

OpenAI Math Digest

CATALOGUE 060

The Global Spherical Shell conjecture

Catalogue guide & one proof overview

Independent reading guide · English edition

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Contents

The catalogue guide is followed by the article. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

[Catalogue 060 · Guide](#)[Read on the website ↗](#)

CATALOGUE 060

The Global Spherical Shell conjecture

This catalogue concerns the claim that every minimal class VII surface with positive second Betti number contains a global spherical shell. Its single manuscript constructs rational curves through weighted analysis on a cyclic cover and reduces the conclusion to established curve criteria.

1. Paper in this catalogue

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 060 · 2026-10-22 checked source snapshot

#

01

Manuscript and overview

Global Spherical Shells on Minimal Surfaces of Class VII →

Short name: Global Spherical Shells

September 24, 2026 · 39 pages

[Original paper \(GitHub\) ↗](#)

Principal claim

Claims that a connected minimal compact class VII surface with $b_2 > 0$ admits a holomorphically embedded neighborhood of the standard S^3 with connected complement.

[Theorem 1.1 · p. 1 ↗](#)

Role and available overview

After the square-zero-divisor case, passes from relative Chern classes on the cyclic cover through sections and compact components to a rational-curve count and a shell criterion.

[Read the proof overview →](#)

2. Connection to another catalogue

Theorem 1.1 of Global Spherical Shells on Minimal Surfaces of Class VII is used in the class VII step of Every complex K3 surface is Oka. Corollary 10.2 of the latter manuscript uses it to place minimal class VII surfaces with $b_2 > 0$ in the Kato classification and then applies established Oka criteria. The input is confined to the class VII part of that corollary; it does not enter the K3 theorem or the Enriques and Kodaira-dimension-zero parts.

Theorem 1.1 · p. 1

[OpenAI26k3, Corollary 10.2, p. 54]

3. Sources and checking scope

The official heading for 060 is The Global Spherical Shell conjecture, in Algebraic and complex geometry. The consulted overview places that subject in the catalogue-number range 032–069 and lists 060 on PDF p.8. Actual entries follow the official list without filling gaps in the numbering.

OpenAI Research Catalog · 060 · p. 8

The manuscript is the September 24, 2026 version, with 39 pages. Comparisons cover the hypotheses and every conclusion of Theorem 1.1, its route through §§2–7, the main external shell-criterion statements, and the use in Corollary 10.2 of catalogue 042. Independent verification of all analytic estimates and original-source comparison of some external inputs remain open, as specified in the [article's checking scope](#). Describing the manuscript's claim is distinct from certifying the complete proof.

Global Spherical Shells on Minimal Surfaces of Class VII

Proving compactness of divisor components on a cyclic cover, then recovering a shell from the number of rational curves.

The manuscript claims a proof of the Global Spherical Shell conjecture for minimal class VII surfaces with positive second Betti number. Its central step excludes noncompact components of effective divisors constructed from compactly supported Chern classes, using weighted Dolbeault cohomology and a cylinder estimate. This article follows that argument; it does not certify the correctness of the complete proof.

Manuscript author: OpenAI · September 24, 2026 · 39 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A global spherical shell

Let X be a connected compact complex surface with

$$b_1(X) = 1, \quad b_2(X) > 0, \quad \kappa(X) = -\infty.$$

Assume also that X is minimal: it contains no smooth rational curve of self-intersection -1 . Then there are an open neighborhood U of

$$S^3 = \{z \in \mathbf{C}^2 : |z| = 1\}$$

in $\mathbf{C}^2 \setminus \{0\}$, an open set $\Sigma \subset X$, and a biholomorphism $\varphi : U \longrightarrow \Sigma$ such that $X \setminus \Sigma$ is connected.

[Theorem 1.1 · p. 1](#)

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Global Spherical Shells](#). External inputs use the following keys.

