

OpenAI Math Digest

CATALOGUE 074

Takeya in three and four dimensions

Catalogue synthesis & 2 proof overviews

Independent reading guide · English edition

Edition: 2026-10-16

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AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 2 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

Catalogue 074 · Synthesis

[Read on the website ↗](#)

CATALOGUE 074

Keakeya in three and four dimensions

The three-dimensional paper addresses density dependence in a maximal estimate; the four-dimensional paper treats Hausdorff dimension for arbitrary sets. Their recorded connection is a weighted plank lemma in the three-dimensional paper, rather than its maximal theorem.

1. Papers in official catalogue order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 074 · 2026-10-16 checked source snapshot

#

01

Manuscript and overview

The Keakeya maximal conjecture in three dimensions →

Short name: The Keakeya maximal conjecture in three dimensions

September 23, 2026 · 97 pages

[Original paper \(GitHub\) ↗](#)

Principal claim

For every $\varepsilon > 0$ and $0 < \delta < 1$, the maximal operator over unit tubes of radius δ maps $L^3(\mathbb{R}^3)$ to $L^3(S^2)$ with norm $C \varepsilon^{-\delta} \varepsilon$.

[Theorem 1.1 · p. 2 ↗](#)

Role and available overview

Weighted configurations retain density dependence; exclusion of stationary extremizers yields the maximal bound. A separate plank lemma also enters the 4D reduction.

[Read the proof overview →](#)

#

02

Manuscript and overview

Every four-dimensional Kakeya set has full Hausdorff dimension →

Short name: Every four-dimensional Kakeya set has full Hausdorff dimension

September 24, 2026 · 175 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Every $K \subset \mathbb{R}^4$ containing a unit segment in each direction has Hausdorff dimension four, without measurability or compactness assumptions on K or its witnesses.

[Theorem 1.1 · p. 3](#) ↗

Role and available overview

Variable-scale covers yield a weighted problem with positive horizon; improvements in three geometric regimes contradict extremality.

[Read the proof overview →](#)

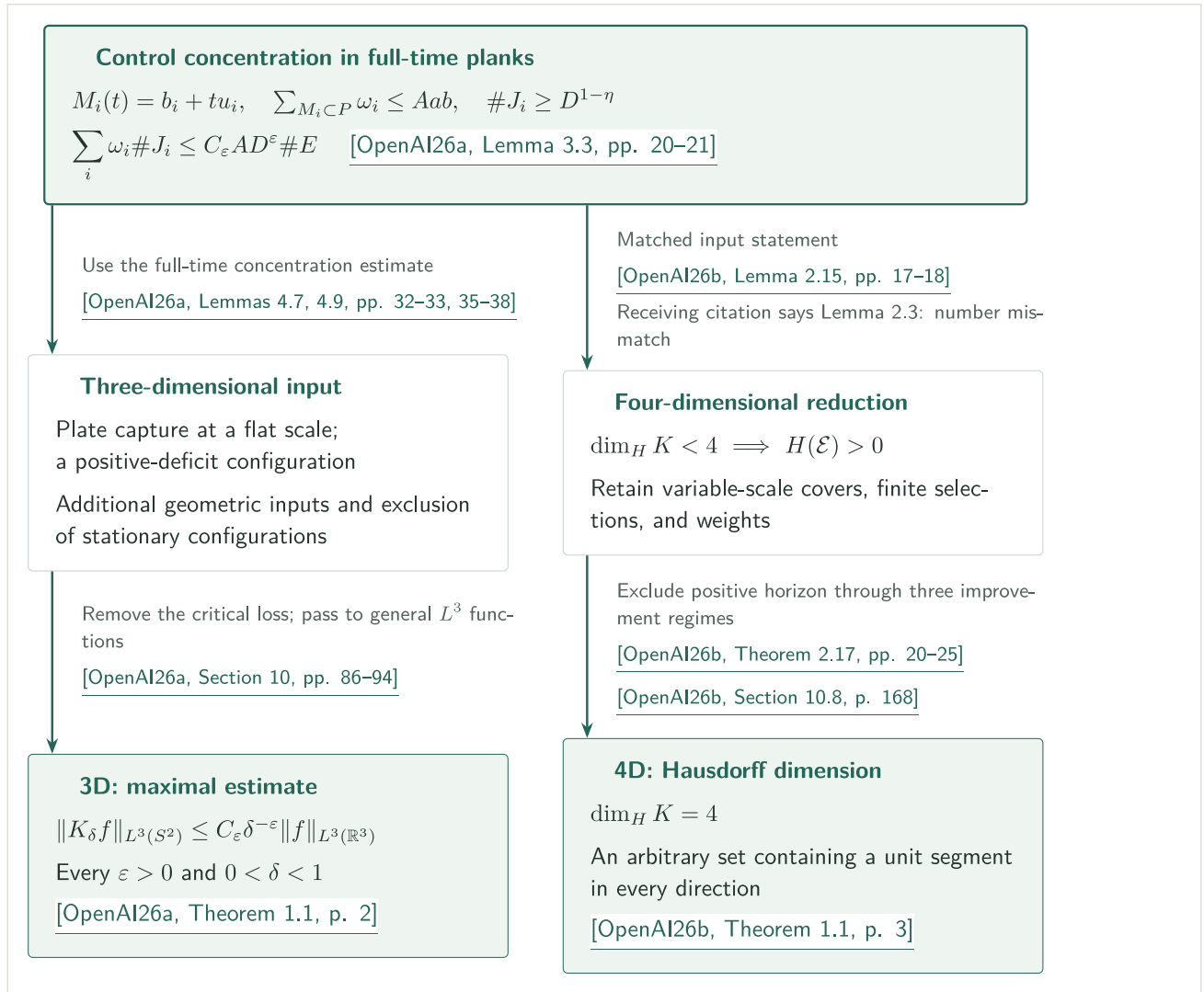
References on the arrows

This guide uses the author-derived keys below and distinguishes manuscript claims from the editorial checking scope.

- **OpenAI26a** — OpenAI, The Kakeya maximal conjecture in three dimensions, September 23, 2026. Role: Weighted configurations retain density dependence; exclusion of stationary extremizers yields the maximal bound. A separate plank lemma also enters the 4D reduction. [[OpenAI26a, Theorem 1.1, p. 2](#)]
- **OpenAI26b** — OpenAI, Every four-dimensional Kakeya set has full Hausdorff dimension, September 24, 2026. Role: Variable-scale covers yield a weighted problem with positive horizon; improvements in three geometric regimes contradict extremality. [[OpenAI26b, Theorem 1.1, p. 3](#)]

2. Two conclusions and the shared plank input

The diagram records the principal connection, not a deduction of either main theorem from the displayed lemma alone. The articles explain the additional geometric inputs and conditions on iteration.



Distinct routes to the maximal estimate and Hausdorff dimension

[Open figure ↗](#) [TeX source ↓](#)

On the three-dimensional side, the weighted full-time plank estimate supplies plate capture at a flat scale and configurations with positive deficit. The main theorem additionally requires exclusion of stationary configurations and the passage from discrete estimates to general L^3 functions; a set-volume bound alone does not replace that passage.

[OpenAI26a, Lemmas 4.7, 4.9, pp. 32–33, 35–38] [OpenAI26a, Section 10, pp. 86–94]

On the four-dimensional side, the matching weighted plank estimate is used as Lemma 2.15 in the reduction from a Hausdorff counterexample to a weighted problem with positive horizon. The subsequent geometric improvements and recovery at the original scale belong to the four-dimensional argument. The conclusion is $\dim_H K = 4$; this guide makes no four-dimensional maximal-operator claim.

[OpenAI26b, Theorem 2.17, pp. 20–25] [OpenAI26b, Section 10.8, p. 168]

Citation-number check. The four-dimensional manuscript cites [24, Lemma 2.3]. In the pinned three-dimensional version, the corresponding full-time plank hypothesis, long-mark condition, and conclusion occur in Lemma 3.3. These interfaces were compared, but the revision history explaining the changed number has not been checked. We retain this discrepancy rather than silently rewriting the receiving citation.

[OpenAI26a, Lemma 3.3, pp. 20–21] [OpenAI26b, Lemma 2.15, pp. 17–18]

See the [three-dimensional article](#) and the [four-dimensional article](#) for the conditions and proof steps.

3. Sources and checking scope

The official subject is **Real and complex analysis** (071–086). This catalogue appears on overview PDF p.9; the two papers are listed in official order.

OpenAI Research Catalog · 074 · p. 9

The source commit is 1581d4cbdd69a0faf34b2089614fe2a814316a6a. The main statements, selected proof passages, and statements and applications of direct inputs were compared. This is not an independent verification of the complete 97- and 175-page proofs; each article identifies unchecked passages. Unlisted connections with other catalogues remain uninvestigated, not absent.

