

OpenAI Math Digest

CATALOGUE 068

Anticanonical nonvanishing under smooth semipositivity

Catalogue synthesis & 7 proof overviews

Independent reading guide · English edition

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AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 7 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`

Anticanonical nonvanishing under smooth semipositivity

The seven manuscripts, in official order, total 270 pages. They approach positive-power anticanonical nonvanishing through distinct results on invariant indices, bounded metrics on klt pairs, chosen-character transfer, metric descent, boundary correction and exact orders. Smooth semipositivity means the existence of a smooth semipositive Hermitian metric; the hypotheses cannot be replaced by nefness alone.

Official catalogue PDF, entry 068: [OpenAI Research Catalog · Catalogue 068 · p. 8](#). Each row links to its manuscript PDF. Catalogue number 068 differs from the internal positions 01–07 below.

Manuscripts in official order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 068 · 2026-10-12 checked source snapshot

#

01

Manuscript and overview

Anticanonical nonvanishing from smooth semipositivity →

Short name: Smooth semipositivity

September 26, 2026 · 27 pages

Original paper (GitHub) ↗

Principal claim

For smooth connected projective X with smoothly semipositive $-K_X$, some positive anticanonical power has a section. A no-forms finite-volume theorem also produces sections of a bounded semipositive target line with poles restricted to a boundary.

Theorems 1.1–1.2 · pp. 2–3 ↗

Role and available overview

Two-metric transfer and dimension induction prove ordinary nonvanishing; an invariant direct-image argument produces naturally invariant sections. The finite cover and norm also supply geometric inputs to other manuscripts.

Read the proof overview →

#

02

Manuscript and overview

Invariant anticanonical indices and conversion of twisted differentials →

Short name: Invariant anticanonical indices

September 26, 2026 · 18 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

The naturally linearized invariant anticanonical Euler index is polynomial on divisible exponents with its actual value at zero. Independently, smooth semipositivity removes a fixed pseudoeffective error from unbounded twists to produce an ordinary positive-power section.

[Theorems 1.1–1.2 · pp. 1–2; Corollary 5.2 · pp. 16–17](#) ↗

Role and available overview

Connects independent localization/deleted-face and bounded-metric/vanishing arguments.

Invariant forms descend to the finite cover before ordinary conversion is applied.

[Read the proof overview](#) →

#

03

[Manuscript and overview](#)

Bounded anticanonical metrics on klt pairs and torus quotients →

Short name: Bounded anticanonical metrics

September 26, 2026 · 20 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

A nef anti-log-canonical divisor on a projective $\mathbb{Q}$ -factorial klt pair with effective rational boundary has a nonzero Cartier multiple if its resolution has a bounded semipositive metric and nonzero Euler characteristic. Smooth RC varieties yield naturally invariant torus sections.

[Theorems 1.1–1.2 · p. 2; Corollary 4.1 · p. 18](#) ↗[Role and available overview](#)

A maximal base and minimal vertical coefficients produce bounded metrics. A stable torus quotient preserves natural linearization and extends quotient sections globally.

[Read the proof overview](#) →

#

04

Manuscript and overview

Cohomological transfer and equivariant anticanonical sections →

Short name: Cohomological transfer

September 26, 2026 · 43 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For any linearization P of a smoothly semipositive anticanonical line, unbounded invariant sections of $M+iP$ with $-M$ pseudoeffective give an invariant section of some positive mP . The natural index and monodromy yield ordinary smooth nonvanishing.

[Theorem 1.1; Corollary 1.2 · p. 2; Theorem 7.2 · pp. 17–19](#) ↗

Role and available overview

A common invariant ratio field, a continuous maximum metric and a singular degree-zero integral remove the error using a fixed, possibly nontrivial multiplier ideal. An independent harmonic mechanism proves higher-direct-image vanishing from a kernel condition.

[Read the proof overview](#) →

#

05

[Manuscript and overview](#)

Metric descent and rank-preserving contractions →

Short name: Metric descent

September 26, 2026 · 76 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Full adjoint rank one at degree zero gives an unweighted integrable anticanonical correction metric downstairs; also proves contractions preserving generic invariant sections and all divisible adjoint ranks, allowed-pole nonvanishing and natural invariant sections.

Theorems 2.1, 4.1, 6.1, 7.1 · pp. 7, 16, 34, 38 ↗

[Role and available overview](#)

Separates toroidal volume from two-metric transfer and controls exceptional correction support, ratios after contraction and boundary poles.

[Read the proof overview](#) →

#

06

[Manuscript and overview](#)

Integrable metrics and effectivity with controlled boundary →

Short name: Controlled boundary

September 26, 2026 · 45 pages

[Original paper \(GitHub\)](#) ↗[Principal claim](#)

A strict singular metric on $-K_Y + C$ with integrable s_C makes every pseudoeffective M rationally effective after correction on C . For smooth semipositive $-K_X$, an unbounded section sequence permits removal of a fixed pseudoeffective error.

[Theorems 1.1, 4.3, 5.1, 9.2 · pp. 2, 9–11, 32–35](#) ↗[Role and available overview](#)

Weighted approximation controls correction support and returns sections from an Iitaka base; unweighted metrics separately supply finite generation of actual multigraded divisor rings.

[Read the proof overview](#) →

#

07

Manuscript and overview

Exact orders and invariant anticanonical linear systems →

Short name: Exact orders

September 26, 2026 · 41 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For smooth semipositive $L = -K_X$ and any fixed line bundle M , effective $N_j - m_j L + M$ in the curvature-null face with normalized centered discrete (or zero) valuation tending to zero give $N \geq 0$, $N \sim_Q L$, $v(N) = 0$. Vanishing higher $H^i(O_X)$ also yields natural invariant sections. Also treats invariant nonvanishing and fixed-error removal when $q = 0$, and a finite rational polyhedral cone containing controlled divisor classes in the RC case.

Theorems 1.1–1.2, 11.1; Propositions 10.1, 12.1 · pp. 3–4, 25, 27, 31 ↗

Role and available overview

Combines the null-polar field, toroidal volume and finite generation; actual boundary-ray generators turn asymptotic order into exact attainment.

[Read the proof overview](#) →

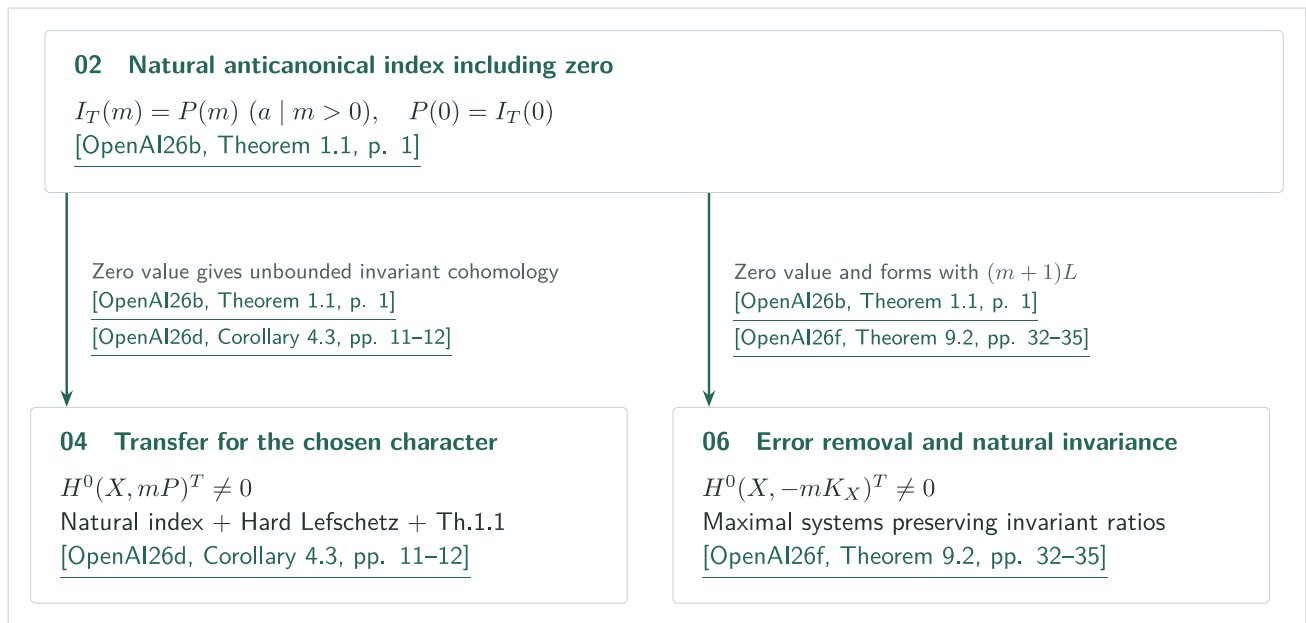
Sources and citation keys

The text and diagrams use the author-derived keys below. All seven are the September 26, 2026 manuscripts at a fixed official-repository commit. GitHub PDF viewers need not jump to the cited page. Within each article, unprefix results belong to that manuscript and keys identify external inputs.

- [OpenAI26a](#) OpenAI — Anticanonical nonvanishing from smooth semipositivity. Two-metric transfer and dimension induction prove ordinary nonvanishing; an invariant direct-image argument produces naturally invariant sections. The finite cover and norm also supply geometric inputs to other manuscripts. September 26, 2026 · 27 pages.

- [OpenAI26b](#) OpenAI — Invariant anticanonical indices and conversion of twisted differentials. Connects independent localization/deleted-face and bounded-metric/vanishing arguments. Invariant forms descend to the finite cover before ordinary conversion is applied. September 26, 2026 · 18 pages.
- [OpenAI26c](#) OpenAI — Bounded anticanonical metrics on klt pairs and torus quotients. A maximal base and minimal vertical coefficients produce bounded metrics. A stable torus quotient preserves natural linearization and extends quotient sections globally. September 26, 2026 · 20 pages.
- [OpenAI26d](#) OpenAI — Cohomological transfer and equivariant anticanonical sections. A common invariant ratio field, a continuous maximum metric and a singular degree-zero integral remove the error using a fixed, possibly nontrivial multiplier ideal. September 26, 2026 · 43 pages.
- [OpenAI26e](#) OpenAI — Metric descent and rank preserving contractions. Separates toroidal volume from two-metric transfer and controls exceptional correction support, ratios after contraction and boundary poles. September 26, 2026 · 76 pages.
- [OpenAI26f](#) OpenAI — Integrable metrics and effectivity with controlled boundary. Weighted approximation controls correction support and returns sections from an Itaka base; unweighted metrics separately supply finite generation of actual multigraded divisor rings. September 26, 2026 · 45 pages.
- [OpenAI26g](#) OpenAI — Exact orders and invariant anticanonical linear systems. Combines the null-polar field, toroidal volume and finite generation; actual boundary-ray generators turn asymptotic order into exact attainment. September 26, 2026 · 41 pages.

Branching uses of the invariant index



The natural index and two downstream uses

[Open figure ↗](#) [TeX source ↓](#)

For the natural anticanonical action, the index theorem in 02 asserts polynomiality on **every positive multiple** of one integer, with the actual invariant index at zero as constant term. The index theorem itself assumes neither projectivity nor positivity. Obtaining sections in 04 and 06 uses additional conditions: a nonzero index at zero, smooth semipositivity, hard Lefschetz and subsequent arguments. [OpenAI26b, Theorem 1.1, p. 1]

04 first proves transfer for a chosen linearization, then sends naturally invariant cohomology through twisted forms and determinants to obtain natural invariant sections as a consequence. 06 uses the same index input with maximal systems preserving the invariant ratio field. Ordinary error removal in 02 does not by itself assert an invariant section. [OpenAI26d, Proposition 4.1; Corollary 4.3, pp. 10–12]; [OpenAI26f, Proposition 9.1; Theorem 9.2, pp. 32–35]

Metric descent and finite generation meet at exact attainment

05 Volume metric from full rank at zero

$$-K_Y + A_0, \quad e^{-\Phi} \in L_{\text{loc}}^1$$

$$p_* f^* A_0 = 0$$

[OpenAI26e, Theorem 2.1, p. 7]

06 Finite generation in two directions

$$R(Y; k_0 P(t_0), k_0 P(t_*))$$

Unweighted integrability and a strict open set

[OpenAI26f, Theorem 4.3, pp. 9–10]

Full rank one at zero and a horizontal strict patch in 07

[OpenAI26e, Theorem 2.1, p. 7]

[OpenAI26g, §§3.2–4, pp. 10–12]

Common correction and denominator give actual generators

[OpenAI26f, Theorem 4.3, pp. 9–10]

[OpenAI26g, §§3.2–4, pp. 10–12]

07 From asymptotic order to exact order zero

$$N_j \sim m_j L + M, \quad v(N_j)/m_j \rightarrow 0$$

$$\implies N \geq 0, \quad N \sim_{\mathbb{Q}} L, \quad v(N) = 0$$

[OpenAI26g, Theorems 1.1–1.2, pp. 3–4]

Unweighted metrics and finite generation give exact order zero

Open figure ↗ TeX source ↓

The main theorem of 07 starts with effective $N_j \sim m_j L + M$ in the curvature-null face, where M is fixed and m_j tends to infinity. If $v(N_j)/m_j$ tends to zero for a centered discrete valuation, including the zero valuation, it obtains $N \geq 0$, $N \sim_{\mathbb{Q}} L$, $v(N) = 0$. Here M is arbitrary; pseudoeffectivity of $-M$ belongs to the separate formal-arc corollary. [OpenAI26g, Theorems 1.1–1.2, pp. 3–4]

07 constructs **full direct-image rank one at degree zero** and a horizontal strict patch, then applies the toroidal volume theorem of 05. This supplies an **unweighted** integrable metric on $-K_Y + A_0$ and an exceptional correction. The finite-generation theorem of 06 is applied to two rational directions with a common correction and denominator. An actual boundary-ray generator in the bigraded divisor ring is selected. The correction is pushed down before evaluation at the valuation center; its own valuation need not vanish. [OpenAI26e, Theorem 2.1, p. 7]; [OpenAI26f, Theorem 4.3, pp. 9–10]; [OpenAI26g, §§3.2–4, pp. 10–12]

Metric descent from 05 is also used in the **alternative route** of 06 §5.3, separate from its main weighted support-control route. Weighted and unweighted integrability, strict positivity on an ordinary open set, and a global Kähler lower bound are distinct premises. [OpenAI26f, §5.3, pp. 13–15]; [OpenAI26f, Theorem 1.1, p. 2]

Result-level connections and independent routes

Input result

Anticanonical nonvanishing from smooth semipositivity · [OpenAI26a, Theorem 2.1; Lemma 2.2, pp. 5, 8]

Recipient and use

Invariant anticanonical indices and conversion of twisted differentials · [OpenAI26b, Proposition 5.1; Corollary 5.2, pp. 16–17]

Type and role

Input to consequences. Descend invariant forms to a finite cover, apply ordinary conversion, then norm; the main nonvanishing theorem of 01 is not used.

Input result

Anticanonical nonvanishing from smooth semipositivity · [OpenAI26a, Theorem 2.1; Lemma 2.2, pp. 5, 8]

Recipient and use

Bounded anticanonical metrics on klt pairs and torus quotients · [OpenAI26c, Corollary 4.1, p. 18]

Type and role

Input to a consequence. Residual compact torus and invariant canonical frames descend the section of Th1.2, then finite étale norm applies.

Input result

Anticanonical nonvanishing from smooth semipositivity · [OpenAI26a, Theorem 2.1; Lemma 2.2, pp. 5, 8]

Recipient and use

Cohomological transfer and equivariant anticanonical sections · [OpenAI26d, §14.1, p. 39]

Type and role

Input to a consequence. Return the natural invariant section of Cor4.3 to general X , without assuming a product decomposition of the finite cover.

Input result

Invariant anticanonical indices and conversion of twisted differentials · [OpenAI26b, Theorem 1.1, p. 1]

Recipient and use

Cohomological transfer and equivariant anticanonical sections · [OpenAI26d, Proposition 4.1; Corollary 4.3, pp. 10–12]

Type and role

Input to a consequence. Actual index at zero gives unbounded cohomology for transfer by Th1.1.

Input result

Invariant anticanonical indices and conversion of twisted differentials · [OpenAI26b, Theorem 1.1, p. 1]

Recipient and use

Integrable metrics and effectivity with controlled boundary · [OpenAI26f, Proposition 9.1; Theorem 9.2, pp. 32–35]

Type and role

Direct input. Produce invariant forms retaining the natural linearization and $(m+1)L$ shift.

Input result

Metric descent and rank-preserving contractions · [OpenAI26e, Theorem 2.1, p. 7]

Recipient and use

Integrable metrics and effectivity with controlled boundary · [OpenAI26f, §5.3, pp. 13–15]

Type and role

Alternative route. Unweighted toroidal metric in §5.3, separate from the main weighted route in §5.2.

Input result

Metric descent and rank-preserving contractions · [OpenAI26e, Theorem 2.1, p. 7]

Recipient and use

Exact orders and invariant anticanonical linear systems · [OpenAI26g, §§3.2–4, pp. 10–12]

Type and role

Direct input. Supply the metric and exceptional correction from full rank at zero and a horizontal strict patch.

Input result

Integrable metrics and effectivity with controlled boundary · [OpenAI26f, Theorem 4.3, pp. 9–10]

Recipient and use

Exact orders and invariant anticanonical linear systems · [OpenAI26g, §§3.2–4, pp. 10–12]

Type and role

Direct input. Use common ε and k_0 for finite generation of the actual two-direction divisor ring.

Input result

Integrable metrics and effectivity with controlled boundary · [OpenAI26f, Theorem 2.2, pp. 6–7]

Recipient and use

Exact orders and invariant anticanonical linear systems · [OpenAI26g, Proposition 12.1; §12.2, pp. 31, 34]

Type and role

Direct input to an additional result. Weighted approximation gives a klt boundary and a removable correction for the finite polyhedral cone of controlled classes, distinct from the finite-generation input to Th1.1.

04 also supplies an independent harmonic mechanism: curvature domination from the base and a vanishing regularization error imply vanishing of higher base cohomology. [OpenAI26d, Theorem 7.2, pp. 17–19] Additional outputs in 07 are invariant nonvanishing under $q = 0$, $\chi \neq 0$, a faithful action and an equivariant very ample bundle, rather than vanishing of every higher $H^i(\mathcal{O})$, invariant fixed-error removal under $q = 0$, and a finite polyhedral cone of controlled classes in the RC case. Their distinct hypotheses are stated in the article. [OpenAI26g, Proposition 10.1; Theorem 11.1, pp. 25, 27]; [OpenAI26g, Proposition 12.1; §12.2, pp. 31, 34]

The main nonvanishing theorem of 01 must be distinguished from its structure theorem and norm. 01 supplies the finite-volume, two-metric route; 03 supplies a bounded-metric and Euler-polynomial route on klt pairs. No use of the main theorem of 03 as an input to transfer in 04 was found in the core passages read. Sharing the smooth nonvanishing conclusion is a comparison, not a proof-dependency arrow. [OpenAI26a, Theorems 1.1–1.2, pp. 2–3]; [OpenAI26c, Theorems 1.1–1.2, p. 2]; [OpenAI26d, Theorem 1.1; Corollary 1.2, p. 2]

The diagrams select two verified patterns of branching and convergence. Structural inputs from 01 and the alternative route remain in the table; the seven manuscripts do not form one dependency chain or an exclusive classification. Unexamined connections are not drawn.

Originals and checking scope

The official overview.pdf, p.8, uses this heading; CONTENTS.md calls the same entry 068 Anticanonical nonvanishing in every dimension. Its official subject is Algebraic and complex geometry. The inventory records the fixed version, SHA-256 hashes, page counts and duplicate check.

Each article is a reading guide based on checking hypotheses and conclusions, selected core proof passages, and the statements and uses of direct external inputs. Authors' theorems are distinguished from editorial checking. A second author's content check is neither an independent verification of every proof nor specialist peer review. Proofs of external inputs have not been recursively verified.

Each article records its read and unread portions. In particular, auxiliary and alternative routes in 04 §§5–13 apart from Th7.2, later sections of 05 §§8–12, alternative routes of 06 §§6–8/10–12, and auxiliary and alternative routes of 07 §§8–14 beyond the added Prop10.1, Th11.1 and Prop12.1 have not received a complete detailed check. The proposed use of 01 in the compact-monodromy descent of 07 is excluded from the verified table.

This is AI-generated exposition; consult the linked originals for important statements, conditions and proofs.

Anticanonical nonvanishing from smooth semipositivity

Finite-adjoint-volume induction and naturally invariant sections

This article follows OpenAI, Anticanonical nonvanishing from smooth semipositivity (September 26, 2026). It separates ordinary section production from the invariance needed for descent. Smooth semipositivity is stronger than nefness; smoothness is also used in the invariant direct-image argument.

Manuscript author: OpenAI · September 26, 2026 · 27 pages, including references

[Read the manuscript on GitHub ↗](#)

Main results

Theorem 1.1 — Anticanonical nonvanishing

Let X be a smooth connected projective complex variety. If $-K_X$ admits a smooth Hermitian metric of semipositive curvature, then for some integer $m > 0$,

$$H^0(X, -mK_X) \neq 0.$$

The assertion neither gives a first-power section nor a dimension-dependent uniform exponent.

[Theorem 1.1 · p. 2](#)

Theorem 1.2 — Finite volume and permitted poles

Let Z be a smooth connected projective complex variety of dimension d , with $H^0(Z, \Omega_Z^q) = 0$ for every $1 \leq q \leq d$. Let $D \geq 0$ be an integral divisor. Suppose $J = -K_Z + D$ has a semipositive singular Hermitian metric h_J such that, for the canonical section $s_D \in H^0(Z, K_Z + J)$,

$$\int_Z |s_D|_{h_J}^2 < \infty.$$

If a line bundle L has a semipositive singular Hermitian metric with locally bounded weights, there are integers $a > 0, b \geq 0$ with

$$H^0(Z, aL + bD) \neq 0.$$

The weights of h_J need not be bounded; those of h_L must be locally bounded on both sides. The conclusion gives a rational section of a positive power of L with poles only on D , without a uniform bound on b .

Theorem 1.2 · p. 3

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Anticanonical nonvanishing from smooth semipositivity](#). External inputs use the following keys.

- [BP08](#) Bo Berndtsson; Mihai Păun — Bergman kernels and the pseudoeffectivity of relative canonical bundles: Relative Bergman metric with singular coefficients, in Lemma 4.1 and Proposition 5.1. [arXiv:math/0703344v4](#), 2008-01-21.
- [Ber09](#) Bo Berndtsson — Curvature of vector bundles associated to holomorphic fibrations: Smooth direct-image L^2 metric; used on the invariant orthogonal summand. [arXiv:math/0511225v2](#), 2007-08-20; published 2009.
- [DPS01](#) Jean-Pierre Demailly; Thomas Peternell; Michael Schneider — Pseudo-effective line bundles on compact Kähler manifolds: Hard Lefschetz with multiplier ideals produces twisted forms. [arXiv:math/0006205v2](#), 2000-10-10; published 2001.
- [CP19](#) Frédéric Campana; Mihai Păun — Foliations with positive slopes and birational stability of orbifold cotangent bundles: Algebraic integrability of foliations of positive minimum movable slope. Published version, 2019, 49 pages.
- [DPS96](#) Jean-Pierre Demailly; Thomas Peternell; Michael Schneider — Compact Kähler manifolds with hermitian semipositive anticanonical bundle: Isometric universal-cover splitting and finite-cover structure; author-manuscript pagination differs from published p.218. Undated author manuscript, 8 pages; published 1996.
- [CDP](#) Frédéric Campana; Jean-Pierre Demailly; Thomas Peternell — Rationally connected manifolds and semipositivity of the Ricci curvature: Structure theorem including rational connectedness of residual factors. [arXiv:1210.2092v3](#), 2018-02-05.
- [GKP](#) Daniel Greb; Stefan Kebekus; Thomas Peternell — Movable curves and semistable sheaves: HN filtrations and tensor slopes for real boundary movable classes. [arXiv:1408.4308v2](#), 2015-04-15.

- **BDPP** Sébastien Boucksom; Jean-Pierre Demailly; Mihai Păun; Thomas Peternell — The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension: Duality of pseudoeffective divisors and movable curves. arXiv:math/0405285v1, 2004.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1 · p. 2](#)

Proof route. Split the universal cover after a finite cover while retaining the torus monodromy on the residual factor. Theorem 1.2 supplies ordinary nonvanishing; Theorem 7.2 supplies a naturally invariant section. A finite étale norm finishes the descent.



The full reduction for the main theorem

[Open figure ↗](#) [TeX source ↓](#)

1. The cover retains monodromy

The structure theorem gives a finite étale cover $S \rightarrow X$ with universal cover $\mathbf{C}^a \times F \times Z$. The Ricci-flat factor F has a parallel nowhere-zero canonical frame; Z has no positive-degree holomorphic forms and smoothly semipositive $-K_Z$. The manuscript passes to a finite-index deck subgroup, kills its action on F , and makes the closure of its action on Z a connected compact torus T . It does not assert that S itself is a product.

Theorem 2.1 · pp. 5–8 [DPS96, Structure Theorem, p. 1] [CDP, Theorem 1.4, p. 3]

2. From finite volume to a naturally invariant section

On the no-forms factor, $D = 0, J = L = -K_Z$ satisfies the finite-volume and bounded-metric assumptions. An ordinary section, however, need not descend under the deck action. Theorem 7.2 takes a rational torus quotient and uses rank one of the invariant adjoint direct image to transfer two metrics to the base. Theorem 1.2 applies on that base, and its section lifts invariantly.

Theorem 1.2 · p. 3 Theorem 7.2 · pp. 24–25

3. Descend using invariance of the natural action

Multiply $s_Z \in H^0(Z, -mK_Z)^T \setminus \{0\}$ by the inverse m th powers of the invariant canonical frames on $\mathbf{C}^a$ and F . The product is deck-invariant, hence descends to a section of $-mK_S$. Its norm, for $e = \deg(S/X)$, gives $H^0(X, -meK_X) \neq 0$. If Z is a point, take $m = 1, s_Z = 1$. The argument requires some positive exponent, without uniform estimates for the cover degree or exponent.

Lemma 2.2; §7.3 · pp. 8, 25–26

Theorem 1.2 — Induction through a smaller base

Proof target : Theorem 1.2 · p. 3

The statement is given in the [main results](#). Assume that every $aL + bD$, with $a > 0, b \geq 0$, has no section.



Ordinary nonvanishing by induction

[Open figure ↗](#) [TeX source ↓](#)

1. Finite volume controls the tangent slope on a supporting face

Since $L + D$ is pseudoeffective but not big, cone duality supplies a nonzero movable class α with $L \cdot \alpha = D \cdot \alpha = 0$. Finite volume enters the relative Bergman metric in Lemma 4.1. Near a multiple fiber $s_1 = z_1^e$, the relative-form coefficient contains $z_1^{-(e-1)}$. A local upper bound after subtracting $(e-1) \log |z_1|^2$ from the Bergman weight extends the metric to the determinant of the saturated conormal sheaf. The positive HN part is a foliation, and positive-slope algebraicity then yields $\mu_{\min, \alpha}(T_Z) \geq -D \cdot \alpha = 0$.

Lemma 4.1; Theorem 4.2; Corollary 4.3 · pp. 11–15 [\[BP08, Theorem 0.1, p. 2\]](#) [\[CP19, Theorem 1.1, p. 2\]](#)
[\[GKP, Corollary 2.27; Theorem 4.2, pp. 8, 12\]](#) [\[BDPP, Theorems 2.2, 2.4, p. 6\]](#)

2. One determinant and a base controlling every section ratio

The no-forms assumption gives $\chi(\mathcal{O}_Z) = 1$. The Riemann–Roch polynomial $\chi(K_Z + mL)$ has constant term $(-1)^d$, hence is nonzero for large m . Bounded weights imply $\mathcal{I}(h_L^m) = \mathcal{O}_Z$, so Hard Lefschetz produces infinitely many maps $-mL \rightarrow \Omega_Z^p$ for one fixed p . Stabilize their generic tail span and take its determinant M . Then $M \cdot \alpha \leq 0$ and there are $s_i \in H^0(Z, M + m_i L)$ with $m_i \rightarrow \infty$.