- [Eno81](#) Ichiro Enoki — Surfaces of class VII_0 with curves: Classify the square-zero-divisor case and link the family to shells in note (3); statement compared. *Tohoku Math. J. (2)* 33 (1981), 453–492; publisher PDF.
- [DOT03](#) Georges Dloussky, Karl Oeljeklaus and Matei Toma — Class VII_0 surfaces with b_2 curves: Final input from b_2 rational curves to a shell; main theorem and shell definition compared. *Tohoku Math. J. (2)* 55 (2003), 283–309; publisher PDF.
- [Tel10](#) Andrei Teleman — Instantons and curves on class VII surfaces: Cross-check the shell formulation of Enoki’s result and the canonical basis. *Annals of Mathematics* 172 (2010), 1749–1804; publisher PDF with front matter.
- [Hem11](#) Gollakota V. V. Hemasundar — Koebe’s general uniformisation theorem for planar Riemann surfaces: Theorem 1.1 embeds a planar open normalization into the sphere; statement compared. *Annales Polonici Mathematici* 100.1 (2011), 77–85; publisher PDF.
- [Don87](#) Simon K. Donaldson — The orientation of Yang–Mills moduli spaces and 4-manifold topology: Negative definite intersection input in Lemma 2.2; original theorem unchecked, application formulation checked in Teleman. *Journal of Differential Geometry* 26 (1987), 397–428; bibliographic record from manuscript.
- [HS74](#) Reese Harvey and Bernard Shiffman — A characterization of holomorphic chains: Analytic-cycle graph limit in Lemma 6.1; complete external statement comparison remains open. *Annals of Mathematics* 99 (1974), 553–587; bibliographic record from manuscript.
- [Tru67](#) Neil S. Trudinger — On imbeddings into Orlicz spaces and some applications: Exponential integrability in Lemma 6.5; original-source comparison of the strip formulation remains open. *Journal of Mathematics and Mechanics* 17 (1967), 473–483; bibliographic record from manuscript.
- [OpenAI26k3](#) OpenAI — Every complex K3 surface is Oka: Downstream use: the shell theorem enters only the class VII part of Corollary 10.2. September 23, 2026; fixed commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

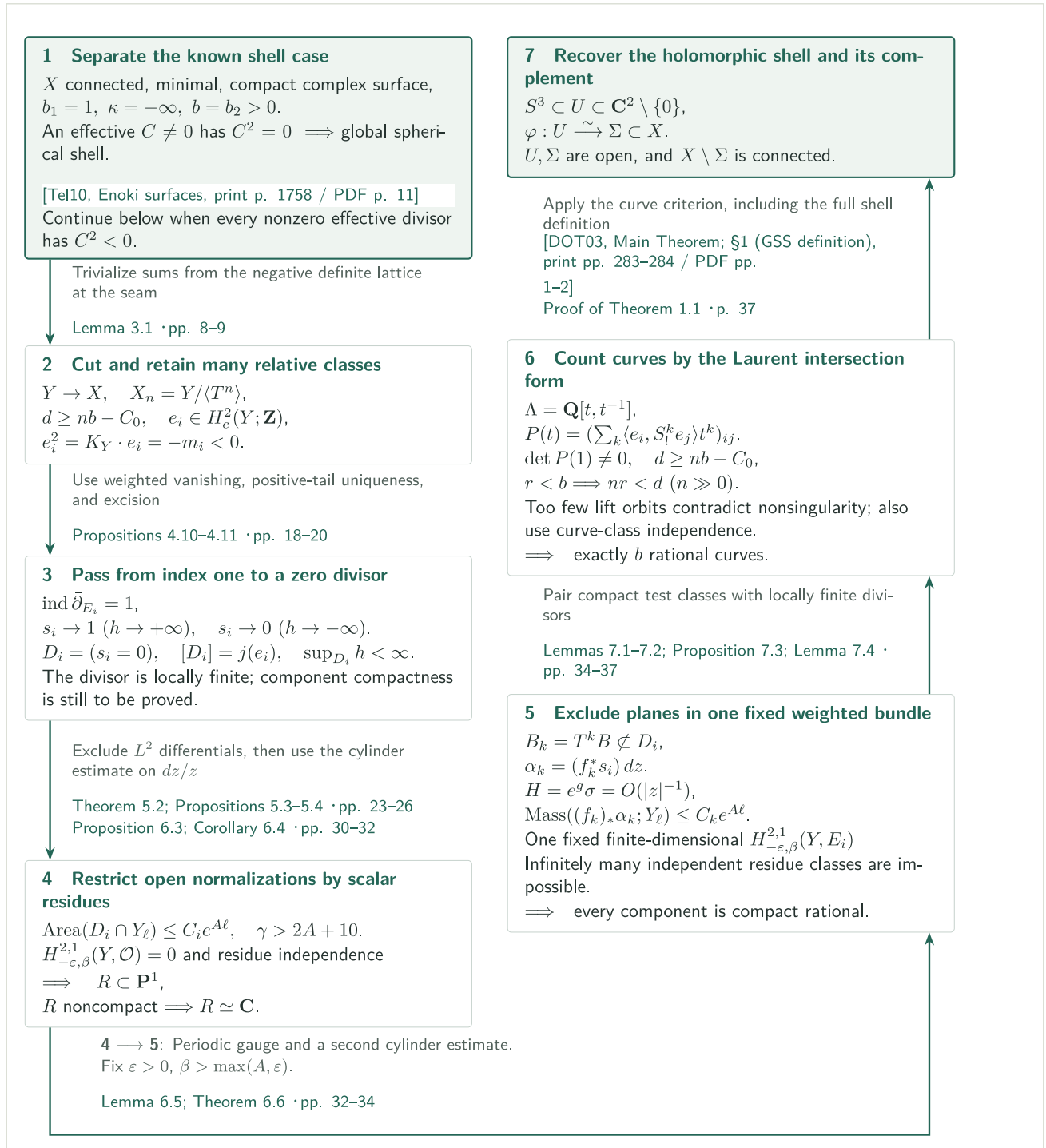
2. Proof of Theorem 1.1 — Divisors, compact components, and curve counting

