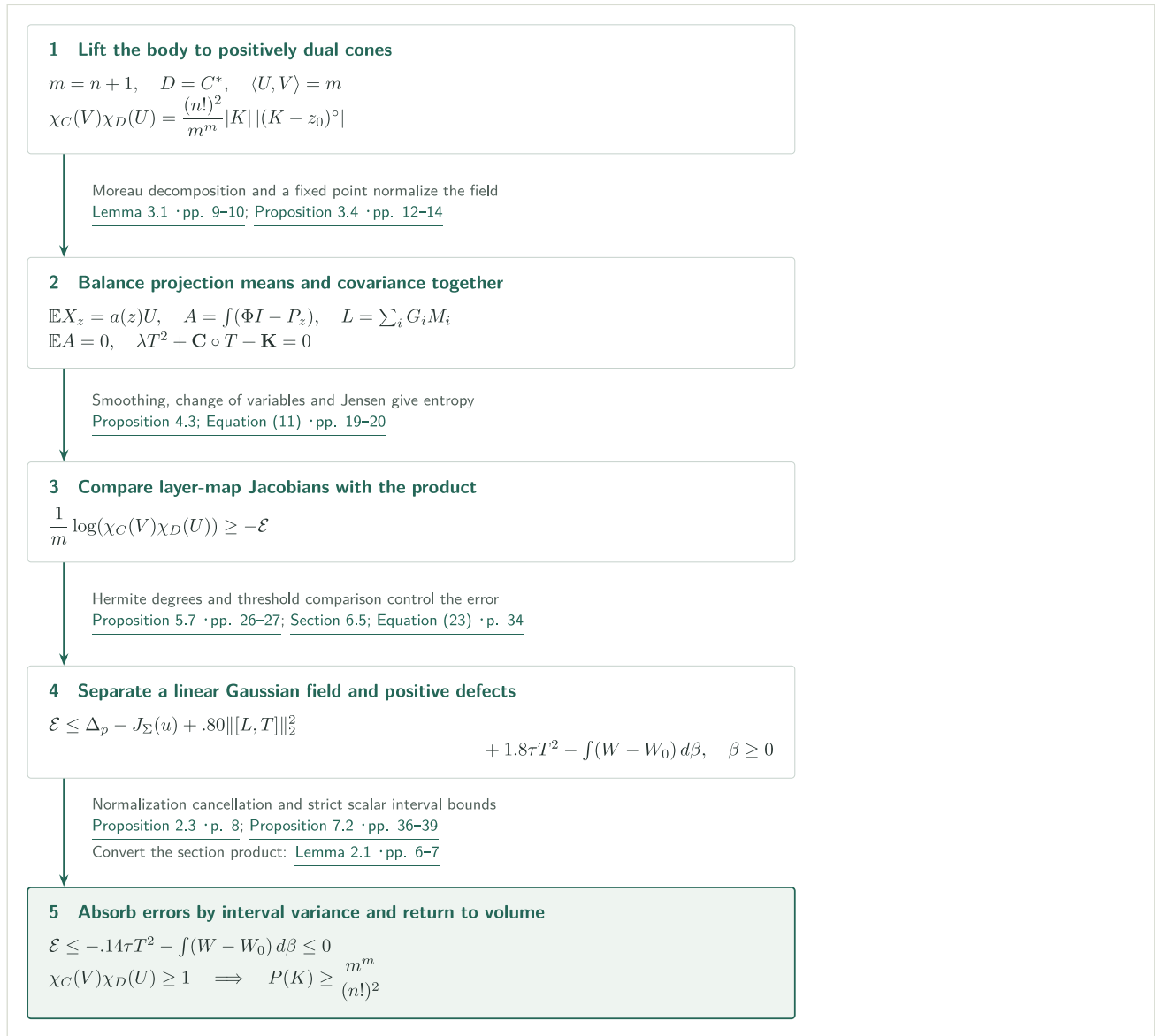
Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Cone-product bound and equality

Proof target : [Theorem 1.1](#) · p. 1

Proof route. Set $m = n + 1$ and transfer the volume product to Laplace integrals on positively dual cones. Simultaneously normalize coordinates and Gaussian covariance, then apply change of variables to maps obtained by integrating projections over layers. Split the remaining error

into a linear Gaussian matrix field and a positive layer defect. A strict interval inequality between eigenvalues absorbs the error. Equality eliminates the defects and separates the projection into individual coordinates in one fixed basis.



Cone, entropy and spectral estimates for general Mahler

[Open figure ↗](#) [TeX source ↓](#)

1. Fix the center and convert volume to Laplace integrals

Fix any $z_0 \in \text{int } K$. Let $C = \{(tx, t) : t \geq 0, x \in K - z_0\}$ and

$D = C^* = \{y : \langle y, x \rangle \geq 0 \ (x \in C)\}$. The cone star denotes **positive duality**, unlike body polarity

◦. For $U = (0, m)$ and $V = (0, 1)$,

$$\chi_C(V)\chi_D(U) = \frac{(n!)^2}{m^m} |K| |(K - z_0)^\circ|, \quad \chi_C(V) = \int_C e^{-\langle V, x \rangle} dx.$$

The height- t slice of D is $-t(K - z_0)^\circ$. This sign and $\int_0^\infty t^n e^{-at} dt = n!a^{-n}$ fix the constant. It suffices to prove that the Laplace product is at least 1 for closed solid pointed convex cones, with interior U, V and $\langle U, V \rangle = m$. The transformation $(C, D, U, V) \mapsto (BC, B^{-T}D, BU, B^{-T}V)$ preserves both the product and the pairing. Klartag states the correspondence using negative duality; the manuscript recalculates it with positive duality.

Lemma 2.1 · pp. 6–7 [Kla18, Lemma 2.1; Corollary 4.1, pp. 6–7, 15]

2. Normalize the projection means and covariance together

Let G be standard Gaussian, $\Sigma = I + T$, $Z = \Sigma^{1/2}G$, and $a(z) = \phi(z) + z\Phi(z)$, where ϕ, Φ are the one-dimensional Gaussian density and distribution function. Choose a bias ξ_z at every layer $z \in \mathbb{R}$ such that

$$X_z = \Pi_C(Z + \xi_z), \quad Y_z = \Pi_D(-Z - \xi_z), \quad \mathbb{E}X_z = a(z)U$$

Moreau's decomposition gives $X_z - Y_z = Z + \xi_z$ and $\langle X_z, Y_z \rangle = 0$. The bias exists by coercivity and strict convexity of $\mathbb{E} \frac{1}{2} |\Pi_C(Z + \xi)|^2 - a(z)U \cdot \xi$. The weak projection derivative $P_z = D\Pi_C(Z + \xi_z)$ satisfies $0 \leq P_z \leq I$, but the layers need not commute or be nested.

Lemma 3.1 · pp. 9–10 [Sol19, Theorem 1.1, p. 1]

Put $A = \int (\Phi(z)I - P_z)dz$, $M_i = \mathbb{E}(G_i A)$, and $L = \sum_i G_i M_i$. Denote the manuscript's fixed scalar profiles by $\mathbf{C}, \mathbf{K}$, and distinguish their matrix averages $\mathbf{C} = \mathbb{E}\mathbf{C}(L)$ and $\mathbf{K} = \mathbb{E}\mathbf{K}(L)$.

Proposition 3.4 simultaneously chooses linear coordinates and T so that

$$\mathbb{E}A = 0, \quad \lambda T^2 + \mathbf{C} \circ T + \mathbf{K} = 0, \quad -\mathbf{C} - 2\lambda T \geq m_* I,$$

Here $\circ$ is the Jordan product, $\lambda = .65$, $m_* = .352$, and $-.022I \leq T \leq .064I$. Operator monotonicity of the square root keeps the covariance update inside this box; an inward coordinate estimate and Brouwer's fixed point theorem also enforce mean balance. Fixing the covariance to I beforehand would lose the matrix cancellation needed later.