The Keakeya maximal conjecture in three dimensions

A density-sensitive critical exponent, stationary plates, and information comparison between two frames

The manuscript claims the three-dimensional Keakeya maximal estimate from L^3 to L^3 , with an arbitrarily small power loss in scale. It sets up an extremal problem that preserves weighted line indices and their marked times, then makes the geometry required by a positive critical exponent incompatible with projection information. The Keakeya set theorem is an input; the passage to maximal estimates, including low-density shadings, requires a separate argument.

Manuscript author: OpenAI · September 23, 2026 · 97 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — The three-dimensional Keakeya maximal estimate

For $a \in \mathbb{R}^3$, $\omega \in S^2$, and $0 < \delta < 1$, put

$$T_\delta(a, \omega) = \{a + t\omega + u : |t| \leq \tfrac{1}{2}, u \cdot \omega = 0, |u| \leq \delta\}, \quad K_\delta f(\omega) = \sup_{a \in \mathbb{R}^3} \frac{1}{\pi\delta^2} \int_{T_\delta(a, \omega)} |f(x)| dx.$$

For every $\varepsilon > 0$ there is a finite C_ε such that, for every $0 < \delta < 1$ and every complex-valued $f \in L^3(\mathbb{R}^3)$,

$$\|K_\delta f\|_{L^3(S^2)} \leq C_\varepsilon \delta^{-\varepsilon} \|f\|_{L^3(\mathbb{R}^3)}.$$

The sphere carries its usual surface measure. The constant is independent of δ , f , and all tube centers and directions.

[Theorem 1.1 · p. 2](#)

Corollary 11.1 — Euclidean and local constant-curvature Nikodym estimates

For a unit line segment $\gamma \subset \mathbb{R}^3$, let $T_\delta^{\text{Euc}}(\gamma)$ be its δ -neighborhood and define

$$N_\delta^{\text{Euc}} f(x) = \sup_{\gamma \ni x} \delta^{-2} \int_{T_\delta^{\text{Euc}}(\gamma)} |f(y)| dy.$$

For $1 \leq p \leq 3$, $q = 2p'$ (with $q = \infty$ at $p = 1$), and $\varepsilon > 0$, one has, for every $0 < \delta < 1$,

$$\|N_\delta^{\text{Euc}} f\|_{L^q(\mathbb{R}^3)} \leq C_{p,\varepsilon} \delta^{1-3/p-\varepsilon} \|f\|_{L^p(\mathbb{R}^3)}.$$

Also fix a smooth three-dimensional Riemannian manifold (M, g) of constant sectional curvature and $U \Subset V$, with $\bar{V}$ compact in a local chart straightening geodesics. Normalize a fixed segment length to one by a fixed metric rescaling. Choose $0 < \delta_0 < 1$ so that the δ_0 -neighborhoods of unit segments contained in U lie in V . For $x \in U$, put

$$N_{\delta,U}^g f(x) = \sup_{\substack{\gamma \ni x \\ \gamma \subset U}} \delta^{-2} \int_{T_\delta^g(\gamma)} |f| dV_g,$$

where the segments are unit geodesic segments and an empty supremum is zero. For the same p, q, ε and $0 < \delta < \delta_0$,

$$\|N_{\delta,U}^g f\|_{L^q(U, dV_g)} \leq C_{p,\varepsilon,U,V,g} \delta^{1-3/p-\varepsilon} \|f\|_{L^p(V, dV_g)}.$$

In particular both operator norms at $p = q = 3$ have loss $\delta^{-\varepsilon}$. The manifold constant depends on the fixed local geometry.

Corollary 11.1 · pp. 94–95

Corollary 11.2 — A local curved Kakeya estimate for a fixed phase

Fix sufficiently small $\rho > 0$, $\Omega = B_\rho^2 \times (-\rho, \rho)$, and $Y = B_\rho^2$. On a neighborhood of this chart, take a smooth real phase

$$\phi(x, t; y) = x \cdot y + \psi(t; y), \quad \psi(0; y) = 0.$$

Assume Hörmander's mixed-rank and curvature nondegeneracy conditions and Bourgain's condition. For this phase form the mixed-rank condition is automatic, and the other conditions become

$$\det \nabla_y^2 \partial_t \psi(t; y) \neq 0, \quad \nabla_y^2 \partial_t^2 \psi(t; y) = c(t; y) \nabla_y^2 \partial_t \psi(t; y).$$

Fix the phase and chart after the shrinking required by the local theorem, and define

$$T_{\delta,y}^\phi(\omega) = \{(x,t) \in \Omega : |\nabla_y \phi(x,t;y) - \nabla_y \phi(\omega,0;y)| < \delta\}, \quad K_\delta^\phi f(y) = \sup_{\omega \in B_\rho^2} \delta^{-2} \int_{T_{\delta,y}^\phi(\omega)} |f| dx dt.$$

There exists $0 < \delta_{\text{curv}} \leq \rho$ such that, for every $1 \leq p \leq 3$, $q = 2p'$ (with $q = \infty$ at $p = 1$), and $\varepsilon > 0$,

$$\|K_\delta^\phi f\|_{L^q(Y)} \leq C_{p,\varepsilon,\phi,\Omega,Y} \delta^{1-3/p-\varepsilon} \|f\|_{L^p(\Omega)}, \quad 0 < \delta < \delta_{\text{curv}}.$$

In particular the $p = q = 3$ loss is $\delta^{-\varepsilon}$. The phase and localization are fixed independently of δ .

Corollary 11.2 · pp. 95–96

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [The Kakeya maximal conjecture in three dimensions](#). External inputs use the following keys.

- [GWZ26](#) Larry Guth; Hong Wang; Joshua Zahl — A streamlined proof of the Kakeya set conjecture in R3: Set estimate with convex clustering and a density threshold, extended to weighted line charts. arXiv:2601.14411v1, 20 January 2026.
- [RW25](#) Kevin Ren; Hong Wang — Furstenberg sets estimate in the plane: Planar Furstenberg input for plate filling and conditional projection bounds. arXiv:2308.08819v3, 19 January 2025.
- [Gut10](#) Larry Guth — The endpoint case of the Bennett–Carbery–Tao multilinear Kakeya conjecture: Multilinear estimate with explicit transversality loss, used for local planiness. Acta Mathematica 205 (2010), 263–286; published version.
- [GLX25](#) Chuanwei Gao; Diankun Liu; Yakun Xi — Curved Kakeya sets and Nikodym problems on manifolds: Transfers to Nikodym and fixed-phase curved Kakeya estimates after the main theorem. arXiv:2503.11574v1, 14 March 2025.
- [OpenAI26b](#) OpenAI — Every four-dimensional Kakeya set has full Hausdorff dimension: Downstream user of the weighted plank estimate; its citation number mismatches the fixed version. September 24, 2026; commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a.

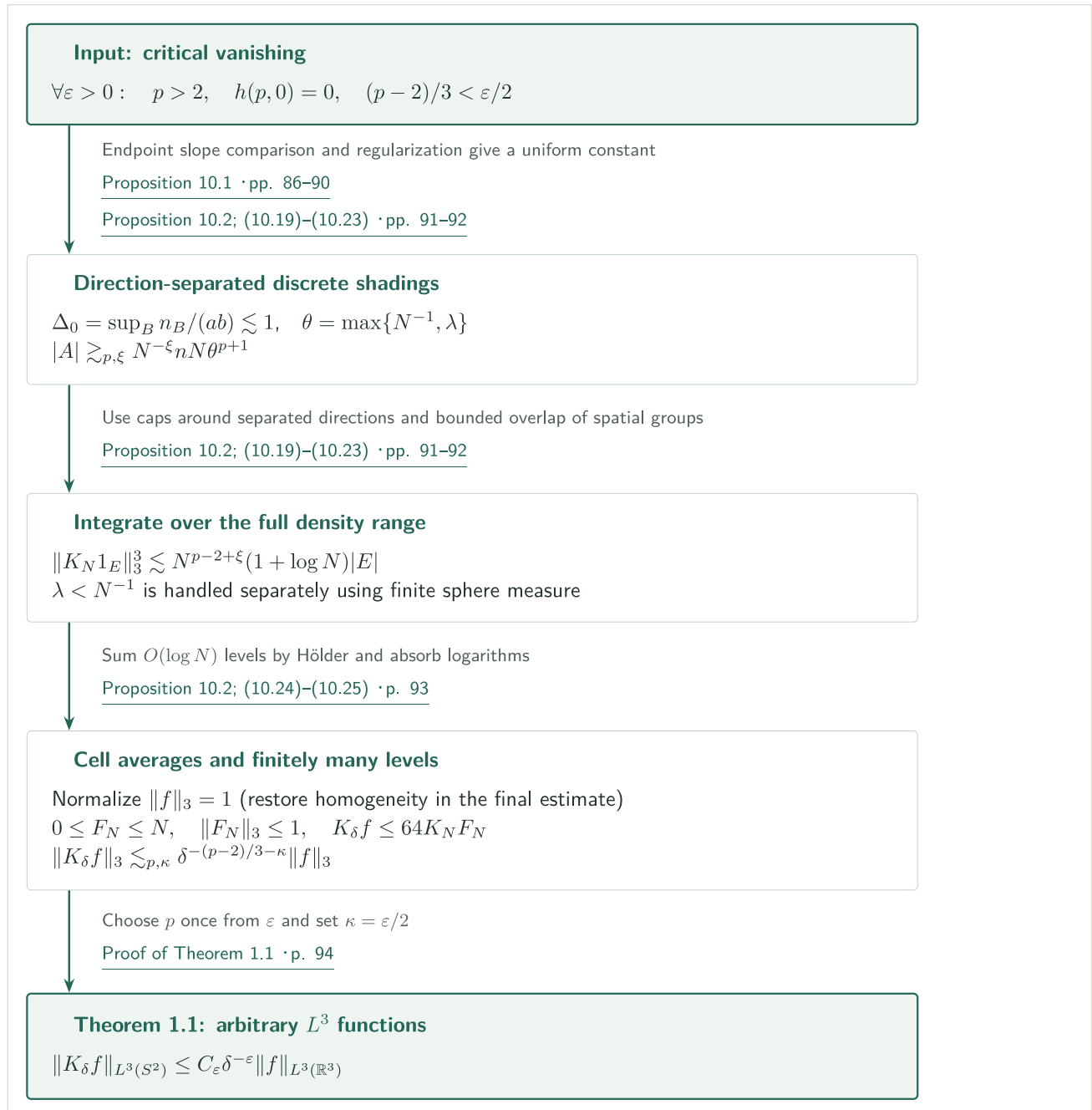
Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — From critical vanishing to general functions