Every complete system $|uL + vM + wD|$, for $u, v, w \geq 0$, has nonpositive degree against α and cannot be generically finite. Choose one with maximal image dimension, and take the relative algebraic closure of its image field in $\mathbf{C}(Z)$ to obtain $\dim S < \dim Z$. Products of sections and maximality put every ratio from these systems in $\mathbf{C}(S)$. The interpolation identity (6.5), together with $m_i \rightarrow \infty$, shows that $\rho = s_2/s_1$ has nonnegative order at horizontal divisors. The same ratio property, after clearing common vertical denominators, gives adjoint rank one for every $j \geq 0$, including $j = 0$.

Lemma 6.1 · pp. 19–20 [DPS01, Theorem 2.1.1, p. 8] Lemma 6.2 · pp. 20–22

3. Transfer two metrics and lift only permitted poles

Proposition 5.1 supplies a bounded semipositive line A and a finite-volume metric on $-K_S + D_S$. Pullback of holomorphic forms is injective, so S also has no positive-degree forms. Induction gives a section of $aA + bD_S$, which lifts to a nonzero section of $akL + cD$, contradicting the assumption. If the base is a point, the absence of horizontal poles already makes ρ a section. The dimension-zero starting case is $a = 1, b = 0$.

Proposition 5.1 · pp. 15–19 §6.3 · pp. 22–23

Proposition 5.1 — Separate the zeroth and high moments

Proof target : Proposition 5.1 · pp. 15–19

Proposition 5.1 — Two-metric transfer

Let X, S, Y be smooth connected projective complex varieties and $\bar{Y}$ normal projective. Let $r : Y \rightarrow \bar{Y}$ and $\bar{\pi} : \bar{Y} \rightarrow X$ be projective birational morphisms, and $\bar{f} : \bar{Y} \rightarrow S$ surjective equidimensional with connected general fiber. Put $\pi = \bar{\pi}r$, $f = \bar{f}r$. Assume the metric conditions on D, J, h_J, L, h_L from Theorem 1.2; the no-forms hypothesis is not required here. Suppose $k > 0$ and a nonzero rational section ρ of $k\bar{\pi}^*L$ have no horizontal poles and, over $\eta = \operatorname{Spec} \mathbf{C}(S)$, for every integer $j \geq 0$,

$$\dim_{\mathbf{C}(S)} H^0(Y_\eta, (K_{Y/S} + \pi^*(J + jkL))|_{Y_\eta}) = 1.$$

After replacing k, ρ by a common positive power, there are a line bundle A , an effective integral divisor D_S on S , and a nonzero regular section $e \in H^0(\bar{Y}, k\bar{\pi}^*L - \bar{f}^*A)$. The line A has a bounded semipositive metric, and $-K_S + D_S$ a semipositive metric of finite adjoint volume. The support of D_S consists only of primes T for which every component of $\bar{f}^*T$ is $\bar{\pi}$ -exceptional or contained in $\operatorname{Supp} \bar{\pi}^*D$. If $a > 0, b \geq 0$ and $H^0(S, aA + bD_S) \neq 0$, then $H^0(X, akL + cD) \neq 0$ for some $c \geq 0$.

Proposition 5.1 · pp. 15–19



Two-metric transfer

[Open figure ↗](#) [TeX source ↓](#)

1. Normalize vertical orders on the equidimensional normal model

Clear denominators in $c_T = \min_{\bar{f}(Q)=T} \text{ord}_Q(\rho) / \text{ord}_Q(\bar{f}^* T)$ and put $A = \mathcal{O}_S(\sum c_T T)$. Dividing ρ by the pullback of its tautological rational section gives e with nonnegative order at every prime divisor. Normality makes it regular. Above each T , at least one component has zero order in e . Equidimensionality is required of the normal model, not of the smooth resolution used for integration.

[Proposition 5.1 · pp. 15–19](#)

2. Zeroth moments carry volume; high moments carry boundedness

Using $\eta_Y = \pi^* s_D$, including its Jacobian, and local frames ds, v , set

$$V_j(s) = \int_{Y_s} |(\eta_Y / f^* ds) e_v^j|^2_{\pi^*(h_J h_L^{j_k})}, \quad \Phi_j = -\log V_j.$$

Rank one identifies a Bergman weight and gives psh representatives of Φ_j . For $j = 0$, a boundary correction D_S yields an adjoint measure equal to $f_*|\eta_Y|^2$, preserving finite volume. In contrast,

$$\Psi = \lim_{\ell \rightarrow \infty} \left(\sup_{j \geq \ell} \frac{\Phi_j}{j} \right)^*$$

is a weight on A . Boundedness of h_L gives $V_j \leq C^j V_0$ and hence a lower bound for Ψ . On a component where e does not vanish, $V_j \geq C_1 C_2^j |s_1|^{2b_T}$ has b_T independent of j . Division by j and passage to the limit remove the boundary term. These two estimates produce bounds on both sides.

§5, (5.3)–(5.10) · pp. 16–19 [BP08, Theorem 0.1, p. 2]

3. The support condition permits lifting

Pull back a base section and multiply by e^a . Poles occur only over bad primes, hence are exceptional over X or supported above D . Exceptional poles disappear under normal pushdown, and multiplication by s_D^c clears the remaining ones. No new forbidden polar divisor survives on X .

Proposition 5.1 · pp. 15–19

Theorem 7.2 — Nonvanishing for the natural linearization

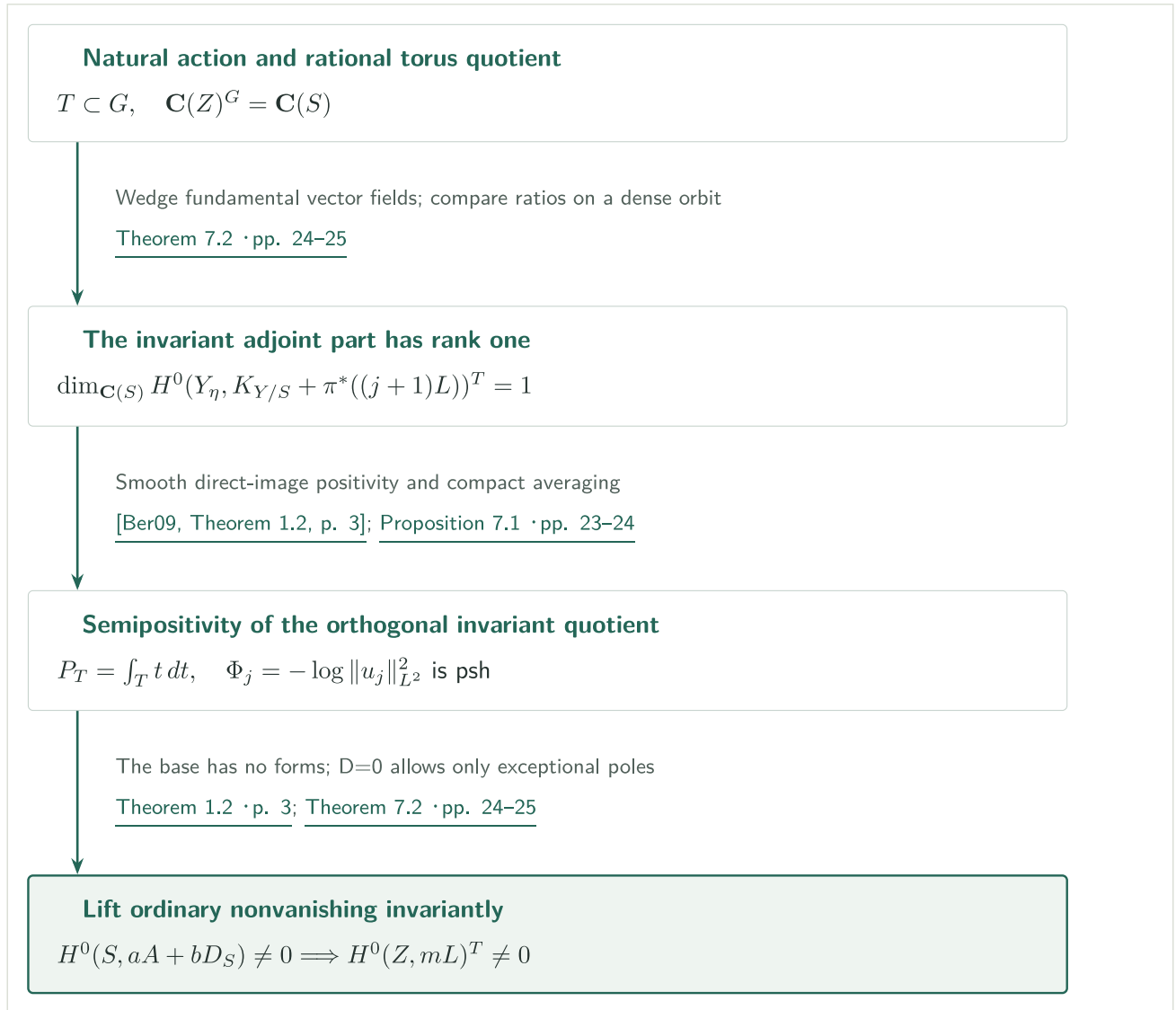
Proof target : Theorem 7.2 · pp. 24–25

Theorem 7.2 — Anticanonical sections fixed by a compact torus

Let Z be smooth connected projective over $\mathbf{C}$, with $H^0(Z, \Omega_Z^q) = 0$ for every $q > 0$.

Suppose $L = -K_Z$ has a smooth semipositive Hermitian metric. Let a compact connected real torus T act continuously by holomorphic automorphisms, and equip L with the natural linearization induced by the differential. Then $H^0(Z, mL)^T \neq 0$ for some integer $m > 0$.

Theorem 7.2 · pp. 24–25



Transfer producing invariant sections

[Open figure ↗](#) [TeX source ↓](#)

1. A rational quotient gives rank one only in the invariant part

The vanishing $H^1(\mathcal{O}_Z) = 0$ extends T to an algebraic torus G and identifies T -fixed and G -fixed sections for the natural action. A general fiber of the Rosenlicht quotient has a dense orbit.

Commutativity makes the fundamental vector fields invariant; their wedge, combined with a base anticanonical frame, gives an invariant ρ without horizontal poles. Ratios of invariant adjoint sections belong to $\mathbf{C}(Z)^G = \mathbf{C}(S)$, so the invariant subspace has dimension one for every $j \geq 0$. The full adjoint space need not have rank one.

Theorem 7.2 · pp. 24–25

2. Positivity comes from an orthogonal quotient of the smooth direct image

Average the smooth semipositive metrics to make them T -invariant. Berndtsson's theorem gives the adjoint direct image its semipositive L^2 metric on the smooth locus. Compact averaging is a holomorphic orthogonal projection onto the invariant line, whose metric is therefore a semipositive quotient metric. The argument does not assume positivity of an arbitrary subbundle. This use of smooth coefficients distinguishes Proposition 7.1 from ordinary Proposition 5.1, which allows singular coefficients.

Proposition 7.1 · pp. 23–24 [Ber09, Theorem 1.2, p. 3]

3. Apply the ordinary theorem downstairs and lift invariantly

Proposition 7.1 uses the boundary estimates of Proposition 5.1 to construct both base metrics. The no-forms condition descends, and Theorem 1.2 gives a section of $aA + bD_S$. The base action is trivial and e is invariant, so the lift remains invariant. Since $D = 0$, every permitted pole is exceptional and disappears on pushdown. If the general orbit has dimension zero, the action is trivial and Theorem 1.2 applies directly; the point case is immediate as well.

Proposition 7.1 · pp. 23–24 Theorem 7.2 · pp. 24–25

Dependencies and catalogue role

This manuscript proves both ordinary and naturally invariant nonvanishing internally, without using another manuscript from catalogue 068 as a direct input. Uses by later manuscripts are distinguished in the catalogue after checking the receiving hypotheses. DPS96/CDP supply structure, BDPP/GKP/CP19 slope theory, DPS01 twisted forms, and BP08 and Ber09 two different positivity mechanisms. Neither nefness alone nor an integrable singular metric alone is substituted for the assumptions of Theorem 7.2.

Return to the sources

- Theorem 2.1 · pp. 5–8 — Finite cover retaining monodromy.
- Lemma 4.1; Theorem 4.2; Corollary 4.3 · pp. 11–15 — Finite-volume cotangent slope bound.
- Proposition 5.1 · pp. 15–19 — Separate uses of zeroth and high moments.
- Lemma 6.2 · pp. 20–22 — Maximal systems give no horizontal poles and rank one in every degree.

- [Proposition 7.1](#) · pp. 23–24 — Invariant direct-image positivity requiring smooth coefficients.
- [Theorem 7.2](#) · pp. 24–25 — Construction of naturally invariant sections.

The editorial reading checked the principal statements, proof passages in §§2 and 4–7, and the indicated statements and uses of eight external sources. The model preparation in §3 was read, but the original sources for flattening, equivariant resolution, the Rosenlicht quotient and Yau's theorem were not separately checked in this draft. External proofs, every psh-envelope extension, and the common-denominator arguments were not independently reproved. This is neither expert peer review nor formal verification; the authors' assertions and this reading scope remain distinct.

Invariant anticanonical indices and conversion of twisted differentials

Two endpoint arguments connected by descent of forms

OpenAI, Invariant anticanonical indices and conversion of twisted differentials (September 26, 2026), proves independently an invariant Euler-index polynomial and an ordinary section theorem removing a fixed pseudoeffective error. In the application, torus-invariant forms descend to a finite cover before conversion. An equivariant conversion theorem is therefore unnecessary.

Manuscript author: OpenAI · September 26, 2026 · 18 pages, including references

[Read the manuscript on GitHub ↗](#)

Main results

Theorem 1.1 — Invariant anticanonical index

Let a compact real torus T act holomorphically on a compact complex manifold F , and equip $L_F = -K_F = \det T_F$ with its natural differential linearization. Put

$$I_T(m) = \sum_{q \geq 0} (-1)^q \dim H^q(F, L_F^m)^T \quad (m \geq 0).$$

There are an integer $a > 0$ and $P \in \mathbf{Q}[t]$ such that

$$\begin{aligned} I_T(m) &= P(m) \quad (m \in a\mathbf{Z}_{>0}), \\ P(0) &= I_T(0). \end{aligned}$$

There is no projectivity, Kählerness or positivity assumption. The natural linearization cannot be replaced by an arbitrary character twist.

Theorem 1.1 · p. 1

Theorem 1.2 — Pseudoeffective-error conversion

Let S be smooth connected projective over $\mathbf{C}$, and suppose $L = -K_S$ has a smooth semipositive Hermitian metric. Let D be a pseudoeffective Cartier divisor. If, for a strictly increasing sequence of positive integers m_j ,

$$H^0(S, \mathcal{O}_S(m_j L - D)) \neq 0,$$

then $H^0(S, \mathcal{O}_S(kL)) \neq 0$ for some integer $k > 0$. The divisor D need not be effective; rational connectedness and an Euler-characteristic hypothesis are unnecessary. The output is an ordinary section.

Theorem 1.2 · p. 2

Corollary 5.2 — Smooth anticanonical nonvanishing

Let X be a smooth connected projective complex variety. If $-K_X$ has a smooth semipositive Hermitian metric, then $H^0(X, -mK_X) \neq 0$ for some integer $m > 0$. Neither a first-power section nor a uniform exponent is asserted.

Corollary 5.2 · pp. 16–17

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Invariant anticanonical indices and conversion of twisted differentials](#). External inputs use the following keys.

- [Ber09](#) Bo Berndtsson — Curvature of vector bundles associated to holomorphic fibrations: Semipositivity of the rank-one adjoint direct-image L^2 metric. arXiv:math/0511225v2, 2007-08-20; published 2009.
- [DPS01](#) Jean-Pierre Demailly; Thomas Peternell; Michael Schneider — Pseudo-effective line bundles on compact Kähler manifolds: Hard Lefschetz with smooth coefficients, producing invariant twisted forms. arXiv:math/0006205v2, 2000-10-10; published 2001.
- [Fuj16](#) Osamu Fujino — Kollár–Nadel type vanishing theorem: Vanishing from a singular metric positive in base directions. arXiv:1607.04684v1, 2016-07-15 (manuscript date 2016-07-10); published 2018.
- [DK01](#) Jean-Pierre Demailly; János Kollár — Semi-continuity of complex singularity exponents and Kähler–Einstein metrics on Fano orbifolds: Small-exponent integrability of psh weights makes the multiplier ideal trivial. arXiv:math/9910118v3, 2013-11-14.
- [ASeg68](#) Michael F. Atiyah; Graeme B. Segal — The index of elliptic operators. II: Lefschetz localization to fixed components for compact-group actions. *Annals of Mathematics* 87 (1968), 531–545; scanned copy with one cover page.

- [ASin68](#) Michael F. Atiyah; Isadore M. Singer — The index of elliptic operators. III: Cohomological fixed-point formula and Riemann–Roch for compact complex manifolds. *Annals of Mathematics* 87 (1968), 546–604; scanned copy with one cover page.
- [LMPTX23](#) Vladimir Lazić; Shin-ichi Matsumura; Thomas Peternell; Nikolaos Tsakanikas; Zhixin Xie — The Nonvanishing Problem for varieties with nef anticanonical bundle: Negative first Chern class of saturated cotangent tensor subsheaves is pseudoeffective. *Documenta Mathematica* 28 (2023), 1393–1440; published PDF.
- [OpenAI26a](#) OpenAI — Anticanonical nonvanishing from smooth semipositivity: Only the finite cover, residual torus, invariant canonical frames, and finite étale norm. September 26, 2026.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

Corollary 5.2 — Proof overview of the main theorem

Proof target : [Corollary 5.2](#) · pp. 16–17

Proof route. Theorems 1.1 and 1.2 are independent. The first produces invariant twisted forms on a compact factor. These descend to a finite cover, where the second converts them into ordinary anticanonical sections. A norm returns a section to X .



Connecting the two independent main results

[Open figure ↗](#) [TeX source ↓](#)

1. From invariant Euler index to invariant forms of one degree

Write the universal cover of a finite cover as $\tilde{S} = \mathbf{C}^b \times C \times F$, with C Ricci-flat and F a compact factor without positive-degree holomorphic forms. Residual monodromy acts through a compact torus T . Hodge symmetry gives $H^q(F, \mathcal{O}_F) = 0$ for $q > 0$, hence $I_T(0) = 1$. The polynomial in Theorem 1.1 is nonzero, so for one fixed q there are invariant classes in $H^q(F, L_F^m)^T$ at unbounded exponents. Hard Lefschetz with coefficient L_F^{m+1} , followed by compact averaging, gives $H^0(F, \Omega_F^p \otimes L_F^{m+1})^T \neq 0$ for one fixed p .

[\[OpenAI26a, Theorem 2.1; Lemma 2.2, pp. 5–8\]](#) [Theorem 2.4](#); [Corollary 2.5](#) · pp. 8–9

[\[DPS01, Theorem 2.1.1, p. 8\]](#)

2. Descend forms before constructing the pseudoeffective error

Multiplication by invariant canonical frames in Proposition 5.1 injects these forms into $H^0(S, \Omega_S^p \otimes (-K_S)^r)$. If $p > 0$, a fixed determinant of their generic span yields a line bundle M saturated in a cotangent tensor power. Nefness makes $-M$ pseudoeffective. Taking $D = -M$ turns the forms into effective divisors $N_j \sim d_j(-K_S) - D$ for unbounded d_j . If $p = 0$, the forms are already anticanonical sections.

Proposition 5.1 · p. 16 Theorem 4.1; Lemma 4.2; Corollary 4.3 · p. 15
[LMPTX23, Theorem 4.1, print p. 1409 / PDF p. 17]

3. Ordinary conversion and the finite étale norm

The anticanonical metric on S is the smooth semipositive pullback metric, so Theorem 1.2 applies. The norm for $d = \deg(S/X)$ sends its ordinary section of $-kK_S$ to a nonzero section of $-dkK_X$. No T -invariance of this section is required. The finite cover is never treated as a global product.

Theorem 1.2 · p. 2 Corollary 5.2 · pp. 16–17 [OpenAI26a, Theorem 2.1; Lemma 2.2, pp. 5–8]

Theorem 1.1 — Retaining the lattice-count endpoint

Proof target : Theorem 1.1 · p. 1

See the [main results](#) for the statement. Two combinatorial lemmas separately provide polynomiality and equality with the actual index at zero.

Natural anticanonical character

$$w = \sum_i \nu_i, \quad [t^0] \text{ in the fixed-component formula}$$

Fixed-component formula; truncate the Chern-root expansion

[\[ASeg68, Theorem 2.12, print p. 540 / PDF p. 11\]; §2.2, \(2.2\)–\(2.4\) · pp. 7–8](#)

Polarization and weighted counts

$$D = \{x_i \geq 0 : \sum \alpha_i x_i = -w\}, \quad A = \bigcup_{i \in I} (D \cap \{x_i = 0\})$$

The character sum places the exterior point u in the same affine space

[Lemmas 2.1–2.2 · pp. 4–6](#)

Retraction onto deleted faces

$$u_i = \pm 1, \quad r(x) = u + t(x)(x - u), \quad \chi(D) - \chi(A) = 0 \ (I \neq \emptyset)$$

Count every divisible exponent; inclusion-exclusion matches $b=0$ at $m=0$

[Lemmas 2.1–2.2 · pp. 4–6; §2.2, \(2.2\)–\(2.4\) · pp. 7–8](#)

Polynomial endpoint equals the actual index at zero

$$I_T(m) = P(m) \ (a \mid m > 0), \quad P(0) = I_T(0)$$

Localization and cancellation of deleted faces

[Open figure ↗](#) [TeX source ↓](#)

1. Fixed-component localization gives all divisible exponents

For a fixed component $Z \subset F^T$, let ν_i be the normal tangent characters and x_i the formal Chern roots. The natural anticanonical character is $w = \sum_i \nu_i$, and the contribution is

$$\int_Z \mathrm{td}(Z) e^{mc_1(L_F|_Z)} t^{mw} \prod_i (1 - t^{-\nu_i} e^{-x_i})^{-1}.$$

Polarize all denominators in one integral direction. The trivial-character coefficient becomes a weighted lattice count with $\sum_i \alpha_i b_i = -mw$, $b_i > 0$ for $i \in I$, and $b_i \geq 0$ otherwise; all α_i are positive under the same linear functional. Lemma 2.1 triangulates closed rational polytopes and uses a fundamental parallelepiped with binomial identity (2.1). This gives the polynomial at every divisible exponent, including small ones. Truncated cohomological degree makes the weights polynomial. Only the complete symmetric sum is interpreted as characteristic classes and integrated.

§2.2, (2.2)–(2.4) · pp. 7–8 Lemmas 2.1–2.2 · pp. 4–6 [ASeg68, Theorem 2.12, print p. 540 / PDF p. 11]
 [ASin68, Theorems 3.9, 4.3, print pp. 562–565 / PDF pp. 18–21]

2. The natural character supplies a retraction onto deleted faces

Set $D = \{x_i \geq 0 : \sum \alpha_i x_i = -w\}$ and $A = \bigcup_{i \in I} (D \cap \{x_i = 0\})$. If $I \neq \emptyset$ and $D \neq \emptyset$, the vector $u_i = -1$ for $i \in I$ and $u_i = 1$ otherwise lies in the same affine equation space. This is where $w = \sum \nu_i$ enters. With $t(x) = \max_{i \in I} (x_i + 1)^{-1}$, the map $r(x) = u + t(x)(x - u)$ retracts D onto A ; the segment homotopy fixes A . Thus $\chi(D) = \chi(A) = 1$, and inclusion–exclusion makes the polynomial constant term $(\chi(D) - \chi(A))W(0, 0) = 0$.

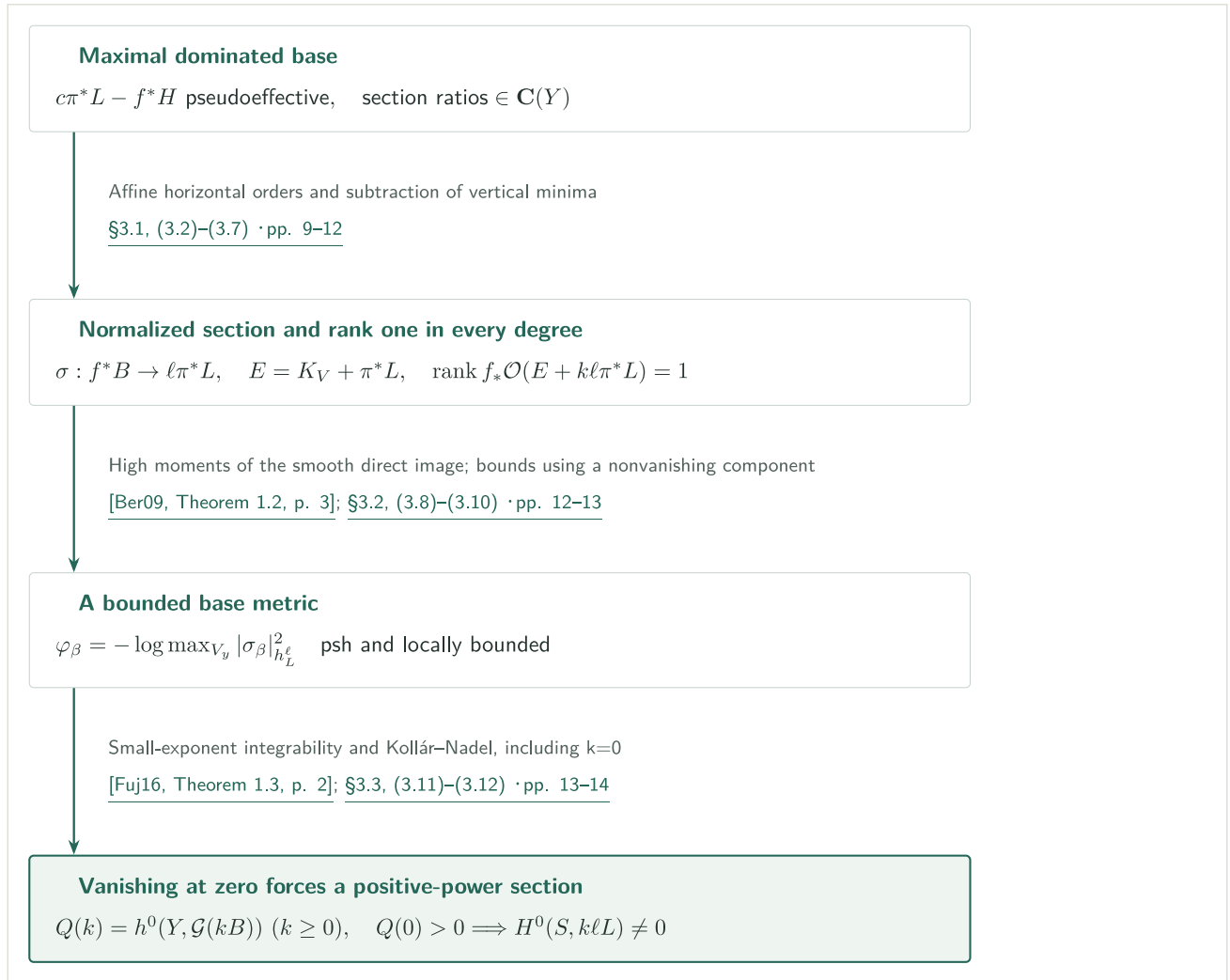
For $I = \emptyset$, the closed count retains $W(0, 0)$. At the actual exponent $m = 0$, positive pairing forces $b = 0$, allowed precisely when $I = \emptyset$. Hence $P(0) = I_T(0)$. If $F^T = \emptyset$, localization and character continuity give $I_T = 0$ at every exponent, so take $P = 0$. Trivial actions and zero-dimensional polytopes are included. The argument uses the difference of Euler characteristics of compact sets, not the ordinary Euler characteristic of the nonclosed complement $D \setminus A$.