Proof target : [Theorem 1.1](#) · p. 1

Proof route. A nonzero effective divisor of square zero is handled by an established shell criterion. In the remaining case, the negative definite intersection forms of finite cyclic quotients supply many compactly supported Chern classes, which are realized by zero divisors of holomorphic sections on the infinite cyclic cover. Residue independence and a cylinder estimate force every component to be compact rational. An intersection argument over a Laurent polynomial ring then yields $b_2(X)$ rational curves downstairs. The final curve criterion supplies both the holomorphic shell and its connected-complement condition.

Write $b = b_2(X)$, let $\pi : Y \rightarrow X$ be the infinite cyclic cover associated with a primitive generator of $H^1(X; \mathbf{Z})$, and let T generate the deck group. The manuscript also uses b for the second weight parameter; we denote that parameter alone by β below.

In the diagram, h is a proper height with $h \circ T = h + 1$ and $Lh > 0$, and $Y_\ell = h^{-1}([-\ell - 1, -\ell])$. The map $j : H_c^2(Y; \mathbf{Z}) \rightarrow H^2(Y; \mathbf{Z})$ forgets the support condition, R denotes a component normalization, and $S = T^n$.



Proof overview from Chern classes through rational curves to a global spherical shell

Open figure ↗ TeX source ↓

1. Remove the square-zero case and construct many classes

Under negative definiteness, a nonzero effective divisor C with $C^2 = 0$ gives a shell by the Enoki criterion in Proposition 2.1(i). The remaining branch assumes $C^2 < 0$ for every nonzero effective divisor. Together with minimality, this gives $K_X \cdot B \geq 0$ for every curve B by Lemma 2.2. In particular, effective divisors cannot have negative canonical degree.

The height is chosen with $h \circ T = h + 1$ and $Lh = c > 0$. Each finite cyclic quotient $X_n = Y/\langle T^n \rangle$ is again minimal class VII, with $b_2(X_n) = nb$. Donaldson diagonalization supplies a canonical basis of $H^2(X_n; \mathbf{Z})/\text{Tors}$ with $u_i \cdot u_j = -\delta_{ij}$ and $K_{X_n} = \sum u_j \pmod{\text{Tors}}$. Choose integral cohomology lifts of the u_j . Restrictions to a fixed cutting hypersurface Z lie in a finite torsion group. Sums of distinct basis elements having the same restriction therefore supply sufficiently many classes trivial on Z . Cutting and extending with the prescribed end frames gives

$$d \geq nb - C_0, \quad e_i \in H_c^2(Y; \mathbf{Z}), \quad e_i^2 = K_Y \cdot e_i = -m_i < 0,$$

where C_0 is independent of n . Using sums of distinct basis elements preserves precisely this intersection identity.