Proof target : [Theorem 1.1](#) · p. 2

Proof route. Proposition 10.1, explained in the [next section](#), gives $h(p, 0) = 0$ for $p > 2$ arbitrarily close to two. Direction separation controls plank concentration. A discrete shading estimate then yields an indicator distribution bound, followed by a finite level decomposition of cell averages. The intermediate density power is λ^{p+1} ; one does not set $p = 2$ at this stage.

Use a dyadic resolution N comparable to δ^{-1} . Below, $|A|$ denotes a cell count, whereas $|E|$ denotes the Lebesgue volume of an indicator set.



Critical vanishing, a density-sensitive discretization, and the general L^3 estimate

[Open figure ↗](#) [TeX source ↓](#)

1. Direction separation controls full-time planks

In a bounded line chart write $M_i(t) = b_i + tu_i$. An index has a positive weight ω_i and marked time bins S_i . Repeated traces with different indices or shadings are not identified. Distinguish total weight $n = \sum_i \omega_i$, incidence $I = \sum_i \omega_i |S_i|$, mean marked count $k = I/n$, and raw multiplicity $m = I/\#E$ on occupied cells. A plank of widths a, b counts the weight n_B of indices whose entire time trace it contains; a single marked point in the plank does not establish membership.

For the maximal reduction, return to unit weights and c/N -separated slopes. Comparing a plank's two endpoint sections places its slopes in an $O(a/N) \times O(b/N)$ rectangle. Thus $n_B \lesssim ab$, and $\Delta_0 = \sup_B n_B/(ab) \lesssim 1$ persists for every subfamily.

Definition 2.3; (2.1)–(2.8) · pp. 6–8 Proposition 10.2; (10.19)–(10.23) · pp. 91–92

2. A uniform discrete bound and the indicator estimate

Assume $h(p, 0) = 0$. If each line has $\gtrsim \max\{1, \lambda N\}$ marked bins, then for every $\xi > 0$,

$$|A| \gtrsim_{p,\xi} N^{-\xi} n N \max\{N^{-1}, \lambda\}^{p+1}.$$

If no uniform constant existed, regularize a violating sequence with

$\theta = \max\{N^{-1}, \lambda\} = N^{-\alpha+o(1)}$. Incidence retention forces $F(1) = \alpha$. At $q = 0$ the temporal cost is exactly $p\alpha$, so the violating multiplicity $N^{p\alpha+\xi-o(1)}$ contradicts the critical bound $N^{p\alpha+o(1)}$. This is the passage from asymptotic exponent notation to a constant uniform over all finite configurations.

For the auxiliary operator K_N on cylinders of length 4 and radius $4/N$, direction charts and grouping centers into unit cubes give

$$\sigma\{K_N 1_E > \lambda\} \lesssim_{p,\xi} N^\xi |E| \max\{N^{-1}, \lambda\}^{-(p+1)}.$$

For $\lambda \geq N^{-1}$ this implies $\lambda^3 \sigma\{K_N 1_E > \lambda\} \lesssim N^{p-2+\xi} |E|$. The smaller-density range is handled separately using the finite surface measure and $|E| \geq N^{-3}$ for a nonempty grid set. Integrating the distribution gives $\|K_N 1_E\|_3^3 \lesssim N^{p-2+\xi} (1 + \log N) |E|$.

Proposition 10.2; (10.19)–(10.23) · pp. 91–92

3. Cell averages, level sets, and an arbitrary power loss

Normalize $\|f\|_3 = 1$ and choose $N^{-1} \leq \delta < 2N^{-1}$. The cell average F_N of $|f|$ on each N^{-1} -cube satisfies $0 \leq F_N \leq N$ and $\|F_N\|_3 \leq 1$. A fixed cylinder enlargement gives $K_\delta f \leq 64K_N F_N$.

Bound the part $F_N < N^{-1}$ directly and split the remaining values into $O(\log N)$ dyadic levels.

The indicator estimate and Hölder for the finite sum give

$$\|K_\delta f\|_3 \lesssim_{p,\xi} N^{(p-2+\xi)/3} (1 + \log N) \lesssim_{p,\kappa} \delta^{-(p-2)/3-\kappa}.$$

For a fixed ε , choose once a p with $h(p, 0) = 0$ and $(p - 2)/3 < \varepsilon/2$, then set $\kappa = \varepsilon/2$. Restoring homogeneity proves the full statement. The source also establishes measurability: continuity of tube integrals in their parameters allows the supremum over centers to be restricted to a countable dense set.

Applying the final theorem to indicators yields $|\bigcup_T Y(T)| \gtrsim_\varepsilon \delta^\varepsilon \lambda^3 \sum_T |T|$ for direction-separated shadings, uniformly over $0 < \lambda \leq 1$. This final cubic density power is distinct from the intermediate $p + 1 > 3$.

Proposition 10.2; (10.24)–(10.25) · p. 93 Proof of Theorem 1.1 · p. 94

3. Proof of Proposition 10.1 — Stationary geometry versus projection constraints

Proof target : Proposition 10.1 · pp. 86–90

Proposition 10.1 — Critical vanishing at the maximal endpoint

Let $0 < \eta < 1/2$, $p > 2$, $z = -q \in [-1, 0]$, and $d = 2 - q$. A temporal profile F is nondecreasing and 1-Lipschitz, with $F(0) = 0$; each occupied block of N^κ bins has $N^{\kappa - F(\kappa) + o(1)}$ marked bins. The manuscript defines

$$D(q, e) = \eta^{-1} \int_{1/2-\eta}^{1/2} \min(q, re) dr, \quad \Pi_F(1) = \int_0^1 [pF'(u) - D(q, F'(u))] du,$$

$$\Delta_A = \sup_B \frac{n_B}{a^{d-1-A} b^{1+A}}, \quad h(p, z) = \min\{A \in [0, d-1] : m \leq_{\text{exp}} N^{A+\Pi_F(1)} \Delta_A \text{ for every uniform-profile sequence}\}.$$

Here $\leq_{\text{exp}}$ means comparison up to $N^{o(1)}$. Proposition 10.1 asserts that $h(p, 0) = 0$ for values $p > 2$ arbitrarily close to two. At $q = 0$, $\Pi_F(1) = pF(1)$ and the conclusion is independent of η .

Definition 2.3; (2.1)–(2.8) · pp. 6–8 Proposition 10.1 · pp. 86–90

Proof route. Positive critical exponent gives an extremal sequence. Local mass bounds, temporal filling, and anisotropic rescaling produce a canonical profile. The coordinate entropy demanded by its packets is compared with projection constraints from two events on one index. This outlines the connections; the complete simultaneous finite-test construction and pinning iteration are outside the checked scope.