Lemmas 2.1–2.2 · pp. 4–6 §2.2, (2.2)–(2.4) · pp. 7–8

Theorem 1.2 — A bounded base metric and vanishing at zero

Proof target : Theorem 1.2 · p. 2

See the [main results](#) for the statement. This proof is independent of the index theorem and uses the anticanonical identity twice to control the Euler polynomial at zero.



Removing the fixed pseudoeffective error

[Open figure ↗](#) [TeX source ↓](#)

1. A maximal dominated base and normalized vertical orders

Choose a rational map with largest base dimension among those satisfying $c\pi^*L - f^*H$ pseudoeffective on a resolution $\pi : V \rightarrow S$, with H ample on the base Y . Make its function field relatively algebraically closed as well. If $c'L - A$ is pseudoeffective, any ratio of two sections of A lies in $\mathbf{C}(Y)$: otherwise its pencil together with the chosen map would give a larger dominated base.

Equation (3.3) for $N_j \sim m_j L - D$ makes horizontal orders affine functions of m_j . Their nonnegativity and $m_j \rightarrow \infty$ imply horizontal effectivity of $N_2 - N_1$. On a normal model, subtract the minimum normalized vertical order above each base prime and clear denominators. This gives $\sigma : f^*B \rightarrow \ell\pi^*L$, with $\ell > 0$, nonvanishing on at least one component above every base prime. A point base already produces a section at this stage.

§3.1, (3.2)–(3.7) · pp. 9–12

2. Rank-one integrals converge to a bounded maximum metric

The anticanonical identity gives an effective exceptional divisor $E = K_V - \pi^*K_S = K_V + \pi^*L$ and its canonical section e . Maximality forces $f_*\mathcal{O}_V(E + k\ell\pi^*L)$ to have generic rank one for every $k \geq 0$. A larger rank would produce, after a sufficiently positive base twist, two sections with a ratio outside the base field, contradicting maximality after pushdown.

For a local frame β of B , define on the smooth-fiber locus

$$\varphi_\beta = -\log \max_{V_y} |\sigma_\beta|_{h_L^\ell}^2.$$

Berndtsson's theorem gives psh weights from $-1/k$ times the logarithm of the L^2 integral of $e\sigma_\beta^k/f^*\tau$. High moments against a positive full-support measure, together with Dini's theorem, converge locally uniformly to φ_β . Properness gives a lower bound for the weight. A local power-map chart at a component where σ is nonzero gives an upper bound. Extension across divisors and then codimension two produces a locally bounded semipositive metric h_B .

§3.2, (3.8)–(3.10) · pp. 12–13 [Ber09, Theorem 1.2, p. 3]

3. Vanishing including $k=0$ forces a nonzero polynomial

Pseudoeffective domination supplies a singular metric positive in base directions. Mix a small proportion of it with smooth h_L . Small-exponent integrability of psh weights gives

$\Theta_h(\pi^*L) \geq \varepsilon f^*\omega_H$ for some $\varepsilon > 0$, with $\mathcal{I}(h) = \mathcal{O}_V$. Boundedness of h_B means multiplication by $f^*h_B^k$ preserves that ideal for each fixed $k \geq 0$. Fujino's theorem with direct-image degree $q = 0$ yields

$$H^i(Y, \mathcal{G} \otimes \mathcal{O}_Y(kB)) = 0 \quad (i > 0, k \geq 0), \quad \mathcal{G} = f_*\mathcal{O}_V(E).$$

Consequently $Q(k) = \chi(Y, \mathcal{G}(kB)) = h^0(Y, \mathcal{G}(kB))$ at every nonnegative exponent, including zero. The section e gives $Q(0) > 0$, so this polynomial is nonzero and gives sections at positive exponents. Multiply one by σ^k and use $\pi_*\mathcal{O}_V(E) = \mathcal{O}_S$ to obtain a section of $k\ell L$. The identity $L = -K_S$ is used both in the rank-one adjoint expression and in the fixed sheaf at exponent zero.

§3.3, (3.11)–(3.12) · pp. 13–14 [DK01, §1.4(8), p. 7] [Fuj16, Theorem 1.3, p. 2]

Proposition 5.1 — Descent of forms before conversion

Proof target : Proposition 5.1 · p. 16

Proposition 5.1 — Descent of invariant twisted forms

Let $\nu : S \rightarrow X$ be the finite cover above, with $\tilde{S} = \mathbf{C}^b \times C \times F$. The deck group acts by translations on $\mathbf{C}^b$, trivially on C , and through a compact torus T on F . After choosing deck-invariant canonical frames on C and $\mathbf{C}^b$, there is, for every $r \geq 0$ and $0 \leq p \leq \dim F$, an injection

$$H^0(F, \Omega_F^p \otimes (-K_F)^r)^T \hookrightarrow H^0(S, \Omega_S^p \otimes (-K_S)^r).$$

For one fixed p , the target is nonzero at unbounded positive integers r .

Proposition 5.1 · p. 16

Invariant forms on the compact factor

$$u \in H^0(F, \Omega_F^p \otimes L_F^r)^T, \quad \tilde{S} = \mathbf{C}^b \times C \times F$$

Translations and the trivial action preserve canonical frames

[\[OpenAI26a, Theorem 2.1; Lemma 2.2, pp. 5–8\]](#)

A deck-invariant tensor

$$\eta^{-r} \otimes \text{pr}_F^* u \in H^0(\tilde{S}, \Omega_{\tilde{S}}^p \otimes K_{\tilde{S}}^{-r})$$

Natural actions agree on the differential and anticanonical factors

[Proposition 5.1 · p. 16](#)

Injection into forms on the finite cover

$$H^0(F, \Omega_F^p \otimes L_F^r)^T \hookrightarrow H^0(S, \Omega_S^p \otimes (-K_S)^r)$$

Use the Hard Lefschetz exponent shift and a saturated determinant

[Theorem 2.4; Corollary 2.5 · pp. 8–9; Theorem 4.1; Lemma 4.2; Corollary 4.3 · p. 15](#)

Retain unbounded exponents for conversion

$$r = m + 1 \longrightarrow \infty, \quad D = -M \text{ pseudoeffective}$$

Invariant-form descent and ordinary conversion

[Open figure ↗](#) [TeX source ↓](#)

1. Natural actions agree on both tensor factors

Let η be the product of the invariant canonical frames. Send an invariant form u to $\eta^{-r} \otimes \text{pr}_F^* u$. The natural actions on the differential and anticanonical factors agree with the deck action, so this tensor descends. Pullback of forms under the projection is injective, tensoring with the frame is invertible, and local covering charts preserve nonvanishing. Projectivity of $\mathcal{S}$ makes the descended holomorphic section algebraic.

Proposition 5.1 · p. 16 [OpenAI26a, Theorem 2.1; Lemma 2.2, pp. 5–8]

2. Keep the exponent shift and determinant saturation

Corollary 2.5, with $I_T(0) = 1$, uses exponent $m + 1$, so apply descent with $r = m + 1$. A point factor gives the same conclusion with $p = 0$. After descent, if $p > 0$, fix a basis of the generic span and replace one basis vector with forms at infinitely many exponents. Their exterior products have exponents m_j plus a fixed sum, still unbounded. The saturated line M in $\bigwedge^{r'} \Omega_S^p$ is reflexive and hence invertible. Its inclusion in a cotangent tensor power preserves saturation because the exterior power is a direct summand. LMPTX23 makes $-M$ pseudoeffective. Ordinary Theorem 1.2 now applies; it is not claimed to convert invariant forms directly into invariant anticanonical sections.

Theorem 2.4; Corollary 2.5 · pp. 8–9 Theorem 4.1; Lemma 4.2; Corollary 4.3 · p. 15
[LMPTX23, Theorem 4.1, print p. 1409 / PDF p. 17]

Dependencies and catalogue connections

From manuscript 01, Anticanonical nonvanishing from smooth semipositivity, this proof uses only the finite-cover Theorem 2.1 and norm Lemma 2.2. Its ordinary nonvanishing Theorem 1.2 and invariant nonvanishing Theorem 7.2 are not inputs to this manuscript's two main theorems. Corollary 5.2 gives another route to the same nonvanishing conclusion; shared geometric inputs must be distinguished from independence of the entire proof.

Return to the sources

- Lemmas 2.1–2.2 · pp. 4–6 — Counts retaining every divisible exponent and the endpoint
- §2.2, (2.2)–(2.4) · pp. 7–8 — Deleted faces determined by the natural anticanonical character
- §3.1, (3.2)–(3.7) · pp. 9–12 — Maximal base, horizontal orders and rank one in every degree

- §3.2, (3.8)–(3.10) · pp. 12–13 — Maximum metric and two-sided boundary bounds
- §3.3, (3.11)–(3.12) · pp. 13–14 — Vanishing including $k=0$ and the nonzero Euler polynomial
- Proposition 5.1 · p. 16 — Descent of invariant forms before conversion

The editorial reading checked the statements and proofs on pp.1–17; the original fixed-component formulas (scan print p.540/PDF p.11 and print pp.562–565/PDF pp.18–21); the indicated Berndtsson, Fujino, Demailly–Kollár, DPS and LMPTX input statements and uses; and manuscript 01's cover and norm. The original Hilbert-scheme model-construction source, every formal-root calculation and external proofs were not recursively verified. Weighted counting and boundary extension were traced rather than independently reproved. This is a reading guide to the authors' assertions, not expert peer review or formal verification.

Bounded anticanonical metrics on klt pairs and torus quotients

Transfer a bounded metric to the base and produce sections from the zero twist

The manuscript proves a bounded-metric nonvanishing criterion for klt pairs and obtains naturally invariant anticanonical sections through a torus quotient. Locally bounded psh weights need not be continuous. The variety upstairs in the quotient construction, however, is assumed to have a smooth semipositive metric. Neither premise is replaced by nefness alone.

Manuscript author: OpenAI · September 26, 2026 · 20 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Bounded-metric nonvanishing

Let (W, D) be a projective $\mathbb{Q}$ -factorial klt pair with effective rational boundary, and put $P = -(K_W + D)$. Assume that P is nef. On a smooth projective resolution $\rho: \widetilde{W} \rightarrow W$, suppose that ρ^*P admits a semipositive metric with locally bounded weights and that $\chi(\widetilde{W}, \mathcal{O}_{\widetilde{W}}) \neq 0$. Then for some integer $m > 0$ with mP Cartier,

$$H^0(W, \mathcal{O}_W(mP)) \neq 0.$$

[Theorem 1.1 · p. 2](#)

Theorem 1.2 — Invariant sections from a stable quotient

Let Z be a smooth rationally connected projective variety whose $-K_Z$ admits a smooth semipositive Hermitian metric. For every algebraic torus T acting on Z , there is an integer $m > 0$ such that

$$H^0(Z, -mK_Z)^T \neq 0.$$

Invariance refers to the natural linearization of $-K_Z = \det T_Z$ induced by the differential. The exponent may depend on the action; this is not a statement for every linearization or every positive exponent.

Theorem 1.2 · p. 2

Corollary 4.1 — A smooth projective application

Let X be a smooth connected complex projective variety. If $-K_X$ admits a smooth semipositive metric, then $H^0(X, -mK_X) \neq 0$ for some integer $m > 0$.

Corollary 4.1 · p. 18

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Bounded anticanonical metrics on klt pairs and torus quotients](#). External inputs use the following keys.

- [DPS01](#) Jean-Pierre Demailly; Thomas Peternell; Michael Schneider — Pseudo-effective line bundles on compact Kähler manifolds: Hard Lefschetz with multiplier ideals supplies twisted differentials. [arXiv:math/0006205v2](#), 10 October 2000.
- [LMPTX23](#) Vladimir Lazić; Shin-ichi Matsumura; Thomas Peternell; Nikolaos Tsakanikas; Zhixin Xie — The Nonvanishing Problem for varieties with nef anticanonical bundle: Pseudoeffectivity of the negative first Chern class of a saturated cotangent tensor subsheaf. *Documenta Mathematica* 28 (2023), 1393–1440.
- [PT14](#) Mihai Păun; Shigeharu Takayama — Positivity of twisted relative pluricanonical bundles and their direct images: Positive curvature of the rank-one adjoint integral metric. [arXiv:1409.5504v1](#).
- [Fuj16](#) Osamu Fujino — Kollár–Nadel type vanishing theorem: Base-direction positivity gives vanishing, including the zero twist. [arXiv:1607.04684v1](#), manuscript version 0.01 (10 July 2016).
- [DWZZ19](#) Fusheng Deng; Zhiwei Wang; Liyou Zhang; Xiangyu Zhou — A new proof of Kiselman’s minimum principle for plurisubharmonic functions: Plurisubharmonicity of the infimum after a separate orbit bound. [arXiv:1810.12733v1](#), 30 October 2018.
- [OpenAI26a](#) OpenAI — Anticanonical nonvanishing from smooth semipositivity: Finite-cover structure retaining torus monodromy and finite étale norm; its nonvanishing theorem is not an input. 2026-09-26; commit [fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb](#).

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — A maximal base and the zero twist

Proof target : Theorem 1.1 · p. 2

Proof route. The Euler premise supplies twisted differentials. Their determinants lead to a base of maximal dimension dominated by the anticanonical class. A divisor decomposition and rank-one direct images produce a bounded base metric. Vanishing including the zero twist then makes an Euler polynomial nonzero. The diagram gives the entire route; the intermediate results below are used as inputs to this theorem.



Bounded metrics, the zero twist, and a positive Cartier multiple

[Open figure ↗](#) [TeX source ↓](#)

1. Produce unbounded twisted differentials

The polynomial $\chi(\widetilde{W}, K_{\widetilde{W}} + t\rho^*(r_0P))$ is nonzero at zero. Nonzero cohomology therefore occurs for infinitely many positive integers, and a subsequence fixes its degree. Positive powers of the bounded metric have trivial multiplier ideals, so hard Lefschetz produces nonzero sections of $\Omega_{\widetilde{W}}^p \otimes n_i\rho^*P$ for a fixed p . The case $p = 0$ finishes immediately. Otherwise, saturating the determinant of their generic span gives effective divisors $N_i \sim_{\mathbb{Q}} M + m_iP$ with $m_i \rightarrow \infty$. The sums of positive twisting exponents are unbounded; numerical nontriviality of P is not added as a premise. The cotangent tensor theorem applies using the klt pair, effective boundary and nef anticanonical class, giving pseudoeffectivity of $-M$.

Lemma 2.1 · pp. 5–6 [DPS01, Theorem 2.1.1, p. 8] [LMPTX23, Theorem 4.1, print p. 1409 / PDF p. 17]

2. Exclude new pencils and subtract minimal vertical ratios

Among diagrams $\pi : S \rightarrow W$ and $f : S \rightarrow Y$ with connected fibers and with $d\pi^*P - f^*A$ pseudoeffective for some $d > 0$ and ample A , maximize $\dim Y$. A pencil dominated by P and an exceptional divisor, nonconstant on the general fiber, would enlarge this base and is excluded. Applying this to ratios of the N_i on a normal equidimensional model makes their horizontal parts affine in m_i . Taking a difference and subtracting, over each base prime, the minimum coefficient-to-multiplicity ratio yields

$$r\pi^*P = f^*V + H, \quad H \geq 0.$$

Above each prime divisor of the base, at least one component is absent from H . This is stronger than an effective decomposition and is what controls the later metric. A zero-dimensional base already gives a section.

Lemmas 2.2–2.3 · pp. 7–8

3. Turn rank-one integrals into a bounded essential-supremum metric

Set $R = K_S - \pi^*(K_W + D)$, $E = \lceil R \rceil$, and $\Xi = E - R$. Then E is effective and exceptional, Ξ has SNC support and coefficients in $[0, 1)$, and $K_S + \pi^*P + \Xi = E$. Maximality forces $f_*\mathcal{O}_S(E + ar\pi^*P)$ to have rank one for every $a \geq 0$. The bundle $B_a = (ar + 1)\pi^*P + \Xi$ is integral and has trivial multiplier ideal.

Positive curvature of the adjoint direct image gives, for a local generating section $s = f^*v s_H$,

$$u_a(y) = -\frac{1}{a} \log \int_{S_y} |s|_{h_P}^{2a} d\mu_y \longrightarrow -\log \operatorname{ess\,sup}_{S_y} |s|_{h_P}^2.$$

The manuscript does not call the intersection of the exponent-dependent base-change opens Zariski open: it uses a common full-measure set and local L^1 compactness of psh functions. Near the component absent from H , the section s is nonzero, bounding the essential supremum

positively from below even toward a missing base divisor. The opposite estimate follows from properness. Psh extension gives a locally bounded semipositive metric on V , without a continuity assumption.

Lemma 2.4; Proposition 2.5 · pp. 9–11 [PT14, Theorems 3.3.4–3.3.5, p. 24]

4. Start at the zero twist and push the section down

Mix a small amount of a positive-current metric associated with $d\pi^*P - f^*A$ into the bounded metric. The coefficient margin in the klt boundary, Skoda integrability at small exponents and Hölder give a metric on $\pi^*P + \Xi$ with strict positivity in base directions and trivial multiplier ideal. Adding the metrics of $af^*V + aH$ gives exactly $\mathcal{O}_S(-aH)$ as multiplier ideal for every $a \geq 0$. The adjoint bundle becomes $E + af^*V$, and Kollár–Nadel vanishing gives

$$H^i(Y, \mathcal{G} \otimes \mathcal{O}_Y(aV)) = 0 \quad (i > 0, a \geq 0), \quad \mathcal{G} = f_*\mathcal{O}_S(E).$$

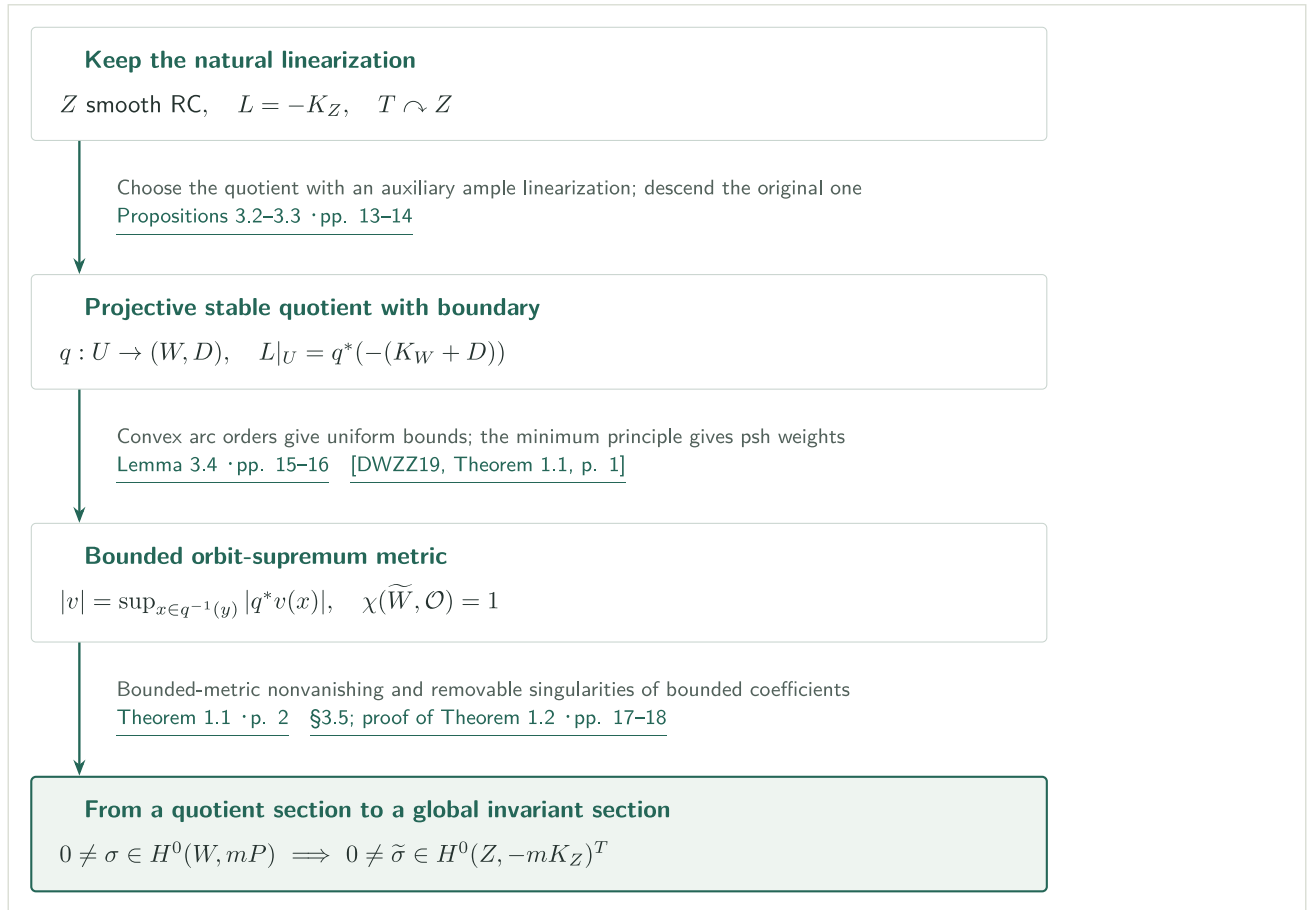
Since $E \geq 0$, vanishing at $a = 0$ yields $Q(0) = h^0(Y, \mathcal{G}) > 0$. Thus $Q(a) = \chi(Y, \mathcal{G}(aV))$ is a nonzero polynomial and provides a section for some $a > 0$. Multiply by s_H^a and use $\pi_*\mathcal{O}_S(E) = \mathcal{O}_W$ to obtain $H^0(W, arP) \neq 0$. The original Euler premise required nonzero value, not positive value.

Lemma 2.6; proof of Theorem 1.1 · pp. 11–12 [Fuj16, Theorem 1.3, p. 2]

3. Proof of Theorem 1.2 — Preserve the natural linearization

Proof target : Theorem 1.2 · p. 2

Proof route. An auxiliary ample linearization selects a complete stable quotient; the original natural anticanonical bundle is the bundle descended. A bounded orbit-supremum metric allows the section from Theorem 1.1 to extend across all omitted divisors.



A bounded quotient metric gives a naturally invariant section

[Open figure ↗](#) [TeX source ↓](#)

1. Separate the auxiliary GIT linearization from the natural bundle

Pass to the effective image and average the smooth metric over the maximal compact subgroup. Positivity of the generic natural anticanonical weight pattern and finiteness of the patterns permit a small character perturbation of $L + \delta J$ with nonempty stable=semistable locus U . The quotient $W = U/T$ is projective. Finite stabilizer slices give a $\mathbb{Q}$ -factorial klt pair and $D = \sum(1 - 1/e_B)B$. The torus adjoint representation is trivial, so the orbit-direction determinant introduces no character. Thus $L|_U = q^*P$ is an isomorphism of naturally equivariant $\mathbb{Q}$ -line bundles. The auxiliary character must not be transferred to L .

[Propositions 3.2–3.3 · pp. 13–14](#) [Remark 3.6 · p. 18](#)

2. Prove the orbit bound before applying the minimum principle

Translate the weak weight inequalities on U into orders of moving lifts along algebraic arcs. Convexity of the orders and the initial weight calculation exclude poles. Compactifying the action graph and selecting a curve upgrades the arcwise conclusion to a locally uniform norm bound. The supremum along an orbit therefore defines a quotient norm. Its negative logarithm is the

infimum of a psh function independent of imaginary exponential coordinates, hence is psh by Kiselman's principle. Slice bounds, the disc test for finite quotients and pullback to a resolution preserve local boundedness. Restriction to curves also establishes nefness of P .

[Lemma 3.4 · pp. 15–16](#) [Proposition 3.5 · pp. 16–17](#) [\[DWZZ19, Theorem 1.1, p. 1\]](#)

3. Supply the Euler premise and extend the invariant section

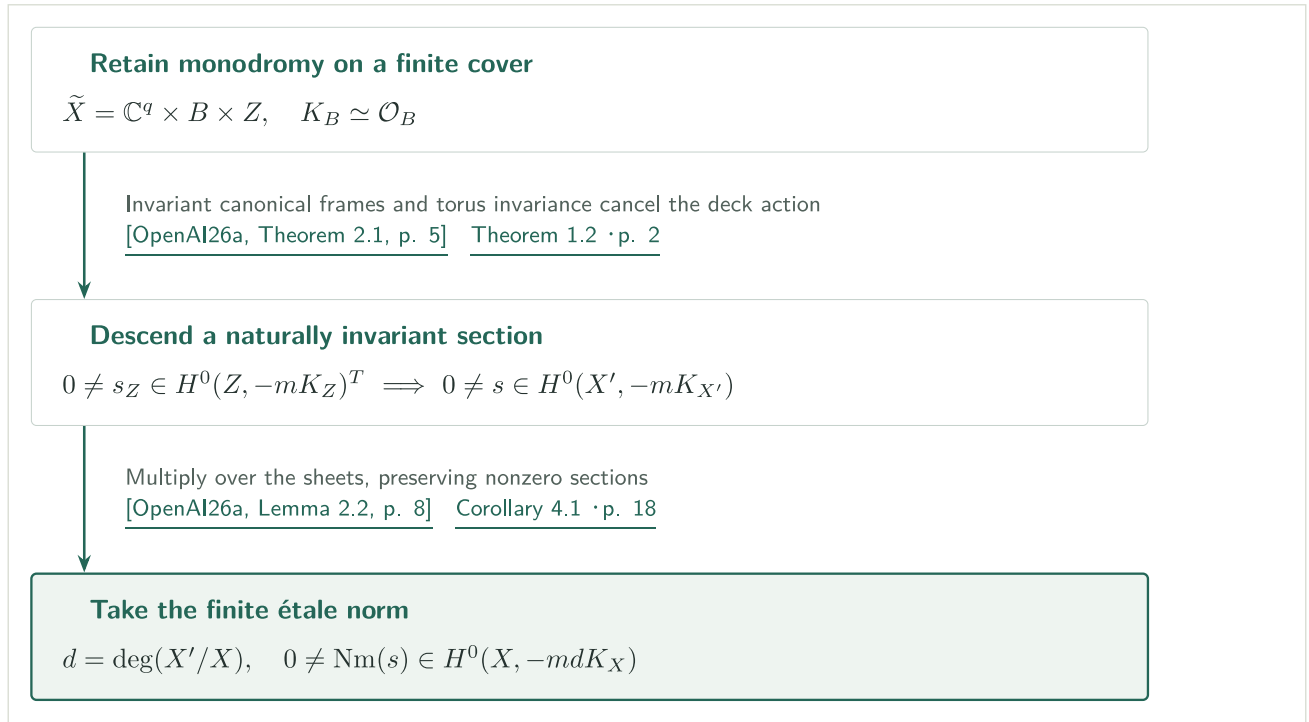
Rational connectedness of Z gives $\chi(\mathcal{O}) = 1$ on a smooth resolution of the quotient. Theorem 1.1 supplies a section of mP ; taking a power clears the denominators in the equivariant identification. On projective W the section has bounded norm for the bounded quotient metric. The orbit-supremum definition bounds its pullback throughout U . Relative to the smooth upstairs metric, removable singularities extend the bounded coefficients across all of $Z \setminus U$, including its divisors. Invariance persists by equality on the dense open set. The manuscript treats trivial actions and a point quotient as well.