Equation (7) · p. 11 Proposition 3.4 · pp. 12–14 [Cha13, Theorem 2.3, p. 4]

3. Split the entropy error into a linear field and layer defects

Integrating X_z, Y_z against positive weights and smoothing produces injective cone-valued maps. Change of variables and Jensen's inequality compare their log Jacobians and linear costs to the Laplace integrals. Proposition 4.3 yields

$$\frac{1}{m} \log(\chi_C(V)\chi_D(U)) \geq -\mathcal{E}$$

where $\mathcal{E}$ is the manuscript's entropy error functional. This estimate alone does not yet prove the product bound.

Proposition 4.3; Equation (11) · pp. 19–20

The Gaussian degrees of $R = A - L$ start at **2**, and those of its odd part start at **3**. The covariance formula, expressed through $(1 + N_G)^{-1}$, controls the coefficients in each degree. It is applied to Gaussian Sobolev **vector fields**, without differentiating the measurable P_z again. Lemma 5.1 reproves the Houdré–Privault semigroup identity by Hermite expansion.

Lemma 5.1 · pp. 20–21 [HP02, Proposition 2.1, Equation (2.4), p. 3]

Comparison with $\Theta_z(L) = 1_{(-\infty, z]}(L)$ defines a positive measure β through the nonnegative quantities $((P_z)_{ii} - 1_{x_i \leq z})(x_i - z)$ in a spectral basis of L . This retains the non-nesting and noncommutation of the layers. Proposition 5.7 and the estimates of Section 6 give

$$\mathcal{E} \leq \Delta_p - J_\Sigma(u) + .80\| [L, T] \|_2^2 + 1.8\tau T^2 - \int (W - W_0) d\beta, \quad W - W_0 > 0.$$

Here $\tau B = m^{-1} \mathbb{E} \operatorname{tr} B$ and $\|B\|_2^2 = \tau(B^T B)$. The terms $\Delta_p, J_\Sigma(u)$ are a profile mean discrepancy and a Dirichlet term evaluated on the linear field. Apart from the final integral, the right side now depends only on L, T .

Section 5.3; Lemma 5.4 · pp. 23–24 Proposition 5.7 · pp. 26–27 Section 6.5; Equation (23) · p. 34

4. Close the bound with strict spectral interval estimates

Divided-difference differentiation of matrix functions produces uniform averages between eigenvalues x_i, x_j . The Daleckii–Kreĭn formula needs no differentiation of eigenvectors; symmetric matrices and smooth profiles suffice. Gaussian integration by parts and the normalization equation cancel the term involving $S - I$, where $S = \sum_i M_i^2$, despite its lack of sign. The strict interval inequality in Proposition 2.3 absorbs the remaining error for distinct endpoints into the interval variance of the profile.

Lemma 7.1 · pp. 35–36 [Nof17, Theorem 2.11, print p. 6 / PDF p. 7] The profile is $u = (\pi, q)$: π is a circular component with angle proportional to $\operatorname{arsinh}(x/s)$, and $q = k - \sqrt{b - r^2 - d}$. The identity $d + |u|^2 = c + 2kq$ reduces the quadratic correction to one component, while variation of the circular component supplies strict variance on every nontrivial interval. That variance simultaneously controls divided-difference errors and the covariance commutator cost.

Proposition 2.3 · p. 8

Proposition 7.2 gives $\Delta_p - J_\Sigma(u) + .80\| [L, T] \|_2^2 \leq -1.94\tau T^2$, hence

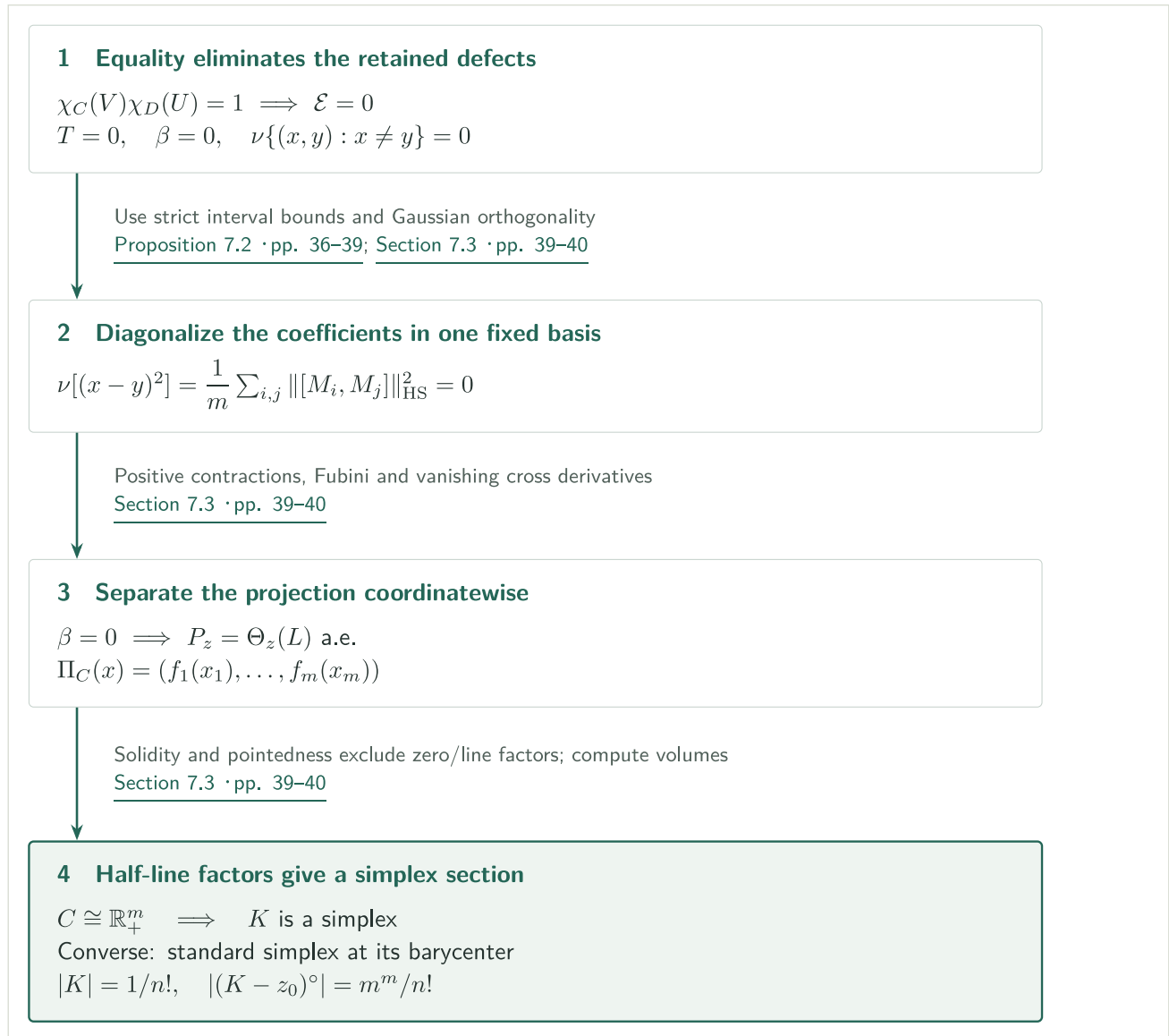
$$\mathcal{E} \leq -.14\tau T^2 - \int (W - W_0) d\beta \leq 0.$$

Returning to the entropy estimate proves that the cone product is at least **1**. The slice formula gives the lower bound at every z_0 , and choosing $z_0 = s(K)$ proves the inequality in Theorem 1.1. Appendices A/B supply rational interval certificates at nodes, analytic interpolation between

nodes and estimates on the unbounded tails. This article has not rerun all certificates or independently checked every estimate; Proposition 2.3 is a substantial internal proof input, not a verified numerical experiment.