[Eno81, Main Theorem; note (3), print p. 454 / PDF p. 2]

[Tel10, Enoki surfaces, print p. 1758 / PDF p. 11]

Proposition 2.1; Lemmas 2.2–2.5 · pp. 5–8

[Tel10, §1.1 (canonical basis), print p. 1756 / PDF p. 9]

Lemma 3.1 · pp. 8–9

2. Use index one to obtain sections with different end limits

The absence of square-zero effective divisors makes the nontrivial character bundles P_λ and $K_X \otimes P_\lambda$ acyclic. Fourier–Laplace transformation in the deck variable transfers this input to Dolbeault complexes with exponential conditions at both ends. Continuing across the trivial character leaves only a scalar constant term. Positive-tail uniqueness, using $Lh > 0$, determines a holomorphic section from its positive constant c_+ and makes canonical-valued holomorphic sections vanish on a positive tail.

Each E_i admits a holomorphic structure with any prescribed finite decay rate γ , and its weighted $H^{0,2}$ vanishes. In weighted spaces allowing positive growth rates at both ends, compact excision and Riemann–Roch give

$$\text{ind } \bar{\partial}_{E_i} = 1 + \frac{1}{2}(e_i^2 - K_Y \cdot e_i) = 1.$$

Thus $\dim H^{0,0} = 1$, and a section s_i can be normalized by $c_+ = 1$. A nonzero negative constant would make its zero divisor D_i compact. Homotopies comparing the section with the prescribed end frames identify its **relative class itself** with e_i . Its effective pushforward would then satisfy $K_X \cdot \pi_* D_i = -m_i < 0$, a contradiction. Consequently $s_i \rightarrow 0$ at the negative end.

What has been obtained at this point is a locally finite divisor $D_i = (s_i = 0)$ bounded above in height, with $[D_i] = j(e_i)$ in ordinary cohomology. Neither compactness of its components nor compactness of the whole divisor follows yet.

Propositions 4.8, 4.10–4.11 · pp. 17–20

3. Use the scalar residue obstruction to restrict open normalizations

Lemma 5.1 bounds the area on negative unit height slabs $Y_\ell = h^{-1}([- \ell - 1, -\ell])$ by $\text{Area}(D_i \cap Y_\ell) \leq C_i e^{A\ell}$. The exponent A depends only on fixed geometry; the holomorphic correction rate is then chosen with $\gamma > 2A + 10$. The constants C_i may vary, while the exponent is common.

A holomorphic differential η on the normalization $f : R \rightarrow B$ of a component defines a $(2, 1)$ -current $f_*\eta$. For $\varepsilon > 0$ and $\beta > \varepsilon$, a mass exponent smaller than β places it in the space with fixed weights $(-\varepsilon, \beta)$. Scalar $H^{2,1}$ vanishes, whereas Proposition 5.3 makes nonzero residue classes on distinct curves linearly independent. A primitive of an exact current first vanishes on the positive tail; analytic continuation then confines its support to the curves. A normal derivative of a distribution supported on a curve cannot produce the required order-zero delta term. This local contradiction is the residue obstruction.

An L^2 holomorphic differential has admissible mass growth by the area estimate and therefore cannot exist. The manuscript constructs such a differential from a handle to prove genus zero, and applies planar uniformization to place an open R inside $\mathbf{P}^1$. If R omits two points, it lies in $\mathbf{C}^*$ and carries dz/z . The cylinder estimate of Proposition 6.3 bounds the corresponding residue mass with the same exponent A , contradicting the scalar obstruction. Any remaining noncompact normalization must therefore be $\mathbf{C}$.

