

OpenAI Math Digest

CATALOGUE 057

Campana's abelianity conjecture and special varieties

Catalogue synthesis & 3 proof overviews

Independent reading guide · English edition

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AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 3 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`

Campana's abelianity conjecture and special varieties

Catalog 057 contains three papers on fundamental groups of special varieties. They concern virtual abelianity of the entire group in the compact Kähler setting, virtual nilpotence of class at most two for linear images in the open setting, and a transfer to fourfold pairs using the compact result. The articles distinguish these targets and explain the manuscripts' proof routes and the scope of editorial checking.

1. All three papers in official order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 057 · 2026-10-11 checked source snapshot

#

01

Manuscript and overview

The abelianity conjecture for special compact Kähler manifolds →

Short name: Special Kähler abelianity

2026-09-23 · 50 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Virtual abelianity of the entire ordinary fundamental group of a special smooth compact Kähler manifold, with consequences for Kodaira dimension zero, convexity and uniformization.

[Theorem 1.1; Corollaries 1.2–1.5 · pp. 2–5](#) ↗

Role and available overview

Chooses minimal quotients of Albanese fibre images; the linear and Hilbert-coefficient branches yield positivity, followed by descent retaining inertia. Supplies the third paper.

[Read the proof overview →](#)

#

02

[Manuscript and overview](#)

Two-step monodromy of special quasi-projective varieties →

Short name: Two-step monodromy

2026-09-24 · 19 pages

[Original paper \(GitHub\)](#) ↗[Principal claim](#)

Linear images for special smooth quasi-projective varieties are virtually nilpotent of class at most two. Under exactness, rational unipotent completion is finite-dimensional and of class at most two; a special open surface can have nonspecial general fibres.

[Theorems 1.1–1.3; Corollary 3.1 · pp. 1–3, 8](#) ↗[Role and available overview](#)

Identifies the fibre image with the commutator in canonical unipotent Lie quotients and combines two Hodge-weight bounds to kill triple commutators. The surface example separates the fibre from its image.

[Read the proof overview](#) →

#

03

Manuscript and overview

A conditional abelianity theorem for special fourfold pairs with a half-weight divisor →

Short name: Half-weight fourfold abelianity

2026-10-05 · 14 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Conditional virtual abelianity of the whole group of a special order-two root orbifold of a smooth projective fourfold with a nonempty smooth connected reduced divisor, assuming compact Kähler abelianity.

[Theorem 1.3 · p. 2](#) ↗

Role and available overview

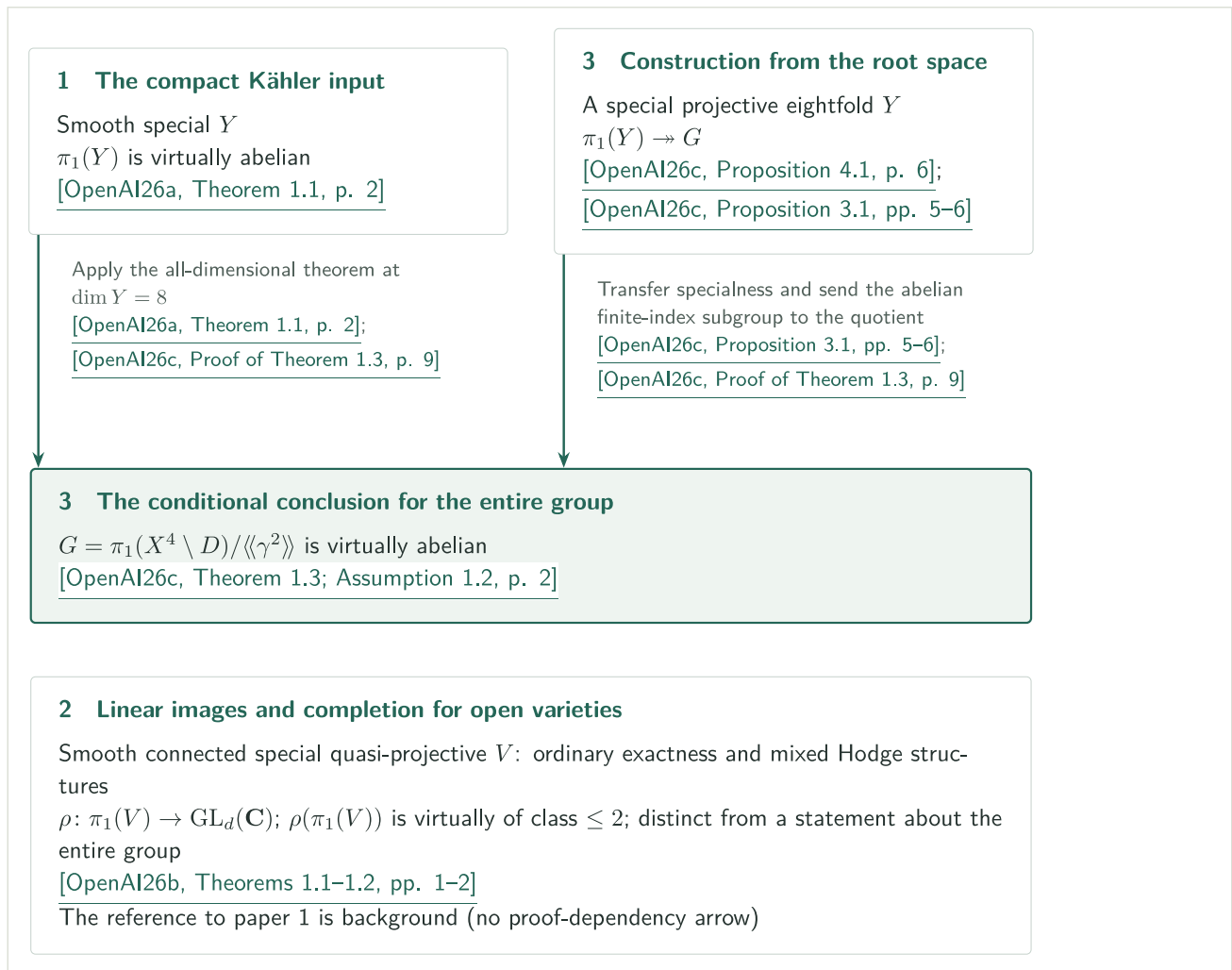
Constructs a projective eightfold with four genus-one factors, transfers specialness and obtains a fundamental-group quotient before applying the first paper.

[Read the proof overview](#) →

2. Where the result for the entire group is used

The diagram keys refer to the following fixed manuscripts. External inputs to each paper are listed in its individual article.

- [OpenAI26a](#) OpenAI — The abelianity conjecture for special compact Kähler manifolds. 2026-09-23; Input for the entire compact Kähler fundamental group.
- [OpenAI26b](#) OpenAI — Two-step monodromy of special quasi-projective varieties. 2026-09-24; Comparison with linear images, completion and nonspecial fibres.
- [OpenAI26c](#) OpenAI — A conditional abelianity theorem for special fourfold pairs with a half-weight divisor. 2026-10-05; Eightfold construction and conditional transfer.



The compact Kähler input and the eightfold construction meet in the root-orbifold conclusion

[Open figure ↗](#) [TeX source ↓](#)

The third paper realizes the root space's fundamental group as a quotient of the fundamental group of a smooth projective eightfold Y . Applying the first paper requires specialness of Y as well as smoothness and projectivity; Propositions 3.1 and 4.1 of the third paper supply these conditions. Virtual abelianity passes to a quotient, giving the conditional conclusion of Theorem 1.3. The third article's overview leads to the arguments establishing these conditions.

[OpenAI26c, Proposition 3.1, pp. 5–6] [OpenAI26c, Proposition 4.1, p. 6]
[OpenAI26c, Proof of Theorem 1.3, p. 9]

The second paper controls linear images and rational unipotent completion through ordinary quasi-Albanese exactness and mixed Hodge structures. It cites the first paper as compact-setting background, not as a direct input to this proof. Its surface example has nonspecial general fibres whose fundamental groups have trivial image in the total group. The [weight argument](#) and [surface calculation](#) explain why these targets must be distinguished.

[OpenAI26b, Theorems 1.1–1.2, pp. 1–2] [OpenAI26b, Theorem 1.3, p. 3]

Source → recipient

Special Kähler abelianity → Half-weight fourfold abelianity

Relation

Explicit premise and direct input

Result, hypotheses and checked use

Virtual abelianity in all dimensions is applied to the constructed special projective eightfold.

Third paper: Assumption 1.2 and final proof on p.9.

Source → recipient

Special Kähler abelianity → Two-step monodromy

Relation

Background

Result, hypotheses and checked use

Comparison with the compact case in §1, p.1 of the second paper; not a proof dependency for Theorems 1.1–1.3.

[OpenAI26a, Theorem 1.1, p. 2] [OpenAI26c, Theorem 1.3; Assumption 1.2, p. 2] [OpenAI26b, §1, p. 1]

3. Sources and checking scope

The official overview title is Campana's abelianity conjecture and special varieties, under Algebraic and complex geometry. CONTENTS instead uses Fundamental groups of special complex varieties and root orbifolds. The order and manuscripts are fixed to official repository commit `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`. In order, the versions are September 23, 2026 (50 pages), September 24, 2026 (19 pages), and October 5, 2026 (14 pages).

[OpenAI Research Catalog · Catalog 057 · p. 7](#)

The articles report the manuscripts' claims. Editorial work checked hypotheses and all stated conclusions, selected main proof passages, and the statements and uses of direct external inputs, followed by review by a different contributor. Complete independent verification does not cover all analytic estimates and descent comparisons in the first paper, the full completed-group-algebra construction in the second, Appendix A's alternative construction in the third, or the complete external proofs. Detailed limits are recorded at the end of each article and in its source records.

The second paper's criticism of fibre specialness in earlier work is restricted to the CDHP arXiv v2 specified in its article. It is not treated as a refutation of that work's separate linear-image theorem. Unchecked generalization or containment relations are not drawn as arrows.

The abelianity conjecture for special compact Kähler manifolds

Minimal quotients of Albanese fibre groups, Hilbert tensors, and descent retaining actual inertia

The manuscript claims virtual abelianity of the entire ordinary fundamental group of a special compact Kähler manifold. On a finite cover, it isolates the image of an Albanese fibre group, derives positivity from an infinite quotient of that image, and descends to a Bogomolov sheaf forbidden by specialness. The argument separates quotients with an infinite finite-dimensional linear image from those whose finite-dimensional linear images are all finite.

Manuscript author: OpenAI · 2026-09-23 · 50 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Virtual abelianity of the ordinary fundamental group

Let X be a connected smooth compact Kähler manifold of arbitrary complex dimension. If X is special, then $\pi_1(X)$ has an abelian subgroup of finite index. Equivalently, there is a connected finite unramified holomorphic cover $X' \rightarrow X$ with abelian fundamental group. The manuscript uses specialness in the sense that no nonzero morphism $L \rightarrow \Omega_X^p$, with $1 \leq p \leq \dim X$, has $\kappa(X, L) = p$. Linearity, residual finiteness and projectivity are not assumed.

[Theorem 1.1 · p. 2](#)

Corollary 1.2 — Kodaira dimension zero

If X is a smooth compact connected Kähler manifold with $\kappa(X, K_X) = 0$, its ordinary fundamental group $\pi_1(X)$ is virtually abelian.

Corollary 1.2 · p. 2

Corollary 1.3 — Conjecture S and bimeromorphic uniformization

Let X be a connected compact Kähler manifold of complex dimension n . Suppose either (i) X is special and has maximal ordinary Γ -dimension $\gamma d(X) = n$, or (ii) its ordinary universal cover is biholomorphic to $\mathbb{C}^n$. In either case, X has a connected finite étale cover X' bimeromorphic to a complex torus. Here γd is the dimension of the Γ -reduction for the identity quotient of $\pi_1(X)$.

Corollary 1.3 · p. 3

Corollary 1.4 — Holomorphic convexity of the universal cover

If X is a connected smooth special compact Kähler manifold, its ordinary universal cover is holomorphically convex. The conclusion does not assert Steinness.

Corollary 1.4 · pp. 3–4

Corollary 1.5 — Compactifiable covers

(i) Let X be a connected normal compact Kähler space, $V \rightarrow X$ a connected infinite étale Galois cover, and Γ its deck group. If V embeds as a Zariski-open subset of a normal compact **complex space**, then Γ is virtually abelian.

(ii) Let X be a connected compact Kähler manifold and U its ordinary universal cover. If U embeds as a Zariski-open subset of a normal compact **Kähler space**, there is a connected finite étale cover $X' \rightarrow X$ whose Albanese map $X' \rightarrow A$ is a locally trivial holomorphic bundle with simply connected compact Kähler fibre F . Writing $q = \dim A$,

$$U \simeq F \times \mathbb{C}^q.$$

Either A or F may be a point. For a nonuniversal cover, (i) does not control the whole group $\pi_1(X)$; (ii) does not assert that X' itself is a product.