[§3.5; proof of Theorem 1.2 · pp. 17–18](#)

4. Proof of Corollary 4.1 — Monodromy and norm

Proof target : [Corollary 4.1 · p. 18](#)

Proof route. Decompose the universal cover of a finite étale cover, descend the naturally invariant section along the deck action, and take a finite norm.



Handle residual torus monodromy and descend by finite norm

[Open figure ↗](#) [TeX source ↓](#)

1. Tensor invariant canonical frames with the section

The companion's structure theorem expresses the universal cover of a finite étale cover $X' \rightarrow X$ as $\mathbb{C}^q \times B \times Z$. It retains invariant canonical frames on the first two factors and residual monodromy on Z inside an algebraic torus. Tensor the section from Theorem 1.2 with their anticanonical frames. The result is deck invariant and descends to X' . The finite cover itself need not be a product.

[\[OpenAI26a, Theorem 2.1, p. 5\]](#) [Theorem 1.2 · p. 2](#)

2. Multiply the exponent by the cover degree

For $d = \deg(X'/X)$, the finite étale norm sends the nonzero section to $H^0(X, -mdK_X)$. This proves nonvanishing at some positive exponent on the original X . The inputs from the companion are its independent structure theorem and norm lemma, not its nonvanishing theorem; the argument does not assume that conclusion circularly.

[\[OpenAI26a, Lemma 2.2, p. 8\]](#) [Corollary 4.1 · p. 18](#)

5. Which papers supply which steps

Input

DPS01 and LMPTX23

Used at

Lemma 2.1, pp. 5–6

Role and checking scope

Hard Lefschetz and the negative Chern class of a saturated cotangent tensor subsheaf.

Statements and applicable hypotheses directly compared.

Input

PT14

Used at

Proposition 2.5, pp. 9–11

Role and checking scope

Positivity of the integral metric with trivial multiplier ideal and rank one. Statement checked; full external proof excluded.

Input

Fuj16

Used at

Proof of Theorem 1.1, pp. 11–12

Role and checking scope

Adjoint direct-image vanishing also at $a = 0$. Statement and the bundle after cancellation compared.

Input

DWZZ19

Used at

Proposition 3.5, pp. 16–17

Role and checking scope

Psh infimum over tube fibers; boundedness comes separately from Lemma 3.4.

Input

Anticanonical nonvanishing from smooth semipositivity

Used at

Corollary 4.1, p. 18

Role and checking scope

Natural monodromy in Theorem 2.1 and norm in Lemma 2.2. An input to the corollary, not to the main theorem.

6. Return to the sources

- [Lemma 2.1](#) · pp. 5–6 — From the Euler premise to unbounded twisted differentials.
- [Lemmas 2.2–2.3](#) · pp. 7–8 — Maximal base and an effective remainder missing a component over each base divisor.
- [Lemma 2.4; Proposition 2.5](#) · pp. 9–11 — Pass from integrals to essential suprema without continuity.
- [Lemma 2.6; proof of Theorem 1.1](#) · pp. 11–12 — The zero twist makes the Euler polynomial nonzero.
- [Lemma 3.4](#) · pp. 15–16 — Upgrade pole-free arcs to locally uniform bounds.
- [Corollary 4.1](#) · p. 18 — Use invariance under residual monodromy.

The editorial reading covered the hypotheses and quantifiers of the principal results and the proof text of §§2–4, pp. 5–18. The five analytic/cotangent inputs above and the companion's structure and norm statements were compared with their uses. Local boundedness versus continuity, the zero twist, and natural versus auxiliary linearizations were kept distinct.

The full external proofs were not independently verified. GIT, Luna slices, the finite-morphism discrepancy formula, Skoda's foundational reference and the psh extension theorems were read as used in this manuscript; direct comparison with their original literature remains incomplete. This is an account of the authors' argument and its checked interfaces, not certification of the complete proof or expert peer review.

Cohomological transfer and equivariant anticanonical sections

Remove a fixed error while preserving its character, then descend invariant sections through monodromy

For a smoothly semipositive anticanonical bundle, invariant sections with one fixed negative pseudoeffective error produce an invariant section in a positive untwisted degree. The main theorem allows any anticanonical linearization. The natural linearization is needed later, when the invariant index supplies the input. This account follows the authors' shortest route, §§2–4 and §14.1, to the main theorem and smooth nonvanishing corollary.

Manuscript author: OpenAI · September 26, 2026 · 43 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Invariant transfer

Let X be smooth connected complex projective, with a smooth semipositive metric on $L = -K_X$, and let an algebraic torus T act on X . Let P, M be linearized line bundles with $P = L$ as ordinary line bundles and with $-M$ pseudoeffective. The linearization of P may differ from the natural one. Suppose that for an unbounded set $I \subset \mathbb{Z}_{>0}$ there are sections

$$0 \neq s_i \in H^0(X, M + iP)^T \quad (i \in I).$$

Then $H^0(X, mP)^T \neq 0$ for some integer $m > 0$. No Euler-characteristic premise or vanishing of $H^1(X, \mathcal{O}_X)$ is assumed.

Theorem 1.1 · p. 2

Corollary 1.2 — Nonvanishing under smooth semipositivity

For every smooth connected complex projective variety X with smoothly semipositive $-K_X$, there is an integer $m > 0$ such that $H^0(X, -mK_X) \neq 0$. The statement gives neither a uniform exponent nor nonvanishing in the first power.

Corollary 1.2 · p. 2

Theorem 7.2 — Higher-direct-image vanishing under a kernel condition

Let $f : V \rightarrow Y$ be a surjective morphism between smooth connected projective complex varieties. Let D_V have a smooth semipositive Hermitian metric with curvature θ , and let the line bundle Q on Y have a positive continuous Hermitian metric with psh local weights. Suppose that

$$\ker \theta \subseteq \ker df$$

on a nonempty ordinary open subset of the submersion locus. Then for **every** integer $k, q \geq 0$ and $p > 0$,

$$H^p(Y, R^q f_*(K_V + D_V) \otimes Q^k) = 0.$$

This includes $k = 0$. Continuity is allowed for the base weights of Q , whereas the coefficient metric on D_V is smooth. Nefness alone does not imply the kernel condition.

Theorem 7.2 · pp. 17–19

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Cohomological transfer and equivariant anticanonical sections](#). External inputs use the following keys.

- [DPS01](#) Jean-Pierre Demailly; Thomas Peternell; Michael Schneider — Pseudo-effective line bundles on compact Kähler manifolds: Equivariant hard Lefschetz supplies unbounded twisted forms. [arXiv:math/0006205v2](#), 10 October 2000.
- [LMPTX23](#) Vladimir Lazić; Shin-ichi Matsumura; Thomas Peternell; Nikolaos Tsakanikas; Zhixin Xie — The Nonvanishing Problem for varieties with nef anticanonical bundle: Pseudoeffectivity of the inverse determinant. *Documenta Mathematica* 28 (2023), 1393–1440.
- [PT14](#) Mihai Păun; Shigeharu Takayama — Positivity of twisted relative pluricanonical bundles and their direct images: Positivity of the singular adjoint direct-image metric and its invariant quotient. [arXiv:1409.5504v1](#).

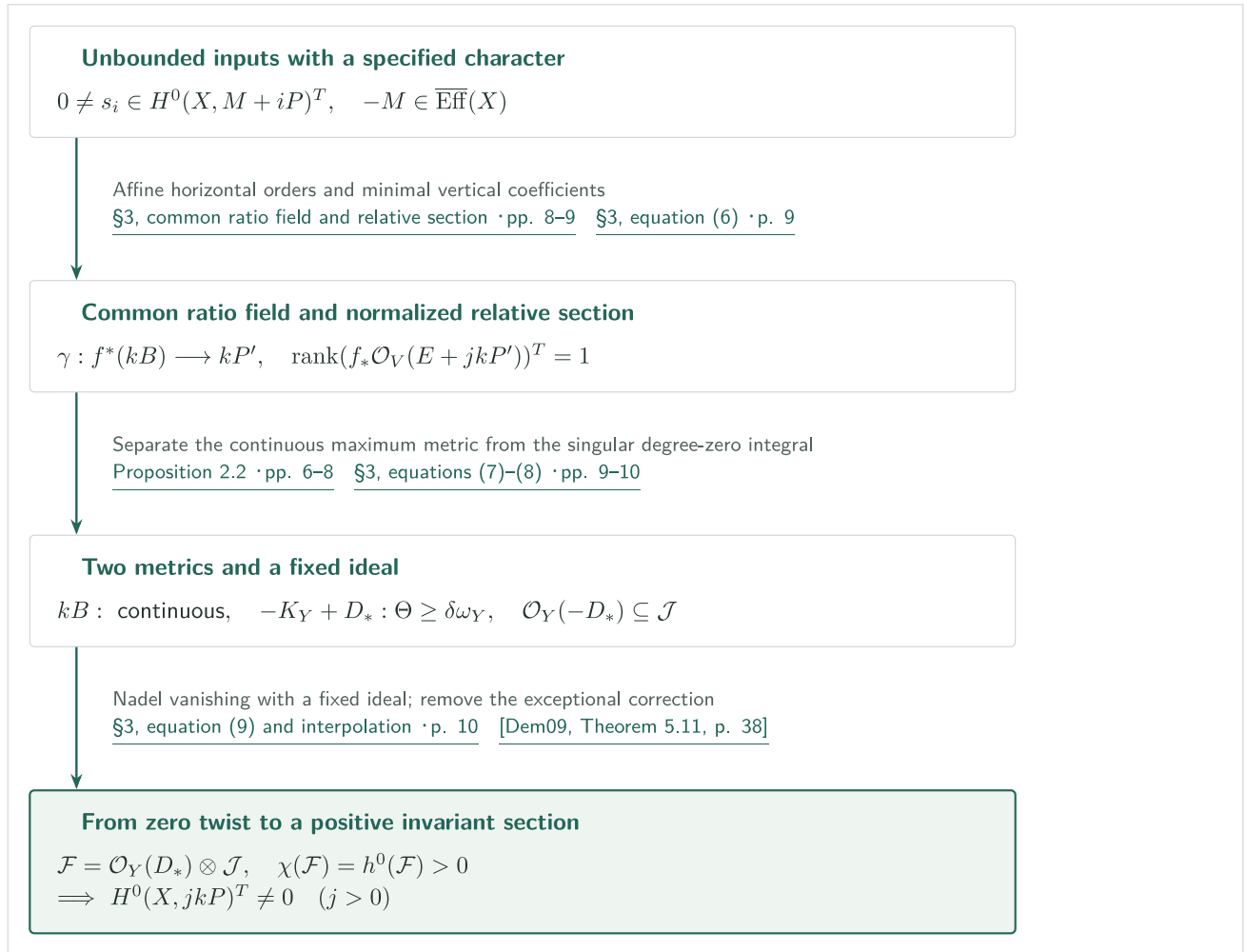
- [OpenAI26a](#) OpenAI — Anticanonical nonvanishing from smooth semipositivity: Finite-cover structure retaining residual monodromy and the finite étale norm. 2026-09-26; commit [fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb](#).
- [Ber09](#) Bo Berndtsson — Curvature of vector bundles associated to holomorphic fibrations: Semipositivity for smooth adjoint integrals and the invariant orthogonal quotient. *Annals of Mathematics* 169 (2009), 531–560.
- [Dem09](#) Jean-Pierre Demailly — Analytic methods in algebraic geometry: Auxiliary statement-level verification of Nadel vanishing used in §3. December 2009 lecture notes, 180-page version; distinct from the July 2011 book cited in the manuscript.
- [OpenAI26b](#) OpenAI — Invariant anticanonical indices and conversion of twisted differentials: Polynomial invariant anticanonical index retaining its actual value at zero. 2026-09-26; commit [fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb](#).
- [Fujisawa15](#) Taro Fujisawa — A remark on the decomposition theorem for direct images of canonical sheaves tensorized with semipositive vector bundles: Derived decomposition for a smooth Nakano semipositive coefficient. [arXiv:1512.03887v1](#); arXiv header 12 December 2015 (PDF internal date June 27, 2021).
- [Dem12](#) Jean-Pierre Demailly — Complex Analytic and Differential Geometry: Richberg smoothing with uniform control and a small curvature error. 21 June 2012; 455-page author PDF.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — Transfer with a fixed ideal

Proof target : [Theorem 1.1](#) · p. 2

Proof route. A common field of ratios of sections of equal linearized degree defines the base. Affine horizontal orders and minimal vertical orders produce a regular relative section; every nonnegative invariant adjoint degree then has rank one. The [maximum metric](#) controls the varying twists. A separate singular metric supplies horizontal positivity and a nonzero Euler value at zero. The [natural index application](#) supplies inputs, and [Corollary 1.2](#) descends sections to general X .



Remove the error with two metrics and a fixed ideal, preserving the specified linearization

[Open figure ↗](#) [TeX source ↓](#)

1. Build a relative section from the common ratio field

Write $N_i = M + iP$. For $p < q < b$, the identity $(b - p)N_q = (b - q)N_p + (q - p)N_b$ preserves linearizations. Multiplication by the same fixed sections in numerator and denominator identifies the invariant ratio fields associated with interior indices. Independent ratios define a projective base; Stein factorization replaces its field by its relative algebraic closure in $\mathbb{C}(X)$. The torus acts trivially on this base.

Take a normal equidimensional model $\mu : W \rightarrow X$, $f_W : W \rightarrow Y$, and a smooth resolution $\pi : V \rightarrow X$, $f : V \rightarrow Y$. Since the ratios come from the base, horizontal section divisors are affine in the exponent; unboundedness makes their slope effective. Subtract the minimum order divided by pullback multiplicity over each base prime and clear denominators, obtaining

$$\gamma \in H^0(W, k\mu^*P - f_W^*(kB))^T, \quad k > 0.$$

At least one component over every base prime is absent from the zero divisor of γ . If the ratio field has transcendence degree zero, its effective slope already gives a regular invariant anticanonical section.

§3, common ratio field and relative section · pp. 8–9

2. Obtain invariant rank one, including degree zero

The effective divisor $E = K_{V/X}$ has its natural invariant Jacobian section. A character shift in $P' = \pi^*P$ does not change this natural action on the Jacobian. For every $j \geq 0$,

$$\text{rank}(f_*\mathcal{O}_V(E + jkP'))^T = 1.$$

The lower bound comes from $1_E\gamma^j$. Two independent invariant sections could be globalized after an ample base twist, then multiplied by input sections and powers of γ to yield an invariant ratio of sections of an equal multiple of N_b . That ratio would belong to the common base field, contradicting independence on the generic fiber. Degree zero is included.

§3, equation (6) · p. 9

3. Separate the metric for varying twists from the degree-zero metric

Proposition 2.2 puts a continuous semipositive metric on kB . Separately, pseudoeffectivity of $-M$ and ample domination on the base give a singular metric on $L' = \pi^*L$ satisfying

$$i\Theta_h \geq \delta f^*\omega_Y, \quad \mathcal{I}(h) = \mathcal{O}_V \quad (\delta > 0).$$

A small mixture with the smooth metric and Skoda integrability give the trivial ideal.

Project the degree-zero adjoint integral to the invariant holomorphic orthogonal quotient, giving a density F_ξ on $-K_Y$ that is locally integrable throughout the base. At a boundary prime with pullback multiplicity ℓ and Jacobian order a , the local estimate is $F_\xi(y) \geq c|y_1|^{2((a+1)/\ell-1)}$. It determines an effective correction D_* allowing extension of the metric on $-K_Y + D_*$. Its support is restricted to base primes whose dominating components are all exceptional. A dimension estimate on the equidimensional model also treats components mapping into codimension at least two. Consequently the entire f^*D_* is exceptional over X .

§3, equations (7)–(8) · pp. 9–10 [PT14, Theorems 3.3.4–3.3.5, p. 24]

4. Keep one possibly nontrivial multiplier ideal

Let $\mathcal{J}$ be the extended metric's multiplier ideal. Local integrability of the density yields

$$\mathcal{O}_Y(-D_*) \subseteq \mathcal{J},$$

without asserting $\mathcal{J} = \mathcal{O}_Y$. Tensoring with the bounded metric on jkB leaves this ideal unchanged. For all $j \geq 0$, Nadel vanishing gives

$$H^v(Y, \mathcal{O}_Y(D_* + jkB) \otimes \mathcal{J}) = 0 \quad (v > 0).$$

The sheaf $\mathcal{F} = \mathcal{O}_Y(D_*) \otimes \mathcal{J}$ contains $\mathcal{O}_Y$, so $\chi(Y, \mathcal{F}) = h^0(Y, \mathcal{F}) > 0$. Its Euler polynomial is not zero and gives a section for some $j > 0$. Pull back, multiply by γ^j , remove the exceptional correction f^*D_* , and push to X . The resulting section of jkP is invariant for the specified linearization.

§3, equation (9) and interpolation · p. 10 [Dem09, Theorem 5.11, p. 38]

3. Proof of Proposition 2.2 — A positive continuous maximum metric

Proof target : Proposition 2.2 · pp. 6–8

Proposition 2.2 — Divisorial normalization and maximum metric

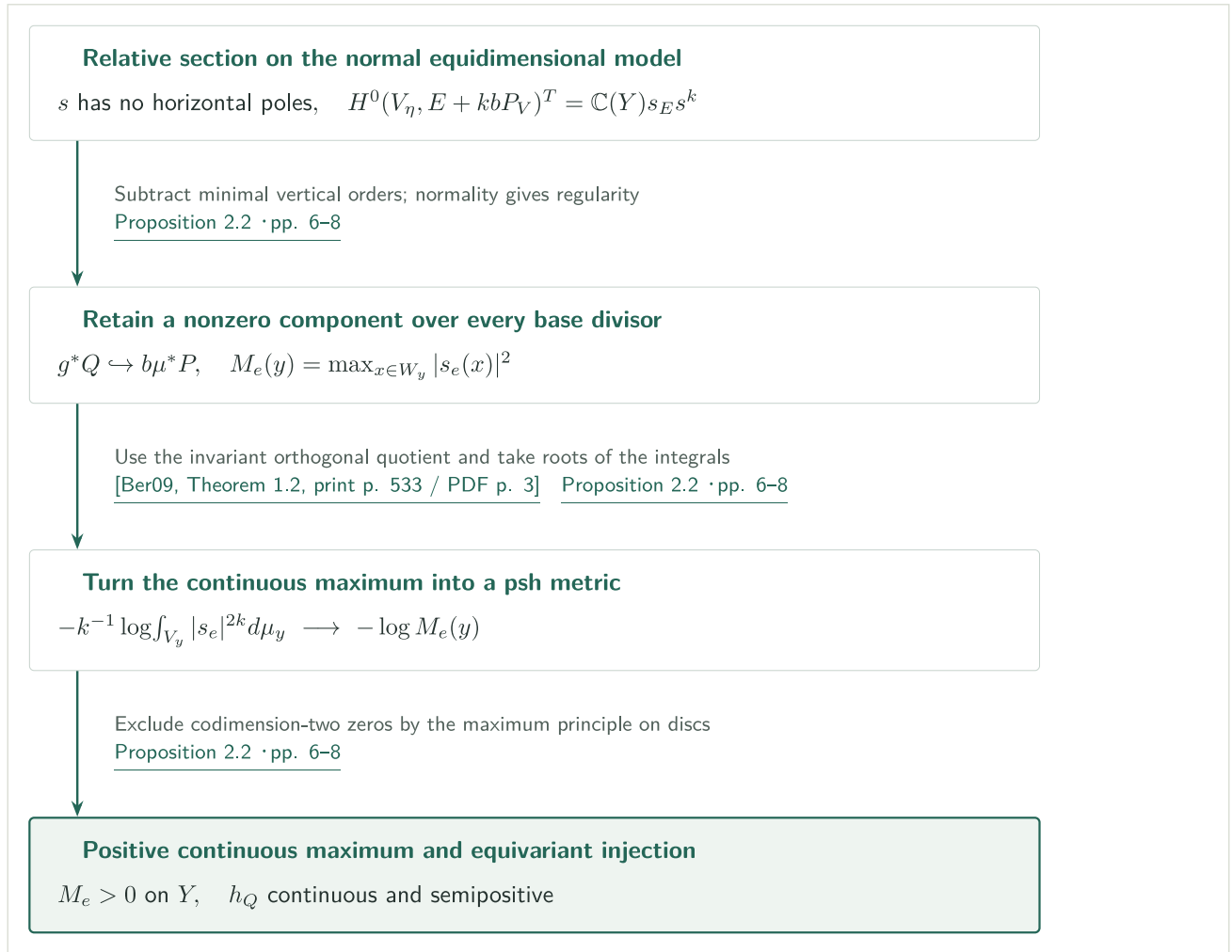
Let X, Y be smooth connected projective and let $L = -K_X$ have a smooth semipositive metric. Suppose $V \xrightarrow{\nu} W \xrightarrow{\mu} X$, $g : W \rightarrow Y$, and $f = g\nu$, where W is normal, g is projective surjective equidimensional, μ is birational and V is a smooth projective resolution. Set $D_V = (\mu\nu)^*L$ and $E = K_{V/X}$. A nonzero rational section s of $b\mu^*L$, $b > 0$, has no horizontal poles and satisfies, for every $k \geq 0$,

$$H^0(V_\eta, E + kbD_V) = \mathbb{C}(Y)s_E s^k.$$

After replacing b, s by a multiple and a power, there is a line bundle Q on the base with a regular injection $g^*Q \rightarrow b\mu^*L$ whose fiber-maximum norms give a positive continuous semipositive metric on Q .

For the equivariant version, a torus acts compatibly on the diagram and trivially on the base, P is any linearization of L , and s is an invariant rational section of $b\mu^*P$. The rank condition may be replaced by $H^0(V_\eta, E + kbP_V)^T = \mathbb{C}(Y)s_E s^k$. Then Q has trivial action and $g^*Q \rightarrow b\mu^*P$ is equivariant.

Proposition 2.2 · pp. 6–8



Use normal equidimensionality and an invariant quotient to obtain a positive psh maximum metric

[Open figure ↗](#) [TeX source ↓](#)

1. Normalize divisors and use equidimensionality for continuity

Subtract the minimal normalized order of s over each base prime, then clear denominators to construct Q and its injection. Its zero divisor misses at least one component over every base prime. For the image s_e of a local frame, $M_e(y) = \max_{W_y} |s_e|^2$ is upper semicontinuous by properness and lower semicontinuous by openness of the normal equidimensional morphism. Its zero locus is algebraic closed and has codimension at least two by the omitted-component property. The rank condition at $k = 0$ also supplies connected generic fibers.

[Proposition 2.2 · pp. 6–8](#)

2. Treat the invariant part as a holomorphic orthogonal quotient

Apply Berndtsson to the smooth adjoint coefficient $(kb + 1)D_V$. Positivity of the invariant line does not follow merely from being a subline. Compact averaging makes the action unitary; the nonzero-weight spaces form the orthogonal complement and define a holomorphic quotient. Normalized rank-one integrals satisfy

$$-\frac{1}{k} \log \int_{V_y} |s_e|^{2k} d\mu_y \longrightarrow -\log M_e(y).$$

On smooth general fibers, the measure has full support and convergence is locally uniform. The negative logarithm of the maximum is therefore psh.

[Ber09, Theorem 1.2, print p. 533 / PDF p. 3] Proposition 2.2 · pp. 6–8

3. Exclude codimension-two zeros and retain equivariance

Continuity extends the psh weight wherever $M_e > 0$. At the remaining codimension-two zero locus, use analytic discs and parallel translates whose boundaries stay off that locus. The maximum principle gives a uniform upper bound for $-\log M_e$, contradicting a zero of the continuous maximum. Hence M_e is positive everywhere, and the frame transformation law defines the continuous semipositive metric on Q . Invariance of the normalized vertical divisors and relative section preserves the equivariant injection and the trivial base action.

Proposition 2.2 · pp. 6–8

4. Proof of Corollary 4.3 — Inputs from the natural index

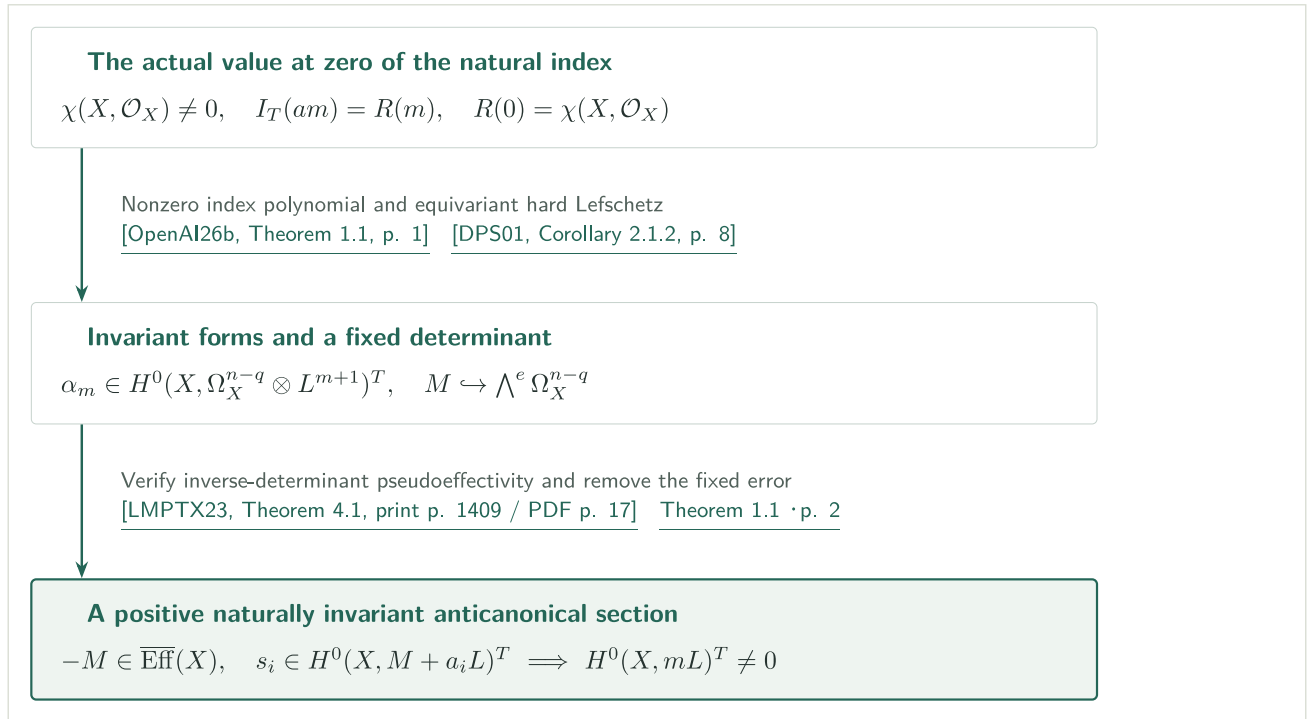
Proof target : Corollary 4.3 · pp. 11–12

Corollary 4.3 — Natural invariant nonvanishing

Let X be smooth connected complex projective with smoothly semipositive $L = -K_X$. If an algebraic torus T acts on X and $\chi(X, \mathcal{O}_X) \neq 0$, then $H^0(X, mL)^T \neq 0$ for some integer $m > 0$, using the natural linearization.

The trivial torus is allowed.

Corollary 4.3 · pp. 11–12



The actual natural index at zero supplies transfer inputs by hard Lefschetz and determinants

[Open figure ↗](#) [TeX source ↓](#)

1. Use the index polynomial with its actual value at zero

A connected torus acts by maps homotopic to the identity, hence trivially on $H^q(X, \mathcal{O}_X)$ through Hodge decomposition. The natural invariant index at zero is therefore $\chi(X, \mathcal{O}_X) \neq 0$. The companion's index theorem identifies this actual value with the constant term of a polynomial on a divisible progression. Eventual polynomiality alone would not exclude the zero polynomial. Fix a cohomological degree along a subsequence, and apply equivariant hard Lefschetz to $mL = K_X + (m+1)L$, obtaining invariant twisted forms.