Assume a positive critical exponent

$$m \leq_{\text{exp}} N^{h+\Pi_F(1)} \Delta_h, \quad h > 0$$

Preserve indices, weights and shadings, including repeated traces

Weighted extensions and localization; retain the density threshold

[\[GWZ26, Theorem 1.1, p. 2\]](#) [\[RW25, Theorem 4.1, p. 9\]](#)

[\[Gut10, Theorem 1.3, print p. 264 / PDF p. 2\]](#)

[Proposition 4.1; Lemma 4.3 · pp. 24–28](#)

Local mass bounds and temporal filling

Convex-clustered set input, planar Furstenberg and transverse triples

A common plane on each short block; positive slack absorbs scale costs

Compare outer and anisotropic inner cuts, preserving inherited weights

[Propositions 5.4, 5.6 · pp. 44–48](#)

Angular nonconcentration gives $\beta < 1$: [Lemma 6.2 · pp. 51–54](#)

Minimum delay produces stationary packets

$$F(s) = \beta(s - \tau)_+, \quad \ell = 1 - \tau, \quad 0 < \beta < 1$$

(X, Y, U, V) have scale speeds $(0, \tau, \ell, 1)$

Compare planar projection and product pinning with short-gap information

[Lemmas 8.4–8.7; \(8.27\) · pp. 73–78](#)

[Proposition 8.1 · pp. 68–69](#)

Projection constraints from two frames

$$Y' = Y + \Delta t V, \quad X' = X + \Delta t U + \Delta \vartheta Y + \Delta t \Delta \vartheta V$$

Actual joint law: hit probability 1; product marginals: D^{-c}

Five cases; strong traces of v_1 localize to its positive set

[Proposition 7.4; Lemmas 7.5–7.6 · pp. 65–68](#)

[Proposition 10.1 · pp. 86–90](#)

A positive gap contradicts criticality

Packet entropy demand is incompatible with projection constraints

$$\exists p \downarrow 2 : h(p, 0) = 0$$

Positive critical exponent leads to canonical packets whose entropy demand contradicts projections

[Open figure ↗](#) [TeX source ↓](#)

1. Weighted set input and local planiness

The set input requires convex clustering $\sum_{T \subset K} |T| \leq \delta^{-\eta_{\text{in}}} |K|$ and mean shading density $\lambda \geq \delta^{\eta_{\text{in}}}$. It is not applied as an arbitrary-density estimate. Lemma 3.2 samples according to weights, controls finitely many box tests, and selects representatives of repeated tubes. Lemma 3.3 converts full-time plank bounds into convex clustering; disjoint small balls at marked times relate cell counts to shading volume.

An excessive local packet mass would contradict criticality after rescaling. The multilinear input's $\theta^{-1/2}$ cost and cap counts become powers of a sufficiently short block scale and are absorbed by positive slack. Blocks with three transverse rich directions can therefore be discarded, leaving, within each short block, a common direction plane along the reference index's retained times. The planar Furstenberg estimate fills plates while preserving the original shading's Frostman conditions.

[GWZ26, Theorem 1.1, p. 2] Theorem 3.1; Lemmas 3.2–3.3 · pp. 18–21

[Gut10, Theorem 1.3, print p. 264 / PDF p. 2] Proposition 4.1; Lemma 4.3 · pp. 24–28

[RW25, Theorem 4.1, p. 9] Theorem 3.4; Corollary 3.5 · pp. 21–22

2. The temporal profile and compatible packets across scales

Replacing the temporal deficit by total density alone would lose the cost after cutting a time block. The manuscript retains F , obtains an initial flat interval by plate filling, and compares an outer scale cut with an inner cut after plate alignment. A minimum-delay choice produces

$$F(s) = \beta(s - \tau)_+, \quad 0 < \tau < 1, \quad \ell = 1 - \tau, \quad 0 < \beta < 1,$$

together with stationary packets preserving the weights of their original indices at every scale. The strict bound $\beta < 1$ uses the later angular nonconcentration conclusion of Lemma 6.2. Stationarity means that repeated cuts reproduce the normalized profile and packet shape.

In an event's normal frame ϑ , use $X = M_i(t)_1 + \vartheta M_i(t)_2$, $Y = M_i(t)_2$, $U = u_{i,1} + \vartheta u_{i,2}$, and $V = u_{i,2}$. Their scale speeds are $(0, \tau, \ell, 1)$. Packet mass lower bounds become a demand on conditional coordinate entropy.

Propositions 5.4, 5.6 · pp. 44–48 Lemma 6.2 · pp. 51–54 Proposition 7.4; Lemmas 7.5–7.6 · pp. 65–68

3. Two events on one index are not independent

The two-frame transformation is

$$\begin{pmatrix} X' & Y' \\ U' & V' \end{pmatrix} = \begin{pmatrix} 1 & \Delta t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & Y \\ U & V \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \Delta \vartheta & 1 \end{pmatrix}.$$

Time and normal differences give scalar projections of coordinate pairs. When the middle speeds tie, $\tau = \ell = 1/2$, the joint Y, U rate must be retained. After fixing these coordinates, the term $\Delta t \Delta \vartheta V$ in X' requires product-slope pinning.

The manuscript compares the actual same-index joint law with the product of its conditional matrix and label marginals. If the recorded support fell below the required projection exponent, its hit probability under the product law would be D^{-c} , while the actual law hits with probability

one because both events use the same matrix. Data processing for relative entropy would require mutual information $\gtrsim \log D$, contradicting the short-gap bound $o(\log D)$ in Lemma 8.5. Bias introduced by pinning is charged through domination bounds for both marginals.

Proposition 8.1 · pp. 68–69 Lemmas 8.4–8.7; (8.27) · pp. 73–78 Sections 9.1–9.2 · pp. 79–85

4. Closing the contradiction with the prescribed loss and limit order

For distinct speeds, the conditional rates a_1, y_1, u_1, v_1 satisfy, among other constraints, $a_1 \geq y_1$, $u_1 \geq v_1$, and $a_1 < 1 \Rightarrow u_1 \leq (a_1 - s_0)_+$, where $s_0 = 1 - \beta > 0$. Five cases, determined by speed ties and the parameter regime, give a positive gap between these constraints and packet demand. Localization to $v_1 > 0$ cannot use weak convergence alone; Lemma 7.5 supplies strong L^1 traces or subsequential compactness.

Restrictions retain $N^{-o(1)}$ of incidence. Finite tests, scales, and positive gaps are fixed before increasing resolution. The smoothing width η is fixed at the start in terms of $p_{\min} - 2$, not readjusted after selecting h, τ, β . With this order, the contradiction excludes a positive critical exponent and supplies the analytic reduction in the preceding section.

Proposition 10.1 · pp. 86–90

4. Proof of Corollary 11.1 — Interpolation and Nikodym transfer

Proof target : Corollary 11.1 · pp. 94–95

Proof route. Interpolate the main L^3 estimate with the elementary $L^1 \rightarrow L^\infty$ estimate to obtain the full range required by the transfer. The same proposition supplies the Euclidean and local constant-curvature conclusions.

Put both endpoints in the same normalization

$$\|\tilde{K}_\delta f\|_3 \lesssim_\eta \delta^{-\eta} \|f\|_3, \quad \|\tilde{K}_\delta f\|_\infty \leq \delta^{-2} \|f\|_1$$

Compare with three cylinders; interpolate with $1/p = 1 - 2\theta/3$

Corollary 11.1 · pp. 94–95

The full interpolation range: $K(3)$

$$1 \leq p \leq 3, \quad q = 2p', \quad \|\tilde{K}_\delta f\|_q \lesssim \delta^{1-3/p-\varepsilon} \|f\|_p$$

Takeya–Nikodym transfer and local geodesic straightening

[GLX25, Proposition 2.3; (6)–(9), pp. 13–14]

Corollary 11.1: two Nikodym estimates

$$\|N_\delta^{\text{Euc}} f\|_q \leq C_{p,\varepsilon} \delta^{1-3/p-\varepsilon} \|f\|_p$$

$$\|N_{\delta,U}^g f\|_{L^q(U)} \leq C_{p,\varepsilon,U,V,g} \delta^{1-3/p-\varepsilon} \|f\|_{L^p(V)}$$

$U \Subset V$ is a fixed local chart; $p = q = 3$ gives $\delta^{-\varepsilon}$

Interpolation followed by Euclidean and constant-curvature Nikodym transfer

[Open figure ↗](#) [TeX source ↓](#)

1. Obtain the full range and transfer to Euclidean Nikodym

The external source uses δ -neighborhoods of unit segments with normalization δ^{-2} . Covering such a neighborhood by three manuscript cylinders gives $\tilde{K}_\delta f \leq 3\pi K_\delta f$. Also

$\|\tilde{K}_\delta f\|_\infty \leq \delta^{-2} \|f\|_1$. Interpolation with $1/p = 1 - 2\theta/3$ and $1/q = \theta/3$ gives $q = 2p'$ and exponent $1 - 3/p - \eta\theta$. The loss $\eta > 0$ is arbitrary; including both endpoints gives $K(3)$.