Corollary 1.5 · pp. 4–5

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [The abelianity conjecture for special compact Kähler manifolds](#). External inputs use the following keys.

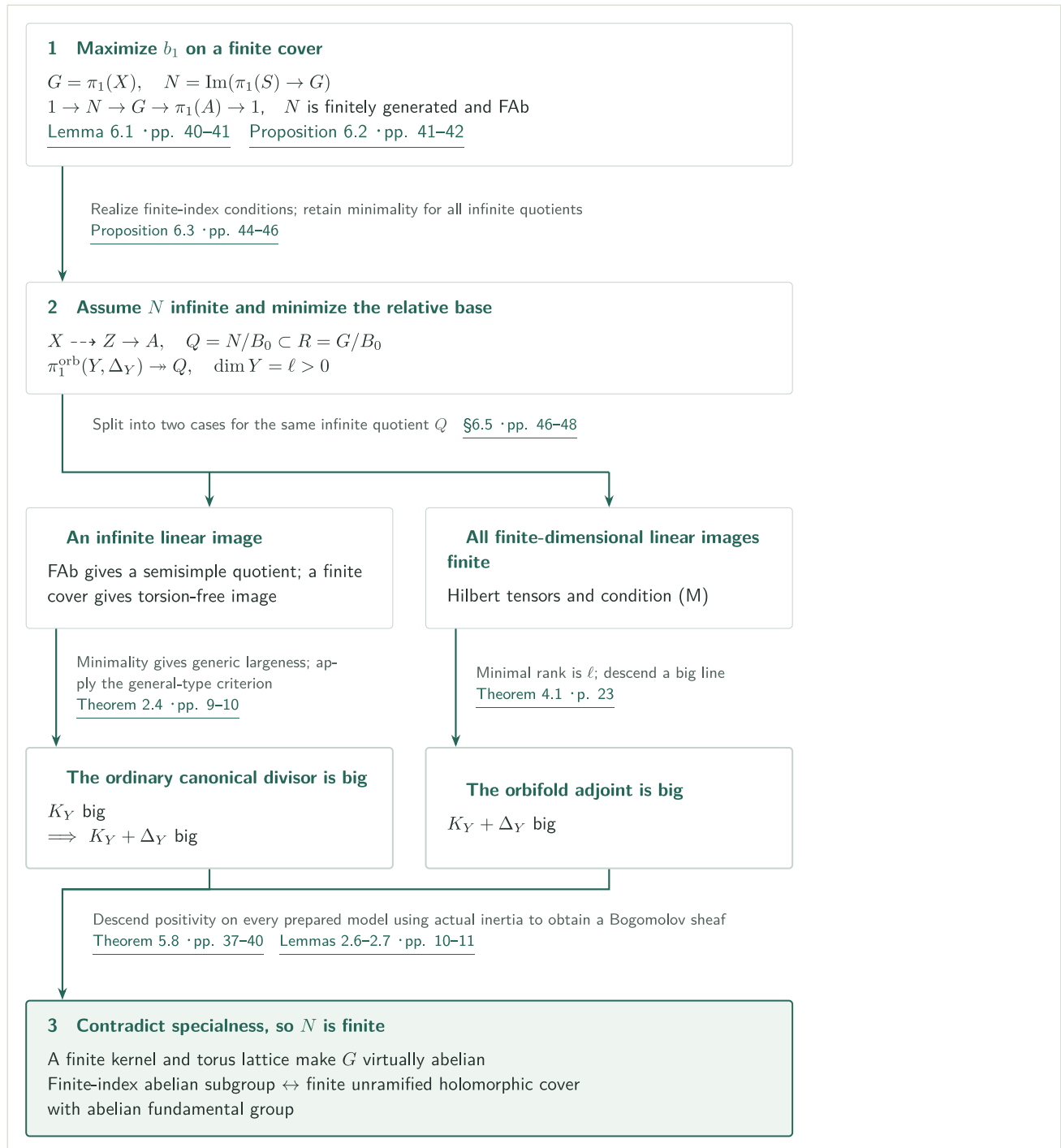
- [Cam04](#) Frédéric Campana — Orbifolds, special varieties and classification theory: Bogomolov-sheaf characterization and preservation of specialness under finite étale covers. arXiv:math/0110051v9.
- [Cam23](#) Frédéric Campana — Albanese map of special manifolds: a correction: Surjectivity and connectedness of Albanese; distinguished from the projective-target condition for eliminating multiple fibres. Ann. Inst. Fourier 73 (2023), 1903–1908; published version.
- [BW16](#) Botong Wang — Torsion points on the cohomology jump loci of compact Kähler manifolds: Isolated rank-one jump-locus characters are torsion, permitting the isogeny in the FAb argument. arXiv:1312.6619v2, 2015-11-30.
- [CCE14](#) Frédéric Campana; Benoît Claudon; Philippe Eyssidieux — Représentations linéaires des groupes kählériens: factorisations et conjecture de Shafarevich linéaire: General type of the semisimple Shafarevich base, and holomorphic convexity for a faithful linear fundamental group. arXiv:1302.5016v3, 2014-02-17; manuscript dated 2014-01-24.
- [Sha06](#) Yehuda Shalom — The algebraization of Kazhdan’s property (T): A finitely generated non-(T) group has a unitary representation with nonzero reduced H^1 . ICM 2006, Vol. II, pp. 1283–1310; full proceedings PDF.
- [Bou02](#) Sébastien Boucksom — On the volume of a line bundle: Positive curvature with strict positivity on an open set gives bigness of the ordinary line. arXiv:math/0201031v1.
- [CP19](#) Frédéric Campana; Mihai Păun — Foliations with positive slopes and birational stability of orbifold cotangent bundles: After projectivity is established, an ordinary big line in an adapted orbifold cotangent tensor gives adjoint bigness. Publ. Math. IHÉS 129 (2019), 1–49; published version.
- [JW21](#) Juanyong Wang — On the Iitaka Conjecture $C_{n,m}$ for Kähler Fibre Spaces: Klt subadditivity over possibly nonprojective tori and Albanese fibrations of log Kodaira dimension zero pairs. Ann. Fac. Sci. Toulouse 30 (2021), 813–897; published version.
- [Cam11](#) Frédéric Campana — Orbifolles géométriques spéciales et classification biméromorphe des variétés kählériennes compactes: Kodaira dimension zero implies specialness, applied with zero boundary. arXiv:0705.0737v8; stamp 2009-07-15; title page 2018-10-25.
- [CH13](#) Benoît Claudon; Andreas Höring; Appendix A by Frédéric Campana — Compact Kähler manifolds with compactifiable universal cover: Specialness of a minimal counterexample, the Albanese bundle, and the appendix implications involving Conjecture S. Bull. Soc. Math. France 141 (2013), 355–375; published version with Appendix A.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1 · p. 2](#) See the hypotheses and conclusion.

Proof route. Pass to a finite cover maximizing b_1 , let $\alpha : X \rightarrow A$ be its Albanese map, and set $G = \pi_1(X)$ and $N = \text{Im}(\pi_1(S) \rightarrow G)$ for a smooth general Albanese fibre S . The goal is finiteness of N ; specialness of S itself is not assumed. If N is infinite, construct a minimal relative base and feed the positivity obtained in two branches into [Theorem 5.8](#).



From the Albanese group through two branches and torus descent to a finite kernel

[Open figure ↗](#) [TeX source ↓](#)

1. Exactness and FAb on finite covers

Finite étale covers preserve specialness, and their Albanese maps are surjective with connected fibres, so $b_1 \leq 2 \dim X$. Choose a cover attaining the maximum; subsequent finite covers have the same first Betti number. Lemma 6.1 establishes

$$1 \longrightarrow N \longrightarrow G \longrightarrow \pi_1(A) \longrightarrow 1.$$

It does not extend the absence of multiple fibres unconditionally to nonprojective tori. If a meridian has nontrivial actual order in G/N , the corresponding effective boundary on the torus is pulled back from an ample divisor on a projective torus quotient. Pulling back a sufficiently small rational multiple through differentials cancels the poles and produces a forbidden Bogomolov sheaf. Thus the meridians vanish and the sequence is exact.

For a further finite cover, Proposition 6.2 considers $V = \mathbf{Hom}(N', \mathbb{C})$ with $N' = N \cap G'$. Nontrivial eigencharacters of the lattice action are isolated points of a rank-one jump locus, hence torsion by [BW16]. An isogeny makes an eigenvector invariant. Inflation–restriction and injectivity of $H^2(A', \mathbb{C}) \rightarrow H^2(X', \mathbb{C})$ would then give extra H^1 , contradicting the Albanese dimension. Lemma 2.7 realizes every prescribed finite-index condition on N by a finite cover of the total space, proving that N is FAb.

[Cam04, Theorems 2.26; 5.12, pp. 27, 56] · [Cam23, Theorem 1.2(1)–(2), print p. 1904 / PDF p. 3] · Lemma 6.1 · pp. 40–41 · Proposition 6.2 · pp. 41–42 · [BW16, Theorem 1.3, p. 2]

2. A minimal relative base carrying an infinite quotient

Assume N is infinite. Across all compatible finite covers and general Albanese fibres S , minimize $\dim T$ among fibrations $S \dashrightarrow T$ for which N/C is infinite, where C is the fibre-group image. Write the minimum as $\ell > 0$. Proposition 6.3 globalizes a minimizer, using cycle families and Γ -reduction, to $X \dashrightarrow Z \rightarrow A$. For the actual fibre-group image B_0 of the first map,

$$Q = N/B_0 \subset R = G/B_0, \quad \dim Y = \ell,$$

and the general torus fibre carries a surjection $\pi_1^{\text{orb}}(Y, \Delta_Y) \twoheadrightarrow Q$. Boundary coefficients use actual meridian orders in R , and $Q \hookrightarrow R$ preserves those orders on restriction. For every finite-index Q_0 , choose **one cover first**, $S' \rightarrow S$, whose image $Q' \subset Q_0$ satisfies $\text{gdim}(S', \psi\nu') \geq \ell$ for **every infinite quotient** of Q' . This is condition (M) of Theorem 4.1; choosing a different cover separately for each infinite quotient would be weaker.

Proposition 6.3 · pp. 44–46 · Lemmas 2.6–2.7 · pp. 10–11

3. The branch with an infinite linear image

An infinite linear image of the FAb group Q cannot be virtually solvable. Pass to finite index, take a simple adjoint factor of the semisimple quotient of its Zariski closure, and use Selberg's lemma and Lemma 2.7 to realize a torsion-free image on a finite cover of the total space. Finite-order meridians die in that image, so the representation extends to the ordinary group $\pi_1(Y)$. If it were not generically large, composing its Γ -reduction with $S' \dashrightarrow Y$ would give an admissible base of dimension below ℓ . The semisimple general-type criterion of [CCE14] makes K_Y , and hence $K_Y + \Delta_Y$, big.

§6.5 · pp. 46–48 · [CCE14, Theorems 1; 6.3, pp. 2, 29]

4. The branch where every finite-dimensional linear image is finite

Ordinary finite-dimensional representations do not exclude this case. The discussion of Theorem 4.1 constructs holomorphic tensors with unitary Hilbert coefficients and forces their minimal coefficient-map rank to be full by (M). This gives bigness of $K_Y + \Delta_Y$. Both branches retain the actual inertia of a fixed group quotient and **choose a fresh very general fibre on each prepared model**, using the same minimum. Bigness on a single model would not supply the descent hypothesis.

Theorem 4.1 · p. 23 · §6.5 · pp. 46–48

5. Descent, a finite kernel, and the covering formulation

The discussion of Theorem 5.8 pulls orbifold pluriforms from a positive-dimensional general-type base to a Bogomolov sheaf on a finite cover, contradicting specialness. Thus N is finite. Lemma 2.6 passes to the finite-index centralizer of the finite kernel and kills its finite central commutator subgroup by taking sufficiently large powers of generators. The resulting abelian subgroup has finite index. This conclusion passes from the initial finite cover back to the original group. Lifting the complex structure to the connected finite topological cover corresponding to that subgroup gives the equivalent unramified holomorphic covering statement.

Theorem 5.8 · pp. 37–40 · Lemmas 2.6–2.7 · pp. 10–11 · §6.5 · pp. 46–48

3. Proof of Theorem 4.1 — From Hilbert coefficients to a big adjoint divisor

Proof target : Theorem 4.1 · p. 23

Theorem 4.1 — Positivity under the minimality condition (M)

Let Y be a connected smooth compact Kähler manifold of dimension $\ell > 0$, let $\Delta = \sum_i (1 - m_i^{-1})D_i$ have SNC support with integers $m_i \geq 2$, and let $\mathcal{Y}$ be its root orbifold. Suppose $q : \pi_1^{\text{orb}}(\mathcal{Y}) \twoheadrightarrow Q$ is surjective, Q is infinite, and every finite-dimensional complex linear representation of Q has finite image.