Theorem 5.2; Propositions 5.3–5.4 · pp. 23–26

[Hem11, Theorem 1.1, print p. 77 / PDF p. 1]

Proposition 6.3; Corollary 6.4 · pp. 30–32

4. Apply the cylinder estimate again to exclude plane components

The cylinder estimate separates nonflat strips with a fixed energy cost from arbitrarily long chains of flat strips whose images lie in small coordinate charts. On the latter, the derivative of a periodic holomorphic map has zero constant angular Fourier mode. The sum of derivative bounds, and hence target-height variation, is controlled independently of chain length. For a multiplier w with uniformly bounded L^2 norms on unit strips, this bounds $\int |df|w$ over a target height slab by area over a fixed enlargement of that slab.

If a plane component B existed, then for all sufficiently large k the curves $B_k = T^k B$ would be distinct and not contained in D_i . On their normalizations $f_k : \mathbf{C} \rightarrow B_k$,

$$\alpha_k = (f_k^* s_i) dz$$

is a nonzero holomorphic differential valued in $f_k^* E_i$. Properness and the upper height bound send $|z| \rightarrow \infty$ to the negative end. A periodic gauge g makes the section coefficient σ into an ordinary holomorphic function $H = e^g \sigma$. The stripwise L^2 bound for e^{Reg} and $\sigma \rightarrow 0$ imply $\int_{S_j} |H|^2 \rightarrow 0$ on unit strips S_j of the logarithmic cylinder; the mean-value estimate then gives uniform convergence $H \rightarrow 0$. It therefore extends with value zero at infinity, so $H(z) = O(|z|^{-1})$, canceling the growth in $dz = z d(\log z)$. Exponential integrability in Lemma 6.5 bounds $e^{-\text{Reg}}$ in L^2 on unit strips. A second application of the cylinder estimate gives all currents $(f_k)_* \alpha_k$ the common mass exponent A .

Fixing $\beta > \max(A, \varepsilon)$ puts their classes in **the same finite-dimensional space with the same coefficient bundle**, $H_{-\varepsilon, \beta}^{2,1}(Y, E_i)$. Residue independence applies to every finite subfamily, contradicting the infinitely many B_k . Their individual constants need not be uniformly bounded in k . Every component of every D_i is therefore compact with rational normalization.

Lemma 6.5; Theorem 6.6 · pp. 32–34

5. Recover the curve count from the Laurent intersection form

Put $S = T^n$, $\Lambda = \mathbf{Q}[t, t^{-1}]$, and $F = \mathbf{Q}(t)$, with t translating supports in the S direction. Organize compact-support intersections as

$$P(a, c) = \sum_{k \in \mathbf{Z}} \langle a, S_{\dagger}^k c \rangle t^k.$$

The sum is finite. Evaluating at $t = 1$ recovers the negative definite Gram matrix on the finite cyclic quotient, so $P(e_i, e_j)$ is nonsingular over F .

If there were only $r < b$ rational curves downstairs, their compact lifts would give at most nr orbits under S . For large n , one has $d \geq nb - C_0 > nr$, leaving a nonzero Λ -linear combination x orthogonal to every orbit. All components of each D_j lie in those orbits, so x is also orthogonal to every $S_{\dagger}^k e_j$. This pairs a **compactly supported test class with a locally finite divisor**; only finitely many components contribute. Nonsingularity gives a contradiction, hence $r \geq b$.

Conversely, Lemma 7.4 shows that distinct curve classes are independent when there is no square-zero effective divisor. There are at most b curves in total. Thus there are exactly b rational curves.

Lemmas 7.1–7.2; Proposition 7.3; Lemma 7.4 · pp. 34–37

6. Obtain the holomorphic shell and connected complement

Connectedness, compactness, minimality, $b_1 = 1$, $\kappa = -\infty$, and $b_2 > 0$ are the original hypotheses on X . With the curve count, every hypothesis of the Dloussky–Oeljeklaus–Toma Main Theorem is satisfied, giving a global spherical shell. The cited definition includes a holomorphic

embedding and a nonseparating condition. The manuscript explains that, if necessary, one shrinks to a small annular neighborhood and pushes the complement of the core sphere onto the complement of that open collar, preserving connectedness. The output is therefore the stated U, Σ, φ with connected $X \setminus \Sigma$, rather than merely a topological sphere.