Proposition 2.3 of GLX25 transfers this full range to the Euclidean Nikodym operator.

[Corollary 11.1 · pp. 94–95](#) [\[GLX25, Proposition 2.3; \(6\)–\(9\), pp. 13–14\]](#)

2. Fix the local geometry on a constant-curvature manifold

Apply the same proposition's geodesic-straightening diffeomorphism on a fixed compact chart.

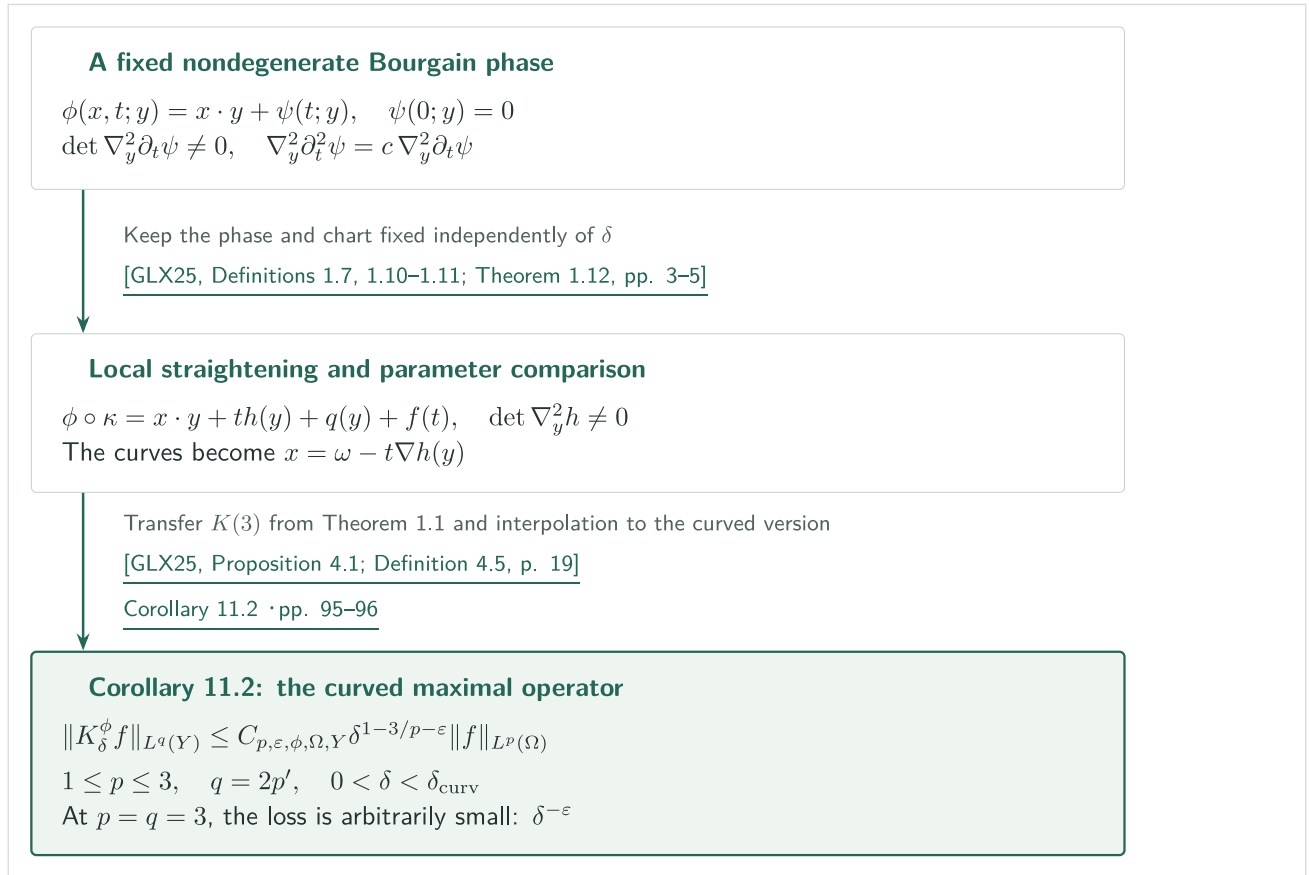
Bounded bi-Lipschitz and volume-comparison constants compare tube averages on $U \Subset V$; the choice of δ_0 keeps the input inside V . Start with a smaller loss to absorb the transfer's arbitrarily small additional loss. Setting $p = q = 3$ yields the local endpoint.

[\[GLX25, Proposition 2.3; \(6\)–\(9\), pp. 13–14\]](#) [Corollary 11.1 · pp. 94–95](#)

5. Proof of Corollary 11.2 — Straightening the phase

Proof target : Corollary 11.2 · pp. 95–96

Proof route. Feed the full $K(3)$ range from the preceding section into the local transfer for a fixed translation-invariant phase satisfying Bourgain's condition.



Straightening a fixed nondegenerate Bourgain phase to obtain the full curved estimate

[Open figure ↗](#) [TeX source ↓](#)

1. Preserve the phase hypotheses and operator normalization

Theorem 1.12 of GLX25 transforms the phase locally into $x \cdot y + th(y) + q(y) + f(t)$ with nondegenerate $\nabla_y^2 h$. Terms depending only on y cancel in the gradient difference, and terms depending only on (x, t) do not contribute to ∇_y . The curves become straight lines $x = \omega - t \nabla h(y)$. Proposition 4.1 makes the ordinary and curved $K(3)$ assertions equivalent, giving the same p, q range and $\delta^{1-3/p-\varepsilon}$ power. Spatial and direction-parameter comparison constants may depend on the fixed phase and chart, which also determine δ_{curv} . The $p = q = 3$ conclusion is included.

[\[GLX25, Definitions 1.7, 1.10–1.11; Theorem 1.12, pp. 3–5\]](#) [\[GLX25, Proposition 4.1; Definition 4.5, p. 19\]](#)
[Corollary 11.2 · pp. 95–96](#)

6. Which papers supply which steps

Source and result

[GWZ26, Theorem 1.1, p. 2]

Used at

Theorem 3.1; Lemmas 3.2–3.3 · pp. 18–21

Role and required hypotheses

Convex clustering and a density threshold; sampling and plank-to-convex-body comparison give a weighted extension

Checking scope

Statement, weighted extension and use compared; complete source proof not checked

Source and result

[RW25, Theorem 4.1, p. 9]

Used at

Theorem 3.4; Corollary 3.5 · pp. 21–22; §9.1

Role and required hypotheses

Frostman conditions on line parameters and shadings; projection exponent $\min\{t, (s + t)/2, 1\}$

Checking scope

Statement and the manuscript's duality explanation compared; complete quantitative reconstruction not checked

Source and result

[Gut10, Theorem 1.3, print p. 264 / PDF p. 2]

Used at

Theorem 3.7; Corollary 3.8 · p. 23; Lemma 4.3

Role and required hypotheses

Determinant lower bound θ for every triple, loss $\theta^{-1/2}$, extension to positive weights and repetitions

Checking scope

Published statement, weighted extension and use compared

Source and result

[GLX25, Proposition 2.3; (6)–(9), pp. 13–14]

Used at

Corollary 11.1

Role and required hypotheses

Full interpolated Takeya range; constant curvature and a fixed local chart

Checking scope

Statement, exponents, normalization and application compared

Source and result

[GLX25, Proposition 4.1; Definition 4.5, p. 19]

Used at

Corollary 11.2

Role and required hypotheses

A fixed nondegenerate translation-invariant Bourgain phase

Checking scope

Statement and locality compared

Input to the four-dimensional manuscript. Lemma 3.3 here assumes, in a bounded line chart, $\sum_{i: M_i \subset P} \omega_i \leq Aab$ for all full-time planks with $1 \leq a \leq b \leq D$, and $\#J_i \geq D^{1-\eta_{\text{in}}}$ for each index. For every target loss $\varepsilon > 0$, a suitable $\eta_{\text{in}} > 0$ gives $\sum_i \omega_i \#J_i \leq CAD^\varepsilon \#E$. Repeated traces and positive weights are allowed. The four-dimensional manuscript's Lemma 2.15 uses this content directly but cites “Lemma 2.3,” a number absent from the present version. The statement correspondence and the unresolved numbering/version discrepancy are recorded separately; this is not a dependency on Theorem 1.1 itself.