Suppose a connected smooth compact Kähler manifold S admits a dominant meromorphic map $a : S \dashrightarrow Y$ and a homomorphism $\nu : \pi_1(S) \rightarrow Q$ with finite-index image, compatible with a and q on a dense analytic Zariski-open set where a is holomorphic and takes values outside the boundary. For every finite-index $Q_0 \leq Q$, suppose there is a connected finite étale cover $S' \rightarrow S$ such that its induced map ν' has image $Q' \subset Q_0$ and

Condition (M):

$$\begin{aligned} \text{gdim}(S', \psi \circ \nu') &\geq \ell, \\ \text{for every } \psi : Q' &\twoheadrightarrow H \text{ with } H \text{ infinite.} \end{aligned}$$

Then $K_Y + \Delta$ is big. The cover may depend on Q_0 , but after it is selected it must work for every infinite quotient.

Theorem 4.1 · p. 23

1 Two analytic inputs

Nonamenable: low scalar spectrum / amenable: reduced $H^1 \neq 0$

Pass low spectrum to a Hilbert ultrapower / obtain cohomology from non-(T)
[Theorem 3.1 · p. 11](#) [\[Sha06, Theorem 4.2, print p. 1300 / PDF p. 1310\]](#)

2 Coefficient lines and positive curvature

$0 \neq \eta \in H^0(\mathcal{Y}, \Omega_{\mathcal{Y}}^p \otimes H_\rho)$

Exterior power gives rank one with infinite coefficient span

Discretize image-germ monodromy; compare finite-volume fibres with (M)
[Proposition 4.6 · pp. 27–30](#) [Lemma 4.7 · pp. 30–31](#)

3 The minimum rank is full

$F : U^\circ \rightarrow \mathbb{P}(B), \quad \text{rank } F = s$
 $\ell \leq \text{gdim}(S', Q' \rightarrow I) \leq s \leq \ell$

Curvature is strictly positive on an open set; the volume criterion gives bigness
[§4.5 · pp. 31–32](#) [\[Bou02, Theorem 1.2, p. 2\]](#)

4 An ordinary big line and projectivity

Descend A by Galois products and powers killing stabilizers
 $A \text{ big} \implies Y \text{ Moishezon} + \text{Kähler} \implies Y \text{ projective}$

Apply only after projectivity and an ordinary big line are established
[\[CP19, Theorem 7.11, p. 42\]](#)

5 Apply positivity on an adapted cover

$\pi^* A \longrightarrow (\pi^* \Omega^1(Y, \Delta))^{\otimes m} \implies K_Y + \Delta \text{ big}$

Hilbert tensors, minimal rank, and the ordinary big line

[Open figure ↗](#) [TeX source ↓](#)

1. A nonzero tensor with infinite-dimensional coefficient span

The nonamenable case uses Theorem 3.1: on a normal cover of a compact Kähler orbifold with infinite deck group, zero lies in the spectrum of the scalar $(p, 0)$ -Dolbeault Laplacian for some $p \in \{0, \dots, n\}$. Under contrary spectral gaps, manuscript §3 constructs weighted inverses and finite-propagation smoothing. They preserve a nonzero pairing of the compact diagonal class, while norm and support-volume estimates force that pairing to zero. Proposition 4.3 folds low-spectrum forms down to regular coefficients and obtains a nonzero holomorphic form in a Hilbert ultrapower.

In the amenable case, an infinite finitely generated group lacks property (T), so [Sha06] supplies nonzero reduced H^1 . Remove the contribution from finite-dimensional subrepresentations and use the Kähler type decomposition to obtain a holomorphic one-form. In either case the coefficient representation can have no nonzero finite-dimensional invariant subspace. Degree zero is excluded because a holomorphic section would be parallel and supply an invariant line, so $p > 0$. Exterior powers produce coefficient rank one with infinite-dimensional span, hence a coefficient-line map and an orbifold line with positive curvature. Mere nonvanishing of a tensor would not contradict specialness.

Theorem 3.1 · p. 11 · Lemma 3.6; §3.6; (35) · pp. 21–23 ·
 [Sha06, Theorem 4.2, print p. 1300 / PDF p. 1310] · Proposition 4.3; Lemmas 4.4–4.5 · pp. 24–27

2. Minimal rank and discrete monodromy

Minimize the rank of such coefficient-line maps on finite orbifold covers, obtaining $s \in \{1, \dots, \ell\}$. Proposition 4.6 works with the manifold of regular image germs and a Lie-group closure of the monodromy. If its identity component moves the image, spectral decomposition of the unitary coefficients produces line tensors of smaller rank. Minimality and the finite-linear-image hypothesis force that component to act trivially, leaving an infinite discrete image.

Lemma 4.7 uses finite mass of the curvature current and coarea to give almost every fibre finite volume, with compact analytic closure. Its group image lies in a finite stabilizer of the discrete action, so the Γ -reduction contracts it and $\mathbf{gdim} \leq s$. Pull back to the single cover S' selected by (M); for the same infinite monodromy quotient, (M) gives $\mathbf{gdim} \geq \ell$. Thus $s = \ell$.

Proposition 4.6 · pp. 27–30 · Lemma 4.7 · pp. 30–31 · §4.5 · pp. 31–32

3. Descending an ordinary line before invoking projectivity

On a Galois orbifold cover, tensor the deck translates of the line and take a power killing chart stabilizers to descend to an ordinary line bundle. Full-rank curvature is strictly positive on a nonempty open set, so the volume criterion of [Bou02] makes this line big. Thus Y is Moishezon and, being Kähler, is projective. Only then is the tensor inclusion interpreted on an adapted projective cover and Theorem 7.11 of [CP19] applied to make $K_Y + \Delta$ big. Neither projectivity nor pseudoeffectivity of $K_Y + \Delta$ was assumed at the outset.

§4.5 · pp. 31–32 · [Bou02, Theorem 1.2, p. 2] · [CP19, Theorem 7.11, p. 42]

4. Proof of Theorem 5.8 — Torus descent retaining actual inertia

Proof target : Theorem 5.8 · pp. 37–40

Theorem 5.8 — Impossibility by descent over a torus

Let M be a special smooth compact Kähler manifold, $\alpha : M \rightarrow A$ a surjection to a complex torus with connected fibres, and $M \dashrightarrow Z \xrightarrow{g} A$ a factorization by fibrations with $d = \dim Z - \dim A > 0$. For a smooth general fibre F_c of the first map, use the actual meridian orders in the fixed quotient

$$R = \pi_1(M) / \text{Im}(\pi_1(F_c) \rightarrow \pi_1(M))$$

to define Δ_Z on each prepared smooth model. It is impossible that, on a cofinal collection of prepared models, $K_Y + \Delta_Z|_Y$ is big for a very general g -fibre Y . The torus may be a point. Preparation resolves maps, makes the boundary SNC and selects a smooth locus; actual orders are read again on new exceptional divisors. Cofinality means that every prepared model is dominated by a member through a holomorphic modification.

Theorem 5.8 · pp. 37–40

1 Fibre positivity on cofinal prepared models

$K_Y + \Delta_Z|_Y$ big, $d = \dim Y > 0$

Use klt subadditivity and preserve sections with an auxiliary boundary
[JW21, Main Theorem A(II), print p. 815 / PDF p. 4] Theorem 5.8 · pp. 37–40

2 A minimum b and the litaka fibration

$b = \min \kappa(Z, K_Z + \Delta_Z) \geq d$
 $K_{Z_1} + \Delta_1 = \mu^*(K_{Z_0} + \Delta_{Z_0}) + E, \quad E \geq 0$

Use the zero-Kodaira litaka fibre and Albanese; an isogeny gives the bundle
[JW21, Theorem C, print p. 818 / PDF p. 7] Theorem 5.8 · pp. 37–40

3 Recover true inertia on a torus bundle

$Z^* = W_1 \times_{A_1/T_1} A_1 \xrightarrow{p} W_1$
 $K_{Z^*} + \Delta_{Z^*} = p^*(K_{W_1} + \Delta_{W_1})$

Equivariant comparison and minimality give bigness; differentials cancel poles on a neat model
Definition 5.1; Lemmas 5.2–5.5 · pp. 33–35 Theorem 5.8 · pp. 37–40

4 From a general-type base to a Bogomolov sheaf

$K_{W_1} + \Delta_{W_1}$ big
 $L \subset \Omega_{M_1}^b, \quad \kappa(M_1, L) = b > 0 \implies \perp$

From positivity on prepared fibres to a general-type base and a Bogomolov sheaf

Open figure ↗ TeX source ↓

1. Minimal log Kodaira dimension and the auxiliary boundary

Klt subadditivity over tori in [JW21] gives $\kappa(Z, K_Z + \Delta_Z) \geq d > 0$. Choose its minimum b in that cofinal collection and resolve the Iitaka map to $h : Z_1 \rightarrow W$. At this step introduce an auxiliary klt boundary Δ_1 , distinct from true inertia, whose exceptional coefficients are sufficiently close to one so that

$$K_{Z_1} + \Delta_1 = \mu^*(K_{Z_0} + \Delta_{Z_0}) + E, \quad E \geq 0 \text{ exceptional.}$$

This preserves sections and the Iitaka map; a general Iitaka fibre H has log Kodaira dimension zero. The argument does not assume birational invariance of the log Kodaira dimension of the true inertial adjoint on arbitrary models.

Theorem 5.8 · pp. 37–40 · [JW21, Main Theorem A(II), print p. 815 / PDF p. 4]

2. A torus-bundle model and recovery of the true boundary

Apply Theorem C and subadditivity of [JW21] to H . Restricting bigness to common general fibres of h and g shows that H maps birationally by an isogeny onto a torus translate. After a finite isogeny, a birational model is the principal torus bundle

$$Z^* = W_1 \times_{A_1/T_1} A_1, \quad p : Z^* \rightarrow W_1.$$

True inertia has no horizontal component, so $\Delta_{Z^*} = p^*\Delta_{W_1}$ and $K_{Z^*} = p^*K_{W_1}$. An equivariant comparison for the finite deck group returns to a cofinal prepared model over the original torus; the minimum b then makes $K_{W_1} + \Delta_{W_1}$ big. A key point is to recover the true boundary rather than substitute the auxiliary exceptional coefficients for it.

Theorem 5.8 · pp. 37–40 · [JW21, Theorem C, print p. 818 / PDF p. 7]

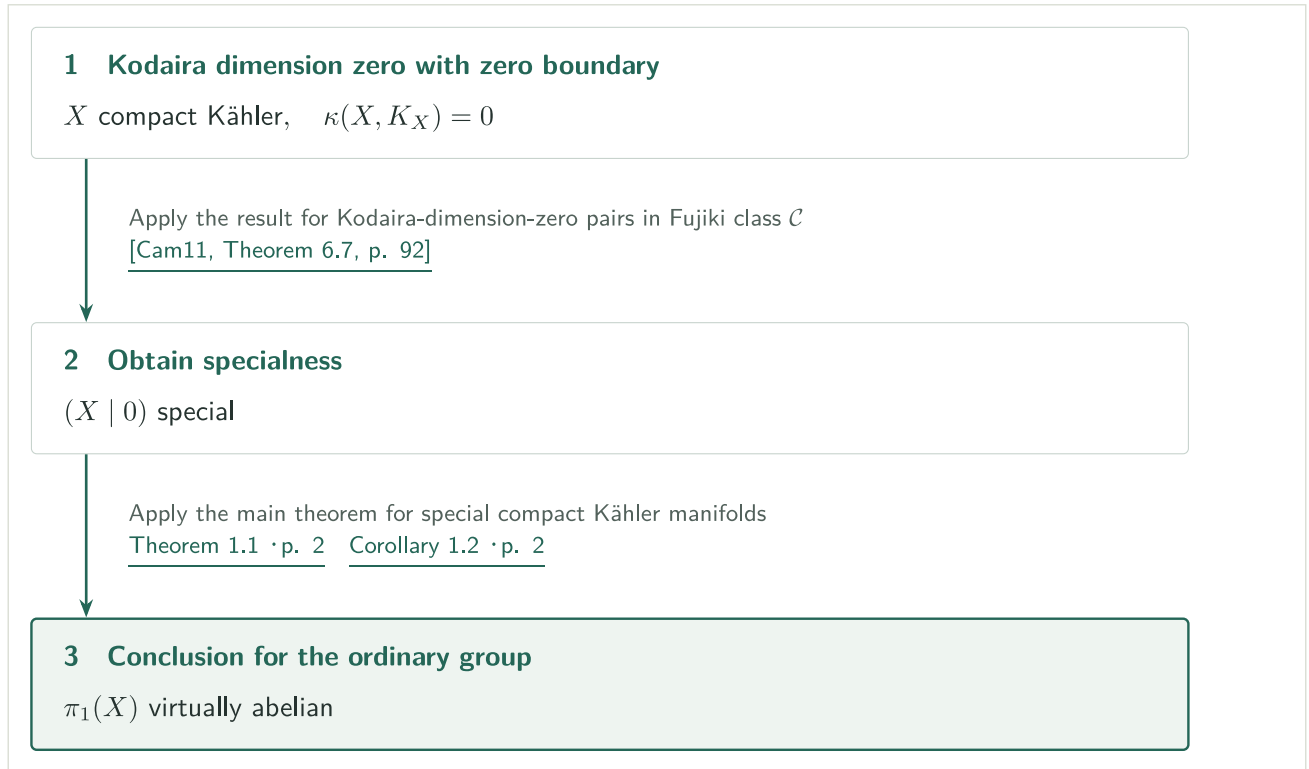
3. Cancellation of poles on a neat model

Bigness remains obtainable after any necessary further modification, so the map $M_1 \dashrightarrow W_1$ can be made neat. Lemma 5.2 ensures that every boundary order divides the multiplicity of every source divisor above it. Lemma 5.5 cancels the poles of orbifold pluriforms with differential vanishing, producing a line in $\Omega^{\dim W_1}$ with Iitaka dimension $\dim W_1 > 0$. This contradicts specialness of the finite étale cover M_1 .