[\[OpenAI26b, Theorem 1.1, p. 1\]](#) [Proposition 4.1; Lemma 4.2 · pp. 10–11](#) [\[DPS01, Corollary 2.1.2, p. 8\]](#)

2. Remove the fixed determinant error by Theorem 1.1

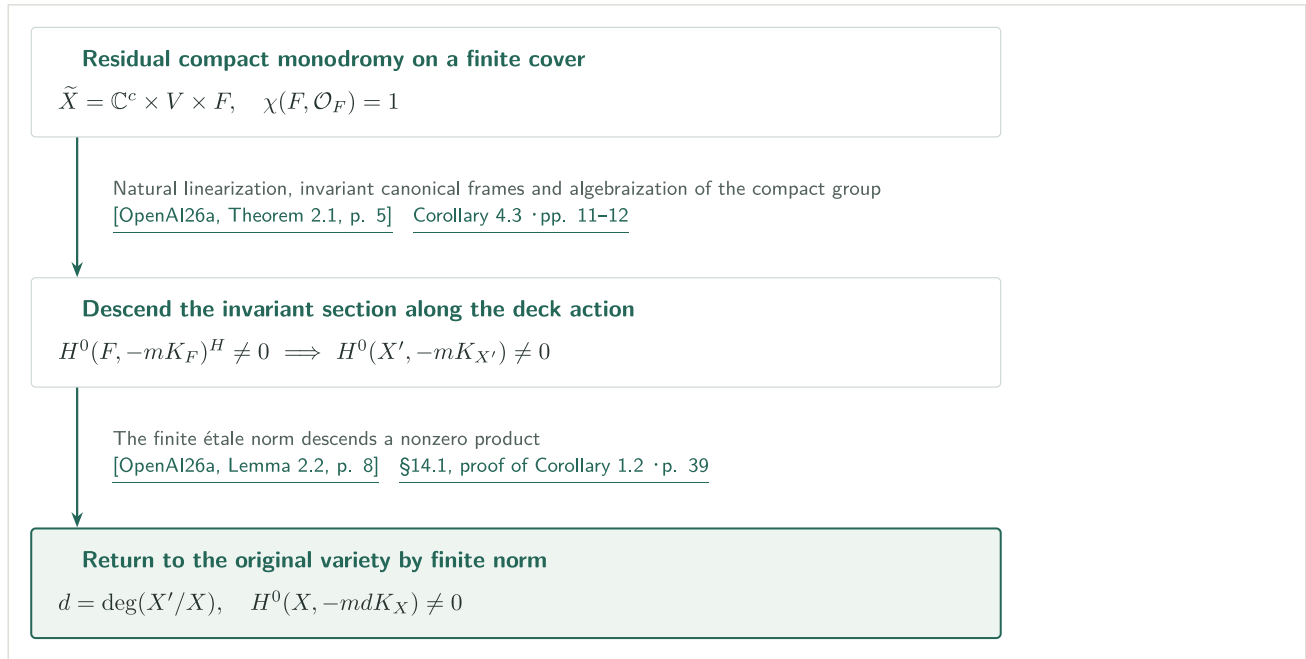
If $q = 0$ or $q = n$, a positive anticanonical section already exists. Otherwise take the generic span of the forms and its saturated determinant line M . Wedge a form of unbounded exponent with a fixed complementary basis to obtain nonzero invariant sections of $M + a_i L$ with $a_i \rightarrow \infty$. The cotangent tensor theorem applies to the smooth zero-boundary pair with nef L , making $-M$ pseudoeffective. Theorem 1.1 then applies with naturally linearized $P = L$.

[Corollary 4.3, proof · pp. 11–12](#) [\[LMPTX23, Theorem 4.1, print p. 1409 / PDF p. 17\]](#)

5. Proof of Corollary 1.2 — Descend invariant sections

Proof target : Corollary 1.2 · p. 2

Proof route. The finite-cover structure gives the universal cover $\mathbb{C}^c \times V \times F$ with residual compact-torus action. Apply Corollary 4.3 on F , attach the invariant canonical frames and descend.



Descend through compact monodromy and take the finite norm back to X

[Open figure ↗](#) [TeX source ↓](#)

1. Descend the compact-factor section to the finite cover

The companion's structure theorem gives a finite étale cover $X' \rightarrow X$ whose deck action is by translations on the Euclidean factor, trivial on V , and through a compact torus H on F . Positive-degree holomorphic forms vanish on F , so Hodge symmetry gives $\chi(F, \mathcal{O}_F) = 1$. Since $\text{Pic}^0(F) = 0$, the compact action can be algebraized through a very ample line. Corollary 4.3 supplies an invariant section for its algebraic-torus closure. Tensoring with invariant canonical frames makes it deck invariant and it descends to X' . For a point factor F , use the constant section.

[OpenAI26a, Theorem 2.1, p. 5] Corollary 4.3 · pp. 11-12 §14.1, proof of Corollary 1.2 · p. 39

2. Take a norm to the original variety

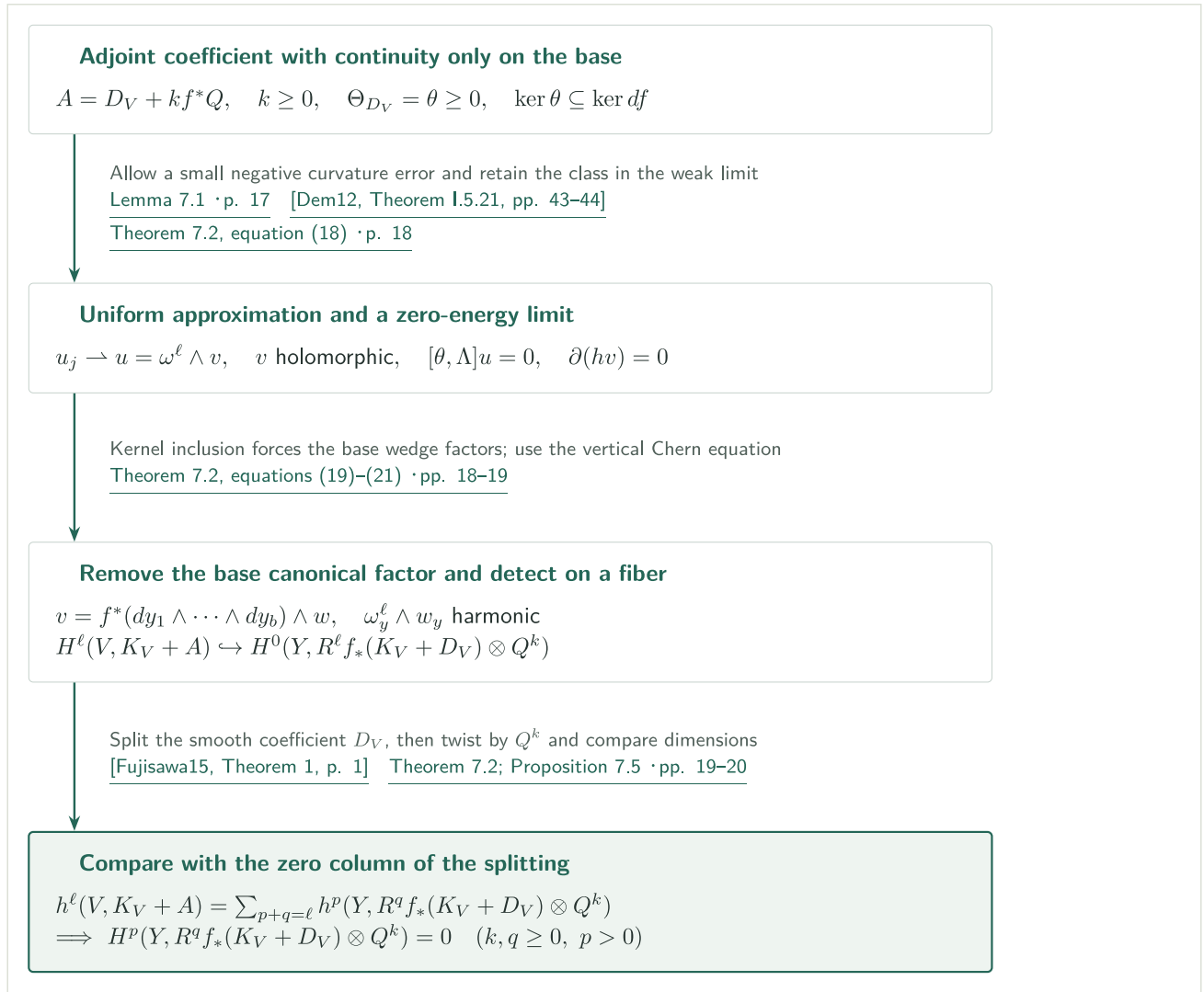
For $d = \deg(X'/X)$, the finite étale norm sends the section to a nonzero section of $-mdK_X$. Common powers and the cover degree may increase the exponent; no uniform exponent is asserted. The structural input does not require the finite cover itself to be a product, and it does not use the companion's nonvanishing theorem.

[OpenAI26a, Lemma 2.2, p. 8] §14.1, proof of Corollary 1.2 · p. 39

6. Proof of Theorem 7.2 — Harmonic restriction through a continuous twist

Proof target : Theorem 7.2 · pp. 17–19

Proof route. Uniformly approximate the continuous base metric and take a limit with zero Bochner energy. The kernel condition forces the limiting form to contain the base canonical covectors, so a nonzero class is detected on a general fiber. Combine the resulting injective edge map with derived decomposition for a smooth coefficient. Decomposition alone does not assert vanishing.



From continuous base weights to higher-direct-image vanishing

[Open figure ↗](#) [TeX source ↓](#)

1. Uniform approximation retains the curvature-null part

Fix $k \geq 0$ and put $A = D_V + kf^*Q$. The Richberg-type approximation in Lemma 7.1 gives smooth metrics h_j on A with $h_j \rightarrow h$ uniformly and $\Theta_{h_j} \geq \theta - o(1)\omega$. A small negative curvature allowance is essential; the approximants are not asserted to be semipositive. The harmonic representatives u_j of a fixed class in $H^\ell(V, K_V + A)$ have uniformly bounded L^2 norms. Bochner–Kodaira gives $\|(D'_j)^*u_j\|^2 + \langle [\theta, \Lambda]u_j, u_j \rangle \rightarrow 0$. A weak limit v of $v_j = *u_j$ is holomorphic, and $u = \omega^\ell \wedge v$ represents the original class. The finite-rank harmonic projection for a fixed reference metric preserves that class through the weak limit.

[Lemma 7.1 · p. 17](#) [\[Dem12, Theorem I.5.21, pp. 43–44\]](#) [Theorem 7.2, equation \(18\) · p. 18](#)

2. Restrict the distributional Chern equation to a general fiber

Zero curvature energy makes every positive-curvature holomorphic covector a wedge factor of v . The kernel inclusion implies that the base covectors are factors on the specified open set; holomorphy extends this property over the smooth fibration locus. Thus

$v = f^*(dy_1 \wedge \cdots \wedge dy_b) \wedge w$. In a local frame, $D'_j v_j = 0$ reads $\partial(h_j v_j) = 0$. Uniform convergence of the metrics and weak L^2 convergence give $\partial(hv) = 0$ in distributions, without convergence of derivatives of the weights.

The continuous base factor is constant under vertical differentiation. On each smooth fiber this yields $D'_{D_V|V_y} w_y = 0$ for its smooth coefficient metric. Restriction here means restriction of a $(0, \ell)$ -form valued in $K_V + A$. Removing the constant factor $K_Y|_y$ gives the fiber-harmonic representative $\omega_y^\ell \wedge w_y$. It is not ordinary pullback of an (n, ℓ) -form to a smaller-dimensional fiber. A nonzero class is nonzero on some general fiber, making the Leray edge map injective.

Theorem 7.2, equations (19)–(21) · pp. 18–19

3. Smooth-coefficient splitting recovers every quantified case

Apply Fujisawa's theorem to the smooth semipositive line D_V : for a line its curvature condition is Nakano semipositivity, and the Kähler and proper-surjective hypotheses hold. The theorem is not applied directly to the continuous metric on $D_V + kf^*Q$. First obtain

$Rf_*(K_V + D_V) \simeq \bigoplus_q R^q f_*(K_V + D_V)[-q]$, then tensor by Q^k on the base. Projection formula gives

$$h^\ell(V, K_V + D_V + kf^*Q) = \sum_{p+q=\ell} h^p(Y, R^q f_*(K_V + D_V) \otimes Q^k).$$

The injective edge map bounds the left side by the $p = 0$ summand, forcing all remaining summands to vanish. For $\ell > \dim V$, the left side itself is zero. Since k was any nonnegative integer, this proves all $k, q \geq 0, p > 0$, including the zero twist. That zero-twist case supplies the later Euler interpolation in Proposition 7.5.

[Fujisawa15, Theorem 1, p. 1] Theorem 7.2; Proposition 7.5 · pp. 19–20

7. Which papers supply which steps

Input

Ber09

Used at

Proposition 2.2, p. 7

Role and checking scope

Smooth adjoint integral positivity. The statement and use of the invariant orthogonal quotient were compared.

Input

PT14

Used at

§3, pp. 9–10

Role and checking scope

Direct-image metric for the singular degree-zero coefficient. Generic multiplier-ideal isomorphism and the holomorphic quotient were checked.

Input

Nadel vanishing, checked in Dem09

Used at

§3, p. 10

Role and checking scope

Strict positivity of $-K_Y + D_* + jkB$, fixed $\mathcal{J}$, and vanishing for every $j \geq 0$. The original Nadel proof was not independently inspected.

Input

Fujisawa15; Dem12

Used at

Theorem 7.2, pp.17–19

Role and checking scope

Uniform smoothing and derived decomposition for a smooth Nakano coefficient; tensor by the continuous base twist after splitting.

Input

Invariant anticanonical indices and conversion of twisted differentials

Used at

Proposition 4.1 → Corollary 4.3, pp. 10–12

Role and checking scope

Natural index and actual zero value in Theorem 1.1. This input is independent of the transfer theorem itself.

Input

DPS01 and LMPTX23

Used at

Lemma 4.2 → Corollary 4.3, pp. 11–12

Role and checking scope

Forms with exponent $m + 1$ and pseudoeffectivity of the inverse determinant; no numerical nontriviality premise is added.

Input

Anticanonical nonvanishing from smooth semipositivity

Used at

§14.1, p. 39

Role and checking scope

Finite-cover structure and norm used in the corollary, without reusing the nonvanishing theorem.

8. Return to the sources

- §3, common ratio field and relative section · pp. 8–9 — Build a common ratio field while retaining the character.
- Proposition 2.2 · pp. 6–8 — Maximum metric on a normal equidimensional model.
- §3, equation (9) and interpolation · p. 10 — A fixed, possibly nontrivial ideal and the zero twist.
- Corollary 4.3, proof · pp. 11–12 — Convert the natural index to transfer inputs.
- §14.1, proof of Corollary 1.2 · p. 39 — Descend invariant sections and take a finite norm.
- Theorem 7.2 · pp. 17–19 — Separate uniform approximation, fiber detection and smooth-coefficient splitting.
- §1.4 · pp. 4–5 — Locate alternative routes not developed in this account.

The editorial reading covered the hypotheses and all conclusions of Theorem 1.1 and Corollary 1.2, Proposition 2.2, Corollary 4.3, and the proof text of §§2–4 and §14.1. The companion's index, structure and norm statements and the analytic inputs above were compared with their uses. Nadel's statement was checked in Demailly's December 2009 lecture notes, distinguished from the 2011 book cited elsewhere in the manuscript.

Within Sections 5–13, Lemma 7.1 and Theorem 7.2 are developed above. The remaining alternative mechanisms include exact source characters, singular injectivity, alteration through a curvature-null field, local strictness and stabilized direct-image orders. They are not treated as premises of the main theorem here, and their full proofs and interfaces are outside this account. The alternative cohomology, form and tensor-holonomy descents in §§14.2–14.4 are likewise not developed. Complete external proofs and the foundational sources for flattening, equivariant resolution, Skoda and psh extension were not independently checked. This is not expert peer review.

Among the additional outputs, Theorem 5.2 (p.13) attains the **full T-character** of the source of every generic cocharacter by an actual section, assuming no positive-degree holomorphic forms and a torus action with a linearized ample bundle. It retains more information than invariant nonvanishing alone. Theorem 11.1 (p.27) gives an alternative conversion of nonzero $H^q(X, mL)$ at one fixed q and unbounded m into ordinary nonvanishing. Proposition 14.1 (p.41) gives a further dichotomy for a compact simply connected nonflat de Rham irreducible Kähler factor of nonnegative Ricci curvature: a parallel holomorphic volume form or vanishing of all positive-degree covariant tensors. We locate and distinguish these outputs while prioritizing the shortest transfer–index–descent route; their proofs and direct inputs are outside this account's detailed checking scope.

[Theorem 5.2 · p. 13](#) [Theorem 11.1 · p. 27](#) [Proposition 14.1 · p. 41](#)

Metric descent and rank-preserving contractions

Adjoint rank at exponent zero, two metrics, and section ratios after contraction

This manuscript separates descent of an anticanonical metric to a base from control of the poles of sections returning to the original variety. Its principal outputs are an unweighted integrable base metric, a contraction preserving a generic section and adjoint ranks, allowed-pole nonvanishing, and naturally invariant anticanonical sections. We present Theorems 2.1, 4.1, 6.1 and 7.1 in their original order.

Manuscript author: OpenAI · September 26, 2026 · 76 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 2.1 — Toroidal volume metric

Let X, Y be smooth connected projective complex varieties, $d = \dim Y > 0$, and suppose $L = -K_X$ carries a smooth semipositive Hermitian metric h with curvature α . Let W be normal and integral, with projective morphisms $p : W \rightarrow X$ birational and $f : W \rightarrow Y$ surjective, equidimensional and toroidal for strict toroidal embeddings. For a smooth projective resolution $\rho : Z \rightarrow W$, put $\pi = p\rho$, $g = f\rho$, $E_Z = K_{Z/X}$ and assume the full direct image satisfies $\mathrm{rk} \, g_* \mathcal{O}_Z(E_Z) = 1$.

For a prime Q on the base and a prime E on W dominating it, put $a_E = \mathrm{ord}_E K_{W/X}$ and $e_E = \mathrm{ord}_E f^* Q$. Then

$$a_Q = \max \left\{ 0, \min_{f(E)=Q} \left(\frac{1 + a_E}{e_E} - 1 \right) \right\}, \quad A_0 = \sum_Q a_Q Q$$

is an effective rational divisor of finite support, and f^*A_0 is p -exceptional. The rational line bundle $-K_Y + A_0$ has a semipositive singular Hermitian metric whose squared frame norms are locally integrable **without a divisor weight**; the metric is smooth on a nonempty Zariski open subset.

Moreover, let $U \subset Y$ be a nonempty Zariski open where g is smooth and adjoint direct-image base change holds. If some nonempty ordinary open patch $V \subset g^{-1}(U) \setminus \text{Supp } E_Z$, Kähler form ω_Y and $c > 0$ satisfy $\pi^*\alpha \geq cg^*\omega_Y$ on V , the constructed metric has strictly positive curvature on a nonempty ordinary open subset of Y . This is not a global Kähler lower bound.

Theorem 2.1 · p. 7

Theorem 4.1 — Rank-preserving contraction

Let X be smooth connected projective over $\mathbb{C}$, with $H^0(X, \Omega_X^q) = 0$ for every $q > 0$, and let $L = -K_X$ carry a smooth semipositive metric. Let an algebraic torus T act on X , with its natural linearization on L . Let $f : X \dashrightarrow Z$ be a dominant T -invariant rational map to a smooth connected projective variety, with $\mathbb{C}(Z)$ relatively algebraically closed in $\mathbb{C}(X)$. On a resolution $\pi : Y \rightarrow X$, $f_Y : Y \rightarrow Z$, put $L_Y = \pi^*L$, $E_Y = K_{Y/X}$ and $\eta = \text{Spec } \mathbb{C}(Z)$.

Assume

$$H^0(Y_\eta, pL_Y|_{Y_\eta})^T \neq 0 \quad \text{for some } p > 0,$$

and that for **every positive multiple** m of some $a > 0$,

$$\dim_{\mathbb{C}(Z)} H^0(Y_\eta, (E_Y + mL_Y)|_{Y_\eta})^T = 1.$$

Then either $H^0(X, rL)^T \neq 0$ for some $r > 0$, or there is a dominant rational map $Z \dashrightarrow W$ to a smooth connected projective variety with $\dim W < \dim Z$. In the latter case $X \dashrightarrow W$ satisfies the same two conditions, possibly with different integers, and $\mathbb{C}(W)$ is relatively algebraically closed in $\mathbb{C}(X)$.

Theorem 4.1 · p. 16

Theorem 6.1 — Allowed-pole nonvanishing

Let V be smooth connected projective over $\mathbb{C}$ with $H^0(V, \Omega_V^j) = 0$ for every $j > 0$. Let $D \geq 0$ be integral. Suppose $A = -K_V + D$ has a semipositive singular metric h_A for which the measure $|s_D|_{h_A}^2$, viewing s_D as an A -valued top form, has finite total mass. If L has a semipositive metric with locally bounded weights, then

$$H^0(V, aL + cD) \neq 0 \quad \text{for some integers } a > 0, c \geq 0.$$

The same holds for a rational line bundle L , with a clearing its denominator. Here L need not be anticanonical.

Theorem 6.1 · p. 34

Theorem 7.1 — Naturally invariant anticanonical sections

Let V be smooth connected projective over $\mathbb{C}$, with $H^0(V, \Omega_V^j) = 0$ for every $j > 0$, and suppose $-K_V$ has a smooth semipositive Hermitian metric. For **each** algebraic torus T acting on V , there exists an integer $m > 0$ such that, for the natural anticanonical action,

$$H^0(V, -mK_V)^T \neq 0.$$

The statement neither fixes $m = 1$ nor allows an arbitrary character twist of the linearization.

Theorem 7.1 · p. 38

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Metric descent and rank-preserving contractions](#). External inputs use the following keys.

- [Ber09](#) Bo Berndtsson — Curvature of vector bundles associated to holomorphic fibrations: Smooth direct-image positivity and its curvature formula, for full rank one or an invariant orthogonal quotient. [arXiv:math/0511225v2](#), 20 August 2007.
- [DPS01](#) Jean-Pierre Demailly; Thomas Peternell; Michael Schneider — Pseudo-effective line bundles on compact Kähler manifolds: Convert nonzero cohomology to twisted forms; bounded weights make all multiplier ideals trivial. [arXiv:math/0006205v2](#), 10 October 2000.
- [CP19](#) Frédéric Campana; Mihai Păun — Foliations with positive slopes and birational stability of orbifold cotangent bundles: Use orbifold quotient slopes for an effective boundary with ample perturbation, not directly for a signed boundary. *Publications Mathématiques de l’IHÉS* 129 (2019), 1–49.
- [HPS](#) Christopher D. Hacon; Mihnea Popa; Christian Schnell — Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun: Metric on the direct image with the actual multiplier ideal; an integrable generator spans the rank-one direct image. Author-hosted 53-page manuscript, retrieved 12 October 2026.
- [PT14](#) Mihai Păun; Shigeharu Takayama — Positivity of twisted relative pluricanonical bundles and their direct images: Underlying singular L2 direct-image positivity, including the generic multiplier-ideal isomorphism condition. [arXiv:1409.5504v1](#), 19 September 2014.

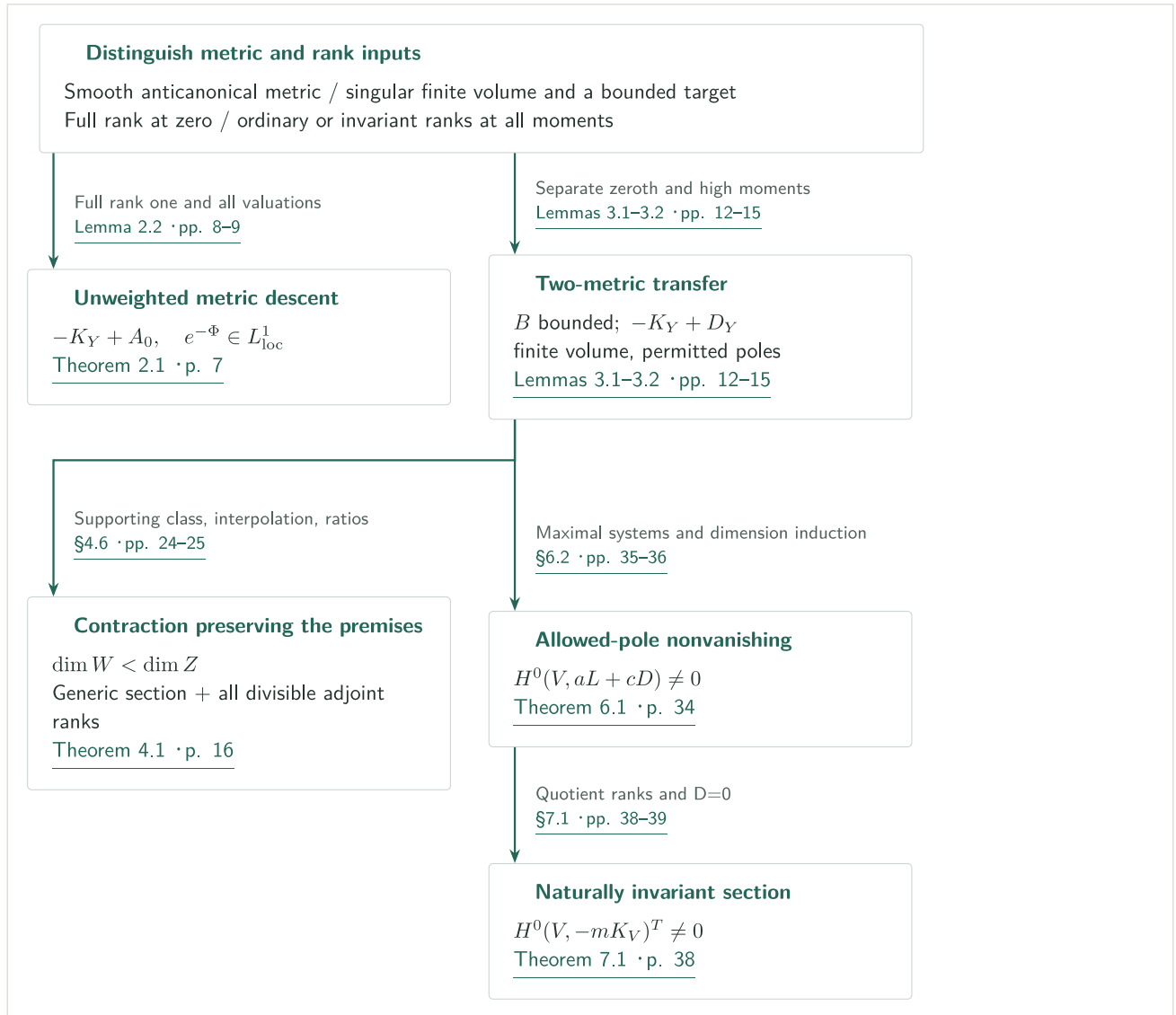
- [Ou23](#) Wenhao Ou — On generic nefness of tangent sheaves: Relative pseudoeffectivity in Druel’s formulation and algebraicity of foliations, used for the finite-volume cotangent estimate. [arXiv:1703.03175v4](#).
- [BDPP13](#) Sébastien Boucksom; Jean-Pierre Demailly; Mihai Păun; Thomas Peternell — The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension: Cone duality supplies a movable class annihilating a nonbig pseudoeffective divisor. [arXiv:math/0405285v1](#), 14 May 2004.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 4.1 — Proof overview of the main theorem

Proof target : [Theorem 4.1](#) · p. 16

Proof route. Metric descent and section descent are separate operations. Section descent must restore adjoint ranks as well as existence so that another contraction is possible. The manuscript also has parallel routes: Theorem 7.1 follows from Theorem 6.1 and two-metric transfer, without Theorem 4.1.