Lemma 3.3 · pp. 20–21 [OpenAI26b, Lemma 2.15, pp. 17–18]

7. Return to the sources

- Definition 2.3; (2.1)–(2.8) · pp. 6–8 — Definitions of weights, full-time planks, density cost and critical exponent
- Theorem 3.1; Lemmas 3.2–3.3 · pp. 18–21 — Weighted extension and rescaling of the set input

- Propositions 5.4, 5.6 · pp. 44–48 — Canonical temporal profile and stationary packets
- Lemmas 8.4–8.7; (8.27) · pp. 73–78 — Same-index two-event law compared with its independent marginals
- Proposition 10.1 · pp. 86–90 — The five-case contradiction and the role of strong traces
- Proposition 10.2; (10.19)–(10.23) · pp. 91–92 — Direction separation and the uniform discrete density bound
- Proposition 10.2; (10.24)–(10.25) · p. 93 — Cell averages and finitely many level sets for general functions
- Corollary 11.1 · pp. 94–95 — Transfer to Euclidean and local constant-curvature Nikodym operators
- Corollary 11.2 · pp. 95–96 — Local curved Kakeya transfer for a fixed phase

The editorial reading covered the weighted model and three geometric inputs, canonical-profile and stationary-data statements, entropy demand and the two-event information comparison, the final five-case contradiction, the discrete-to-general-function reduction, and both transfer corollaries. Input statements and use were compared with four external sources. The simultaneous finite tests, complete regularization and compactness details in §§2,4–9, and the full product-pinning iteration remain unchecked. This is not a certification of the complete 97-page proof.

The radial-projection references in §9 are methodological precedents for pinning; the required finite-scale version is proved within this manuscript. The spherical Fourier restriction route cited in §1.3 is an alternative proof. Neither relationship is turned into a direct-input arrow here. The four-dimensional manuscript's citation-number mismatch with the fixed version remains unresolved.

074 / 02

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CATALOGUE 074 · PAPER 02 / 2 Overview

Every four-dimensionalakeya set has full Hausdorff dimension

From variable-scale covers to positive horizon, then contradiction by improving quadratic local fits

The manuscript translates the Hausdorff dimension problem for four-dimensionalakeya sets into the exponent required to localize weighted trajectories in time. It selects a problem whose density decreases at a rate below one, improves quadratic fits, and contradicts the optimized time exponent or spatial width. A good local approximation must also return to the original problem as a chart with substantial incidence mass and a short list of labels recoverable from the original observation.

Manuscript author: OpenAI · September 24, 2026 · 175 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Hausdorff dimension of arbitrary four-dimensionalakeya sets

Let $K \subset \mathbb{R}^4$. Suppose that for every $e \in S^3$ there is $a_e \in \mathbb{R}^4$ such that

$$\{a_e + te : 0 \leq t \leq 1\} \subset K.$$

The manuscript proves

$$\dim_H K = 4.$$

Neither K nor the choice $e \mapsto a_e$ is assumed measurable. Compactness and stickiness are not assumed either. Hausdorff dimension is defined using countable covers of arbitrary sets. The conclusion concerns variable-scale covers; Theorem 1.1 does not assert a four-dimensionalakeya maximal-function estimate.

[Theorem 1.1 · p. 3](#)

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Every four-dimensional Kakeya set has full Hausdorff dimension](#). External inputs use the following keys.

- [OpenAI26a](#) OpenAI — The Kakeya maximal conjecture in three dimensions: Weighted full-time plank bound; the text records the discrepancy between the cited number 2.3 and the pinned version's 3.3.. September 23, 2026; fixed commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a.
- [CV13](#) Anthony Carbery; Stefán Ingi Valdimarsson — The endpoint multilinear Kakeya theorem via the Borsuk–Ulam theorem: Determinant-weighted multilinear Kakeya for independent directions.. arXiv:1205.6371v1, 29 May 2012.
- [WZ25](#) Hong Wang; Joshua Zahl — Sticky Kakeya sets and the sticky Kakeya conjecture: Restricted-triple dot-product alternative for the scalar-normal first jet.. arXiv:2210.09581v2, 7 October 2025.
- [OSS24](#) Tuomas Orponen; Nicolas de Saxcé; Pablo Shmerkin — On the Fourier decay of multiplicative convolutions: Multiplicative-convolution Fourier decay and flat generated coefficients for planar rigidity.. arXiv:2309.03068v2, 27 February 2024.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1](#) · p. 3

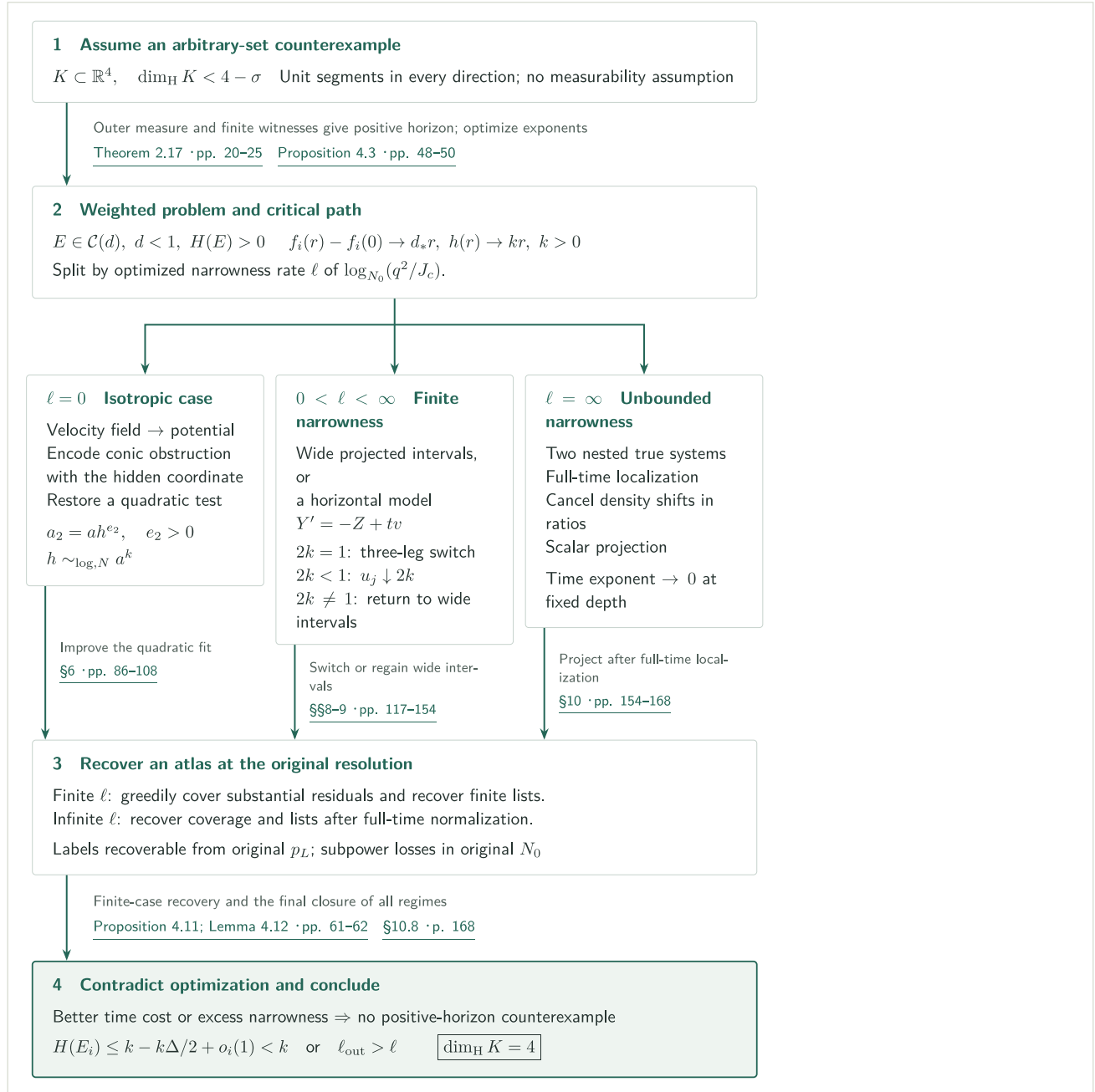
Proof route. Assume $\dim_{\mathbb{H}} K < 4$. The [reduction in the next section](#) produces a weighted problem with positive horizon. Successive optimization of density, time, and spatial narrowness leaves three cases for the optimized narrowness ℓ . Each case gives local improvements, which must return to chart systems in the original problem before contradicting optimality.

Write $N_0 \rightarrow \infty$ for the original resolution and $a_r = N_0^{-r}$ for thickness at depth r . Trajectories have the form $(t, y_0 + tV, w_0 + tW)$, with $y \in \mathbb{R}^2$. The observation p_r retains exact (t, y) and bins w at width a_r/B , where B is the hidden-derivative scale. Write $n(r) = N_0^{-f(r)}$ for its typical density. A chart of time length q , spatial Jacobian J_c , and selected mass m_c is **true** when

$$\frac{m_c}{q} \geq N_0^{-o(1)} n(r) J_c.$$

The spatial box and quadratic small-value conditions hold along assigned trajectories throughout the whole time block; only the lower hidden-derivative bound is required on selected incidences. For true terminal systems at depth L covering mass $N_0^{-o(1)}$, the horizon $H(E)$ is the infimum of the exponents in $q = N_0^{-h+o(1)}$. Subsequences and further permitted selections are included in this infimum.