Definition 5.1; Lemmas 5.2–5.5 · pp. 33–35 · Theorem 5.8 · pp. 37–40

5. Proof of Corollary 1.2 — Applying the theorem when $\kappa=0$

Proof target : Corollary 1.2 · p. 2 See the hypotheses and conclusion.



Kodaira dimension zero, specialness, and virtual abelianity

[Open figure ↗](#) [TeX source ↓](#)

1. Input specialness with zero boundary

Apply Theorem 6.7 of [Cam11] to $(X \mid 0)$ and then Theorem 1.1. The hypothesis is $\kappa = 0$; it is not replaced by $c_1(X) = 0$. The manuscript also mentions Orbifold Iitaka in catalogue 033 as another record of this established specialness implication. It is not an additional direct input to this proof.

[\[Cam11, Theorem 6.7, p. 92\]](#) · [Theorem 1.1 · p. 2](#) · [Corollary 1.2 · p. 2](#)

6. Proof of Corollary 1.3 — Maximal Γ -dimension and Albanese

Proof target : [Corollary 1.3 · p. 3](#) See the hypotheses and conclusion.



Two hypotheses leading to bimeromorphic Albanese on a finite cover

[Open figure ↗](#) [TeX source ↓](#)

1. Specialness and maximal ordinary Γ -dimension

Theorem 1.1 gives a finite étale cover X' with torsion-free abelian fundamental group. Specialness and maximal ordinary Γ -dimension persist: a positive-dimensional subvariety with finite group image upstairs would, under the finite map, yield one with finite group image downstairs. The map from $\pi_1(X')$ to its Albanese lattice is an isomorphism. A smooth general Albanese fibre therefore has trivial group image, so maximal Γ -dimension forces it to be zero-dimensional. Connectedness makes it a single point, proving that $X' \rightarrow \text{Alb}(X')$ is bimeromorphic.

[Corollary 1.3 · p. 3](#) · [\[Cam23, Theorem 1.2\(1\)–\(2\), print p. 1904 / PDF p. 3\]](#)

2. Affine space as universal cover

The manuscript uses Remarks A.4(2)–(3) in Appendix A of [CH13]. Abelianity yields Conjecture S as above, and Conjecture S gives the bimeromorphic Iitaka conclusion for $U \simeq \mathbb{C}^n$. The latter is a separate geometric implication supplied by the appendix, rather than a direct assertion that the shape of the universal cover makes the Albanese map an isomorphism.

7. Proof of Corollary 1.4 — Faithful linearization and holomorphic convexity

Proof target : Corollary 1.4 · pp. 3–4 See the hypotheses and conclusion.



A finite-index abelian subgroup, faithful induction, and linear Shafarevich

[Open figure ↗](#) [TeX source ↓](#)

1. Construct a faithful representation of the whole group

The finite-index abelian subgroup $H \simeq \mathbb{Z}^r \oplus T$ of the finitely generated group G has a faithful finite-dimensional complex representation σ . Its induced representation $\mathrm{Ind}_H^G \sigma$ is faithful: a kernel element preserves the summand indexed by the coset H , hence lies in H , and acts there through the faithful σ . Corollary 5.9 of [CCE14] applies to this faithful linearity of G , giving holomorphic convexity of the ordinary universal cover. Neither projectivity nor maximal Γ -dimension is added.

Corollary 1.4 · pp. 3–4 · [CCE14, Corollary 5.9, p. 26]

8. Proof of Corollary 1.5 — A minimal counterexample and bundle trivialization

Proof target : Corollary 1.5 · pp. 4–5 See the two compactification hypotheses.



Compactifiable deck groups and the Albanese bundle under Kähler compactification

[Open figure ↗](#) [TeX source ↓](#)

1. (i) Exclude a minimal-dimensional counterexample

Choose a counterexample with non-virtually-abelian deck group and minimal base dimension. Proposition 1.4 of [CH13] makes the base smooth and special. Theorem 1.1 makes its fundamental group virtually abelian, as is its quotient deck group, a contradiction. This step requires neither a Kähler compactification nor extension of the deck action to it.

[\[CH13, Proposition 1.4, print p. 357 / PDF p. 4\]](#) · [Corollary 1.5 · pp. 4–5](#)

2. (ii) The Albanese bundle and simply connected fibre

If $\pi_1(X)$ is infinite, apply (i) to the universal cover; if finite, it is already virtually abelian. Now the **Kähler compactification** hypothesis and Theorem 1.5 of [CH13] give a locally trivial holomorphic Albanese bundle with simply connected fibre on a finite étale cover. The fibre is compact Kähler; projectivity is not asserted.

[CH13, Theorem 1.5, print p. 358 / PDF p. 5] · Corollary 1.5 · pp. 4–5

3. The product decomposition of the universal cover

Pull the bundle back along $\mathbb{C}^q \rightarrow \mathbf{A}$. With structure group the automorphism group of the compact complex manifold F , the bundle is topologically trivial over this contractible base. The Oka–Grauert principle over the Stein base makes it holomorphically trivial. Since F is simply connected, $F \times \mathbb{C}^q$ is the ordinary universal cover of X' , hence also of X . This follows the connection on manuscript pp. 4–5, separately from the bundle conclusion supplied by [CH13].

Corollary 1.5 · pp. 4–5

9. Which papers supply which steps

Input

Cam04, Cam23

Used in the manuscript

Theorem 2.1; §6.1

Role and checking scope

Preservation of specialness on finite covers and surjectivity/connectedness of Albanese checked in the source statements. No absence of multiple fibres is assumed over a nonprojective base.

Input

BW16

Used in the manuscript

Proposition 6.2

Role and checking scope

Torsion of isolated characters and invariant vectors after an isogeny.

Input

CCE14

Used in the manuscript

Theorem 2.4; §6.5; Corollary 1.4

Role and checking scope

General type for torsion-free semisimple images and convexity for faithful linear groups are distinct inputs.

Input

Sha06, Bou02, CP19

Used in the manuscript

Proposition 4.3; §4.5

Role and checking scope

Hilbert coefficients, an ordinary big line, and bigness of the projective orbifold adjoint checked in the respective statements.

Input

JW21

Used in the manuscript

Theorem 5.8

Role and checking scope

Applicability to smooth klt pairs and arbitrary complex tori checked.

Input

Cam11, CH13

Used in the manuscript

Corollaries 1.2, 1.3, 1.5

Role and checking scope

Inputs to consequences, distinguishing $\kappa=0$, maximal Γ -dimension and the two compactification hypotheses.

The reference list before the diagrams and the citations at each proof step associate these keys with results and source pages. Checking those connections is separate from rechecking the complete external proofs.

10. Return to the sources

- Theorem 1.1 · p. 2 — The target is the full ordinary fundamental group.
- Lemma 6.1 · pp. 40–41 — Why meridians vanish over a nonprojective Albanese torus.
- Proposition 6.2 · pp. 41–42 — From torsion jump-locus points to finite abelianization on all covers.
- Proposition 6.3 · pp. 44–46 — Choose the finite cover before all further infinite quotients.
- Theorem 4.1 · p. 23 — Minimal rank, discrete monodromy and the big line.
- Theorem 5.8 · pp. 37–40 — Descent distinguishing true inertia from the auxiliary boundary.
- Corollary 1.5 · pp. 4–5 — The two compactification hypotheses and the bundle/product conclusions.

The principal statements, the central connections in §§4–6, and the reduction to the diagonal pairing in the latter part of §3 were read in the fixed manuscript. The specified external result statements were compared with their uses. This article explains the manuscript's proof strategy; it does not report completion of independent proof verification or expert peer review.

The full constants and limit operations in Theorem 3.1, including its potential construction and elliptic estimates, the analytic details of the image-germ space and direct integral in Proposition 4.6, and every model comparison in Theorem 5.8 have not been independently reproved. Complete external proofs, their further dependencies, and a separate primary source for the Oka–Grauert principle are outside the checking scope. The account should not be read as verification of the entire proof.

Two-step monodromy of special quasi-projective varieties

Identify the fibre image with the commutator, then use mixed Hodge weights to stop at step two

The manuscript proves virtual nilpotence of class at most two for every complex linear representation image of a smooth special complex quasi-projective variety, using ordinary quasi-Albanese exactness and unipotent completion. It also constructs special surfaces with nonspecial general quasi-Albanese fibres. The object controlled by the argument is the image of the fibre group, rather than the fibre group itself.

Manuscript author: OpenAI · 2026-09-24 · 19 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Two-step linear monodromy

Let V be a connected smooth special complex quasi-projective variety. For every positive integer d and every representation $\rho : \pi_1(V) \rightarrow \mathrm{GL}_d(\mathbb{C})$, the image has a finite-index subgroup of nilpotency class at most two. Here π_1 is the ordinary topological fundamental group, with no additional orbifold relations imposed on boundary meridians.

Specialness uses the open-variety convention of §1: on allowed proper birational modifications, every dominant algebraic fibration with connected general fibre onto a positive-dimensional normal quasi-projective base has orbifold-base Kodaira dimension strictly below the base dimension, computed birationally with infimum multiplicities.

[Theorem 1.1 · p. 1](#)

Theorem 1.2 — Completion under quasi-Albanese exactness

Let V be a connected smooth complex quasi-projective variety and $a : V \rightarrow A = \text{Alb}(V)$ its **algebraic** quasi-Albanese map. Assume a is dominant, choose a smooth connected general fibre F and $v \in F$, and suppose

$$\pi_1(F, v) \longrightarrow \Gamma := \pi_1(V, v) \xrightarrow{a_*} \pi_1(A, a(v)) \longrightarrow 1$$

is exact. Every canonical rational unipotent quotient then has class at most two. The full rational unipotent completion is a **finite-dimensional** unipotent algebraic group of class at most two; every finite-dimensional complex unipotent representation also has image of class at most two. Injectivity of the first arrow, properness of a , and specialness of V or F are not assumed.

Theorem 1.2 · p. 2

Theorem 1.3 — A special surface with nonspecial general fibre

Let E be a complex elliptic curve, let $q_1, \dots, q_n \in \mathbb{P}^1$ be distinct with $n \geq 3$, and choose arbitrary points $e_1, \dots, e_n \in E$. Set

$$b : Y = \text{Bl}_{\{(e_i, q_i)\}_{i=1}^n} (E \times \mathbb{P}^1) \longrightarrow E \times \mathbb{P}^1.$$

Let D_i be the strict transform of $E \times \{q_i\}$, $D = \sum_i D_i$, and $X = Y \setminus D$. **The exceptional curves are not added to the boundary.** The manuscript proves:

1. (Y, D) is a smooth projective special pair, including every upper model in Definition 4.1: every elementary birational orbifold morphism $u : (Y', D') \rightarrow (Y, D)$ and every positive-dimensional neat fibration on it are tested.
2. After choosing origins, $a = (\text{pr}_E \circ b)|_X$ is the algebraic quasi-Albanese map. All its fibres are connected, and its general fibre is $F \simeq \mathbb{P}^1 \setminus \{q_1, \dots, q_n\}$. Its log canonical degree is $n - 2 > 0$, so F is not special.
3. $a_* : \pi_1(X) \xrightarrow{\sim} \pi_1(E) \simeq \mathbb{Z}^2$, and $\pi_1(F) \rightarrow \pi_1(X)$ is trivial.