[DOT03, Main Theorem; §1 (GSS definition), print pp. 283–284 / PDF pp. 1–2]

Proof of Theorem 1.1 · p. 37

3. Which papers supply which steps

Source and result

Enoki's square-zero-divisor criterion, also stated by Teleman

Used at

Proposition 2.1(i), step 1

Role and required hypotheses

Minimal class VII, $b_2 > 0$, and a nonzero effective divisor of square zero give a shell

Checking scope

Enoki's Main Theorem and note (3), and Teleman's formulation, compared; full proofs outside scope

Source and result

Dloussky–Oeljeklaus–Toma, Main Theorem

Used at

Proposition 2.1(ii), step 6

Role and required hypotheses

Exactly b_2 rational curves give a holomorphic global spherical shell

Checking scope

Conclusion, definition, and application hypotheses compared; the full input proof is outside scope

Source and result

Donaldson diagonalization

Used at

Lemma 2.2 and Lemma 2.5 on finite cyclic quotients

Role and required hypotheses

Diagonalize the integral intersection lattice of a negative definite closed four-manifold

Checking scope

Application formulation compared in Teleman §1.1; original Donaldson theorem unchecked

Source and result

Planar uniformization, analytic-cycle compactness, and Trudinger exponential integrability

Used at

Proposition 5.4, Lemma 6.1, Lemma 6.5

Role and required hypotheses

Planarity of open normalizations, graph limits, and the periodic-gauge multiplier estimate

Checking scope

Uniformization Theorem 1.1 source-compared; complete statements of the other two inputs remain unchecked

Taubes and Mrowka–Ruberman–Saveliev provide background for periodic-end methods. The Dolbeault Fredholm, continuation, and excision arguments required here are presented in §4; a background citation alone is not represented as direct application of a theorem from those papers.

Input to a consequence in catalogue 042. The relevant section of Every complex K3 surface is Oka explicitly uses the present Theorem 1.1 in the class VII part of Corollary 10.2. It brings minimal class VII surfaces with $b_2 > 0$ into the Kato classification and combines this with established Oka criteria. This is not an input to the K3 theorem, nor to the Enriques or Kodaira-dimension-zero parts of the corollary.

[OpenAI26k3, Corollary 10.2, p. 54]

4. Return to the sources

- Theorem 1.1 · p. 1 — Hypotheses and every shell conclusion
- Lemma 3.1 · pp. 8–9 — Torsion at the seam and many relative classes
- Propositions 4.8, 4.10–4.11 · pp. 17–20 — Index one, end frames, and the zero negative coefficient
- Theorem 5.2; Propositions 5.3–5.4 · pp. 23–26 — Fixed weighted spaces and residue obstruction
- Proposition 6.3; Corollary 6.4 · pp. 30–32 — The target-height-slab estimate
- Lemma 6.5; Theorem 6.6 · pp. 32–34 — Excluding planes with one fixed coefficient bundle
- Lemmas 7.1–7.2; Proposition 7.3; Lemma 7.4 · pp. 34–37 — Locally finite divisors and compact test classes
- Proof of Theorem 1.1 · p. 37 — Final passage to a holomorphic shell and connected complement

External comparisons cover Enoki’s Main Theorem and note (3) (print p.454 / PDF p.2), the DOT Main Theorem and shell definition (print pp.283–284 / PDF pp.1–2), Teleman’s formulations (print pp.1756, 1758 / PDF pp.9, 11), and Hemasundar Theorem 1.1 (print p.77 / PDF p.1). Complete original-source comparison remains open for Donaldson’s theorem, Harvey–Shiffman cycle convergence, and the bounded-strip Trudinger formulation.

The hypotheses and every conclusion of the main theorem, the route above through §§2–7, and the use in Corollary 10.2 of catalogue 042 were read and compared in the manuscript bodies. The account distinguishes the relative class e_i from its ordinary image $j(e_i)$, compactness of components from compactness of a whole divisor, and fixed weights and coefficient bundles from individual mass constants. The complete analytic estimates and proofs of external inputs have not been independently verified; no expert peer review or formal verification is claimed. Corollaries 1.2–1.3 are not included among this article’s main results, and an overview of all their conclusions is outside its scope.