Proposition 7.2 · pp. 36–39 Appendix A.2; Proposition B.1 · pp. 43–45, 54–55

5. Recover the simplex from zero defects and compute the converse



Equality, commuting coefficients, separated projections and simplices

[Open figure ↗](#) [TeX source ↓](#)

If the cone product equals $\mathbf{1}$, then $0 \geq -\mathcal{E} \geq 0$, so $\mathcal{E} = 0$. The strictness retained in Proposition 7.2 forces $T = 0$, $\beta = 0$, and diagonal support for the eigenvalue-pair measure ν . Moreover,

$$\nu[(x - y)^2] = \frac{1}{m} \sum_{i,j} \|[M_i, M_j]\|_{\text{HS}}^2 = 0$$

All M_i therefore diagonalize in one fixed orthonormal basis. The vanishing of β , together with $0 \leq P_z \leq I$, gives $P_z = \Theta_z(L)$ almost everywhere. Fubini fixes one layer, and the everywhere positive Gaussian density makes $D\Pi_C$ diagonal Lebesgue-almost everywhere in that same basis.

A Lipschitz projection with vanishing cross weak derivatives separates coordinatewise. Its image C is thus a product of one-dimensional closed convex cones. Solidity excludes a zero factor and pointedness excludes a line, leaving half-lines. Undoing the linear change gives $m = n + 1$ independent rays; the height-one section is a simplex. Equality for $P(K)$ enters this argument at its attaining point $s(K)$.

Conversely, for $K = \text{conv}(0, e_1, \dots, e_n)$ and $z_0 = \mathbf{1}/m$, the polar is $\text{conv}(m\mathbf{1}, -me_1, \dots, -me_n)$, of volume $m^m/n!$, while $|K| = 1/n!$. Together with the lower bound this proves attainment and minimality at that center; affine invariance covers every simplex. In dimension 1, the direct formula $(b-a)((z-a)^{-1} + (b-z)^{-1}) \geq 4$ gives equality at the midpoint of the interval.

Section 7.3 · pp. 39–40 Section 2.1 · pp. 5–6

3. Corollary 1.2 — From all dimensions to the functional bound

Proof target : Corollary 1.2 · p. 4

Proof route. Use the Fradelizi–Meyer implication through auxiliary bodies in increasing dimensions. The translation-minimized functional product controls the origin-centered Legendre product, and a separate calculation supplies an extremizer.

1 Use the geometric bound in every dimension

$$P(K) \geq \frac{(d+1)^{d+1}}{(d!)^2} \quad (d \geq 1)$$

All-dimensional implication for general log-concave functions
[\[FM08, Proposition 2\(a\), print pp. 1434–1435 / PDF pp. 6–7\]](#)

2 Pass from auxiliary bodies to functional products

$$P(f) = (\int f) \inf_z \int f^z \geq e^n$$

$$\lim_{k \rightarrow \infty} \frac{k^n P(\Delta_{n+k})}{P(\Delta_k)} = e^n$$

Every centered product exceeds its infimum; integrate the example
[Section 8.1 · pp. 40–41](#)
 Equality classification does not pass through the limit:
[\[FM08, Remark 2, print p. 1437 / PDF p. 9\]](#)

3 Origin-centered Legendre product and attainment

$$f^0 = e^{-\varphi^*} \implies (\int e^{-\varphi})(\int e^{-\varphi^*}) \geq e^n$$

$$\varphi = \sum_i x_i + \iota_{[-1, \infty)^n}, \quad \int e^{-\varphi} = e^n, \quad \int e^{-\varphi^*} = 1$$

All-dimensional geometric Mahler, the functional limit and sharpness

[Open figure ↗](#) [TeX source ↓](#)

1. Use the translation-minimized functional product

For $f = e^{-\varphi}$ define $f^z(y) = \inf_{x: f(x) > 0} e^{-\langle x-z, y-z \rangle} / f(x)$ and $P(f) = (\int f) \inf_z \int f^z$. Fradelizi–Meyer Proposition 2(a) derives $P(f) \geq e^n$ from general geometric Mahler in every dimension. With an auxiliary simplex in dimension k , the ratio $k^n P(\Delta_{n+k}) / P(\Delta_k)$ tends to e^n . Theorem 1.1 supplies exactly the all-dimensional hypothesis.

Since $f^0 = e^{-\varphi^*}$, the product at the origin is at least the minimized product $P(f)$. This adds no centering assumption on φ . An infinite dual integral makes the requested inequality immediate.

[\[FM08, Proposition 2\(a\), print pp. 1434–1435 / PDF pp. 6–7\]](#) [Section 8.1 · pp. 40–41](#)

2. Separate attainment from classification

Writing ι_A for the convex indicator, take

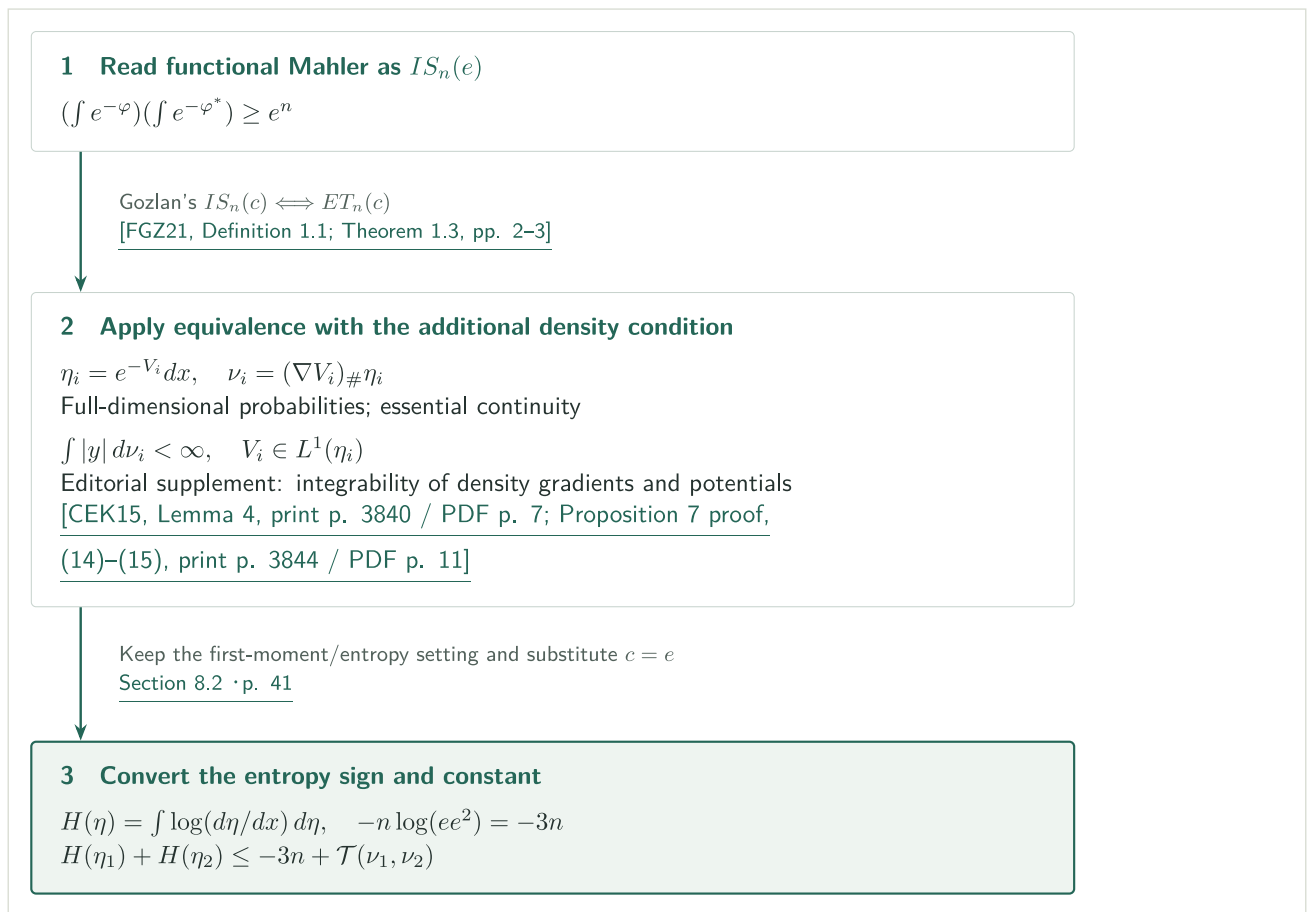
$$\varphi(x) = \sum_i x_i + \iota_{[-1, \infty)^n}(x), \quad \varphi^*(y) = n - \sum_i y_i + \iota_{(-\infty, 1]^n}(y).$$

Then $\int e^{-\varphi} = e^n$ and $\int e^{-\varphi^*} = 1$, proving sharpness. The high-dimensional limit in Fradelizi–Meyer does not transfer equality cases, so geometric simplex rigidity does not yield a complete functional classification.