Theorem 1.3 · p. 3

Corollary 3.1 — Log Kodaira dimension zero

Let V be a connected smooth complex quasi-projective variety with $\bar{\kappa}(V) = 0$. Every finite-dimensional complex linear representation image is virtually nilpotent of class at most two. Its full rational unipotent completion is finite-dimensional of class at most two, and every

finite-dimensional complex unipotent representation image has class at most two. The finite-index subgroup in the first assertion may depend on the representation. Virtual class-two nilpotence of the entire ordinary discrete group is not asserted.

Corollary 3.1 · p. 8

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Two-step monodromy of special quasi-projective varieties](#). External inputs use the following keys.

- [CDY25](#) Benoît Cadorel; Ya Deng; Katsutoshi Yamanoi — Hyperbolicity and fundamental groups of complex quasi-projective varieties (III): applications: Linear-image decomposition, specialness under finite étale covers, and quasi-Albanese dominance and exactness. [arXiv:2512.20360v1](#), 2025-12-23.
- [Fuj24](#) Osamu Fujino — On quasi-Albanese maps: First-homology identification induced by the quasi-Albanese map. [arXiv:2402.04595v2](#), 2024-03-21.
- [Hain87](#) Richard M. Hain — The de Rham homotopy theory of complex algebraic varieties I: Pointed functorial mixed Hodge structures and the bracket. *K-Theory* 1 (1987), 271–324; author-hosted scan.
- [Hain15](#) Richard Hain — Genus 3 mapping class groups are not Kähler: Extension of the ground field for unipotent completion. [arXiv:1305.2052v2](#).
- [Del71](#) Pierre Deligne — Théorie de Hodge. II: Strictness and weights on first homology of smooth open varieties. *Publications Mathématiques de l’IHÉS* 40 (1971), 5–57; Numdam scan.
- [Cam](#) Frédéric Campana — Orbifolddes géométriques spéciales et classification biméromorphe des variétés kählériennes compactes: Specialness at log Kodaira dimension zero, neat models and composed base divisors. [arXiv:0705.0737v8](#); [arXiv stamp 2009-07-15](#), title date 2018-10-25.
- [CDHP26](#) Junyan Cao; Ya Deng; Christopher D. Hacon; Mihai Păun — Two-step nilpotent monodromy of local systems on special varieties: Conventions for special pairs and neat models; prior linear-image result and comparison with fibre specialness. [arXiv:2603.14539v2](#), 2026-04-07; comparison restricted to this version.
- [Groth](#) Alexander Grothendieck et al. — Revêtements étales et groupe fondamental (SGA 1): Algebraization of the finite cover corresponding to a finite-index subgroup. [arXiv:math/0206203v2](#), 2004-01-04; updated edition.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1 · p. 1](#) See the statement.

Proof route. On a finite étale cover, split the identity component of the linear image closure as $U \times T$. Apply ordinary quasi-Albanese exactness on that same cover and the [weight argument in Theorem 1.2](#) to bound the image in U . Combine it with the torus factor. No specialness of the general fibre enters this route; the surface example explains why this distinction matters.

1 Split the image on a finite cover

$$L = \overline{\rho(\pi_1(V))}^{\text{Zar}}, \quad \Gamma_0 = \rho^{-1}(L^0), \quad L^0 = U \times T.$$

Take the connected finite étale cover $V_0 \rightarrow V$ corresponding to Γ_0 .

Image decomposition, algebraization and specialness of the cover

[\[CDY25, Theorem A, p. 2\]](#) [\[CDY25, Theorem 3.1, p. 5\]](#)

[\[Groth, Exposé XII, Theorem 5.1, print p. 251 / PDF p. 267\]](#)

Quasi-Albanese dominance, connectedness and exactness

[\[CDY25, Lemma 4.4, p. 9\]](#) [\[CDY25, Proposition 4.13, p. 12\]](#)

2 Re-establish exactness on that cover

V_0 is special. Its quasi-Albanese is dominant, and

$$\pi_1(F_0) \longrightarrow \pi_1(V_0) \longrightarrow \pi_1(\text{Alb}(V_0)) \longrightarrow 1$$

is exact, with F_0 a smooth connected general fibre.

Apply the completion theorem to the image in U ; T is abelian

[Theorem 1.2 · p. 2](#)

Return to the original image, retaining finite index

[§3 · pp. 7–8](#)

3 Triple commutators vanish in both factors

$$\gamma_3 \rho(\Gamma_0) = 1, \quad [\rho(\pi_1(V)) : \rho(\Gamma_0)] < \infty.$$

[Theorem 1.1](#): the linear image is virtually of class ≤ 2 .

From exactness on the finite cover to the class-two linear image

[Open figure ↗](#) [TeX source ↓](#)

1. Choose the cover and obtain exactness there

Put $L = \overline{\rho(\pi_1(V))}^{\text{Zar}}$ and take the connected finite étale cover $V_0 \rightarrow V$ corresponding to $\Gamma_0 = \rho^{-1}(L^0)$. CDY gives $L^0 = U \times T$. The cover remains special; its quasi-Albanese map is dominant, has connected general fibre, and satisfies the ordinary exact sequence. The weakly special hypothesis in [Lemma 4.4](#) follows from [Theorem 3.1](#). Generic smoothness supplies a smooth general fibre.

[\[CDY25, Theorem A, p. 2\]](#) [\[CDY25, Theorem 3.1, p. 5\]](#) [\[CDY25, Lemma 4.4, p. 9\]](#)

[\[CDY25, Proposition 4.13, p. 12\]](#) [\[Groth, Exposé XII, Theorem 5.1, print p. 251 / PDF p. 267\]](#)

2. Kill triple commutators in both factors

Theorem 1.2 applied to $\pi_1(V_0) \rightarrow U$ kills triple commutators. Their projection to the abelian group T also vanishes, hence they vanish in the image in $U \times T$. This image has finite index in the original image. Exactness must be re-established on V_0 : completion need not be invariant under passage to finite index. The manuscript credits Cao–Deng–Hacon–Păun with the prior class-two linear-image conclusion and presents an independent proof.

[CDHP26, Theorem A, p. 1]

§3 · pp. 7–8

3. Proof of Theorem 1.2 — Weights, centrality and stabilization

Proof target : Theorem 1.2 · p. 2 See the hypotheses and three conclusions.

Write $\mathfrak{m}_V$ for the complete Lie algebra of rational unipotent completion, $\mathfrak{g}_c = \mathfrak{m}_V / \gamma_{c+1} \mathfrak{m}_V$, and $U_c = \exp(\mathfrak{g}_c)$. The lower-central terms are closed in the inverse-limit topology. The argument takes place on canonical finite stages carrying mixed Hodge structures.

1 Canonical finite stages

$$\mathfrak{g}_c = \mathfrak{m}_V / \overline{\gamma_{c+1} \mathfrak{m}_V}, \quad U_c = \exp(\mathfrak{g}_c)$$

The pointed inclusion $F \hookrightarrow V$ and bracket are MHS morphisms.

First-homology identification and elimination of torsion; preserve pointed MHS

[Fuj24, Lemmas 3.11–3.12, pp. 20–21]

[Hain87, Theorem 6.3.1, print p. 321 / PDF p. 51]

§2.2; (2.3)–(2.5) · pp. 5–6

2 The fibre image is the commutator

$$N = \ker a_*, \quad N/[\Gamma, \Gamma] = \mathrm{Tor} H_1(V, \mathbb{Z})$$

$$K_c = \overline{\mathrm{im}(N)}^{\mathrm{Zar}} = [U_c, U_c], \quad \mathfrak{k}_c = [\mathfrak{g}_c, \mathfrak{g}_c].$$

Fibre weights are $-1, -2$; commutator weights are at most -2

[Del71, Corollary 3.2.15(ii), print p. 39 / PDF p. 36]

[Del71, Theorem 2.3.5, print p. 30 / PDF p. 27]

§2.3; (2.6)–(2.10) · p. 6

3 Combine two weight bounds

$$H_1(F, \mathbb{Q}) \twoheadrightarrow B_c = \mathfrak{k}_c / [\mathfrak{k}_c, \mathfrak{k}_c]$$

$$W_{-3}B_c = 0, \quad W_{-2}B_c = B_c, \quad \mathfrak{g}_c \otimes \mathfrak{k}_c \longrightarrow B_c \text{ is } 0.$$

Nilpotence, separated inverse limit, field extension and universality

§2.3; (2.6)–(2.10) · p. 6 §2.3 · p. 7

[Hain15, §4, p. 8]

4 Stabilize at step two; pass to every unipotent image

$$\gamma_3 \mathfrak{g}_c \subseteq \gamma_4 \mathfrak{g}_c \subseteq \gamma_3 \mathfrak{g}_c \Rightarrow \gamma_3 \mathfrak{g}_c = 0$$

$\mathfrak{m}_V \simeq \mathfrak{g}_2$ is finite-dimensional; complex unipotent images have class ≤ 2 .

Two weight bounds on the fibre image stabilize unipotent completion

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1. Identify the fibre image with the commutator

Exactness gives $N = \mathrm{im}(\pi_1(F) \rightarrow \Gamma) = \ker a_*$. The first-homology property of quasi-Albanese yields

$$[\Gamma, \Gamma] \subseteq N, \quad N/[\Gamma, \Gamma] \simeq \mathrm{Tor} H_1(V, \mathbb{Z}).$$

Do not identify the two discrete subgroups. Instead take the Zariski closure K_c of N in U_c . Since U_c^{ab} is a torsion-free additive group, $K_c \subseteq [U_c, U_c]$. Conversely the dense image of Γ in U_c/K_c is abelian, so the commutator morphism vanishes and the reverse inclusion follows. Thus

$\mathfrak{k}_c := \mathrm{Lie} K_c = [\mathfrak{g}_c, \mathfrak{g}_c]$. No left-exactness of completion is invoked.

Hain's construction applied to the pointed algebraic inclusion $F \hookrightarrow V$ makes $\mathfrak{g}_c(F) \rightarrow \mathfrak{g}_c(V)$ a mixed-Hodge morphism, and gives its image $\mathfrak{k}_c$ the induced structure. An arbitrary representation quotient is not assumed to carry such a structure.

§2.2; (2.3)–(2.5) · pp. 5–6 [Fuj24, Lemmas 3.11–3.12, pp. 20–21]
[Hain87, Theorem 6.3.1, print p. 321 / PDF p. 51]

2. Combine the fibre's lower weight bound with the commutator's upper bound

Iterated brackets give weights at most $-r$ on $\gamma_r \mathfrak{g}_c$. Meanwhile

$$H_1(F, \mathbb{Q}) \twoheadrightarrow B_c := \mathfrak{k}_c / [\mathfrak{k}_c, \mathfrak{k}_c]$$

and strictness restrict the weights of B_c to $-1, -2$. Its position in $\mathfrak{k}_c = \gamma_2 \mathfrak{g}_c$ therefore makes it pure of weight -2 . The bracket

$$\mathfrak{g}_c \otimes \mathfrak{k}_c \longrightarrow B_c$$

has source of weights at most -3 and consequently vanishes. Hence

$$\gamma_3 \mathfrak{g}_c \subseteq [\gamma_2 \mathfrak{g}_c, \gamma_2 \mathfrak{g}_c] \subseteq \gamma_4 \mathfrak{g}_c \subseteq \gamma_3 \mathfrak{g}_c.$$

All subsequent lower-central terms coincide, and nilpotence forces $\gamma_3 \mathfrak{g}_c = 0$. This proves the assertion for each canonical quotient.

§2.3; (2.6)–(2.10) · p. 6 [Del71, Theorem 2.3.5, print p. 30 / PDF p. 27]
[Del71, Corollary 3.2.15(ii), print p. 39 / PDF p. 36]

3. Pass to full completion and complex unipotent representations

The Lie algebra $\mathfrak{m}_V$ is the separated inverse limit of its canonical quotients. Vanishing of γ_3 in every quotient gives $\gamma_3 \mathfrak{m}_V = 0$. Thus $\mathfrak{m}_V \simeq \mathfrak{g}_2$: the tower stabilizes and full completion inherits finite-dimensionality of $\mathfrak{g}_2$, which follows from finite generation of Γ . A uniform class bound on the finite stages alone would not explain this conclusion.