Definitions 2.1, 2.6, 2.8 · pp. 8–12; Definition 2.14 · p. 17



A counterexample yields positive horizon; improvements in all three narrowness regimes contradict optimization

Open figure ↗ TeX source ↓

1. From positive horizon to linear critical paths

The reduction gives a cap class $\mathcal{C}(d)$ satisfying, for fixed $S, \eta > 0$,

$$f(L) - f(r) \leq d(L - r), \quad d < 1,$$

and a joint density ceiling $n(L)N_0^{-Su+o(1)}$ when a velocity bin is appended to the terminal observation, for $0 < u \leq \eta L$. Lemma 3.5 reduces the time exponent from h to $h - e$ at the cost of at most Ce in thickness depth. Its constant is independent of the individual problem's complexity, so the maximal time rate stays $0 < k \leq 1$ as one approaches the least density exponent d_* permitting positive horizon.

After normalizing length to one, optimize d_* first, then $H(E)/L$, then narrowness. Proposition 4.3 produces paths with relative density $f_i(r) - f_i(0) \rightarrow d_* r$ and cumulative time exponent kr . Narrowness is $\log_{N_0}(q^2/J_c)$; for finite ℓ , its cumulative value approaches ℓr . No bound on the absolute density exponent $f_i(0)$, and no single limiting problem realizing all optima, is required.

Lemma 3.5 · p. 36; Proposition 4.3; Lemmas 4.1–4.5 · pp. 47–51

2. In the isotropic case, construct a potential and restore the quadratic class

When $\ell = 0$, scalar-normal estimates provide polynomial fits along trajectories. Multilinear Kakeya for independent velocities and interpolation assemble them into a polynomial field. Switching between trajectories yields approximate conservativity and hence a potential. Its degree need not initially be at most two.

Proposition 6.8 identifies the cubic and quartic obstruction when broad directions remain close to a conic. The existing hidden coordinate encodes these terms as a quadratic correction, restoring the allowed chart test. Writing a for local thickness and $h = N^{o(1)}a^k$ for block length, the resulting thickness is $a_2 = ah^{e_2}$, with $e_2 > 0$, and mass divided by block length is at least $N^{-o(1)}\nu_Q(a_2/a)^{d_*}$. The fit holds throughout the same time block. Here N is the local tangent resolution, and ν_Q is the original trajectory assignment mass, not a prior reconditioned on success. The scalar version of this argument first proves $H = 0$ in Corollary 6.9 and supplies the later projection theorem.

Theorem 6.1 · p. 86; Proposition 6.8; Corollary 6.9 · pp. 104–108

3. Finite narrowness leads through the horizontal model

For $0 < \ell < \infty$, scalar projection yields quadratic graphs fitting whole trajectories. If projected intervals are sufficiently wide at three nearby depths, comparison gives a time improvement directly. Otherwise planar rigidity, retaining original incidence edges, reduces to the horizontal model $Y' = -Z + tv$. Its narrowness rate equals k , and its natural spatial widths are H and H^2 .

At the critical rate $2k = 1$, a path with three refreshed line segments fills enough three-dimensional base volume to improve the fit. For $2k > 1$, the argument returns to wide projected intervals. For $2k < 1$, it first improves full-time predictions on parameter boxes of width a_s^u . The iteration in Proposition 9.5 is

$$u_{j+1} = \frac{u_j}{1 + \alpha(u_j - 2k)}, \quad u_0 = 1, \quad \alpha = \frac{1 - d_*}{10000}, \quad u_j \downarrow 2k.$$

Only after absorbing the losses from each finite iteration does the iteration count increase slowly. This constructs actual labels and mass floors at the endpoint width $h_s^2 = a_s^{2k}$. Proposition 9.6 then restores the wide-interval hypotheses of Proposition 8.4. Substitution of a limiting exponent alone would not give these objects.

Propositions 8.4–8.5 · pp. 124–130; Propositions 9.2, 9.5–9.6 · pp. 138–154

4. Infinite narrowness and the return from local improvements to the original scale

When $\ell = \infty$, two nested true systems localize trajectories for the entire time interval. Further normalization may shift absolute density exponents substantially, but these shifts cancel in the density ratios used. Scalar projection and label recovery produce atlases at a fixed positive depth whose time exponents tend to zero, contradicting positive k .

For finite ℓ , finding one local candidate also does not suffice. Proposition 4.11 assumes that every substantial residual supplies a candidate; greedy assignment obtains coverage, scalar-sheet comparison shortens label lists, and Lemma 4.12 recovers them in the original law. Success is at least $p_{\text{path}} / \log N_0$, where p_{path} is the original path mass, and lists and slacks are subpower in the original resolution. Local $N^{o(1)}$ is not simply treated as $N_0^{o(1)}$: the original-scale exclusion is fixed before the diagonal is chosen.

The resulting atlas either has excess narrowness at the same depth and time, or gains depth $\Delta > 0$ at time cost at most $k\Delta/2$. In the latter case, completion through the remaining depths gives

$$H(E_i) \leq h_i + \frac{k\Delta}{2} + k(d_i)(1 - r_i - \Delta) \leq k - \frac{k\Delta}{2} + o_i(1),$$

contradicting $H(E_i) \rightarrow k$. All three cases are excluded, so $\dim_{\mathbb{H}} K \geq 4$; the ambient upper bound gives equality.

Lemma 4.5 · p. 51; Proposition 4.11; Lemma 4.12 · pp. 61–62; Proposition 10.1; §10.8 · pp. 154–168

3. Proof of Theorem 2.17 — From arbitrary sets to positive horizon

Proof target : Theorem 2.17 · p. 20

Theorem 2.17 — From a counterexample to a positive-horizon problem

Suppose that a Kakeya set $K \subset \mathbb{R}^4$ with $\dim_{\mathbb{H}} K < 4$ exists. Then there are $0 < d < 1$, $S, \eta > 0$, and a tempered homogeneous version-2 exact problem $E \in \mathcal{C}(d)$ such that $H(E) > 0$. The density and angular conditions defining $\mathcal{C}(d)$ are those stated above.

Theorem 2.17 · p. 20

This reduction supports the main theorem's absence of measurability assumptions. In its proof, take $\dim_{\mathbb{H}} K < 4 - \sigma$, with $0 < \sigma < 1$. In this section k denotes a dyadic covering-scale index, distinct from the critical time rate above.

1 Variable-scale Hausdorff cover

$$\dim_{\mathbb{H}} K < 4 - \sigma, \quad \sum_i r_i^{4-\sigma} \leq 1$$

Common time interval and bounded intercept box; positive outer slope measure

Countable outer subadditivity, maximal separated set, finite selection

[Theorem 2.17, finite selection · pp. 20–21](#)

2 Choose finitely many witnesses at one scale

$$\delta = 2^{-k}, \quad ck^{-2}\delta^{-3} \leq M \leq C\delta^{-3}$$

Covered time $\gtrsim k^{-2}$ per segment; no measurable choice over all directions

Use full-time plank weight caps and long marked sets

[\[OpenAI26a, Lemma 3.3, pp. 20–21\]](#)

Restated here as: [Lemma 2.15 · pp. 17–18](#)

3 Density profile and angular cap

$$g = r - f, \quad g(1, 0) \geq \sigma, \quad g(0, 0) \leq 0, \quad g(r, u) \leq r + 2 - 2u$$

Control smoothed slope density; the plank estimate gives $g(0, 0) \leq 0$

Transport one-sided maximizing inequalities through the crop

[Theorem 2.17, \(2.24\)–\(2.32\) · pp. 21–23](#)

4 Select the endpoint and angular scale

$$(r_0, u_0) \in \arg \max(g - Rr + Su), \quad g(r_0, u_0) \geq \sigma/2$$

$$L = r_0, \quad d = 1 - R < 1, \quad E \in \mathcal{C}(d)$$

Cancel crop-prior denominators; use the monic-cubic Jacobian

[Theorem 2.17, \(2.33\)–\(2.39\) · pp. 23–25](#)

5 Compare the mass floor and slope volume

$$N_0^{\sigma/2} \leq m(r_0, u_0) \leq N_0^{o(1)} q^{-3}$$

$$q = N_0^{-h+o(1)} \implies H(E) \geq \sigma/6 > 0$$

A lower bound for terminal charts surviving further selections and subsequences

Finite witnesses from a Hausdorff cover, then density caps and chart mass comparisons produce positive horizon

[Open figure ↗](#) [TeX source ↓](#)

1. Use outer measure to choose a common scale, then select only finitely many witnesses

Restrict directions to a graph-slope chart. A countable decomposition by a common rational-endpoint time interval and bounded intercept box supplies a slope set $\mathcal{S}_0$ of positive outer measure. Normalize its time interval to $[0, 1]$.