4. Corollary 8.1 — From functional products to entropy–transport

Proof target : Corollary 8.1 · p. 41

Proof route. Read Corollary 1.2 as functional inverse Santaló $IS_n(c)$ with $c = e$, then apply Gozlan’s equivalence to essentially continuous densities.



Functional inverse Santaló and the entropy transport constant

Open figure ↗ TeX source ↓

1. Preserve the hypotheses and convert the constant

Definition 1.1 and Theorem 1.3 of Fradelizi–Gozlan–Zugmeyer identify $IS_n(c)$ with

$$H(\eta_1) + H(\eta_2) \leq -n \log(ce^2) + \mathcal{T}(\nu_1, \nu_2)$$

for full-dimensional log-concave probability measures with essentially continuous densities.

Matching these measure and moment-measure conventions to Corollary 8.1 and setting $c = e$ gives $-n \log(e^3) = -3n$. The manuscript also asserts finite moment-measure first moments and

entropy. The following is an editorial supplement explaining finiteness, not a route explicitly cited in manuscript §8.2. Lemma 4 of Cordero-Erausquin–Klartag gives $\int |\nabla \rho| < \infty$ for an integrable log-concave density $\rho = e^{-V}$. Hence for $\nu = (\nabla V)_\# \eta$, $\int |y| d\nu = \int |\nabla V| e^{-V} dx < \infty$. Equations (14)–(15) in the proof of their Proposition 7 show $V \in L^1(\eta)$ for convex V with $0 < \int e^{-V} < \infty$; thus $H(\eta) = -\int V d\eta$ is finite. Applying these facts to each i supplies the finiteness conclusions. The direct statements and their application were checked; the complete external equivalence and integrability proofs were not independently verified.

[CEK15, Lemma 4, print p. 3840 / PDF p. 7; Proposition 7 proof, (14)–(15), print p. 3844 / PDF p. 11]

[FGZ21, Definition 1.1; Theorem 1.3, pp. 2–3] Section 8.2 · p. 41

5. Which papers supply which steps

Input

Klartag’s cone correspondence

Used at

Lemma 2.1

Role and hypotheses

Section volumes and Laplace products for proper convex cones; signs and factorials recalculated for positive duality

Checking scope

Original statement and manuscript calculation compared

Input

Moreau decomposition, as stated by Soltan

Used at

Lemma 3.1

Role and hypotheses

Orthogonal projection decomposition for a closed convex cone and its negative polar; convert signs for positive D

Checking scope

Theorem 1.1 and the manuscript’s reproof compared

Input

Chansangiam's Löwner–Heinz inequality

Used at

Proposition 3.4

Role and hypotheses

Monotonicity of square roots of positive matrices without commutation

Checking scope

Statement at $r = 1/2$ and covariance update compared

Input

Houdré–Privault covariance representation

Used at

Lemma 5.1

Role and hypotheses

Gaussian Sobolev covariance through an inverse generator

Checking scope

Semigroup formula and manuscript's Hermite reproof compared

Input

Daleckiĭ–Kreĭn formula recorded by Noferini

Used at

Lemma 7.1

Role and hypotheses

Differentiate smooth symmetric-matrix functions by divided differences

Checking scope

Statement and application compared

Input

Fradelizi–Meyer Proposition 2(a)

Used at

Section 8.1

Role and hypotheses

All-dimensional general geometric bound implies the functional bound

Checking scope

Statement, high-dimensional limit and equality caveat compared

Input

Cordero-Erausquin–Klartag Lemma 4 and Proposition 7 proof

Used at

Editorial supplement to Corollary 8.1, not explicitly cited in manuscript §8.2

Role and hypotheses

Finite moment-measure first moments and entropy

Checking scope

Direct integrability statements and application checked; full proofs unchecked

Input

Gozlan equivalence, Fradelizi–Gozlan–Zugmeyer Theorem 1.3

Used at

Section 8.2

Role and hypotheses

Entropy–transport under essential continuity

Checking scope

Equivalence statement, sign and constant compared; external proof unchecked

These checks do not independently verify the full external proofs. The symmetric Mahler manuscript in catalog 087 treats a stronger constant on the symmetric class and is a comparison, not an input to this manuscript’s Theorem 1.1. No direct dependence on Symplectic Balls in Symmetric Polar Products is recorded in the passages read here.

6. Return to the sources

- Lemma 2.1 · pp. 6–7 — Fix the center, dual sign and factorials.
- Proposition 3.4 · pp. 12–14 — The simultaneous coordinate/covariance fixed point.
- Proposition 4.3; Equation (11) · pp. 19–20 — Reduce the logarithmic product to the error functional.
- Proposition 7.2 · pp. 36–39 — Interval estimates and normalization cancellation.
- Section 7.3 · pp. 39–40 — Extract fixed-basis commutation and a simplex from equality.
- Appendix A.2; Proposition B.1 · pp. 43–45, 54–55 — Rational certificates, interpolation and tails underlying the constants; independent checking incomplete.
- Section 8.1 · pp. 40–41 — The all-dimensional limit and extremizing example.
- Section 8.2 · p. 41 — Essential continuity and transport/entropy sign conventions.

The source is the 72-page manuscript at fixed commit 1581d4cbdd69. Reading covered the principal statements, center and cone conversion, simultaneous normalization, the principal entropy-to-linear-field connections, equality recovery in Section 7, and both consequences in Section 8. External statements and their applications were compared as recorded above.

Outstanding work includes rerunning the rational certificates and checking all interpolation/tail estimates in Appendices A/B, independently recalculating every integrability and noncommutative-kernel estimate in Sections 4–6, and verifying full external proofs. These are important ingredients supporting the sharp constant and equality classification; completing this overview does not certify the manuscript’s correctness. Written anew from the original sources by explicitly assigned gpt-6-astra.

Symplectic Balls in Symmetric Polar Products

Combining holomorphic vanishing with a uniform lens estimate to embed balls in polar products

OpenAI's manuscript of September 22, 2026 constructs smooth symplectic embeddings of every standard ball of capacity below 4 into the Lagrangian product of an origin-symmetric convex body and its polar. The actual embeddings giving the lower bound, the cylinder obstruction giving the upper bound, and the Mahler inequality obtained by volume preservation are separate parts of the argument.

Manuscript author: OpenAI · 2026-09-22 · 19 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Let $K \subset \mathbb{R}^n$ be compact and convex, with nonempty interior and $K = -K$, and put $K^\circ = \{p : \langle q, p \rangle \leq 1 \ (q \in K)\}$. In position and conjugate momentum coordinates, set

$$\omega_0 = \sum_{j=1}^n dq_j \wedge dp_j, \quad U_K = \text{int } K \times \text{int } K^\circ, \quad B^{2n}(c) = \{\pi(|q|^2 + |p|^2) < c\}.$$

Here c is capacity, the radius is $\sqrt{c/\pi}$, and the volume is $c^n/n!$. The Gromov width c_G is the supremum of capacities admitting a smooth symplectic ball embedding.