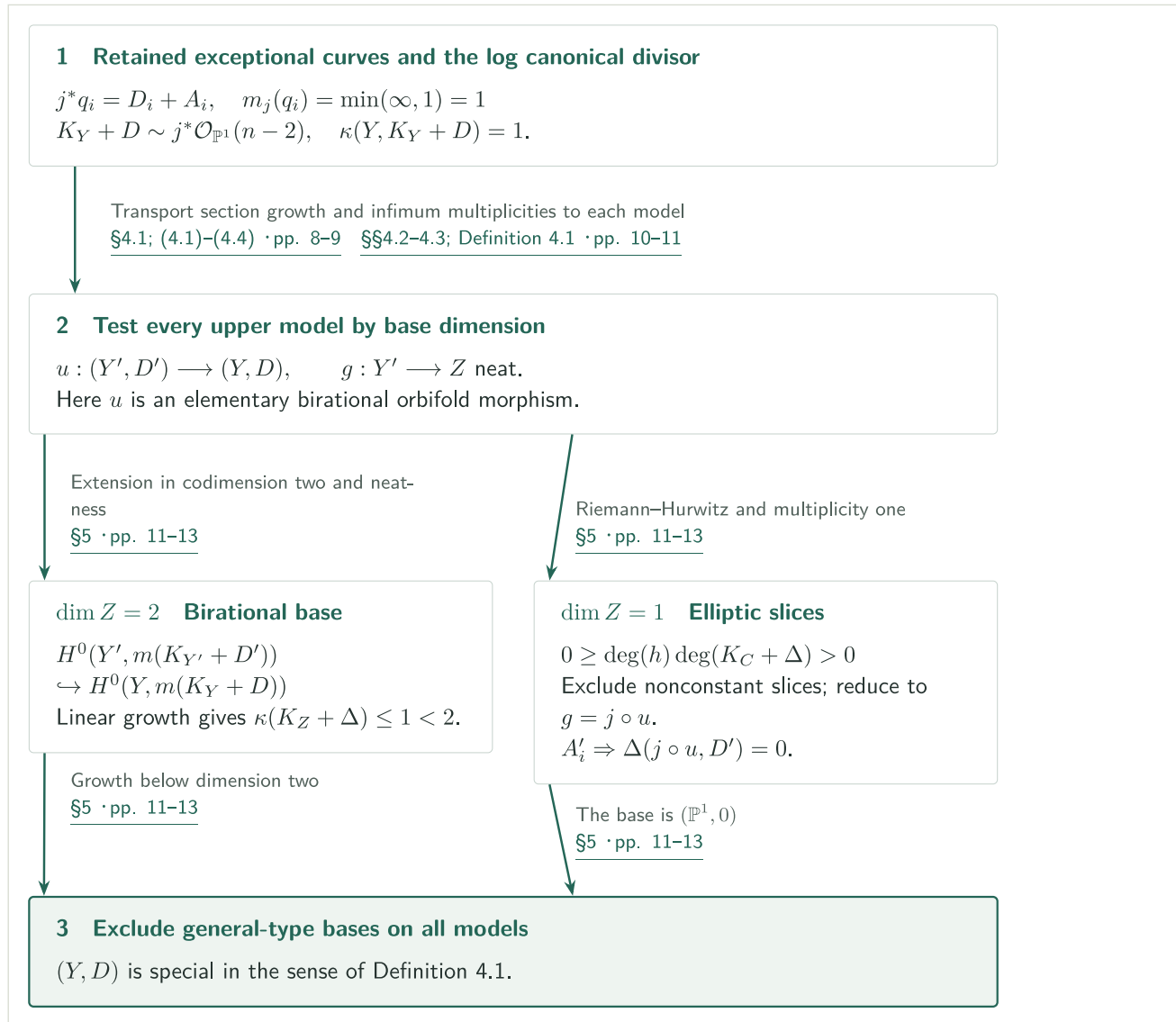
Completion commutes with characteristic-zero field extension. Extend from $\mathbb{Q}$ to $\mathbb{C}$ and use universality to factor each complex unipotent representation. This proves the third conclusion. Injectivity of $\Gamma \rightarrow \Gamma^{\text{un}}(\mathbb{Q})$ is not supplied; killing $\gamma_3 \Gamma$ in completion does not identify its elements inside the discrete group.

§2.3 · p. 7 [Hain15, §4, p. 8] Question 8.1; §8 · p. 18

4. Proof of Theorem 1.3(i) — Exclude general-type bases on every upper model

Proof target : Theorem 1.3 · p. 3 See the construction and assertion (i).

Set $j = \text{pr}_{\mathbb{P}^1} \circ b$ and $A_i = b^{-1}(e_i, q_i)$. The affine line $A_i^\circ = A_i \cap X \simeq \mathbb{A}^1$ is retained, while $j^*q_i = D_i + A_i$ has both coefficients one. Thus $m_j(q_i) = \min(\infty, 1) = 1$ and $\Delta(j, D) = 0$. At the same time $K_Y + D \sim j^*\mathcal{O}_{\mathbb{P}^1}(n-2)$, so $\kappa(Y, K_Y + D) = 1$. These are distinct computations.



Exclude surface and curve bases on every upper model

[Open figure ↗](#) [TeX source ↓](#)

1. Exclude a surface base by the growth of sections

Fix any allowed $u : (Y', D') \rightarrow (Y, D)$ and neat fibration $g : Y' \rightarrow Z$. If $\dim Z = 2$, then g is birational. Neatness and canonical dimension bound the base Kodaira dimension by $\kappa(Y', K_{Y'} + D')$. For sufficiently divisible m , restrict sections away from the finite exceptional image of u and extend by normality to obtain

$$H^0(Y', m(K_{Y'} + D')) \hookrightarrow H^0(Y, m(K_Y + D)).$$

The target grows linearly, so the base has Kodaira dimension at most one. Birational invariance under arbitrary boundary changes is not assumed.

§4.1; (4.1)–(4.4) · pp. 8–9 §5 · pp. 11–13

[CDHP26, Definitions 2.1–2.9; Lemma 2.7; Proposition 2.8, pp. 7–9]

2. Exclude a curve base using elliptic slices and retained components

Suppose $Z = C$ with $\deg(K_C + \Delta(g, D')) > 0$. A sufficiently general fibre C_t of $j \circ u$ is an unchanged complete elliptic curve avoiding the boundary and exceptional locus. If $h = g|_{C_t}$ is nonconstant, its ramification index satisfies $e_x \geq m_c$ for the infimum multiplicity m_c . Infinite m_c is impossible, and Riemann–Hurwitz gives

$$0 = \deg K_{C_t} \geq \deg(h) \deg(K_C + \Delta) > 0.$$

Thus g factors through $j \circ u$. Connected general fibres force the intervening map of curves to have degree one, hence $C \simeq \mathbb{P}^1$. The strict transform of each A_i still has coefficient zero in D' and ordinary multiplicity one above q_i . Strict transforms of the other elliptic fibres do likewise. Therefore $\Delta(j \circ u, D') = 0$: additional exceptional components cannot increase an infimum already equal to one. This proves specialness on every upper model.

§5 · pp. 11–13

5. Proof of Theorem 1.3(ii)–(iii) — Universality and the ordinary group

Proof target : Theorem 1.3 · p. 3 See assertions (ii)–(iii).

1 Two calculations on the same surface

$$X = Y \setminus \sum D_i, \quad A_i^\circ \simeq \mathbb{A}^1 \subset X, \quad a : X \longrightarrow E.$$

The intersection matrix forces constant units; the section makes factorization unique
[Lemma 6.1; §6.1 · pp. 13–14](#) [§4.1; \(4.1\)–\(4.4\) · pp. 8–9](#)

2 Algebraic universality from the section and units

$$s : E \longrightarrow X, \quad a \circ s = \text{id}, \quad \mathcal{O}(X)^* = \mathbb{C}^*$$

$$f : X \longrightarrow G \text{ semiabelian} \Rightarrow f = (f \circ s) \circ a.$$

All fibres are connected; general $F = \mathbb{P}^1 \setminus \{q_i\}$ is nonspecial.

Compute the group independently; identify meridians in the blowup chart
[§6.2; \(6.3\)–\(6.4\) · p. 15](#)

3 Discs through the retained curves

$$U = X \setminus \bigcup A_i^\circ \simeq E \times (\mathbb{P}^1 \setminus \{q_i\})$$

$$b(u, t) = (ut, t), \quad m_i = \partial\{u = 0, |t| \leq \varepsilon\} \text{ in } X.$$

Kill all puncture loops; retain elliptic generators using the section
[§6.2; \(6.3\)–\(6.4\) · p. 15](#)

4 Ordinary fundamental group and fibre image

$$a_* : \pi_1(X) \xrightarrow{\sim} \pi_1(E) \simeq \mathbb{Z}^2, \quad \text{im}(\pi_1(F) \rightarrow \pi_1(X)) = 1.$$

The section and units give universality; local discs kill the fibre image

[Open figure ↗](#) [TeX source ↓](#)

1. The section and units determine quasi-Albanese universality

Choose a complete section $s : E \rightarrow X$ using $q_* \notin \{q_i\}$. The divisor of a unit $v \in \mathcal{O}(X)^*$ is $\sum b_i D_i$. Intersecting with A_j , using $D_i \cdot A_j = \delta_{ij}$ and zero intersection for a principal divisor, gives $b_j = 0$. Consequently every unit is constant.

For an algebraic morphism $f : X \rightarrow G$ into a semiabelian extension $1 \rightarrow T \rightarrow G \rightarrow B \rightarrow 1$, its projection to B is constant on the general punctured rational fibre. The difference $f - (f \circ s) \circ a$ therefore takes values in T . It is constant by the units calculation and zero on the section, hence zero everywhere. After fixing origins, $f \circ s : E \rightarrow G$ is a homomorphism: its projection to B is a homomorphism, and its group-law defect is a map from the projective variety $E \times E$ to an affine torus, hence constant and zero. The section proves uniqueness. This establishes the algebraic universal property, beyond a dimension count of logarithmic one-forms.

Every special fibre consists of the strict transform of a vertical curve with retained affine exceptional lines attached at individual points, and is connected. The general fibre is the n -punctured projective line; its log canonical degree $n - 2 > 0$ proves it is nonspecial.

Lemma 6.1; §6.1 · pp. 13–14 §4.1; (4.1)–(4.4) · pp. 8–9

2. Discs inside X kill every puncture meridian

We have $U = X \setminus \bigcup A_i^\circ \simeq E \times (\mathbb{P}^1 \setminus \{q_i\})$. The inclusion induces a surjection on fundamental groups whose kernel is normally generated by meridians of the A_i° . In the local blowup chart $b(u, t) = (ut, t)$, a meridian of $A_i^\circ = \{t = 0\}$ corresponds to the puncture loop m_i about q_i and bounds the actual disc $\{u = 0, |t| \leq \varepsilon\} \subset X$. Hence

$$\pi_1(X) \simeq \frac{\pi_1(E) \times \pi_1(\mathbb{P}^1 \setminus \{q_i\})}{\langle\langle m_1, \dots, m_n \rangle\rangle} \simeq \mathbb{Z}^2.$$

The map a_* is this isomorphism; the section also shows that the elliptic generators survive. Since puncture loops generate the free fibre group, its image in the total group is zero. No additional orbifold relations have been imposed.

§6.2; (6.3)–(6.4) · p. 15

6. Proof of Corollary 3.1 — Three conclusions from log Kodaira dimension zero

Proof target : Corollary 3.1 · p. 8 See the statement.

1 A compactification of log Kodaira dimension zero
 $V = Y \setminus D, \quad (Y, D) \text{ smooth projective SNC}, \quad \kappa(Y, K_Y + D) = 0.$

Campana's theorem and comparison on a neat model
[\[Cam, Theorem 6.7, p. 92\]](#) [\[Cam, Corollary 4.11, p. 49\]](#)
[§§4.2–4.3; Definition 4.1 · pp. 10–11](#)

2 From pair specialness to open-variety specialness
 $(Y, D) \text{ special}, \quad D_{\text{open}} \leq D_{\text{high}}$
 $\Delta(g, D_{\text{open}}) \leq \Delta(g, D_{\text{high}}) \Rightarrow V \text{ special}.$

Use Theorem 1.1 for linear images; Theorem 1.2 with exactness on V itself
[Theorem 1.1 · p. 1](#) [Theorem 1.2 · p. 2](#)
[Corollary 3.1 · p. 8](#)

3 Both the linear-image and completion routes

Linear images: class ≤ 2 after finite index.
 Full rational completion: finite-dimensional, class ≤ 2 .
 Complex unipotent images: class ≤ 2 .

Specialness of the pair leads to linear-image and completion conclusions

[Open figure ↗](#) [TeX source ↓](#)

1. Compare specialness conventions and apply both theorems

Choose a smooth projective SNC compactification (Y, D) with $\kappa(Y, K_Y + D) = 0$. Apply Campana's specialness theorem and the calculation on neat models. The comparison after Definition 4.1 resolves an open fibration so that $D_{\text{open}} \leq D_{\text{high}}$. Infimum multiplicities then give $\Delta(g, D_{\text{open}}) \leq \Delta(g, D_{\text{high}})$. This supplies the required implication to open-variety specialness.

Theorem 1.1 yields the finite-index linear-image assertion. For completion, apply ordinary quasi-Albanese exactness to V itself and then Theorem 1.2. This gives finite-dimensionality and class two for full rational completion, together with class two for complex unipotent images. The last two conclusions do not follow merely from the finite-index assertion for linear images.