Connections among metric descent, rank preservation, allowed poles and invariant sections

[Open figure ↗](#) [TeX source ↓](#)

1. Two base metrics and two exits

Theorem 2.1 uses full adjoint rank one at exponent zero and an explicit A_0 to construct an unweighted integrable metric on $-K_Y + A_0$. Lemmas 3.1–3.2 instead transfer two lines separately: a finite-volume donor and a bounded-metric target. Theorem 4.1 uses their smooth invariant version and ratios of divisors annihilated by a supporting movable class to choose a smaller base. Theorem 6.1 uses the ordinary singular version to close dimension induction. Their rank hypotheses are distinct.

Lemmas 3.1–3.2 · pp. 12–15

2. The route to nonvanishing

On the generic quotient fiber in Theorem 7.1, a wedge of infinitesimal action fields gives an invariant section, and invariant section ratios descend to the quotient field. This supplies invariant adjoint rank one, including exponent zero. Apply Theorem 6.1 on the transferred base; the original allowed boundary $D = 0$ ensures that no poles remain. This route does not assume the contraction theorem's premises without constructing them.

§7.1 · pp. 38–39

3. Proof of Theorem 2.1 — Curvature and unweighted integrability

Proof target : Theorem 2.1 · p. 7

The statement appears above. Equidimensionality of W is an assumption; the smooth resolution Z need not be equidimensional.



Full rank at exponent zero and toroidal orders produce an unweighted integrable metric

[Open figure ↗](#) [TeX source ↓](#)

1. Exceptional support and all valuations

If $a_Q > 0$, every prime above Q has $a_E > 0$ and is exceptional over X . Equidimensionality makes every vertical prime dominate a base prime, preventing support from being hidden over higher codimension. Put $\Theta = -K_{W/X} + f^* A_0 \leq 0$ and let D_W be the full toroidal boundary. The effective rational Cartier divisor $B = D_W - \Theta$ contains D_W , and

$$A_{W,\Theta}(v) = A_{W,D_W}(v) + v(B) > 0$$

for every divisorial valuation. Thus $\gamma_F = \mathrm{coeff}_F(K_{Z'/X} - (g')^* A_0) > -1$ also for new exceptional divisors above boundary intersections. Checking only the primes already visible on W would not suffice.

[Lemma 2.2 · pp. 8–9](#)

2. Identify the integrated density and extend its metric

Write $g_*\pi^*\Omega_h = H(y)dV_y$. Full rank one and the Jacobian section identify

$g_*(K_{Z/Y} + \pi^*L) \simeq -K_Y$ on a smooth open, so H is the L^2 norm on the direct-image line.

Smooth direct-image positivity gives plurisubharmonicity of $-\log H$. This does not restrict a positive metric to an arbitrary subline.

The corrected weight is $\Phi = -\log H + \sum a_Q \log |t_Q|^2$. A **lower bound for the density** from ramification orders gives the local upper bound for Φ needed for psh extension. Separately,

$$e^{-\Phi} dV_y = g'_* \left(\prod_Q |t_Q \circ g'|^{-2a_Q} (\pi')^* \Omega_h \right).$$

The inequalities $\gamma_F > -1$ make the upstairs measure a bounded positive factor times $\prod |z_j|^{2\gamma_j}$, hence locally integrable. Properness supplies local integrability downstairs. This is integrability of the frame norm itself, not integrability after multiplication by s_{A_0} . The original smooth open retains a smooth density.

[Ber09, Theorem 1.2, p. 3]

§§2.2–2.3; (2.7)–(2.11) · pp. 9–11

3. Strict positivity on an ordinary open

On the extra patch V , the Jacobian generator is nonzero. The curvature formula for a Chern-normal generator separates the integral of coefficient curvature from a nonnegative primitive term. Integrating $\pi^*\alpha \geq cg^*\omega_Y$ over this patch supplies a positive contribution in each base tangent direction; smoothness extends strictness to an ordinary neighborhood. This calculation occurs away from the correction and supplies no global strict lower bound.

[Ber09, Proposition 4.2; (4.8), pp. 14, 17]

4. Proof of Theorem 4.1 — Restore the ranks after contraction

Proof target : Theorem 4.1 · p. 16

Refer to the statement above. A generic section and rank one in positive divisible degrees are different premises; together they imply rank one at exponent zero.



Vertical minima, horizontal interpolation and section ratios preserve the ranks

[Open figure ↗](#) [TeX source ↓](#)

1. Normalize the section and locate the permitted poles

On a toroidal equidimensional model $Y_0 \rightarrow Z$, write the generic section as $\text{div}(s)/p = H + f_0^*V$. Coefficients of V are minima of orders divided by vertical multiplicities; $H \geq 0$, and above each base prime some component has coefficient zero in H . Let G sum the base primes for which all such components are exceptional over X . For small $\delta > 0$, $(H - \delta f_0^*G)_-$ is exceptional. A section of $V + \delta G$, multiplied by the normalized section and pushed down, therefore gives a global invariant section on X .

Multiplication by the generic section and the positive rank condition give rank one at zero. Compact averaging identifies the invariant direct image with an orthogonal quotient, permitting smooth direct-image positivity. High-moment limits give V bounded semipositive weights. The zeroth density gives $K_Z + B^+ + M \sim_{\mathbb{Q}} B^-$ with M nef and $\text{Supp } B^- \subset G$. This signed boundary is not an effective klt boundary.

§§4.1–4.3; Lemma 4.3 · pp. 16–21

2. Choose a smaller base from zero-degree systems

If $V + G$ is big, the preceding exit applies. Otherwise cone duality supplies a nonzero movable class β with $V \cdot \beta = G \cdot \beta = B^- \cdot \beta = 0$. Add a general effective representative of $M + tA$ to B^+ , apply nonnegativity of orbifold cotangent quotients for this effective klt boundary, and let $t \downarrow 0$. Every cotangent tensor line N then has $c_1(N) \cdot \beta \leq 0$. The identity $\chi(\mathcal{O}_Z) = 1$, Riemann–Roch, hard Lefschetz with multiplier ideals and determinant extraction give

$$D_j \sim m_j V + N, \quad m_j \longrightarrow \infty, \quad D_j \cdot \beta = 0.$$

Maximize the image dimension of a zero-degree linear system and take its Stein factorization. The resulting base has $\dim W < \dim Z$, and ratios in every such system belong to $\mathbb{C}(W)$.

[BDPP13, Theorem 2.2, p. 6]

[CP19, Theorem 1.3, p. 3]

[DPS01, Theorem 2.1.1, p. 8]

Lemmas 4.4–4.5 · p. 22

3. Recover existence and uniqueness from horizontal equality

In the three-index relation

$$(m_\ell - m_j)D_k \sim (m_\ell - m_k)D_j + (m_k - m_j)D_\ell,$$

the ratio comes from the base, so the horizontal parts over W are equal as **actual divisors**.

Horizontal multiplicities are affine in m_j ; effectivity and $m_j \rightarrow \infty$ make their slopes nonnegative. This gives a generic effective representative of V , which combines with the normalized section to restore a generic invariant section of L .

Restoring rank needs further pole control. Clear the poles of two new generic adjoint sections with the same ample base twist, then divide by the Jacobian and a power of the normalized section. The old rank condition puts the results on the old base. Outside G , a nonexceptional component with $H = 0$ prohibits poles. The remaining ratio belongs to one zero-degree system, hence to $\mathbb{C}(W)$. The rank is at most one; the new section and Jacobian supply a nonzero element in every sufficiently divisible positive degree, so it is exactly one. Relative algebraic closure is preserved through Stein factorization and the field inclusions.

Lemmas 4.6–4.7 · p. 23

§4.6 · pp. 24–25

5. Proof of Theorem 6.1 — Dimension induction retaining two metrics

Proof target : Theorem 6.1 · p. 34

Refer to the statement above. We follow the first proof in §6.2.



Twisted forms give a generic section and transfer both metrics to a smaller base

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1. Turn twisted forms into effective divisors

Bounded weights on L make the multiplier ideals of all powers trivial. Absence of positive-degree forms gives $\chi(\mathcal{O}_V) = 1$, so $\chi(K_V + mL)$ is a polynomial with nonzero constant term. Hard Lefschetz converts nonzero cohomology in unbounded degrees into twisted forms. The determinant of their generic span supplies $N_i \sim M + m_i L$.

Finite **weighted** volume produces effective approximants to $-K_Y + D + A/m$ whose subtraction by D is sub-plt. Relative pseudoeffectivity of foliations and slope arguments give pseudoeffectivity of $-M + kD$. This step does not add smooth anticanonical semipositivity or a global strict curvature premise.

Lemma 3.3 · pp. 15–16

Proposition 5.3 · p. 27

[Ou23, Propositions 2.2, 3.1, pp. 5–6]

2. Maximal admissible systems force rank one

Call $|P|$ admissible if $aL + cD - P$ is pseudoeffective for some $a > 0, c \geq 0$. Products remain admissible, so choose a system of maximal image dimension. Full-dimensional image makes $aL + cD$ big and concludes the argument. Otherwise choose the smaller connected-fiber base Y ; every admissible section ratio belongs to $\mathbb{C}(Y)$.

Horizontal parts of the three-index relations among N_i produce a pole-free generic section ρ . If $J + p^*D + mp^*L$ had two independent generic sections, clearing their poles by a base twist would give an admissible ratio enlarging the chosen image. Thus in multiples of the degree of ρ the generic ranks are one, including exponent zero.

§6.2 · pp. 35–36

3. Transfer finite volume and boundedness separately

Lemma 3.2 checks that $u_m = s_{J+p^*D}\rho^m$ belongs to the direct image with the **actual** multiplier ideal using its finite integral; rank one makes it a generator. Positivity of this L^2 direct image gives psh zeroth densities and high moments. Their normalized limit is a fiber supremum. Together with vertical-minimum normalization in Lemma 3.1 it gives bounded semipositive weights on a base line B . The different line $-K_Y + D_Y$ retains finite weighted volume.

The base inherits absence of positive-degree forms, so induction applies. Transfer of its section of $aB + cD_Y$ leaves poles only on D or exceptional support. After pushdown an integral multiple of D clears the finitely many poles. For rational L , clear denominators at the beginning and end. Identifying the two metrics would lose the hypotheses needed for this induction.

[HPS, Theorem 21.1, pp. 31–32]

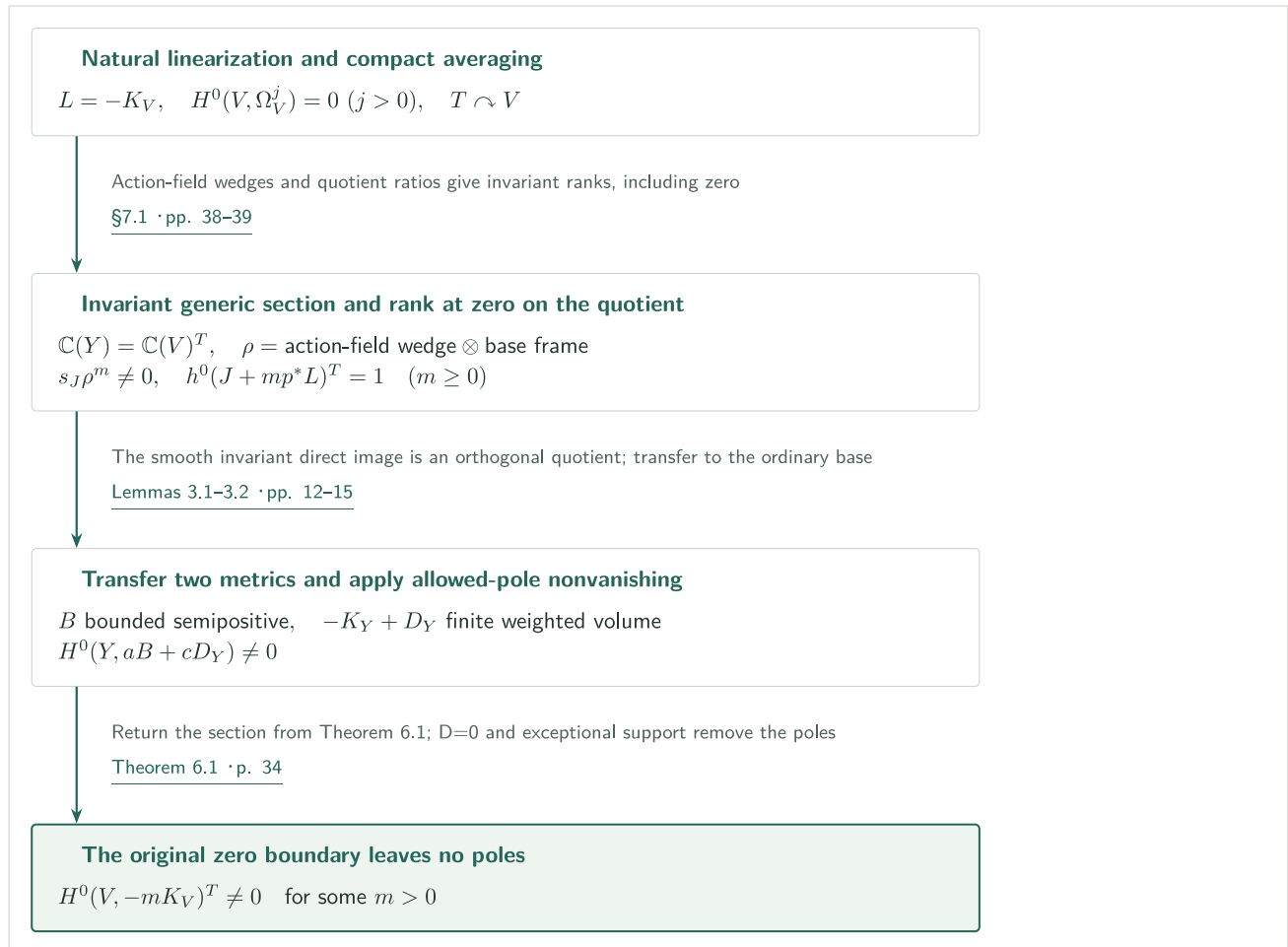
[PT14, Theorem 3.3.5, p. 24]

Lemmas 3.1–3.2 · pp. 12–15

6. Proof of Theorem 7.1 — From the invariant quotient to a regular section

Proof target : Theorem 7.1 · p. 38

Refer to the statement above. We use the rank-one quotient construction in §7.1.



Natural action-field wedge, two-metric transfer and removal of allowed poles

[Open figure ↗](#) [TeX source ↓](#)

1. Construct quotient ranks starting at zero

Average metric weights over a compact real form. For a trivial action apply Theorem 6.1 with $D = 0$, $A = L = -K_V$. Otherwise take a smooth projective model Y of $\mathbb{C}(V)^T$. Connectedness of the torus makes this field relatively algebraically closed, and the general fiber has a dense orbit. Wedge independent infinitesimal action fields, then use a rational anticanonical base frame and the Jacobian to obtain a naturally invariant generic anticanonical section ρ .

Ratios of invariant adjoint sections are quotient functions. The sections $s_J \rho^m$ supply nonzero elements at all $m \geq 0$, including zero, so the invariant adjoint ranks are one. With smooth coefficient metrics, compact averaging is an orthogonal holomorphic projection and the invariant metric equals a quotient metric. No positivity of an arbitrary invariant subbundle with singular metric is assumed.

§7.1 · pp. 38–39

[Ber09, Theorem 1.2, p. 3]

2. Use the zero allowed boundary at the last step

For a point quotient the full wedge of action fields already gives a global section. Otherwise transfer both metrics and apply Theorem 6.1 on the base, which has no positive-degree forms. The return map multiplies by base functions, preserving natural invariance. Since the original boundary is $D = 0$, all poles disappear into exceptional support, giving a global section on smooth V .

Section 7.2 supplies an alternative proof of the two psh weights by logarithmic orbit integrals and suprema. Our principal route is §7.1; the alternative is not an additional necessary input.

Theorem 6.1 · p. 34

Lemma 7.2; §7.2 · pp. 39–42

7. Which papers supply which steps

- Smooth direct-image curvature supplies the density and strictness in Theorem 2.1 and the smooth invariant versions in Lemmas 3.2/4.3. Ordinary singular transfer uses HPS with the actual multiplier ideal and a separate integrability check for its generator.
- Hard Lefschetz supplies forms in Lemma 3.3. It uses trivial multiplier ideals for every power, furnished by bounded weights; nefness alone is not substituted.
- Orbifold cotangent positivity enters Lemma 4.4. Druel–Ou relative pseudoeffectivity enters the finite-volume slope bound in Proposition 5.3. Signed boundaries, effective boundaries and unweighted integrals remain distinct.
- Within this catalogue, Integrable metrics and effectivity with controlled boundary uses Theorem 2.1 in an alternative unweighted-transfer proof. We do not make the present Theorems 6.1/7.1 depend on that companion's main theorem. See the catalogue connection record for the precise interface.

8. Return to the sources

- [Lemma 2.2 · pp. 8–9](#) — Check all valuations, including boundary intersections, and exceptional support.
- [§§2.2–2.3; \(2.7\)–\(2.11\) · pp. 9–11](#) — Separate curvature of the density from unweighted integrability.
- [Lemmas 3.1–3.2 · pp. 12–15](#) — Track the two metrics and the permitted polar support.
- [§4.6 · pp. 24–25](#) — Check how ratios return to the smaller base field.
- [§6.2 · pp. 35–36](#) — Read maximal admissible systems and dimension induction.
- [§7.1 · pp. 38–39](#) — Locate the natural invariant section and rank at exponent zero.

The editorial comparison covered the hypotheses, quantifiers and all conclusions of the four displayed theorems, the principal steps in §§2–4, Proposition 5.3, §6.2 and §7.1. We compared the statements and application conditions of Berndtsson's Theorem 1.2 and curvature formula, DPS Theorem 2.1.1, Campana–Păun Theorem 1.3, HPS Theorem 21.1, Păun–Takayama Theorem 3.3.5, Ou Propositions 2.2/3.1 and BDPP Theorem 2.2. This is an account of the manuscript's arguments, not independent verification of complete proofs or completed expert review.

The ratio-field variants, contractions with other boundary/rank data and finite-flat norm construction in §§8–12 lie outside this detailed account. We have not rechecked the original weak toroidalization/equidimensionalization sources, independently verified every valuation calculation or psh limit, or recursively verified the external proofs. Theorem 2.1 is stated with its toroidal equidimensional model as a hypothesis.

Integrable metrics and effectivity with controlled boundary

Turn integrability into a klt boundary, then remove an exceptional correction to return an anticanonical section

The basic route derives effectivity with controlled correction support from integrability of a distinguished section, and then removes a fixed pseudoeffective error. Smooth semipositivity means that $L = -K_X$ admits a smooth Hermitian metric of semipositive curvature; nefness alone is not substituted for this hypothesis. The singular metrics on auxiliary bases satisfy two distinct conditions: weighted and unweighted integrability.

Manuscript author: OpenAI · September 26, 2026 · 45 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Effectivity modulo a prescribed boundary

Let Y be a smooth projective complex variety of positive dimension and $C \geq 0$ an integral divisor. Suppose $-K_Y + C$ has a singular Hermitian metric h whose curvature dominates a Kähler form and for which $\mathcal{J}(h) \supseteq \mathcal{O}_Y(-C)$. For every pseudoeffective rational divisor M , there is an effective rational divisor C' supported on C such that $M + C'$ is rationally linearly equivalent to an effective rational divisor.

[Theorem 1.1 · p. 2](#)

Theorem 4.3 — Finite generation near a correction divisor

Let Y be smooth projective of positive dimension and A_0 a rational divisor. Suppose $-K_Y + A_0$ has a semipositive singular metric whose squared nonvanishing local frame norms are locally integrable, smooth and strictly positive on a nonempty ordinary open set. For any finite list of rational divisors $P_1, \dots, P_s$, there are a common $\epsilon \in \mathbb{Q}_{>0}$ and an integer $k_0 > 0$ such that the actual divisors $B_i = k_0(A_0 + \epsilon P_i)$ are Cartier and

$$R(Y; B_1, \dots, B_s) = \bigoplus_{(a_1, \dots, a_s) \in \mathbb{N}^s} H^0\left(Y, \mathcal{O}_Y\left(\sum_i a_i B_i\right)\right)$$

is a finitely generated $\mathbb{C}$ -algebra. Effectivity of A_0 is not assumed here.

Theorem 4.3 · pp. 9–10

Theorem 5.1 — Removing one fixed pseudoeffective error

Let X be smooth connected projective over $\mathbb{C}$, and let $L = -K_X$ be smoothly semipositive. Fix a pseudoeffective Cartier divisor P . If effective integral divisors $N_j \sim m_j L - P$ exist for unbounded positive integers m_j , then $H^0(X, mL) \neq 0$ for some positive integer m . Neither effectivity of P nor $L \not\equiv 0$ is assumed.

Theorem 5.1 · p. 11

Corollary 5.6 — Conversion of twisted differentials of fixed degree

For X, L as in Theorem 5.1, if $H^0(X, (\Omega_X^1)^{\otimes p} \otimes mL) \neq 0$ for one fixed integer $p \geq 0$ and unbounded positive integers m , a positive multiple of L has a nonzero section. The same conclusion holds with Ω_X^p in place of the tensor power. This is ordinary nonvanishing; the conclusion does not assert invariance.

Corollary 5.6 · p. 16

Theorem 9.2 — Invariant nonvanishing for the natural linearization

Let X be smooth connected projective over $\mathbb{C}$ with $\chi(X, \mathcal{O}_X) \neq 0$. If an algebraic torus T acts on X and $L = -K_X$ is smoothly semipositive, then $H^0(X, mL)^T \neq 0$ for some positive integer m , for the natural tangent-determinant linearization.

Theorem 9.2 · pp. 32–35

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Integrable metrics and effectivity with controlled boundary](#). External inputs use the following keys.

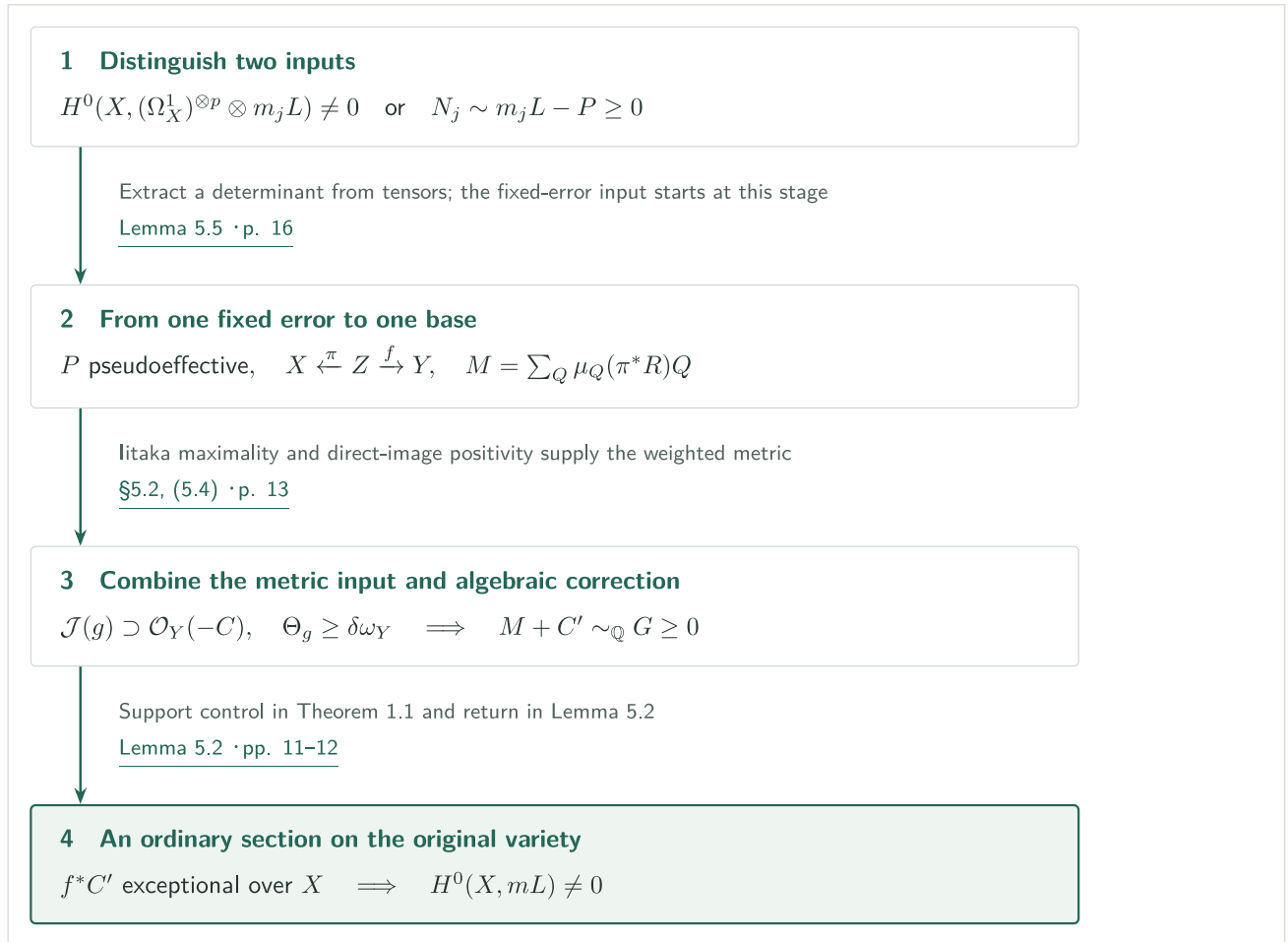
- [BCHM10](#) C. Birkar, P. Cascini, C. D. Hacon, J. McKernan — Existence of minimal models for varieties of log general type: Nonvanishing for an adjoint with big klt boundary. [arXiv:math/0610203v2](#).
- [CL13](#) A. Corti, V. Lazić — New outlook on the Minimal Model Program, II: Finite generation of multigraded adjoint rings and Veronese subrings. [arXiv:1005.0614v3](#).
- [PT14](#) M. Păun, S. Takayama — Positivity of twisted relative pluricanonical bundles and their direct images: Extension of direct-image metrics when the multiplier-ideal inclusion is generically an isomorphism. [arXiv:1409.5504v1](#) (2014); published in 2018.
- [OpenAI26b](#) OpenAI — Invariant anticanonical indices and conversion of twisted differentials: Invariant index including its value at zero. 2026-09-26.
- [OpenAI26e](#) OpenAI — Metric descent and rank-preserving contractions: Toroidal volume metric for the alternative ordinary conversion. 2026-09-26.
- [LMPTX23](#) V. Lazić, S. Matsumura, T. Peternell, N. Tsakanikas, Z. Xie — The Nonvanishing Problem for varieties with nef anticanonical bundle: Pseudoeffectivity of the negative first Chern class of saturated cotangent-tensor subsheaves. *Documenta Mathematica* 28 (2023), 1393–1440.
- [DPS01](#) J.-P. Demailly, T. Peternell, M. Schneider — Pseudo-effective line bundles on compact Kähler manifolds: Hard Lefschetz surjectivity with smooth semipositive coefficients. [arXiv:math/0006205v2](#).