Theorem 1.1 — Gromov width of a symmetric polar product

For every integer $n \geq 2$ and every origin-symmetric convex body $K \subset \mathbb{R}^n$, one has $c_G(U_K) = 4$. More precisely, for each $0 < c < 4$ there exists $B^{2n}(c) \hookrightarrow U_K$. No smoothness, strict convexity, or coordinatewise sign symmetry of K is assumed. An embedding at the supremum 4 is not asserted.

[Theorem 1.1 · p. 1](#)

Corollary 5.3 — Symmetric Mahler inequality

For every integer $n \geq 1$ and every origin-symmetric convex body $K \subset \mathbb{R}^n$,

$$\mathrm{vol}_n(K) \mathrm{vol}_n(K^\circ) \geq \frac{4^n}{n!}.$$

The cube and its crosspolytope polar attain this constant. The manuscript does not classify all equality cases.

Corollary 5.3; Remark 5.4 · pp. 16–17

Corollary 5.5 — A sufficient finite packing criterion

Let $n \geq 3$ and $k \geq 1$ be integers, let $K \subset \mathbb{R}^n$ be an origin-symmetric convex body, and suppose positive capacities $R_1, \dots, R_k$ satisfy

$$\sum_{i=1}^k R_i^n < 4^n, \quad R_i + R_j < 4 \quad (i < j).$$

There are symplectic embeddings of open neighborhoods of the closed balls $\overline{B}^{2n}(R_i)$ into U_K with pairwise disjoint images. The pairwise condition is vacuous for $k = 1$. These are sufficient conditions for the polar product, not a claimed necessary and sufficient criterion.

Corollary 5.5 · p. 17

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Symplectic Balls in Symmetric Polar Products](#). External inputs use the following keys.

- [AAKO14](#) Shiri Artstein-Avidan, Roman Karasev, Yaron Ostrover — From symplectic measurements to the Mahler conjecture: Supporting-square cylinder, used for the upper bound. [arXiv:1303.4197v2](#).
- [Gro85](#) Mikhael Gromov — Pseudo holomorphic curves in symplectic manifolds: Nonsqueezing bounds the capacity of an embedded ball. *Inventiones Mathematicae* 82 (1985), 307–347; [published PDF](#).
- [Gross19](#) Renan Gross — A conformal Skorokhod embedding: Uniform-distribution coordinate, rotated here; needed properties are proved locally. [arXiv:1905.00852v1](#).
- [Mos65](#) Jürgen Moser — On the Volume Elements on a Manifold: Deformation of forms; compact support is verified locally. *Transactions of the American Mathematical Society* 120 (1965), 286–294; [published PDF](#).

- [OpenAI26p](#) OpenAI — Symplectic Ball Packings in Higher Dimensions: Higher-dimensional packing for the corollary only; capacity and neighborhood conventions checked. September 23, 2026; commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Comparing two scales and the cylinder obstruction

Proof target : [Theorem 1.1](#) · p. 1

Proof route. For a body defined by finitely many symmetric strips, Theorem 2.1 supplies balls of capacity below πk , while Proposition 3.2 bounds the required momentum enlargement by $(1 + o(1))\pi k/4$. Proposition 4.1 combines these inputs; finite approximation on the polar side then handles arbitrary K . A supporting cylinder and nonsqueezing independently give the upper bound. The two analytic inputs are explained in the following proof sections.

1 Encode finite strips holomorphically

$$n \geq 2, \quad \text{span}_{\mathbb{R}}\{b_j\} = \mathbb{R}^n$$

$$K = \{x : |b_j \cdot x| \leq 1\}, \quad f_j = g(b_j \cdot z)^k, \quad \tau = \sum_j |f_j|^2$$

$$V = \{\tau < 1\}, \quad \omega = dd^c(|z|^2 + \tau)$$

Combine the isolated zero and compact sublevels with the uniform slice bound
[Theorem 2.1 · p. 4](#) [Proposition 3.2 · p. 9](#)

2 Supply a ball and control momentum

$$B^{2n}(d) \hookrightarrow (V, \omega) \quad (d < \pi k)$$

$$\Phi_k(z) = (x, -u - \sum_j J_k(b_j \cdot u, b_j \cdot x)b_j)$$

$$\Phi_k^* \omega_0 = \omega, \quad h_K(p)/k \leq C_K/k + (\pi/4)\tau + m\epsilon_k$$

Horizontal monotonicity gives injectivity; the support function bounds the image
[Proposition 4.1, Steps 1–2 · pp. 12–13](#)
[Proposition 4.1, Steps 3–4; Equations \(4.4\)–\(4.6\) · p. 14](#)

3 Cancel the two conformal factors

$$S = (1 + \eta)\pi k/4, \quad (1 + \eta)c < 4 \implies Sc < \pi k$$

$$D_{\sqrt{S}}(z) = \sqrt{S}z, \quad L_S(q, p) = (q, p/S)$$

$$L_S \circ \Phi_k \circ E \circ D_{\sqrt{S}} : B^{2n}(c) \hookrightarrow U_K$$

After finite polar approximation, dilate the source by $\sqrt{\lambda}$ and position by $1/\lambda$
[Lemma 5.1; proof of Theorem 1.1; Remark 5.2 · pp. 15–16](#)

4 Obtain the lower bound for every body

$$K \subset K' \subset \lambda K, \quad (K')^\circ \subset K^\circ, \quad \lambda c < 4$$

$$B^{2n}(c) \hookrightarrow U_K \quad (0 < c < 4), \quad c_G(U_K) \geq 4$$

Supporting conjugate coordinates and an area-preserving map; apply nonsqueezing
[\[AAKO14, Remark 4.2, pp. 14–15\]](#)
[\[Gro85, §0.3.A, Corollary, print p. 310 / PDF p. 4\]](#)

5 Combine with the independent cylinder obstruction

$$U_K \hookrightarrow (-1, 1)^2 \times \mathbb{R}^{2n-2} \simeq B^2(4) \times \mathbb{R}^{2n-2}$$

$$c_G(U_K) \leq 4 \implies c_G(U_K) = 4$$

The two inputs for the capacity lower bound and the independent upper bound

[Open figure ↗](#) [TeX source ↓](#)

1. From finite strips to phase space

Write $K = \{x : |b_j \cdot x| \leq 1, 1 \leq j \leq m\}$, where the b_j span $\mathbb{R}^n$. Using the lens D and its inverse coordinate $g : D \rightarrow \mathbb{D}$ below, put $\ell_j(z) = b_j \cdot z$, $\Omega = \bigcap_j \ell_j^{-1}(D)$, $f_j = g(\ell_j)^k$, and $\tau = \sum_j |f_j|^2$. The domain Ω is bounded and the common zero is 0. On $\{\tau \leq s\}$ each $\ell_j(z)$ lies in $F(\{|w| \leq s^{1/(2k)}\}) \Subset D$. Injectivity of the linear map $(\ell_j)_j$ makes these closed sublevels compact, verifying the hypotheses of Theorem 2.1.

Writing $z = u + ix$, the convention is $dd^c|z|^2 = \sum du_j \wedge dx_j$. Thus target position is $q = x$ and the original momentum is $-u$. Correct it by

$$\Phi_k(z) = (q, p), \quad q = x, \quad p = -u - \sum_j J_k(b_j \cdot u, b_j \cdot x) b_j.$$

The identity $\partial_v J_k = |(g^k)'|^2$ gives $\Phi_k^* \omega_0 = dd^c(|z|^2 + \tau)$. This local identity alone does not prove embedding. For two points with the same x , monotonicity on the horizontal slices gives

$$(p_2 - p_1) \cdot (u_2 - u_1) \leq -|u_2 - u_1|^2,$$

which proves global injectivity. The comparison uses the entire horizontal interval and remains valid even if a fiber of the sublevel domain is disconnected.