[\[Cam, Theorem 6.7, p. 92\]](#) [\[Cam, Corollary 4.11, p. 49\]](#) [§§4.2–4.3; Definition 4.1 · pp. 10–11](#)
[Corollary 3.1 · p. 8](#)

7. Which papers supply which steps

Source

CDY25

Used at

Theorem 1.1; Corollary 3.1

Role and hypotheses

The smooth special total-space hypothesis persists on the cover; use image decomposition and ordinary exactness

Checking scope

Statements in the specified v1 and their use in §3 checked

Source

Fuj24, Hain87, Del71

Used at

Theorem 1.2

Role and hypotheses

First homology, pointed algebraic functoriality, weights and strictness

Checking scope

Statements and use checked; the full completed-group-algebra construction was not independently verified

Source

Hain15, Groth

Used at

Theorems 1.2, 1.1

Role and hypotheses

Field extension for completion and algebraization of finite covers

Checking scope

Relevant statements and use checked

Source

Cam, CDHP26

Used at

Corollary 3.1; surface example

Role and hypotheses

Pair specialness, neat upper models and infimum multiplicities

Checking scope

Conventions and statements checked; the CDHP comparison is restricted to v2

In §7 the manuscript compares the direct calculation $\Delta(j, D) = 0$ with $\Delta(\text{pr}_{\mathbb{P}^1}, \sum H_i) = \sum q_i$ after blowdown. This illustrates why equality in Campana's composition rule is conditional. The authors present it as a counterexample to CDHP26 Theorem 7.15 and Claim 7.17. Editorial checking covered the statements in the specified v2, the point where Claim 7.17 changes flatness over J' to flatness over a different intermediate space, and the displayed multiplicity calculation. This comparison is not extended to other versions or to that paper's separate linear-image theorem.

§7; (7.1)–(7.3) · pp. 15–18 [Cam, Proposition 3.13, p. 44] [CDHP26, Theorem 7.15, p. 70]
[CDHP26, Claim 7.17, pp. 73–74]

The first paper in catalogue 057 is cited as compact Kähler background. It is not a direct input to the proof routes for Theorems 1.1–1.3 described here.

8. Return to the sources

- §3 · pp. 7–8 — Reapply the inputs on the finite cover
- §2.2; (2.3)–(2.5) · pp. 5–6 — Handle discrete torsion and identify the commutator
- §2.3; (2.6)–(2.10) · p. 6 — Use two weight bounds to force centrality
- §2.3 · p. 7 — Stabilization and passage to complex representations
- §5 · pp. 11–13 — Exclude curve and surface bases on every upper model
- Lemma 6.1; §6.1 · pp. 13–14 — The algebraic quasi-Albanese universal property
- §6.2; (6.3)–(6.4) · p. 15 — Kill meridians using discs inside X
- §7; (7.1)–(7.3) · pp. 15–18 — Compare base divisors with the specified v2
- Question 8.1; §8 · p. 18 — Separate the kernel invisible to completion

The principal statements and relevant arguments in §§2–8 were read, and the reductions in the diagrams, coverage of each conclusion, and statements and use of the direct external inputs were checked in the specified versions. Hain's pointed functoriality was also inspected on the image of printed page 321; Morgan's older functoriality assertion is not substituted for it. Complete proofs of the external results, the entire completed-group-algebra construction, recursive dependencies, expert refereeing and formal verification are outside this checking scope. The ordinary-group conclusion remains Question 8.1 in the manuscript and is kept distinct from the linear-image and completion assertions.

A conditional abelianity theorem for special fourfold pairs with a half-weight divisor

A genus-one construction on the root space realizes its orbifold group as a quotient of an ordinary fundamental group

The manuscript takes abelianity for smooth compact Kähler manifolds as an input and transfers it to the order-two root orbifold of a smooth projective fourfold pair. Its central construction is a projective eightfold whose general fibre is a product of four genus-one curves. Descent of differential forms and the meridian relations separately provide specialness of this eightfold and a surjection from its fundamental group.

Manuscript author: OpenAI · 2026-10-05 · 14 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.3 — Conditional virtual abelianity

Input hypothesis (Assumption 1.2). Every connected smooth special compact Kähler manifold, in every dimension, has virtually abelian fundamental group. The manuscript obtains this input from Theorem 1.1 of the first paper in this catalog.

Assumption 1.2 · p. 2 [\[OpenAI26a, Theorem 1.1, p. 2\]](#)

Let X be a connected smooth complex projective fourfold and $D \subset X$ a nonempty smooth connected reduced divisor. Put $\mathcal{X} = \sqrt[2]{(X, D)}$. If $\mathcal{X}$ is special, then, under the hypothesis above,

$$G = \pi_1(X \setminus D, x) / \langle\langle \gamma^2 \rangle\rangle$$

contains an abelian subgroup of finite index, where γ is a meridian of D . This assertion concerns the entire discrete group G ; it gives neither a uniform index bound nor torsion-freeness.

Theorem 1.3 · p. 2

Specialness here is Definition 1.1 of the manuscript: for $1 \leq p \leq \dim V$, there is no nonzero map from a line bundle L to Ω_V^p with $\kappa(L) = p$. On the root space, forms and global sections are interpreted equivariantly on its local covers.

Definition 1.1 · p. 2

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A conditional abelianity theorem for special fourfold pairs with a half-weight divisor](#). External inputs use the following keys.

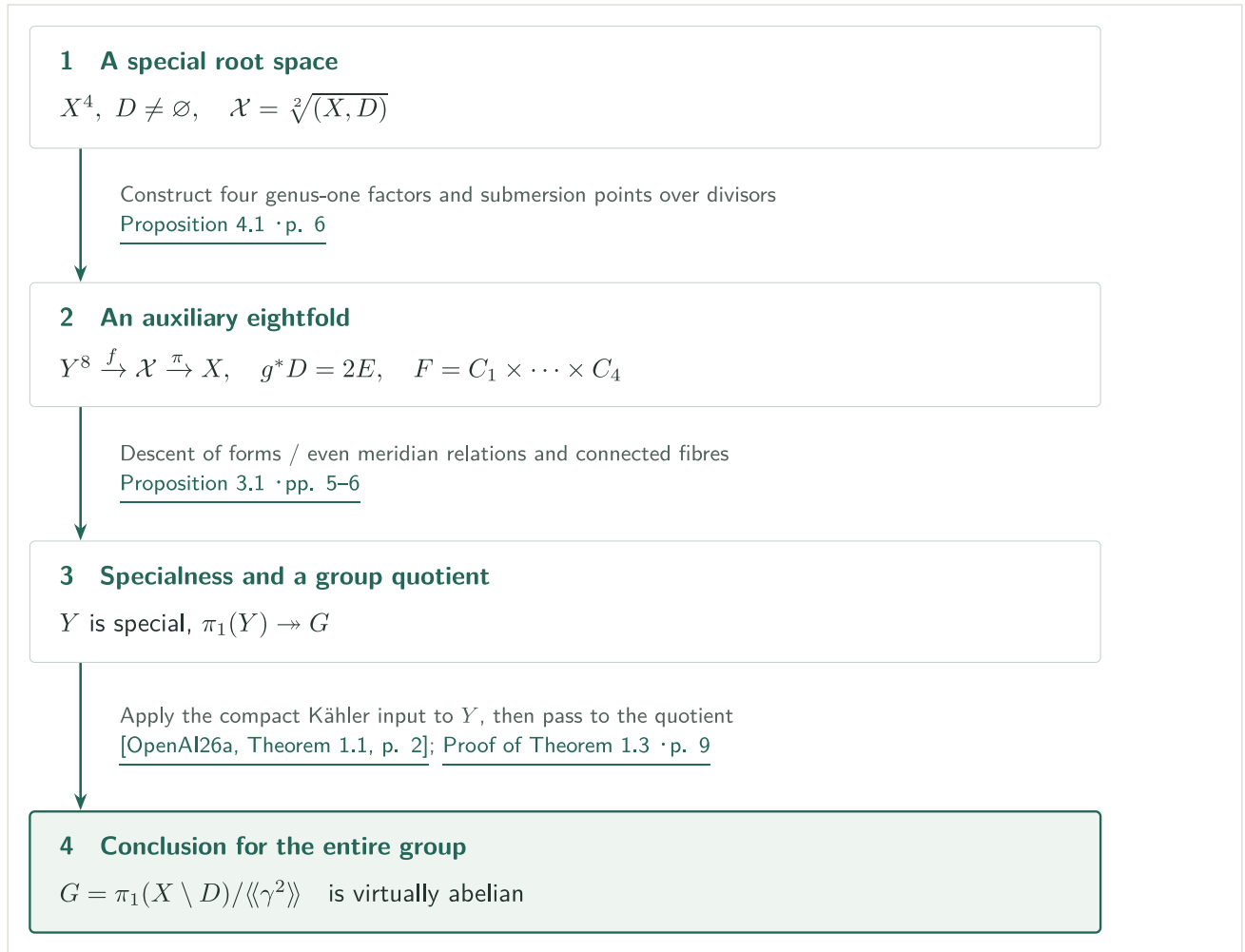
- [OpenAI26a](#) OpenAI — The abelianity conjecture for special compact Kähler manifolds: Virtual abelianity of the entire fundamental group of a special smooth compact Kähler manifold; supplies Assumption 1.2. 2026-09-23.
- [Ehr52](#) Charles Ehresmann — Les connexions infinitésimales dans un espace fibré différentiable: Local triviality for submersions with connected compact fibres, used for surjectivity on fundamental groups. 1952, Séminaire Bourbaki, exp. 24 (lecture 1950), pp. 153–168.
- [Ser56](#) Jean-Pierre Serre — Géométrie algébrique et géométrie analytique: Algebraization of a finite analytic map through its coherent algebra and structure maps. 1956, Annales de l'Institut Fourier 6, pp. 1–42.
- [Dem12](#) Jean-Pierre Demailly — Complex Analytic and Differential Geometry: Closedness of holomorphic differential forms in Lemma 2.1. June 21, 2012.
- [StacksM](#) The Stacks project authors — The Stacks Project — Morphisms of Schemes: Projectivity of finite morphisms. version ed88ff78, compiled July 14, 2026.
- [StacksC](#) The Stacks project authors — The Stacks Project — Cohomology of Schemes: Cohomology of projective space in the Koszul calculation. version ed88ff78, compiled July 14, 2026.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.3 — Proof overview of the main theorem

Proof target : [Theorem 1.3 · p. 2](#)

Use the hypotheses and notation of [Theorem 1.3 above](#). Construct an auxiliary variety, establish its specialness and the group quotient, and then apply the compact Kähler input.



From an auxiliary eightfold to the fundamental group of the root orbifold

[Open figure ↗](#) [TeX source ↓](#)

The construction and the two transfers

[Proposition 4.1](#) constructs a connected smooth projective eightfold Y and a surjection $g : Y \rightarrow X$. Its general fibre F is a product of four genus-one curves, so Ω_F^1 is trivial. Moreover, $g^*D = 2E$ and g lifts to $f : Y \rightarrow \mathcal{X}$. Suitable submersion points over general points of every divisor are essential for the next step.

[Proposition 4.1 · p. 6](#)

[Proposition 3.1](#) shows that a differential line bundle on Y with $\kappa(L) = p$ would descend to $\mathcal{X}$. Specialness of $\mathcal{X}$ therefore implies specialness of Y . A separate topological argument uses even ramification multiplicities and connected general fibres to obtain $\pi_1(Y) \twoheadrightarrow G$.

[Proposition 3.1 · pp. 5–6](#)

Apply the input to the entire group and pass to its quotient

Since Y is smooth projective, it is compact Kähler. Apply Assumption 1.2 to this **eight-dimensional** Y to obtain an abelian finite-index subgroup $A \subset \pi_1(Y)$. Its image in G is abelian and has index at most $[\pi_1(Y) : A]$. This proves the full conclusion of the main theorem. The argument does not assume that the root space admits a finite manifold cover.

[OpenAI26a, Theorem 1.1, p. 2] Proof of Theorem 1.3 · p. 9

3. Proposition 3.1 — Transfer of specialness and fundamental groups

Proof target : Proposition 3.1 · pp. 5–6

Proposition 3.1 — Four conditions for transfer

Let X be a connected smooth projective variety of positive dimension, D a nonempty smooth connected reduced divisor, Y a connected smooth projective variety, and $g : Y \rightarrow X$ surjective. Assume the following.

1. $g^*D = 2E$ for an effective integral divisor E , giving a lift $f : Y \rightarrow \mathcal{X}$.
2. Over a nonempty Zariski open subset $B \subset X \setminus D$, the map g is smooth with connected fibres F and trivial Ω_F^1 .
3. Over a general point of each prime divisor $P \neq D$, there is a submersion point of g .
4. Over a general point of D , a local lift to a root chart has a submersion point.