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 5.1 — Proof overview of the main theorem

Proof target : [Theorem 5.1](#) · p. 11

Proof route. Start with the fixed-error input of [Theorem 5.1](#). Corollary 5.6 first adds determinant extraction when starting from twisted tensors. The analytic input on the base meets the supported algebraic output of Theorem 1.1; returning after removing the exceptional correction completes the argument.



Overview of ordinary conversion among the main results

[Open figure ↗](#) [TeX source ↓](#)

1. What remains fixed and what is converted on the base

The determinant fixes one pseudoeffective error P , while the litaka field places the ratios of all N_j over one base. The vertical-minimum divisor M and the metric on $-K_Y + C$ supplied by rank-one direct image have different roles and meet in Theorem 1.1. Details are separated into supported effectivity, fixed-error conversion, and tensor extraction.

[Theorem 5.1 · p. 11](#)

2. Treat rings and invariance as distinct outputs

[Theorem 4.3](#) produces a finitely generated ring from unweighted integrability, supplying the subsequent Exact orders manuscript. This is not an output required by the main proof of ordinary conversion in 5.1. [Theorem 9.2](#) chooses an invariant ratio field and corrects its rational section by invariant base functions. Its [invariant return](#) is followed separately; invariance is not inferred from ordinary conversion.

[Theorem 4.3 · pp. 9–10](#)

[Theorem 9.2 · pp. 32–35](#)

3. Proof of Theorem 1.1 — From weighted approximation to supported nonvanishing

Proof target : [Theorem 1.1 · p. 2](#)

Proof route. Subtract the boundary and Jacobian from the approximation's fixed part, producing a signed divisor. Place its positive part in a big klt boundary. This effective boundary, rather than the signed divisor itself, is the nonvanishing input.

1 Weighted metric

$$\Theta_h \geq \omega, \quad |s_C|^2 e^{-\varphi} \in L^1_{\text{loc}}$$

Compare fixed, Jacobian and boundary orders

[Lemma 2.1; Theorem 2.2 · pp. 5–7](#)

2 Signed approximation

$$\Psi = G/r - \sigma^* C - K_{Y'/Y}, \quad \text{coeff } \Psi < 1$$

Absorb tM in the ample reserve for small rational $t > 0$

[Theorem 1.1 \(proof\); Lemma 3.1 · pp. 7–8](#)

3 Big klt boundary

$$K_{Y'} + \Delta_t \sim_{\mathbb{Q}} \Psi^- + t\sigma^* M$$

Nonvanishing, rational recovery and pushforward

[\[BCHM10, Theorem D, p. 13\]](#)

4 Keep the support

$$M + C' \sim_{\mathbb{Q}} D \geq 0, \quad C' = \sigma_* \Psi^- / t$$

Weighted integrability and supported correction

[Open figure ↗](#) [TeX source ↓](#)

1. Integrability gives strict coefficient inequalities

Subtract a small ample rational divisor A from the metric and resolve $|r(-K_Y + C - A)|$ into its fixed part G and a free part. Lemma 2.1 uses Nadel vanishing, regularity, and comparison from point extension. At a prime with fixed, boundary, and Jacobian orders a, j, k , integrability gives $a/r - j - k < 1$. Thus the positive part of $\Psi = G/r - \sigma^*C - K_{Y'/Y}$ is klt, while its negative part is supported on the boundary and exceptional locus. The condition $|s_C|^2 e^{-\varphi} \in L^1$ must not be replaced by $e^{-\varphi} \in L^1$.

Lemma 2.1; Theorem 2.2 · pp. 5–7

2. Insert M into the ample reserve and retain only the correction

Choose small rational $t > 0$ with $A + tM$ ample. General free and ample representatives with small coefficients give $\Delta_t \geq 0$ such that (Y', Δ_t) is klt, Δ_t is big, and $K_{Y'} + \Delta_t \sim_{\mathbb{Q}} \Psi^- + t\sigma^*M$ is pseudoeffective. Big-boundary nonvanishing gives real-linear effectivity. A rational solution to finitely many rational coefficient inequalities recovers rational linear equivalence. Pushforward gives $C' = \sigma_*\Psi^-/t$ supported on C . The approximation can be fixed before M , whereas t and C' may depend on M .

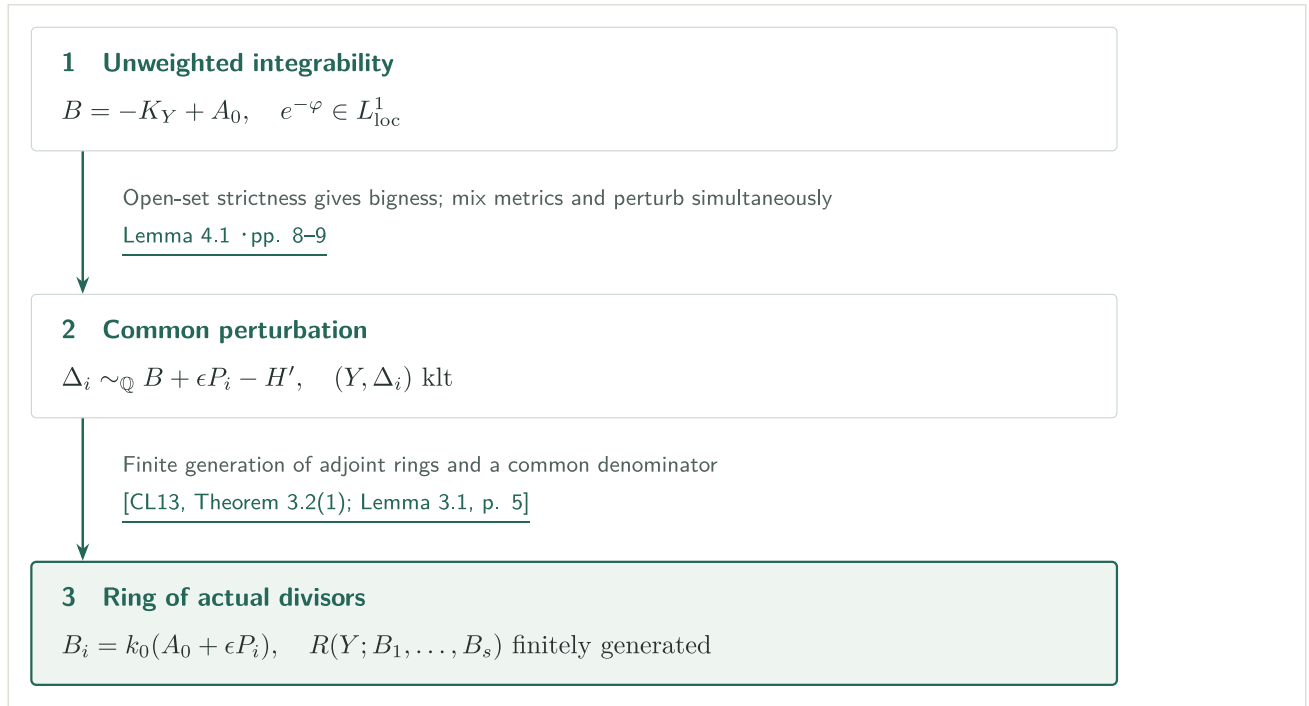
[BCHM10, Theorem D, p. 13]

Theorem 1.1 (proof); Lemma 3.1 · pp. 7–8

4. Proof of Theorem 4.3 — From an unweighted metric to adjoint rings

Proof target : Theorem 4.3 · pp. 9–10

Proof route. Strictness on an ordinary open set gives bigness. Mixing metrics produces a global strict lower bound. Simultaneous perturbations retaining unweighted integrability supply several klt boundaries with a common ample reserve.



Unweighted integrability and finite generation

[Open figure ↗](#) [TeX source ↓](#)

1. A metric allowing one common perturbation

Lemma 4.1 uses the volume criterion, strong openness, small-exponent integrability, and Hölder's inequality to mix the original metric with a big metric by a small rational fraction. Global strictness and frame integrability coexist, so smooth metrics in the finitely many directions P_i can be added or subtracted. A common ample H' and $\epsilon > 0$ give effective klt divisors $\Delta_i \sim_{\mathbb{Q}} -K_Y + A_0 + \epsilon P_i - H'$. The weighted hypothesis of Theorem 1.1 alone does not replace this step.

[Lemma 4.1 · pp. 8–9](#)

2. Transfer finite generation to actual Cartier divisors

Apply multigraded adjoint finite generation to $K_Y + \Delta_i + H'$. A common k_0 clears divisors and rational linear equivalences. Multiplication by the rational functions defining these principal adjustments identifies the desired ring with a Veronese ring, compatibly with all multidegrees. Numerical equivalence would not supply this identification. This is the finite-generation input supplied to the subsequent Exact orders manuscript.

[\[CL13, Theorem 3.2\(1\); Lemma 3.1, p. 5\]](#)

5. Proof of Theorem 5.1 — Collecting the error in one litaka field

Proof target : Theorem 5.1 · p. 11

Proof route. One litaka base for N_1 expresses the differences of all N_j using base principal divisors. Vertical minima define a pseudoeffective divisor M . A rank-one direct image supplies the metric for Theorem 1.1; its exceptional correction can then be removed.



The route from a fixed error to an anticanonical section

[Open figure ↗](#) [TeX source ↓](#)

1. Ratio field, vertical minima, and rank one

After flattening, prepare $X \xleftarrow{\pi} Z \xrightarrow{f} Y$ so that primes mapping into base codimension at least two are π -exceptional. Flatness of the final smooth Z is not asserted. With

$R = (N_1 - N_0)/(m_1 - m_0)$, ratios in the same linear system belong to the base field, giving

$$\pi^* N_j = \pi^* N_0 + (m_j - m_0)(\pi^* R + f^* Q_j), \quad Q_j \sim_{\mathbb{Q}} 0.$$

The horizontal part is nonnegative. The divisor M formed from

$\mu_Q(\pi^*R) = \min_{D \rightarrow Q} \text{ord}_D(\pi^*R) / \text{ord}_D(f^*Q)$ is pseudoeffective by dividing a fixed effective error by $m_j - m_0$ and taking limits. If Y is a point, effectivity is already obtained; if $\dim Y = \dim X$, L is big. In intermediate dimension, direct-image rank at least two for $E = K_{Z/X}$ would, after an ample base twist, produce a ratio enlarging the litaka field, a contradiction.

§5.2, (5.1)–(5.3) · p. 12

2. Build a strict metric and remove the correction's poles

The identities $b\pi^*N_1 \sim f^*H + D_1$ and $N_1 \sim m_1L - P$ give a singular metric on π^*L dominating a positive base curvature. A small mixture with its smooth semipositive metric yields

$\Theta_h \geq \delta f^*\omega_Y$ and $\mathcal{J}(h) = \mathcal{O}_Z$ on the **total space**. The multiplier-ideal direct-image inclusion is therefore an isomorphism, so Păun–Takayama extension applies to the ordinary direct image. Its rank-one reflexive hull is $\mathcal{O}_Y(-K_Y + C)$. Here C is integral and effective, f^*C is exceptional, and Fubini gives integrability of s_C . Theorem 1.1 provides $M + C' \sim_{\mathbb{Q}} G \geq 0$. Push forward $\pi^*R + f^*(G - M - C')$: possible negative coefficients lie only on exceptional primes, yielding an actual nonzero section on X .

[PT14, Setup 3.2.1; Theorem 3.3.5, pp. 20, 24]

§5.2, (5.4) · p. 13

Lemma 5.2 · pp. 11–12

The alternative route in §5.3 retains the original smooth coefficient metric. The toroidal volume theorem supplies a rational correction A_0 and an unweighted integrable metric, requiring full direct-image rank one and a horizontal strictness patch. Strictness on an ordinary open set is not a global Kähler lower bound; the metric mixing in Corollary 5.4 bridges the distinction.

[OpenAI26e, Theorem 2.1, p. 7]

§5.3; Corollary 5.4 · pp. 13–15

6. Proof of Corollary 5.6 — Extracting a fixed error by determinants

Proof target : Corollary 5.6 · p. 16

Proof route. Saturate the span inside one fixed tensor bundle and fix its determinant. Replacing one generic basis vector by a section with large twist preserves unbounded exponents.

1 Twisted tensors of fixed degree

$$H^0(X, (\Omega_X^1)^{\otimes p} \otimes m_j L) \neq 0$$

Vary one generic basis vector to retain unbounded exponents

[Lemma 5.5 · p. 16](#)

2 Determinant of the saturated span

$$A = \det S, \quad P = -A \text{ pseudoeffective}, \quad N_j \sim m'_j L - P \geq 0$$

The case $p = 0$ is immediate; exterior forms embed by alternation

[Corollary 5.6 · p. 16](#)

3 Ordinary nonvanishing

$$H^0(X, mL) \neq 0 \quad (m > 0)$$

Conversion through one fixed determinant error

[Open figure ↗](#) [TeX source ↓](#)

1. Fix one determinant

Saturate the generic span of the images of $\mathcal{O}_X(-mL) \rightarrow (\Omega_X^1)^{\otimes p}$ to obtain S , and set $A = (\bigwedge^{\text{rk} S} S)^{**}$. The manuscript uses the cotangent-subsheaf consequence of generic tangent nefness to make $P = -A$ pseudoeffective. Replacing one member of a finite generic basis produces determinant sections valued in this same A at unbounded positive twists. Saturation and reflexive extension ensure that the sections are global.

[LMPTX23, Theorem 4.1, print p. 1409 / PDF p. 17]

[Lemma 5.5 · p. 16](#)

2. Distinguish tensors, exterior forms, and degree zero

For $p > 0$, apply Theorem 5.1. For $p = 0$, an input section already proves the conclusion. In characteristic zero, alternation embeds exterior forms in tensor powers. The output is nonvanishing for some positive multiple, not every positive integer. Even invariant determinant sections do not allow the ordinary Theorem 5.1 to assert invariant nonvanishing.

[Corollary 5.6 · p. 16](#)

7. Proof of Theorem 9.2 — Retaining an invariant rational section throughout

Proof target : Theorem 9.2 · pp. 32–35

Proof route. The invariant index retaining its zero value supplies invariant forms. Keep their invariant ratio field and an invariant rational section, and perform the correction using invariant base functions.



Invariant index and return retaining the character

[Open figure ↗](#) [TeX source ↓](#)

1. From the invariant index to base curvature

A connected torus acts trivially on $H^q(X, \mathcal{O}_X)$, so the zero-index value is $\chi(X, \mathcal{O}_X) \neq 0$. The index theorem and compact-averaged hard Lefschetz give invariant sections in

$H^0(X, \Omega_X^p \otimes (m+1)L)^T$ for fixed p and unbounded m . The shift by one is required by $K_X + (m+1)L = mL$. Determinant sections σ_j define the invariant ratio field; its relative

algebraic closure remains invariant by connectedness. Lemma 9.3 and semipositive linear algebra put the curvature kernel in the vertical directions and give a horizontal strictness patch. The invariant direct image has rank one; full direct-image rank one is not asserted here.

[OpenAI26b, Theorem 1.1, p. 1]

[DPS01, Theorem 2.1.1; Corollary 2.1.2, p. 8]

Proposition 9.1 · p. 32

2. Check all valuations and return the invariant section

On a separate simplicial toroidal model W , put $V = (\operatorname{div}(p^*\sigma_2) - \operatorname{div}(p^*\sigma_1))/(m_2 - m_1)$ and take its vertical minima M . Then $V \geq f^*M$, and M is pseudoeffective. The smooth orthogonal quotient metric of the invariant direct image gives a volume density ρ . Its correction F_0 satisfies $f^*F_0 \leq E_W = K_{W/X}$. On a resolution $h : W' \rightarrow W$, all coefficients of

$$K_{W'/W} + h^*(E_W - f^*F_0)$$

are greater than -1 : the first term uses the klt property of W , and the second is effective. This verifies unweighted integrability including exceptional divisors over higher-codimension centers. Corollary 5.4 gives $F_0 + \epsilon M \sim_{\mathbb{Q}} G \geq 0$. Return a power of σ_2/σ_1 multiplied by a base function. Both factors are invariant, so the extended section after removing the exceptional correction remains invariant for the natural linearization.

§§9.1–9.2 · pp. 33–35

8. Which papers supply which steps

Input

BCHM big-boundary nonvanishing

Application

Theorem 1.1; Corollary 5.4

Role and checking scope

Matched the klt, big-boundary and pseudoeffective-adjoint hypotheses; input proof not verified.

Input

Corti–Lazić adjoint rings

Application

Theorem 4.3

Role and checking scope

Finite generation with a common ample part and klt boundaries; statement checked.

Input

Păun–Takayama singular direct images

Application

Theorem 5.1

Role and checking scope

$\mathcal{J}(h) = \mathcal{O}_Z$ gives the required generic isomorphism; statement and application checked.

Input

Invariant anticanonical indices

Application

Proposition 9.1 $\rightarrow$ Theorem 9.2

Role and checking scope

Index polynomial retaining the zero value; direct input to invariant conversion.

Input

Metric descent and rank-preserving contractions

Application

Alternative proof in §5.3

Role and checking scope

Toroidal metric with full rank one; separate from the main ordinary-conversion proof.

9. Return to the sources

- Lemma 2.1; Theorem 2.2 · pp. 5–7 — From weighted integrability to strict coefficient bounds.
- Theorem 1.1 (proof); Lemma 3.1 · pp. 7–8 — Nonvanishing retaining correction support.
- Theorem 4.3 · pp. 9–10 — From an unweighted metric to finite generation.
- §5.2, (5.1)–(5.3) · p. 12 — One Iitaka field and vertical minima.
- §5.2, (5.4) · p. 13 — Global strict lower bound and integrability of the distinguished section.
- §§9.1–9.2 · pp. 33–35 — Invariant ratio field and all-valuation integrability.

This article reports the authors' statements and proof structure. The editorial reading covered statements and proof text in §§2–5.5 and §9, and matched the statements and uses of BCHM, Corti–Lazić, Păun–Takayama, and the same-catalogue inputs. The generic-nefness input and hard Lefschetz statement were also checked. Not all external statement checks for point extension and approximation, the volume criterion and strong openness, and toroidal preparation are complete. The alternative proofs and Fano-type model in §§6–8, and the alternative invariant constructions, descent, and extremal-character method in §§10–12 are outside this article's detailed checking scope. This is neither independent verification of every proof nor expert peer review.

Exact orders and invariant anticanonical linear systems

Attain zero order on a boundary ray of a finitely generated ring, then pass from orbit orders to invariant characters

The limit $\text{ord}(N_j)/m_j \rightarrow 0$ does not exclude vanishing at the center in every degree. This manuscript combines a curvature-null condition with finite generation to obtain an effective divisor in the actual anticanonical $\mathbb{Q}$ -linear class with valuation exactly zero. It then translates generic orbit orders into characters. Smooth semipositivity is not replaced by nefness alone.

Manuscript author: OpenAI · September 26, 2026 · 41 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Exact attainment of zero valuation

Let X be a smooth connected projective complex variety of positive dimension n , and fix a smooth semipositive Hermitian metric on $L = -K_X$. Let its curvature form α represent $c_1(L)$ and have maximum complex rank ν . For an ample line bundle G_0 , set

$$\ell(D) = \begin{cases} D \cdot L^\nu \cdot G_0^{n-\nu-1} & \nu < n, \\ 0 & \nu = n. \end{cases}$$

Let v be a discrete valuation of $\mathbb{C}(X)$ centered on X , or the zero valuation, and let M be **any fixed line bundle**. Suppose effective integral divisors N_j and positive integers $m_j \rightarrow \infty$ satisfy

$$\begin{aligned} N_j &\sim m_j L + M, \\ \ell(N_j) &= 0, \\ \frac{v(N_j)}{m_j} &\longrightarrow 0. \end{aligned}$$

Then there is a rational divisor $N \geq 0$ with $N \sim_{\mathbb{Q}} L$ and $v(N) = 0$. Pseudoeffectivity of $-M$, vanishing irregularity, and an Euler-characteristic condition are not hypotheses of this theorem.

Theorem 1.1 · p. 3

Theorem 1.2 — Invariant sections for the natural anticanonical linearization

Let X be smooth connected projective over $\mathbb{C}$, with $-K_X$ smoothly semipositive and $H^i(X, \mathcal{O}_X) = 0$ for every $i > 0$. For every algebraic torus T acting on X , there is a positive integer m with $H^0(X, -mK_X)^T \neq 0$ for the natural anticanonical linearization.

Theorem 1.2 · p. 4

Corollary 5.1 — Zero order along a generic formal arc

Let X be smooth integral projective over $\mathbb{C}$, with $L = -K_X$ smoothly semipositive. Let M be a line bundle with $-M$ pseudoeffective, and let $\gamma : \operatorname{Spec} F[[t]] \rightarrow X$ be a generic formal arc, meaning its fraction point maps to the generic point of X . If effective integral divisors $N_j \sim M + m_j L$, with positive $m_j \rightarrow \infty$, satisfy $v_\gamma(N_j) = o(m_j)$, there are an integer $r > 0$ and an effective integral divisor $D \sim rL$ with $v_\gamma(D) = 0$. A point variety is included.

Corollary 5.1 · p. 12

Proposition 10.1 — Invariant nonvanishing with nonzero Euler number

Let W be smooth connected projective over $\mathbb{C}$, with $q(W) = 0$ and $\chi(W, \mathcal{O}_W) \neq 0$. Assume $A = -K_W$ has a smooth semipositive Hermitian metric, an algebraic torus T acts faithfully on W , and an equivariantly linearized very ample line bundle is available. For the natural linearization, $H^0(W, rA)^T \neq 0$ for some integer $r > 0$.

Proposition 10.1 · p. 25

Theorem 11.1 — Removing a fixed pseudoeffective equivariant error

Let X be smooth connected projective over $\mathbb{C}$, with $q(X) = 0$ and smoothly semipositive $A = -K_X$. Let an algebraic torus T act on X and use the natural linearization on A . Fix a pseudoeffective T -linearized line bundle G . If an unbounded set I of positive integers has nonzero sections $\delta_j \in H^0(X, jA - G)^T$ for every $j \in I$, then $H^0(X, mA)^T \neq 0$ for some integer $m > 0$.

Theorem 11.1 · p. 27

Proposition 12.1 — A finite cone containing controlled invariant primes

Let V be smooth connected rationally connected projective over $\mathbb{C}$, with smoothly semipositive $L = -K_V$. Let an algebraic torus T act on V , and give L its natural linearization. There is a rational polyhedral cone in $\text{Pic}^T(V)_{\mathbb{R}}$, generated by invariant effective rational divisor classes, containing the divisor-linearized class $[D]^T$ of **every invariant prime** D for which $aL - D$ is pseudoeffective for some integer $a > 0$. The generators themselves need not satisfy this control condition.

Proposition 12.1 · p. 31

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Exact orders and invariant anticanonical linear systems](#). External inputs use the following keys.

- [OpenAI26e](#) OpenAI — Metric descent and rank-preserving contractions: Toroidal volume metric and a disappearing rational correction. 2026-09-26.
- [OpenAI26f](#) OpenAI — Integrable metrics and effectivity with controlled boundary: Unweighted finite generation on the main route and weighted approximation on the cone route. 2026-09-26.
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Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — From the null-polar field to exact attainment

Proof target : [Theorem 1.1](#) · p. 3

Proof route. Collect functions with null polar divisors in one relatively algebraically closed field. Its base supports a rank-one volume metric and a finitely generated ring. Two fixed input divisors express the sequence, and an actual generator on the boundary ray supplies the endpoint. The next section details the final removal of the correction by pushforward.



The full exact-attainment route

[Open figure ↗](#) [TeX source ↓](#)

1. The curvature-null condition forces full rank one

The set $k = \{0\} \cup \{h \in \mathbb{C}(X)^\times : \ell((h)_\infty) = 0\}$ is a field. The poles of coefficients in a monic equation bound the poles of an algebraic element, proving relative algebraic closedness and finite generation. Take $X \xleftarrow{\pi} Z \xrightarrow{g} Y$ with $\mathbb{C}(Y) = k$. Bounding the poles of ratios of a very ample base system by one null divisor gives $\ell(\pi_* g^* H) = 0$. Dividing any generic-fiber section of $E = K_{Z/X}$ by its Jacobian section again gives a function with null poles, hence an element of k . Thus the **full adjoint direct image at exponent zero** has rank one.

[Lemmas 3.1–3.2 · pp. 9–10](#)

2. Turn a horizontal strictness patch into integrability and finite generation

For $\nu < n$, the intersection $\int_Z (\pi^* \alpha)^\nu \wedge g^* \omega_H \wedge \pi^* \omega_0^{n-\nu-1}$ vanishes. Semipositivity makes the top-degree form itself zero and gives $\ker \pi^* \alpha \subseteq \ker dg$ on the maximum-rank locus. For $\nu = n$, the inclusion is automatic. A smaller patch then satisfies $\pi^* \alpha \geq cg^* \omega_H$. If $\nu = 0$, a positive-dimensional base cannot occur.

Prepare an equidimensional toroidal model $X \xleftarrow{p} W \xrightarrow{f} Y$ by birational modifications without extending the field. Metric descent, Theorem 2.1, uses full rank one and this patch to give an effective rational correction A_0 with $p_* f^* A_0 = 0$ and an unweighted integrable metric on $-K_Y + A_0$. Its strictness on an ordinary open set allows Controlled boundary, Theorem 4.3, to supply common ϵ, k_0 and a finitely generated ring in finitely many directions. Patch strictness is not itself described as a global lower bound.

§3.2, (3.4)–(3.6) · pp. 10–11

[OpenAI26e, Theorem 2.1, p. 7]

[OpenAI26f, Theorem 4.3, pp. 9–10]

3. Reduce an arbitrary fixed auxiliary line to two directions

Choose input divisors N_1, N_2 with distinct exponents u_1, u_2 . Solve $u_1 t_j^1 + u_2 t_j^2 = 1$ and $t_j^1 + t_j^2 = 1/m_j$. For $D(\mathbf{t}) = t^1 N_1 + t^2 N_2$, this gives

$$N_j/m_j = D(\mathbf{t}_j) + \operatorname{div}_X(h_j), \quad D(\mathbf{t}_*) \sim_{\mathbb{Q}} L.$$

Here $h_j \in k^\times \otimes_{\mathbb{Z}} \mathbb{Q}$ denotes formal rational powers of rational functions, not an extension obtained by adjoining roots. Bounding the principal corrections' poles by N_1, N_2 places them in this field. If $k = \mathbb{C}$, the fixed-support limit already gives the desired divisor. Otherwise, vertical minima are minima of finitely many rational affine functions, and become affine on a short rational segment at the endpoint. Apply finite generation in the two endpoint directions and Proposition 2.2 to obtain $N \sim_{\mathbb{Q}} L$ with $v(N) = 0$.

§4, (4.1) · pp. 11–12

3. Proof of Proposition 2.2 — Remove the correction before comparing valuations

Proof target : Proposition 2.2 · pp. 7–8

Proposition 2.2 — Birational return of a boundary generator

Let X, Y be smooth integral projective complex varieties, W normal projective, $p : W \rightarrow X$ birational, and $f : W \rightarrow Y$ surjective and equidimensional. Let v be a centered discrete valuation on X , or the zero valuation. Let $A \geq 0$ be rational on Y with $p_* f^* A = 0$, and let D_0, D_1 be rational Cartier divisors on X . Put $D(t) = (1 - t)D_0 + tD_1$ and suppose

$$P(t) = \sum_Q \min_{E: f(E)=Q} \frac{\text{coeff}_E p^* D(t)}{\text{ord}_E(f^* Q)} Q = (1 - t)P(0) + tP(1)$$

for $0 \leq t \leq 1$. Suppose rational $t_j \in [0, 1]$ tend to zero and $h_j \in \mathbb{C}(Y)^\times \otimes \mathbb{Q}$ satisfy $N_j = D(t_j) + \text{div}_X(h_j) \geq 0$ and $v(N_j) \rightarrow 0$. If some $\epsilon \in \mathbb{Q}_{>0}$ and integer $k > 0$ make $B_i = k(A + \epsilon P(i))$, $i = 0, 1$, Cartier and $R(Y; B_1, B_0)$ finitely generated, then there is a rational divisor $N \geq 0$ with $N \sim_{\mathbb{Q}} D_0$ and $v(N) = 0$.