Proposition 4.1, Steps 1–2 · pp. 12–13

2. Polar control and cancelling dilations

For the support function h_K , one has $h_K(\pm b_j) \leq 1$. Proposition 3.2 therefore yields

$$\frac{h_K(p)}{k} \leq \frac{C_K}{k} + \frac{\pi}{4} \tau + m\epsilon_k, \quad \epsilon_k \rightarrow 0.$$

Fix K, m, C_K before increasing k . For any $\eta > 0$, the image lies in $\text{int } K \times S \text{int } K^\circ$ with $S = (1 + \eta)\pi k/4$ once k is large enough. If $(1 + \eta)c < 4$, then $Sc < \pi k$, so Theorem 2.1 supplies $B^{2n}(Sc)$. Precomposing by $D_{\sqrt{S}}(z) = \sqrt{S}z$ and postcomposing by $L_S(q, p) = (q, p/S)$ cancels the factors S and S^{-1} in the symplectic forms. This gives $B^{2n}(c) \hookrightarrow U_K$.

Proposition 4.1, Steps 3–4; Equations (4.4)–(4.6) · p. 14

3. Finite approximation for an arbitrary body

For each $c < 4$, choose $\lambda = 1 + \alpha > 1$ with $\lambda c < 4$. Lemma 5.1 takes the polar of the symmetric convex hull of a finite net in K° to obtain a finite strip body K' satisfying

$$K \subset K' \subset \lambda K, \quad (K')^\circ \subset K^\circ.$$

View $B^{2n}(\lambda c) \hookrightarrow \text{int } K' \times \text{int}(K')^\circ$ as an embedding into $\lambda \text{int } K \times \text{int } K^\circ$. A source dilation by $\sqrt{\lambda}$ and a position contraction by $1/\lambda$ again cancel their conformal factors. The choices are made in the order c, α, K' and its finite constraint list, η , then k . Each prescribed capacity uses one finite construction; no limiting embedding or smoothing of the convex boundary is required.

Lemma 5.1; proof of Theorem 1.1; Remark 5.2 · pp. 15–16

4. The supporting-cylinder upper bound

Choose $q_0 \in K$ and $p_0 \in K^\circ$ with $\langle q_0, p_0 \rangle = 1$. Complete q_0 by a basis of $\ker p_0$ to obtain a matrix A . The symplectic change $Q = A^{-1}q$, $P = A^T p$ gives $|Q_1|, |P_1| < 1$. Thus U_K embeds into the cylinder over a square of area 4 in the first conjugate coordinate pair. An area-preserving map sends that square to the disk of area 4, and Gromov's nonsqueezing bounds the capacity of any embedded ball by 4. The lower bound is not inferred from the value of the Hofer–Zehnder capacity.

Proof of Theorem 1.1, upper bound · p. 16 [AAKO14, Remark 4.2, pp. 14–15]
[Gro85, §0.3.A, Corollary, print p. 310 / PDF p. 4]

3. Theorem 2.1 — Turning a holomorphic zero into a ball

Proof target : Theorem 2.1 · p. 4

Theorem 2.1 — A ball from a holomorphic zero

Let $n, m \geq 1$, let $\Omega \subset \mathbb{C}^n$ be open with $0 \in \Omega$, and let $f : \Omega \rightarrow \mathbb{C}^m$ be holomorphic. Suppose some integer $k \geq 1$ satisfies $f^{-1}(0) = \{0\}$ and $|f(z)| = O(|z|^k)$, and that $\{|f|^2 \leq s\} \Subset \Omega$ for every $0 < s < 1$. Set $\tau = |f|^2$, $V = \{\tau < 1\}$, $d^c \phi(X) = -\frac{1}{4} d\phi(iX)$, and $\omega = dd^c(|z|^2 + \tau)$. Then $(B^{2n}(c), \omega_0) \hookrightarrow (V, \omega)$ for each $0 < c < \pi k$.

Theorem 2.1 · p. 4

Proof route. Convert vanishing order into a logarithmic slope, replace the singularity near 0 by a smooth radial potential, and use the compactly supported difference to obtain a complete Moser flow. The ball visible in the radial model is transported back to the original form.

1 Turn vanishing order into logarithmic slope

$0 < a < k, \quad a/k < b < 1, \quad a < \mu < bk$
 $H = \chi(\log \tau) \leq bk \log |z|^2 + C, \quad H = \tau \text{ near the outer region}$

Use $\mu < bk$ for inner domination; choose δ after fixing r_1
[Theorem 2.1, Steps 1–2 · pp. 4–6](#)

2 Replace only the singularity

$P_\delta = \mu \log(|z|^2 + \delta) - M, \quad \tilde{H} = \text{rmax}(H, P_\delta)$
 $\tilde{H} = P_\delta \text{ on } B_{r_1}, \quad \text{supp}(\tilde{H} - \tau) \Subset V$

Fixed compact support ensures existence of the flow on the noncompact domain
[\[Mos65, §4, Theorem 2 and deformation equation, print pp. 292–293 / PDF pp. 7–8\]](#)
[Theorem 2.1, Steps 3–4 · pp. 6–7](#)

3 Transport forms by a compactly supported flow

$\omega_s = dd^c(|z|^2 + (1-s)\tau + s\tilde{H})$
 $\iota_{Y_s} \omega_s = -d^c(\tilde{H} - \tau), \quad \phi_1^* \omega_1 = \omega$

Use radial monotonicity and the explicit pullback identity
[Theorem 2.1, Steps 3–4 · pp. 6–7](#)

4 Pull back the ball in the radial model

$R(z) = \sqrt{1 + \mu/(|z|^2 + \delta)} z, \quad R^* \omega_0 = \omega_1$
 $\pi(r_1^2 + \mu r_1^2/(r_1^2 + \delta)) > \pi a$
 $\phi_1^{-1} \circ \iota \circ R^{-1} : B^{2n}(\pi a) \hookrightarrow (V, \omega)$

Vanishing order, replacement of the singularity, and compactly supported transport

[Open figure ↗](#) [TeX source ↓](#)

1. Logarithmic slope and the regularized maximum

Write the capacity as $c = \pi a$. Choose $0 < a < k$, $a/k < b < 1$, and $a < \mu < bk$. An increasing convex function χ makes $H = \chi(\log \tau)$ agree with τ near the outer part of the domain and satisfy $H \leq bk \log |z|^2 + C$ near 0. For $P_\delta = \mu \log(|z|^2 + \delta) - M$, first arrange $P_\delta < H - 2$ on an outer sphere, then $P_\delta > H + 2$ on an inner ball whose radius is independent of δ . The strict inequality $\mu < bk$ provides the latter comparison. Only afterwards is $\delta > 0$ chosen small.

A regularized maximum produces a smooth plurisubharmonic $\tilde{H}$, equal to P_δ inside and to H outside. The support of $\tilde{H} - \tau$ lies in some $\{\tau \leq \sigma\}$ with $\sigma < 1$, compact by hypothesis.

[Theorem 2.1, Steps 1–2 · pp. 4–6](#)

2. Moser transport and the radial capacity

The forms $\omega_s = dd^c(|z|^2 + (1-s)\tau + s\tilde{H})$ remain nondegenerate. The solution of $\iota_{Y_s}\omega_s = -d^c(\tilde{H} - \tau)$ has fixed compact support, so its flow exists for $0 \leq s \leq 1$ even though V is noncompact, and satisfies $\phi_1^*\omega_1 = \omega$. The manuscript proves the needed local version of Moser's deformation method; it does not apply the original compact-manifold theorem to a noncompact domain without a support argument.

Inside, $\Psi(t) = t + \mu \log(t + \delta)$ and $R(z) = \sqrt{\Psi'(|z|^2)}z$ carry the radial form to the standard one. The image of the radius- r_1 ball has capacity $\pi(r_1^2 + \mu r_1^2/(r_1^2 + \delta)) > \pi a$. Pull the smaller standard ball back by R^{-1} , inclusion, and ϕ_1^{-1} . Neither a regular-value condition on the sublevel boundary nor connectedness of V is needed.