Then specialness of $\mathcal{X}$ implies specialness of Y . Without assuming specialness, there is a surjection $\pi_1(Y) \twoheadrightarrow G$. Consequently, if $\mathcal{X}$ is special and Assumption 1.2 holds, G is virtually abelian.

Proposition 3.1 · pp. 5–6

1 Assume a violation of specialness

$$L \longrightarrow \Omega_Y^p, \quad \kappa(L) = p, \quad u_a = s_a/s_0$$

Trivial graded pieces make ratios fibrewise constant; connectedness gives descent
[Lemma 2.1 · p. 3](#); [Lemma 2.4 · p. 4](#)

2 Descend ratios and their direction

$$u_a = g^*v_a, \quad \sigma = dv_1 \wedge \cdots \wedge dv_p \neq 0$$

Detect divisorial poles by submersions and extend in the saturated direction
[Lemma 2.2 · pp. 3–4](#); [Proposition 3.1 · pp. 5–6](#)

3 A forbidden line bundle

$$K \longrightarrow \Omega_X^p, \quad \kappa(K) = p \implies \perp$$

4 The separate topological argument

$$g^*D = 2E, \quad g^{-1}B \longrightarrow B \text{ has connected fibres}$$

Meridians map to γ^{2e} ; local triviality gives surjectivity on π_1
[\[Ehr52, §1, Proposition, print p. 154 / PDF p. 3\]](#); [Proposition 3.1 · pp. 5–6](#)

5 Kill meridians and retain surjectivity

$$\pi_1(Y) \twoheadrightarrow \pi_1(X \setminus D) / \langle\langle \gamma^2 \rangle\rangle$$

Two transfers: descend ratios and exclude poles; then use the meridian relations

[Open figure ↗](#) [TeX source ↓](#)

Descend ratios of sections to the base

Suppose $L \rightarrow \Omega_Y^p$ is nonzero with $\kappa(L) = p$. Choose sections $s_0, \dots, s_p$ of a power L^m with algebraically independent ratios $u_a = s_a/s_0$. Lemma 2.1 identifies the generic direction of L with the wedge of the differentials of these ratios. Its input is closedness of holomorphic forms on a resolution of the cover obtained by simultaneously adjoining m th roots of the sections.

[Lemma 2.1 · p. 3](#) [\[Dem12, VI, Proposition 8.9, p. 312\]](#)

On a general fibre, the cotangent sequence is $0 \rightarrow \mathcal{O}_F^{\dim X} \rightarrow \Omega_Y^1|_F \rightarrow \Omega_F^1 \rightarrow 0$. Its exterior powers have a filtration with trivial graded pieces. A component of the first nonzero graded projection of $L|_F$ gives a section t of $L^{-1}|_F$. On a general fibre where $t^m s_0$ is a nonzero constant, all ratios s_a/s_0 are constant. Rational descent along connected general fibres then gives $u_a = g^*v_a$. Thus $\sigma = dv_1 \wedge \cdots \wedge dv_p \neq 0$, and $p \leq \dim X$.

[Lemma 2.4 · p. 4](#)

Detect poles at submersion points in root charts

Write $\pi : \mathcal{X} \rightarrow X$ and saturate the direction of $\tau = \pi^*\sigma$ inside $\Omega_{\mathcal{X}}^p$ to obtain a line bundle K . Lemma 2.2 uses primitive generators in codimension one and Hartogs extension; it does not assume that the resulting map is a subbundle everywhere.

Lemma 2.2 · pp. 3–4

First write the images of s_a as $h_a(g^*\sigma)^m$. Over an open set where σ is holomorphic and nonvanishing and g is a submersion, h_a is holomorphic and therefore constant on compact connected fibres. Lemma 2.4 gives $h_a = g^*b_a$, so the images are $g^*b_a(g^*\sigma)^m$. The submersion points in conditions 3 and 4 admit local sections. Any divisorial pole downstairs would therefore be detected after pullback to Y . Consequently $(\pi^*b_a)\tau^m$ is holomorphic on every root chart and extends across codimension two. Independence of the ratios v_a gives $\kappa(K) \geq p$. On the other hand, $f^*K \rightarrow \Omega_Y^p$ and Lemma 2.1 give $\kappa(K) \leq \kappa(f^*K) \leq p$. This produces the forbidden line bundle with $\kappa(K) = p$ on $\mathcal{X}$.

Proposition 3.1 · pp. 5–6

Even meridian powers yield the group surjection

For $Y \setminus g^{-1}D \rightarrow X \setminus D$, a meridian of a component of multiplicity $2e$ in g^*D maps to γ^{2e} and hence dies in G . The inclusion of this complement in Y induces a surjection on fundamental groups, whose kernel is normally generated by divisor meridians. The map therefore factors through $\pi_1(Y) \rightarrow G$. The proper smooth map $g^{-1}B \rightarrow B$ has connected compact fibres and is locally trivial, so it is surjective on fundamental groups. General position also gives surjectivity of $\pi_1(B) \rightarrow \pi_1(X \setminus D)$, proving the required surjection.

[Ehr52, §1, Proposition, print p. 154 / PDF p. 3] Proposition 3.1 · pp. 5–6

4. Proposition 4.1 — Constructing a projective eightfold

Proof target : Proposition 4.1 · p. 6

Proposition 4.1 — A construction with four genus-one factors

Let X be a connected smooth complex projective fourfold and $D \subset X$ a nonempty smooth connected reduced divisor. There exist a connected smooth projective eightfold Y and a surjective map $g : Y \rightarrow X$ satisfying all four conditions of Proposition 3.1. Its general fibre is a product of four smooth connected genus-one curves. The construction itself does not require specialness of $\mathcal{X}$.

Proposition 4.1 · p. 6

1 Eight equations on the root space

$$T^2 = \pi^* \mathcal{O}_X(D), \quad q_{j\nu} = \sum_{i=1}^4 a_{j\nu i} z_{ji}^2 = 0 \\ j = 1, \dots, 4, \quad \nu = 1, 2$$

Independent evaluations give smoothness; four determinants exclude fixed points
[Lemma 4.2 · pp. 7–8](#)

2 A smooth space without fixed points

Parameter conditions at a fixed $x \in D$:
 $\text{codim} = 4 > \dim D = 3$

Square coordinates; algebraize the coherent algebra; use projectivity of finite maps
[\[Ser56, §12, Théorèmes 2–3, print pp. 19–20 / PDF pp. 20–21\];](#)
[\[StacksM, Lemma 45.16 \(Tag 0B3I\), p. 108\]](#)

3 Algebraize over an ordinary projective space

$$S \rightarrow Q = (\mathbb{P}_X(\mathcal{O}_X^{\oplus 2} \oplus \mathcal{O}_X(D)^{\oplus 2}))_X^4 \text{ finite}$$

Koszul gives connectedness; rank conditions give submersion points over divisors
[\[StacksC, Lemma 8.1 \(Tag 01XT\), pp. 16–17\];](#) [Lemma 4.4 · p. 9](#)

4 Good fibres and control along divisors

$$F = C_1 \times \dots \times C_4, \quad \Omega_F^1 \simeq \mathcal{O}_F^4, \quad \text{codim}_X Z \geq 2$$

Connected general fibres give a unique dominating component
[Lemma 4.4 · p. 9;](#) [Proposition 4.1 · p. 6](#)

5 Take the dominating component

$$Y^8 \rightarrow X^4, \quad g^*D = 2E \quad \text{and all four conditions of Proposition 3.1}$$

Four pairs of quadrics avoid the fixed locus; algebraization produces a projective variety

[Open figure ↗](#) [TeX source ↓](#)

Diagonal quadrics on the root space and avoidance of fixed points

Set $T = \mathcal{O}_{\mathcal{X}}(\mathcal{D})$, so $T^2 = \pi^* \mathcal{O}_X(D)$. Choose M with both M and $M(-D)$ globally generated. On the fourfold fibre product of $\mathbb{P}_{\mathcal{X}}(\mathcal{O}^{\oplus 2} \oplus T^{\oplus 2})$, impose two diagonal quadrics in each factor $j = 1, \dots, 4$:

$$q_{j\nu} = \sum_{i=1}^4 a_{j\nu i} z_{ji}^2 = 0 \quad (\nu = 1, 2).$$

For $i = 1, 2$ the coefficients lie in $H^0(X, M)$, and for $i = 3, 4$ in $H^0(X, M(-D))$. The latter are evaluated after tensoring with T^2 ; they are not automatically zero along D .

Independent coefficient evaluations for the eight equations give smoothness for general parameters. Over D , a fixed point of μ_2 lies in one of two eigenspace copies of $\mathbb{P}^1$ in every factor. A common zero of the corresponding two quadrics requires the relevant 2×2 determinant to vanish. The four factors impose four independent conditions. Since $\dim D = 3$, general parameters avoid every fixed point. This dimension count explains the use of four factors.

Lemma 4.2 · pp. 7–8

Squaring coordinates provides projectivity

The smooth analytic space S obtained from the free root charts maps finitely, by squaring coordinates, to the ordinary projective bundle product

$$Q = (\mathbb{P}_X(\mathcal{O}_X^{\oplus 2} \oplus \mathcal{O}_X(D)^{\oplus 2}))_X^4.$$

GAGA algebraizes the finite coherent algebra together with its multiplication and unit. Taking relative Spec gives an algebraic finite morphism to Q . Finite morphisms are projective and Q is projective, so S is projective.

Lemma 4.3 · p. 8 [Ser56, §12, Théorèmes 2–3, print pp. 19–20 / PDF pp. 20–21]
[StacksM, Lemma 45.16 (Tag 0B3I), p. 108]

Arrange connected fibres and submersion points over divisors

For general parameters there is a nonempty open B where all two-column minors of every 2×4 coefficient matrix are nonzero. The intersection of its two quadrics is smooth there. The Koszul sequence and vanishing of intermediate cohomology on $\mathbb{P}^3$ give $H^0(C, \mathcal{O}_C) = \mathbb{C}$, while adjunction gives $K_C \simeq \mathcal{O}_C$. Thus each curve is connected of genus one.

[StacksC, Lemma 8.1 (Tag 01XT), pp. 16–17] Lemma 4.4 · p. 9

The rank-drop condition for a 2×3 matrix obtained by deleting any column has codimension two. The corresponding bad set $Z \subset X$ can therefore be chosen to have codimension at least two. At $x \notin Z$, the kernel contains a vector with every coordinate nonzero. Taking its

coordinatewise square roots yields a point where the vertical derivative of each pair of quadrics has rank two. It is a submersion point to the root chart, giving the required condition over general points of every prime divisor, including D .

Connected general fibres give a unique component Y of S dominating X , and the submersion points above belong to it. Finally, $g^*D = 2f^*\mathcal{D}$ and $\dim Y = 4 + 4 = 8$. All hypotheses of Proposition 3.1 have now been supplied.

[Lemma 4.2 · pp. 7–8](#) [Lemma 4.4 · p. 9](#)

5. Dependencies and their uses

The direct input from the first paper concerns the **entire fundamental group** of a special compact Kähler manifold. Its analytic and group-theoretic proof is discussed in [that article](#), not reproved here. The second paper's linear-monodromy results are not substituted for this input.

Closedness of differential forms enters Lemma 2.1; Ehresmann's result enters the group surjection in Proposition 3.1; GAGA and projectivity of finite morphisms enter Lemma 4.3; projective-space cohomology enters Lemma 4.4. We checked these statements and their uses. The original resolution-of-singularities input in Lemma 2.1 and complete proofs of the external results lie outside this checking scope.

6. Return to the sources

- [Theorem 1.3 · p. 2](#) — The conditional conclusion and the group it concerns.
- [Proposition 3.1 · pp. 5–6](#) — Separate specialness transfer from the meridian argument for the group quotient.
- [Lemma 4.2 · pp. 7–8](#) — Distinguish codimension four for fixed-point avoidance from codimension two for the excluded set.
- [Lemma 4.4 · p. 9](#) — The connected genus-one product and submersion points in every factor.
- [\[OpenAI26a, Theorem 1.1, p. 2\]](#) — The assertion about the entire fundamental group used at the last step.
- [Appendix A · pp. 10–13](#) — An alternative 14-dimensional construction, outside the detailed checking scope here.

The source is the October 5, 2026 manuscript, 14 pages. The conclusions above are the manuscript's claims. Editorial checking covers the main proofs on pp. 3–9, the hypotheses of Theorem 1.3 and Propositions 3.1 and 4.1, the path to each conclusion, and the statements and uses of the direct inputs. Appendix A's alternative 14-dimensional construction was not checked in detail. This is not a complete independent reconstruction of every local calculation on the root space or of the external proofs.

The source version, SHA-256, page mappings and unresolved items are recorded in [sources.json](#); timing and further edits are in [status.md](#). This is an AI-generated exposition; consult the linked originals for research use.