Proposition 2.2 · pp. 7–8

1 Sections approaching a boundary ray

$$s_j = f_j x^{a_j} y^{b_j}, \quad a_j / (a_j + b_j) \rightarrow 0$$

Choose a monomial of valuation no greater than that of the sum
[Lemma 2.1 · pp. 5–6](#)

2 One generator on the boundary ray

$$rx^0 y^b, \quad v(r) / (\epsilon kb) \leq -v(D_0)$$

Use vertical minima and limits of horizontal coefficients
[Proposition 2.2 · pp. 7–8](#)

3 Effectivity with correction

$$p^* D_0 + f^*(\text{div}(h_*) + A/\epsilon) \geq 0$$

Remove A by $p_* f^* A = 0$, then use centeredness
[Proposition 2.2 · pp. 7–8](#)

4 Zero value after pushforward

$$N = D_0 + \text{div}(h_*) \geq 0, \quad 0 \leq v(N) \leq 0$$

Selecting a boundary generator and removing the exceptional correction

[Open figure ↗](#) [TeX source ↓](#)

1. Select among finitely many actual generators

Write $t_j = a_j/d_j$, $d_j = a_j + b_j$, and clear denominators so that $s_j = h_j^{\epsilon k d_j}$ is a section in bidegree (a_j, b_j) . Express each section as a sum of generator monomials. At least one monomial has valuation no greater than that of the sum. Every off-ray generator consumes at least one unit of first degree, so their total contribution is $O(a_j)$. Since $a_j/d_j \rightarrow 0$, some degree- $(0, b)$ generator r satisfies $v(r)/(\epsilon k b) \leq -v(D_0)$. A finitely generated numerical cone alone would not provide this rational function.

Lemma 2.1 · pp. 5–6

2. Do not assume the correction has valuation zero

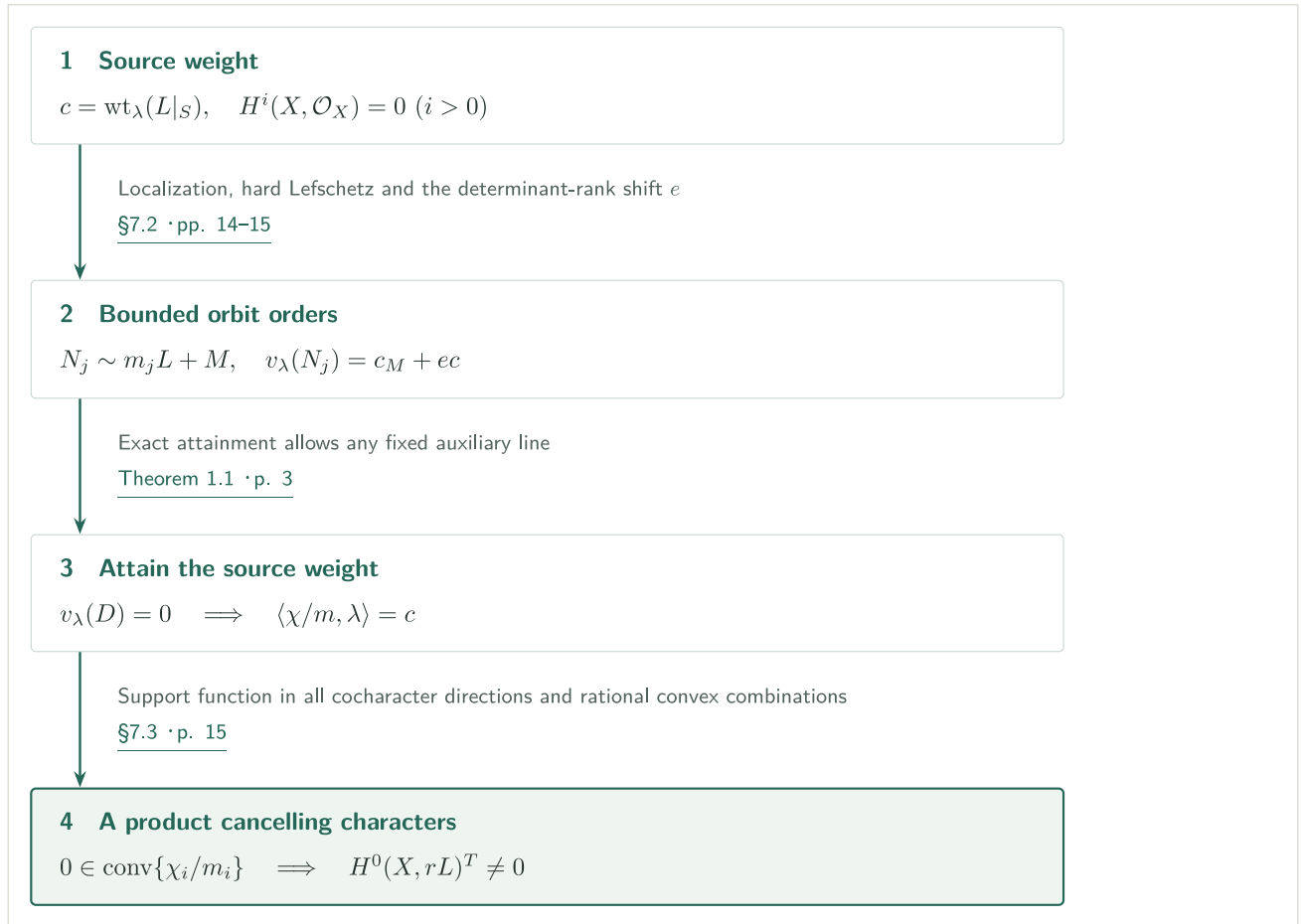
Set $h_* = r^{1/(\epsilon k b)}$. The generator's section inequality and the vertical minima give $p^*D_0 + f^*(\text{div}(h_*) + A/\epsilon) \geq 0$; horizontal coefficients are nonnegative by passage to the limit. Push forward by p to remove the correction, obtaining $N = D_0 + \text{div}_X(h_*) \geq 0$. Only now use nonnegativity of a centered valuation to conclude $0 \leq v(N) \leq 0$. The exceptional correction may have positive valuation on a higher model.

Proposition 2.2 · pp. 7–8

4. Proof of Theorem 1.2 — From source weights to an invariant product

Proof target : Theorem 1.2 · p. 4

Proof route. Extract the source weight in each cocharacter direction from cohomology. Hard Lefschetz and determinants give divisors of bounded orbit order. Exact attainment realizes the source weight by an actual normalized section character, and a finite product cancels characters.



From orbit orders to invariant sections

[Open figure ↗](#) [TeX source ↓](#)

1. Lift an extremal source weight to forms

Use the left action $(g \cdot s)(x) = g(s(g^{-1}x))$. If c is the natural weight of L on the source S of a cocharacter λ , then $c > 0$ for a nonzero effective cocharacter. Concavity of the orbit logarithmic norm for the semipositive metric makes c the largest fixed-component weight. The highest-weight coefficient in localization is $\chi(S, (l-1)L|_S)$, with value one at zero. For large l , cohomology of weight $(l-1)c$ is therefore nonzero. Hard Lefschetz with smooth coefficient lL lifts it to eigensections of $\Omega_X^a \otimes lL$ with fixed a .

§7.1 · p. 14

[DPS01, Theorem 2.1.1; Corollary 2.1.2, p. 8]

2. Retain the determinant-rank shift

Let the generic span of the images have rank e , and saturate its determinant in $\bigwedge^e \Omega_X^a$ to fix a line M . The manuscript uses Calabi–Yau metrics and the subbundle curvature inequality to obtain $\ell(M) \leq 0$. Fixing $e-1$ basis vectors and varying the last produces eigensections of $M + m_j L$ of weight $(m_j - e)c$. If c_M is the weight of M on S , then

$$v_\lambda(N_j) = m_j c + c_M - (m_j - e)c = c_M + ec$$

is bounded. Effectivity and the intersection inequality give $\ell(N_j) = 0$. This construction uses Theorem 1.1 for arbitrary fixed M , without requiring $-M$ pseudoeffective. Replacing the weight by $m_j c$ would lose the necessary rank shift.

§7.2 · pp. 14–15

3. Turn exact order into a finite product of characters

The generic orbit formula $v_\lambda(\operatorname{div} s) = \mu - w$ shows that a zero-order section has normalized character χ/m with $\langle \chi/m, \lambda \rangle = c$. Varying the cocharacter identifies the support function of the closed convex hull of normalized section weights with the source-weight function c . Positivity in every nonzero direction puts zero in the interior, hence in the convex hull of finitely many actual rational weights. Clear a rational convex combination's denominators and multiply the corresponding eigensections to obtain a nonzero invariant section in a positive anticanonical multiple. The trivial torus and point cases are immediate. For an ineffective action, pass to the effective quotient: its entire kernel, including its finite part, acts trivially on the natural anticanonical line, so invariance lifts to T .

Lemmas 6.1–6.2 · p. 13

§7.3 · p. 15

5. Proof of Corollary 5.1 — Turn a formal arc into a centered valuation

Proof target : Corollary 5.1 · p. 12

Proof route. Pseudoeffectivity of $-M$ gives the null condition. Genericity embeds the function field in a Laurent-series field, producing a discrete valuation or the zero valuation.

1 A generic formal arc
 $\gamma : \text{Spec } F[[t]] \rightarrow X, \quad -M \text{ pseudoeffective}$

Normalize a nonzero valuation; retain the zero valuation case

[Corollary 5.1 · p. 12](#)

2 Centered valuation and null condition
 $\mathbb{C}(X) \hookrightarrow F((t)), \quad 0 \leq \ell(N_j) = \ell(M) \leq 0$

Apply Theorem 1.1 and clear a common denominator

[Theorem 1.1 · p. 3](#)

3 Zero order for an integral divisor
 $N \sim_{\mathbb{Q}} L, \quad v_{\gamma}(N) = 0 \quad \implies \quad D = rN \sim rL$

Applying exact attainment to a formal arc

[Open figure ↗](#) [TeX source ↓](#)

1. Verify the null condition and the center

For $\nu < n$, ℓ is nonnegative on the pseudoeffective cone and $\ell(L) = 0$, so

$0 \leq \ell(N_j) = \ell(M) \leq 0$. For $\nu = n$, $\ell = 0$ by definition. The generic arc gives $\mathbb{C}(X) \hookrightarrow F((t))$, and the image of ord_t is zero or $d\mathbb{Z}$. At the closed point's image ξ , the local ring maps into $F[[t]]$ and units have order zero, giving centeredness.

[Corollary 5.1 · p. 12](#)

2. Clear denominators while preserving zero order

Divide a nonzero valuation by d , and retain the zero valuation as permitted in Theorem 1.1. This fixed normalization preserves the $o(m_j)$ condition. Clear the coefficients of the resulting $N \sim_{\mathbb{Q}} L$ and its rational linear equivalence by a common $r > 0$. Then $D = rN \sim rL$ is integral and effective, and its arc order remains exactly zero. For a point, use the zero divisor.

[Corollary 5.1 · p. 12](#)

6. Proof of Proposition 10.1 — Approximating vertices from the Euler number

Proof target : [Proposition 10.1 · p. 25](#)

See Proposition 10.1 above. With weaker cohomological assumptions than Theorem 1.2, the original proof uses the approximate arc transfer of Theorem 9.1; exact attainment of order zero is unnecessary.



From the maximal source weight and fixed order to an invariant product

[Open figure ↗](#) [TeX source ↓](#)

1. Retain the nonzero Euler number at the source

Let P be the convex hull of fiber characters of $A = -K_W$ on the fixed components. For a nonzero cocharacter λ , nefness and the attracting decomposition identify the source weight μ_+ with the support function of P . Faithfulness and the natural tangent-determinant character give $\mu_+ > 0$, so 0 lies in the interior of P .

Expand the fixed-point formula at infinity. The coefficient of the maximal cohomological weight $m\mu_+$ is $Q(m) = \chi(S_+, mA|_{S_+})$; a negative normal weight at every other component removes its contribution to this coefficient. The connected torus acts trivially on $H^j(W, \mathcal{O}_W)$, giving $Q(0) = \chi(W, \mathcal{O}_W) \neq 0$. Riemann–Roch and hard Lefschetz with coefficient $(m+1)A$ produce eigenforms of one degree at unbounded exponents.

§10, (10.1)–(10.2) · pp. 25–27 [DPS01, Theorem 2.1.1; Corollary 2.1.2, p. 8]

2. The determinant-rank shift becomes a fixed arc order

Let s be the rank of the generic span of these forms and M its saturated determinant. The cotangent-tensor inclusion and nef anticanonical bundle make $-M$ pseudoeffective; for $p = 0$ the underlying line is trivial. Completing each varying form with a fixed finite choice of basis forms gives

$$N_i \sim M + m_i A, \quad \text{section weight} = (m_i - s)\mu_+.$$

For the generic orbit arc, order is fiber weight minus section weight. Hence $v(N_i) = \mu_M + s\mu_+$ is constant. The rank shift s is essential to this computation.

§10, (10.1)–(10.2) · pp. 25–27 [LMPTX23, Theorem 4.1, print p. 1409 / PDF p. 17]
Lemmas 6.1–6.2 · p. 13

3. Multiply actual characters close to each vertex

Theorem 9.1 supplies sections of rA with arbitrarily small $v(N)/r$. One eigencomponent has no greater order, and its normalized character approaches μ_+ in the chosen direction. Nonnegative orders for all cocharacters put actual normalized characters in P . An exposing direction for each vertex then supplies characters approaching that vertex. Choose finitely many close enough that their convex hull still contains zero. Clear a rational convex combination and multiply the corresponding nonzero sections to obtain a naturally invariant section. For the trivial torus, approximate transfer already supplies the section.

The approximate-transfer input removes the fixed divisor from the arc order, then extends first jets after nef reduction to prove bigness. Its full analytic extension proof is not reproduced here.

Theorem 9.1 · p. 21 §9, (9.3)–(9.7) · pp. 22–24 Lemmas 6.1–6.2 · p. 13

7. Proof of Theorem 11.1 — Extending an invariant quotient class

Proof target : Theorem 11.1 · p. 27

See Theorem 11.1 above. This route preserves invariance throughout and does not pass through vertex approximation or first-jet generation.

1 Invariant field with controlled poles

$$F = \{u \in \mathbb{C}(X)^T : bA - (u)_\infty \text{ pseudoeffective for some } b > 0\} \cup \{0\}$$

Ratios in F force invariant adjoint rank one for every $l \geq 1$

[§11, \(11.1\)–\(11.3\) · pp. 27–29](#); [Lemmas 8.2–8.4 · pp. 17–20](#)

2 Invariant rank gives a bounded metric and nef descent

$$\tau \in H^0(S, aA_S - g^*L)^T, \quad L \text{ nef}, \quad q(Y) = 0$$

Concentrated curvature and transverse nonintegrability detect the class

[§11, \(11.4\)–\(11.5\) · pp. 29–30](#)

3 A nonzero invariant quotient class on one fiber

$$\mathcal{I}(h_J) = \mathcal{O}_S(-kP_\tau), \quad [s_E(\tau g^*e_L)^{p_0}] \neq 0$$

Lift through the quotient surjection; averaging preserves its nonzero image

[\[CDM17, Theorem 1.1, p. 2\]](#); [§11, \(11.4\)–\(11.5\) · pp. 29–30](#)

4 Extend, average, and remove the exceptional factor

$$H^0(S, mA_S + E)^T \neq 0 \implies H^0(X, mA)^T \neq 0$$

A controlled invariant field, nef descent, and a nonzero invariant quotient

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1. The invariant field forces rank one at every positive exponent

Let F consist of zero and invariant rational functions u for which $bA - (u)_\infty$ is pseudoeffective for some $b > 0$. Pole inequalities and triviality of finite orbits of a connected torus make F relatively algebraically closed and finitely generated. Its base Y has trivial action, and every ample base divisor H satisfies a domination $bp^*A - f^*H$ pseudoeffective.

Three-index invariant ratios belong to F , forcing nonnegative horizontal orders of δ_s/δ_r . Subtract vertical minima on a normal equidimensional model to obtain a base line B and $\tau \in H^0(ap^*A - f^*B)^T$, with order zero on a component above each base prime. Globalize invariant adjoint sections by a sufficiently ample base twist. Their ratios again have anticanonically controlled poles and belong to F , so the invariant adjoint rank is one for every $l \geq 1$. Ordinary rank one is not assumed.

[§11, \(11.1\)–\(11.3\) · pp. 27–29](#)

2. A bounded metric, nef reduction, and an isolated pole

Lemma 8.2 uses smooth coefficient metrics and the orthogonal quotient given by compact averaging. The limit of high moments gives a bounded semipositive metric on B . The condition $q(X) = 0$ implies $q(Y) = 0$. In Lemma 8.3, flattening the nef reduction ensures numerical triviality on every contracted curve. Numerical pullback and removal of torsion using $\text{Pic}^0 = 0$ then descend B up to rational linear equivalence to a nef line L on a smaller base.

If $\dim Z = b > 0$, Lemma 8.4 concentrates curvature of $B_k = kL + \epsilon H$ at one point. For sufficiently large $m = ap_0$, decomposition (11.4) gives an invariant metric h_J on $J = (m+1)A_S$ and a function Ψ with neat analytic singularities satisfying

$$i\Theta_{h_J} + (1+t)i\partial\bar{\partial}\Psi \geq 0 \quad (0 \leq t \leq 1), \quad \mathcal{I}(h_J) = \mathcal{O}_S(-kP_\tau).$$

The ideal equality combines sufficiency from integrability of the remaining singular factor with necessity from nonintegrability of uncanceled integral divisorial poles.

Lemmas 8.2–8.4 · pp. 17–20 [Leh15, Theorem 1.2, p. 1] [EKL95, Theorem 3.1, p. 14]
§11, (11.4)–(11.5) · pp. 29–30

3. Average a lift of a nonzero class and remove the exceptional factor

The Cao–Demailly–Matsumura theorem lifts global classes in $\mathcal{I}(h_J)/\mathcal{I}(h_J e^{-\Psi})$ with coefficient $K_S + J$. Near the chosen fiber, $s_E(\tau g^* e_L)^{p_0}$ belongs to the larger ideal and glues with zero outside the fiber. At a submersion point its coefficient is a unit, while the transverse pole exponent $3b/2 \geq b$ forces nonintegrability. Its quotient class is therefore nonzero.

The base and frame actions are trivial, so this class is invariant. Compact averaging of a lift preserves the same nonzero class. The resulting invariant section of $mA_S + E$ descends after removing the exceptional Jacobian factor, and extends across codimension two on smooth X . This gives a nonzero invariant section of mA . A point base is handled directly by pushing down τ .

[CDM17, Theorem 1.1, p. 2] §11, (11.4)–(11.5) · pp. 29–30

8. Proof of Proposition 12.1 — A finite effective cone retaining characters

Proof target : Proposition 12.1 · p. 31

See Proposition 12.1 above. This bounds the classes of all controlled invariant primes in one finite cone, rather than attaining zero for one valuation.

1 Send all but finitely many primes to one base

$$P \subset \mathbb{C}(V), \quad p^\diamond = \pi_* f^*, \quad p^\diamond[J]^T = 0$$

Singular full rank-one metric and weighted approximation with coefficients < 1
[\[PT14, Setup 3.2.1; Theorem 3.3.5, pp. 20, 24\]](#); [\[OpenAI26f, Theorem 2.2, pp. 6–7\]](#)

2 Weighted integrability and a disappearing boundary

$$|s_J|^2 e^{-\varphi} \in L^1, \quad D_0 = \Psi_-, \quad p^\diamond[D_0]^T = 0$$

Adjoint finite generation, weight-zero projection, and Gordan's lemma
[\[CL12, Theorem A, p. 2\]](#); [§12.3, \(12.10\) · pp. 34–35](#)

3 Finite generation in a neighborhood including characters

$$[D_0]^T \in \text{int conv}\{U_i^T\}, \quad R^T = \mathbb{C}[u_1, \dots, u_b]$$

Express the degree of $D_0 + tR_0$, remove the correction, add finite exceptions
[§12.3, \(12.10\) · pp. 34–35](#)

4 From equality of degrees to a finite effective cone

$$p^\diamond[R_0]^T \in \text{cone}\{p^\diamond[\text{div}(u_i)]^T\}$$

Controlled polar field, disappearing boundary, and a multigraded invariant ring

[Open figure ↗](#) [TeX source ↓](#)

1. Pull back all but finitely many primes from one base

Let P be the field of functions with poles supported on controlled primes. Rational connectedness gives $\text{Pic}^0(V) = 0$. Choose finitely many controlled prime classes spanning the rest. A principal relation involving any further prime has its function in P , making that prime vertical over a model of P . A flat intermediate model ensures that no prime of V maps to codimension at least two on the base.

The rational pullback $p^\diamond = \pi_* f^*$ kills divisors exceptional for base modifications, **including their divisor linearizations**. Apart from finitely many exceptions, a controlled prime D equals $p^\diamond Q$ for a base prime Q . Lemma 12.2 supplies full adjoint rank one at zero and an invariant effective correction J satisfying $p^\diamond[J]^T = 0$.

[§12.1; Lemma 12.2 · pp. 31–33](#)

2. Weighted approximation gives a disappearing correction

Mix an anticanonically dominated ample base metric with the smooth metric by a small coefficient. This produces $\mathcal{I}(h) = \mathcal{O}_U$ and a strict base-direction lower bound. The multiplier-ideal condition supplies the generic isomorphism required by Păun–Takayama. Rank-one fiber integration gives a strict metric on $-K_Y + J$, but the integrable object is $|s_J|^2 e^{-\varphi}$.

Theorem 2.2 of Controlled boundary makes every coefficient of $\Psi = G/r - \sigma^*J - K_{Y'/Y}$ less than one and supports $D_0 = \Psi_-$ on σ^*J and the exceptional locus. Thus $p^\diamond[D_0]^T = 0$. Pushing down the canonical invariant rational section shows that the character component also vanishes. The ordinary rational linear equivalence in the approximation theorem is not automatically treated as equivariant.

[PT14, Setup 3.2.1; Theorem 3.3.5, pp. 20, 24] [OpenAI26f, Theorem 2.2, pp. 6–7]
 §12.2, (12.5)–(12.9) · pp. 33–34

3. Turn invariant finite generation into a cone of classes

Choose a rational polytope around $[D_0]^T$ in $\text{Pic}^T(Y')_{\mathbb{R}}$. A small exceptional adjustment supplies an ample reserve, and each underlying vertex class becomes adjoint to a common ample part and an SNC klt boundary. Adjoint finite generation on a common divisible lattice makes the multigraded ring R finitely generated. The actual linearized line bundles define its torus action; general boundary representatives need not be invariant.

Split generators into eigencomponents. Exact projection to weight zero and Gordan's lemma make R^T finitely generated by multihomogeneous elements u_i . For any invariant effective base divisor R_0 , express $[D_0]^T + t[R_0]^T$ as a rational convex combination of the vertices for sufficiently small $t > 0$. Clear denominators and torsion, express its canonical section in the u_i , and choose a nonzero monomial of the same multidegree. Only **equality of degrees** is needed, not equality of zero divisors with the original section.

Apply $p^\diamond[D_0]^T = 0$. Then $p^\diamond[R_0]^T$ belongs to the cone of $p^\diamond[\text{div } u_i]^T$. Adjoin the finitely many invariant controlled exceptions identified earlier to obtain the required rational polyhedral cone. Its generators are effective but need not be controlled.

[CL12, Theorem A, p. 2] §12.3, (12.10) · pp. 34–35

9. Which papers supply which steps

Input

Metric descent, Theorem 2.1

Application

§3.2; Theorem 1.1

Role and hypotheses

Full adjoint rank one, an equidimensional toroidal model and a strict horizontal patch give an unweighted metric and disappearing correction.

Input

Controlled boundary, Theorem 4.3

Application

§§3.2–4; Theorem 1.1

Role and hypotheses

Common ϵ, k_0 in two rational divisor directions give finite generation of the actual divisor ring.

Input

DPS hard Lefschetz

Application

§7.1; Theorem 1.2

Role and hypotheses

Smooth semipositivity of lL lifts weights in cohomology with coefficient $(l - 1)L$ to forms.

This exact-attainment route does not use the weight-zero index theorem of Invariant indices. The extremal-weight localization calculation is performed in the manuscript. The added sections explain the distinct invariant-conversion and controlled-prime-cone routes. Sections 13–14 descend using geometric structure theorems. These are not direct inputs to Theorem 1.1.

Additional outputs and the limits of this account

Proposition 10.1, Theorem 11.1 and Proposition 12.1 are stated and explained above. The following entries locate remaining alternative routes; their full proofs are not presented.

- [Theorem 9.1 · p. 21](#) assumes $q(W) = 0$, smooth anticanonical semipositivity, pseudoeffectivity of $-M$, and bounded arc orders, and gives normalized order $< \delta$ for every $\delta > 0$. [Corollary 5.1](#) gives a stronger exact conclusion under these hypotheses; the later result supplies a different proof through nef reduction.
- [Theorem 12.4 · p. 36](#) gives natural invariant nonvanishing in the rationally connected case, an alternative proof of a conclusion included in Theorem 1.2. Sections 13 (p. 37) and 14 (pp. 38–39) give two descents to ordinary nonvanishing; their details remain unchecked here.

[§§13–14 · pp. 37–39](#)

10. Return to the sources

- [Lemma 2.1 · pp. 5–6](#) — Select an actual boundary-ray generator.
- [Proposition 2.2 · pp. 7–8](#) — Compare valuations after removing the exceptional correction by pushforward.
- [Lemmas 3.1–3.2 · pp. 9–10](#) — The null-polar field and full rank one.
- [§4, \(4.1\) · pp. 11–12](#) — Express the fixed auxiliary line through two signed divisor directions.
- [§7.1 · p. 14](#) — Extremal cohomological source weight.
- [§7.3 · p. 15](#) — From exact order to actual characters and an invariant product.
- [§10, \(10.1\)–\(10.2\) · pp. 25–27](#) — From the Euler number and determinant-rank shift to vertex approximation.
- [§11, \(11.4\)–\(11.5\) · pp. 29–30](#) — Lift the invariant quotient class without losing nonvanishing.
- [§12.3, \(12.10\) · pp. 34–35](#) — Turn equality of multidegrees into a finite cone retaining characters.

The authors' statements and proof route are organized from §§1–7 of the fixed version. Editorial checks covered the arbitrary fixed M , centered and zero valuations, full rank one, common finite-generation ring, valuation comparison after removing the correction, and the left-action sign and determinant-rank shift. The companion statements on the unweighted ring, toroidal metric and weighted approximation, and DPS hard Lefschetz, were checked. The supplement

checks the stated inputs and uses of CDM17 quotient extension, CL12 multigraded adjoint finite generation, Leh15 numerical pullback, EKL95 Seshadri bounds, the PT14 generic-isomorphism condition, and the LMPTX23 negative determinant. External statements on localization and attracting decompositions, Calabi–Yau metrics, and toroidalization have not all been independently read. The supplement also reads the three results in §§10–12.3, their §8 inputs and the core transfer in §9. External nef-reduction existence, Hilbert/flattening and Picard-torsion statements have not all been rechecked; detailed proofs of Theorem 12.4 and §§13–14 remain outside scope. This is neither independent verification of every proof nor expert peer review.