Theorem 2.1, Steps 3–4 · pp. 6–7

[Mos65, §4, Theorem 2 and deformation equation, print pp. 292–293 / PDF pp. 7–8]

4. Proposition 3.2 — Uniform horizontal integrals up to the tips

Proof target : Proposition 3.2 · p. 9

Proposition 3.2 — The uniform slice estimate

Use the biholomorphism of Lemma 3.1,

$$F(w) = \frac{8}{\pi^2} \sum_{\ell \geq 0} \frac{(-1)^\ell w^{2\ell+1}}{(2\ell+1)^2} : \mathbb{D} \longrightarrow D, \quad g = F^{-1}.$$

The domain D is bounded, with horizontal slice $(-V(t), V(t)) + it$ at each $-1 < t < 1$.

For integers $k \geq 2$, define

$$J_k(v, t) = k^2 \int_0^v |g(h + it)|^{2k-2} |g'(h + it)|^2 dh.$$

The function J_k is smooth on D , odd in v , and satisfies $\partial_v J_k = k^2 |g|^{2k-2} |g'|^2 \geq 0$. There are $\epsilon_k \geq 0$, depending only on F , with $\epsilon_k \rightarrow 0$, such that simultaneously for every $v + it \in D$,

$$\frac{|J_k(v, t)|}{k} \leq \frac{\pi}{4} |g(v + it)|^{2k} + \epsilon_k.$$

Proposition 3.2 · p. 9

Proof route. Rotate Gross's uniform-distribution coordinate in source and target, $F(w) = i\psi(iw)$, without scaling it. Convert horizontal integration into radial integration in the disk, separating the leading constant $\pi/4$ from a uniform error at the tips.

[Gross19, §3.2, p. 9]

1 Parametrize a horizontal slice by radius

$$F(w) = i\psi(iw), \quad g = F^{-1}, \quad F(ir) = iT(r) \\ r_0 = T^{-1}(|t|), \quad R = |g(v + it)|, \quad A = \Re(wF'(w)) > 0$$

Change variables using $|g'|^2 dv = (r/A) dr$; rotation preserves the scale
[Gross19, §3.2, p. 9] [Lemma 3.1; Equation \(3.11\) · pp. 7–10](#)

2 Extract the leading coefficient $\pi/4$

$$J_k(v, t)/k = k \int_{r_0}^R \frac{r^{2k-1}}{A} dr \\ 1/A = \pi/2 + (1/A - \pi/2)$$

Bound the tip loss using $T(r) - |t| \geq L_*(r - r_a)$
[Equations \(3.15\)–\(3.21\) · pp. 10–12](#)

3 Control the moving endpoint uniformly

$$r_a = \max(r_0, r_*), \quad L_* = \inf_{r \geq r_*} T'(r) \rightarrow \infty \\ k \int_{r_a}^1 r^{2k-1} \sqrt{\frac{1-r}{r-r_a}} dr \leq 1 + \pi/e$$

First choose r_* close to 1; then let $k \rightarrow \infty$ at fixed r_*
[Equations \(3.15\)–\(3.21\) · pp. 10–12](#)

4 Make the error vanish at every height

$$\frac{|J_k|}{k} - \frac{\pi}{4} |g|^{2k} \leq C_* k r_*^{2k-2} + \frac{C_0(1 + \pi/e)}{\sqrt{L_*}} \\ |J_k|/k \leq (\pi/4) |g|^{2k} + \epsilon_k, \quad \epsilon_k \rightarrow 0$$

The order of choices that controls moving slice endpoints uniformly

[Open figure ↗](#) [TeX source ↓](#)

1. Radial integration and the leading coefficient

Write $F(ir) = iT(r)$, with $T'(r) = 8 \operatorname{arctanh} r / (\pi^2 r) \rightarrow \infty$. On a positive horizontal slice, write $w = re^{i\theta(r, t)}$. Its radius increases from $r_0 = T^{-1}(|t|)$ to $R = |g(v + it)|$, and

$$A(r, \theta) = \Re(wF'(w)) = \frac{4}{\pi^2} \arctan \frac{2r \cos \theta}{1 - r^2}, \quad \frac{J_k(v, t)}{k} = k \int_{r_0}^R \frac{r^{2k-1}}{A(r, \theta(r, t))} dr.$$

The constant part $\pi/2$ of $1/A$ contributes $\frac{\pi}{4}(R^{2k} - r_0^{2k})$. A boundary limit at fixed t alone would not control the remainder when $t = t_k \rightarrow \pm 1$.

Lemma 3.1; Equation (3.11) · pp. 7–10

2. Moving endpoints and the order of limits

Split at $r_* \in (1/2, 1)$. The inner contribution is bounded by $C_* k r_*^{2k-2}$. Outside, put $r_a = \max(r_0, r_*)$ and $L_* = \inf_{r_* \leq r < 1} T'(r)$. Then

$$\frac{1}{A} \leq \frac{\pi}{2} + C_0 \sqrt{\frac{1-r}{T(r)-|t|}}, \quad T(r) - |t| \geq L_*(r - r_a).$$

For the remaining integral $I = k \int_{r_a}^1 r^{2k-1} \sqrt{(1-r)/(r-r_a)} dr$, the substitution $y = k(1-r)$ and $H = k(1-r_a)$ yields $I \leq 1 + \pi/e$ uniformly for every $H > 0$. The excess is therefore at most

$$C_* k r_*^{2k-2} + \frac{C_0(1 + \pi/e)}{\sqrt{L_*}}.$$

First choose $r_* \uparrow 1$ to make the second term small; then hold that cutoff fixed and let $k \rightarrow \infty$.

This permits the height and both endpoints to vary with k . Smoothness follows by differentiating over compact horizontal segments, and oddness and monotonicity follow from symmetry and nonnegativity of the density. These conclusions also enter the global embedding in §4.

Equations (3.15)–(3.21) · pp. 10–12

5. Corollary 5.3 — The volume consequence

Proof target : Corollary 5.3; Remark 5.4 · pp. 16–17

Proof route. In dimensions $n \geq 2$, approach the supremum through ball volumes. In dimension one, compute interval lengths directly. Exhibiting an equality example is separate from classifying equality cases.

1 Identify interior and closed-body volumes

$$tC \subset \text{int } C \quad (0 < t < 1), \quad t \uparrow 1$$

$$\text{vol}_{2n}(U_K) = \text{vol}_n(K) \text{vol}_n(K^\circ)$$

The embedding in the main theorem preserves $\omega_0^n/n!$
[Theorem 1.1 · p. 1](#) [Corollary 5.3; Remark 5.4 · pp. 16–17](#)

2 Bound the volume below with each ball

$$n \geq 2, \quad 0 < c < 4: \quad B^{2n}(c) \hookrightarrow U_K$$

$$c^n/n! \leq \text{vol}_{2n}(U_K)$$

Let $c \uparrow 4$; no capacity-4 embedding is needed. Compute $n = 1$ directly
[Corollary 5.3; Remark 5.4 · pp. 16–17](#)

3 Take the numerical capacity limit; include dimension one

$$\text{vol}_n(K) \text{vol}_n(K^\circ) \geq 4^n/n!$$

$$n = 1: \quad [-a, a] \text{ and } [-a^{-1}, a^{-1}] \implies 4$$

Decompose the polar into 2^n simplices
[Corollary 5.3; Remark 5.4 · pp. 16–17](#)

4 Check the sharp example

$$\text{vol}([-1, 1]^n) = 2^n, \quad \text{vol}(\{\sum |p_j| \leq 1\}) = 2^n/n!$$

An equality example, not a classification of every equality case

Interior volume, ball volume, and the sharp constant

[Open figure ↗](#) [TeX source ↓](#)

1. Interiors and the capacity limit

For a convex body C containing 0 in its interior, $tC \subset \text{int } C$ for $0 < t < 1$. Letting $t \uparrow 1$ shows that its interior has the same volume. Thus $\text{vol}_{2n}(U_K) = \text{vol}_n(K) \text{vol}_n(K^\circ)$. A symplectic embedding preserves $\omega_0^n/n!$, so $c^n/n! \leq \text{vol}_{2n}(U_K)$ for every $c < 4$. Letting the number c increase to 4 is sufficient; no capacity-4 embedding or convergence of embeddings is required.