Split a ball cover with $\sum_i r_i^{4-\sigma} \leq 1$ into dyadic scales, with unions U_k . Every eligible segment spends at least ck^{-2} time in some U_k . Let $\mathcal{S}_k$ consist of slopes for which such a segment exists. Countable outer subadditivity gives a scale also satisfying $|\mathcal{S}_k|^* \gtrsim k^{-2}$. For $\delta = 2^{-k}$, a finite maximal δ -separated set has cardinality

$$ck^{-2}\delta^{-3} \leq M \leq C\delta^{-3}.$$

Only now choose one successful witness for each selected slope. There is no measurable selection over all directions. Powers of k lost as $k \rightarrow \infty$ are subpower in $N_0 = \delta^{-1}$.

Theorem 2.17, finite selection · pp. 20–21

2. Control the density profile by smoothing and the plank estimate

Give the finite line family uniform weights and independently smooth every intercept and slope coordinate at scale $O(\delta)$. Selected mass is at least $N_0^{-o(1)}$, and the three-dimensional slope density is at most $N_0^{o(1)}$. Jointly homogenize hidden depth r and two-dimensional velocity-bin depth u , and put

$$n(r, u) = N_0^{-f(r, u)}, \quad g(r, u) = r - f(r, u).$$

The cover count, slope density, and three-dimensional weighted plank estimate give

$$g(1, 0) \geq \sigma, \quad g(0, 0) \leq 0, \quad g(r, u) \leq r + 2 - 2u.$$

The middle bound is the direct input from the three-dimensional paper. A cap on the weight of trajectories wholly contained in full-time planks controls average multiplicity for long marked sets. It supplies the starting point of the density profile, rather than replacing the original variable-scale cover by a single-scale volume conclusion.

Choose $R, S > 0$ with $R + 4S < \sigma/2$, and maximize $g(r, u) - Rr + Su$ on $[0, 1] \times [0, 4]$. A maximizer (r_0, u_0) has $r_0 > 0$, $u_0 < 4$, and $g(r_0, u_0) \geq \sigma/2$. Cropping to the associated spatial and velocity boxes and normalizing yields a cap-class problem of length $L = r_0$ with $d = 1 - R < 1$. The denominator introduced by conditioning the crop prior cancels exactly on return to original mass.

Theorem 2.17, (2.24)–(2.32) · pp. 21–23

3. Compare a chart's mass floor with the slope-volume ceiling

Let a terminal chart have time length q and physical spatial Jacobian J_{phys} . Returning its true floor to the original unnormalized prior gives, with $m(r_0, u_0) = N_0^{g(r_0, u_0)}$,

$$\frac{m_{\text{old},c}}{q} \geq N_0^{-o(1)} m(r_0, u_0) a_{r_0} J_{\text{phys}}.$$

The set Σ of slopes that can satisfy its geometric tests obeys

$$|\Sigma| \leq N_0^{o(1)} \frac{a_{r_0} J_{\text{phys}}}{q^3}.$$

Choose eligible intercepts semialgebraically with fixed complexity, $\mathbf{b} = \mathbf{b}(s)$. The Jacobian of $s \mapsto \mathbf{b}(s) + ts$ is $\det(D\mathbf{b}(s) + tI_3)$, a monic cubic in t . On good times of total length comparable to q , this Jacobian is bounded below by a constant times q^3 , and the hidden derivative also has a lower bound. The area formula and quadratic sublevel estimate then control the image volume. Semialgebraic complexity gives a uniform multiplicity bound.

Compare with the original slope-density ceiling and cancel $a_{r_0} J_{\text{phys}}$ to obtain $m(r_0, u_0) \leq N_0^{o(1)} q^{-3}$. Every true terminal system therefore has $h \geq \sigma/6$, hence $H(E) \geq \sigma/6 > 0$. The same estimate survives all permitted selections and subsequences, so it bounds the horizon infimum itself.

Theorem 2.17, (2.33)–(2.39) · pp. 23–25

4. Which papers supply which steps

Source and result

The Kakeya maximal conjecture in three dimensions, Lemma 3.3

Used at

Lemma 2.15, Proposition 2.16, Theorem 2.17

Role and required hypotheses

Positive weights and repeated indices; a full-time plank cap and at least $D^{1-\eta}$ marks per trajectory yield $\sum_i \omega_i \# J_i \lesssim AD^\varepsilon \# E_{\text{vis}}$. Applied after finite selection and smoothing.

Checking scope

Statement and application hypotheses compared. The four-dimensional paper cites “Lemma 2.3”; the matching statement is Lemma 3.3 in the pinned three-dimensional version. This does not independently verify the input's whole proof.

Source and result

Carbery–Valdimarsson, determinant multilinear Kakeya

Used at

§6.3, (6.10), Proposition 6.2

Role and required hypotheses

Converts independent-velocity contributions to a weighted volume estimate controlling physical support and parameter mass.

Checking scope

The external Theorem 1 was compared with p. 90 here: full-line tubes, nonnegative weights, equal wedge and ambient dimensions, and the radius-scaling factor r^n .

Source and result

Wang–Zahl, restricted-triple dot-product alternative

Used at

Theorem 5.5, Proposition 5.4

Role and required hypotheses

Under an upper covering condition, either restricted triple images expand or line concentration occurs. Scalar support exponent $d < 1$ excludes expansion.

Checking scope

External Definition 4.2 and the §8 restatement of Theorem 5.2 were compared with p. 76 here. All downstream calculations were not checked.

Source and result

Orponen–de Saxcé–Shmerkin, multiplicative convolution

Used at

Lemma 7.4, Proposition 7.6

Role and required hypotheses

Produces coefficients with small interval probabilities from the planar ratio law, then localizes retained original incidence edges to parallel strips.

Checking scope

External Theorem 1.5 was compared with pp. 112–113 here: normalization to $[-1, 1]$, $ns > 1$, and smoothed-energy conditions.

[OpenAI26a, Lemma 3.3, pp. 20–21]

[CV13, Theorem 1, p. 2]

[WZ25, Definition 4.2, p. 17; Theorem 5.2, §8 restatement, p. 49]

[OSS24, Theorem 1.5, p. 2]

This direct input is not a dependence on the entire three-dimensional maximal-function theorem. Planar rigidity additionally uses graph Balog–Szemerédi–Gowers and fixed-iterate sumset calculus; the slope-volume argument uses parameter-uniform semialgebraic choice and cell decomposition. Proofs of these foundational results lie outside this article's checking scope.

5. Return to the sources

- Theorem 1.1 · p. 3 — Quantification over arbitrary sets and the Hausdorff dimension convention.
- Theorem 2.17 · pp. 20–25; §10.8 · p. 168 — Variable-scale-cover reduction and the final closure of all three branches.
- Lemma 3.5 · p. 36; Proposition 4.3; Lemmas 4.1–4.5 · pp. 47–51 — Order of density, time, and narrowness optimization and composition.
- Proposition 4.11; Lemma 4.12 · pp. 61–62 — Conditions for returning local candidates to original coverage and finite lists.
- Theorem 6.1 · p. 86; Proposition 6.8; Corollary 6.9 · pp. 104–108 — Isotropic improvement, restoration of quadratic degree, and prior exclusion of the scalar model.
- Proposition 9.5 · pp. 143–149 — The iteration $u_j \rightarrow 2k$ and construction of actual endpoint labels.
- Proposition 10.1 · pp. 154–168 — Nested localization, cancellation in density ratios, and recovery at the original scale.

We checked the main theorem's hypotheses and conclusion, the arbitrary-set reduction in §2.6, optimization in §4.1, the recovery conditions in Proposition 4.11, the output statements of the three branches, closure of the iteration in Proposition 9.5, and the connection to the conclusion in §10.8. The three-dimensional plank input was compared with Lemma 3.3 of the pinned version; the citation-number discrepancy is recorded above. The external inputs of Carbery–Valdimarsson, Wang–Zahl, and Orponen–de Saxcé–Shmerkin were also compared with the formulations and application conditions here, within the scope in the table.

This does not independently verify the full scalar-sheet comparison, conditional higher-jet estimates, construction of the potential and restoration of quadratic degree, or every exceptional-set and diagonal loss. The article explains the manuscript's argument with those results as inputs; it does not certify the entire 175-page proof. The geometric consequences in §11 are not proof targets here, and their separate transfer theorems remain unexamined.

[Theorem 2.17](#) · pp. 20–25; [§10.8](#) · p. 168