2. Dimension one and sharpness

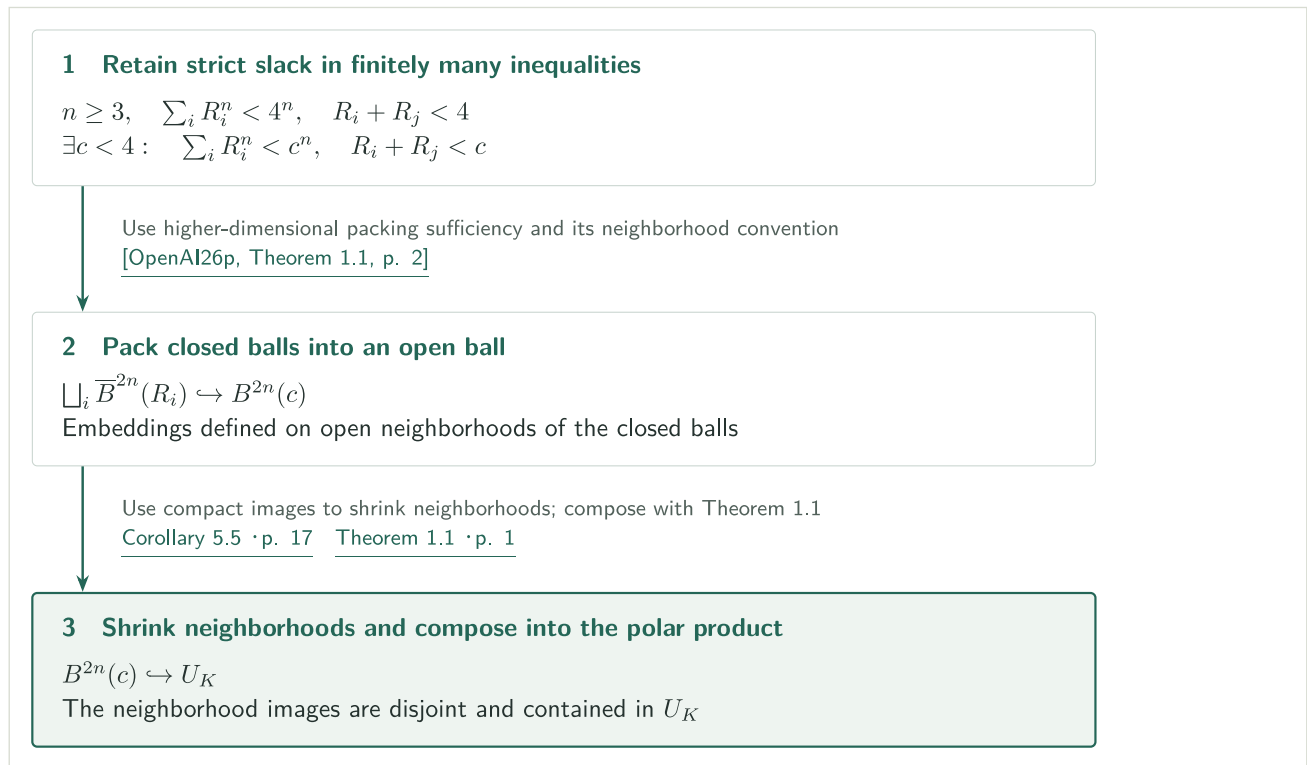
For $n = 1$, $K = [-a, a]$ and $K^\circ = [-a^{-1}, a^{-1}]$ have length product 4. In every dimension, the cube has volume 2^n and its crosspolytope polar has volume $2^n/n!$, by decomposing it into one simplex per orthant. Their product is $4^n/n!$. This example does not classify all other equality cases.

[Corollary 5.3; Remark 5.4 · pp. 16–17](#)

6. Corollary 5.5 — Transporting ball packings to the polar product

Proof target : Corollary 5.5 · p. 17

Proof route. Use the slack in the two strict inequalities to choose one $c < 4$, then compose the external ball-packing theorem with Theorem 1.1.



Composition of packings while retaining neighborhoods of the closed balls

[Open figure ↗](#) [TeX source ↓](#)

1. Strict slack and shrinking neighborhoods

Finiteness and strictness allow $c < 4$ with $\sum_i R_i^n < c^n$ and $R_i + R_j < c$ for every pair. The external Theorem 1.1 packs the closed balls into $B^{2n}(c)$ for $n \geq 3$, by embeddings defined on open neighborhoods. Since their compact images are disjoint and lie in the open ball, shrink these finitely many neighborhoods to preserve both disjointness and containment. Compose with $B^{2n}(c) \hookrightarrow U_K$ from Theorem 1.1. The external theorem's necessity statement concerns a ball target; this composition does not establish necessity for U_K .

Corollary 5.5 · p. 17 [OpenAI26p, Theorem 1.1, p. 2]

7. Which papers supply which steps

Source

Gross

Use

Lemma 3.1, §3

Content and connection

Rotate the uniform-distribution conformal map; needed lens and slice properties are proved locally

Checking scope

Series in §3.2 and rotation checked; the full probabilistic proof is outside scope

Source

Moser

Use

Theorem 2.1, Step 3

Content and connection

Transport a path of forms by a vector field; fixed compact support is proved here

Checking scope

Deformation equation in §4 and support argument on pp.5–6 checked

Source

Artstein-Avidan–Karasev–Ostrover; Gromov

Use

Upper bound in Theorem 1.1

Content and connection

Supporting square cylinder and nonsqueezing

Checking scope

Input statements and application on p.16 checked; full nonsqueezing proof not verified

Source

OpenAI, Symplectic Ball Packings in Higher Dimensions

Use

Corollary 5.5

Content and connection

 $n \geq 3$, strict inequalities, embeddings on neighborhoods of closed balls

Checking scope

Theorem 1.1 and capacity/neighborhood conventions checked; packing proof not verified

The same-catalogue manuscript The symmetric Mahler conjecture and its equality cases provides background with a common lens construction. This proof does not use its volume inequality or equality classification. Conversely, Corollary 5.3 supplies the same symmetric volume lower bound, without classifying equality. No direct input from the general-convex-body manuscript is recorded in the proof passages read here; this is not a claim to have investigated every possible connection.

Background and prior constructions · p. 2

8. Return to the sources

- Theorem 1.1 · p. 1 — Capacity convention and scope of the theorem.
- Theorem 2.1 · p. 4 — Hypotheses converting vanishing order into ball capacity.
- Equations (3.15)–(3.21) · pp. 10–12 — Uniform error with moving endpoints.
- Proposition 4.1, Steps 1–2 · pp. 12–13 — Local form identity and global injectivity.
- Lemma 5.1; proof of Theorem 1.1; Remark 5.2 · pp. 15–16 — Approximation and the order of choices.
- Corollary 5.3; Remark 5.4 · pp. 16–17 — Volume deduction and sharpness.
- Corollary 5.5 · p. 17 — The consequence using the external packing theorem.

The fixed manuscript's §§2–5 (pp.3–17) were read to check the deformation from vanishing order to a ball, the tip-uniform estimate, global injectivity for finite strips, both cancelling rescalings, the supporting cylinder, and the volume and packing deductions. External inputs were checked to the extent listed above: statements and uses. This is editorial checking of the manuscript's argument, not independent certification of its all-dimensional claims. Complete external proofs, recursive verification of the bibliography, and equality classification remain outside scope.