

OpenAI Math Digest

CATALOGUE 050

A counterexample to Griffiths' positivity conjecture

Catalogue guide & one proof overview

Independent reading guide · English edition

Edition: 2026-10-22

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Contents

The catalogue guide is followed by the article. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

00 · Catalogue guide	3
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01 · Ample rank-two bundles on the quadric surface without Griffiths-positive metrics	6
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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

A counterexample to Griffiths' positivity conjecture

The manuscript constructs a rank-two bundle on $\mathbf{P}^1 \times \mathbf{P}^1$ whose family $E_m = f_m^* G \otimes \mathcal{O}(1, 1)$ is ample for every positive integer m , while admitting no smooth strictly Griffiths-positive Hermitian metric for all sufficiently large m . This guide distinguishes the manuscript's claims from the scope of editorial checking.

1. Paper in this catalogue

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 050 · 2026-10-22 checked source snapshot

#

01

Manuscript and overview

Ample rank-two bundles on the quadric surface without Griffiths-positive metrics →

Short name: Ample rank-two bundles on the quadric surface

September 24, 2026 · 18 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For an explicit rank-two G , the bundle $E_m = f_m^* G \otimes \mathcal{O}_S(1, 1)$ is ample for every $m \geq 1$. There is an m_0 such that, for every $m \geq m_0$, it has no smooth strictly Griffiths-positive Hermitian metric.

[Theorem 1.1 · p. 2](#) ↗

Role and available overview

Connects a symmetric-power filtration from two quotients to an invariant-measure obstruction from the graph identities.

[Read the proof overview](#) →

2. Entry into the proof

The individual article starts with the explicit construction of G along three graphs. It then follows ampleness through two quotient sequences and a measure obstruction to a quadratic psh limit on the dual bundle. The graph identities connect the two arguments. The contradicted conjecture concerns a smooth strictly Griffiths-positive Hermitian metric on the ample bundle itself; neither exclusion of semipositive metrics nor an effective threshold is asserted.

[Theorem 1.1 · p. 2](#)

There is one paper in this catalogue. Direct external inputs are collected in the article: removable singularities and decreasing limits for psh functions, Čech comparison, Künneth, and projective-bundle constructions. Comparisons with other counterexample manuscripts are not treated as proof dependencies.

3. Sources and checking scope

The official subject is Algebraic and complex geometry and the official catalogue number is 050. The internal article ID is separate from that number.

OpenAI Research Catalog · Algebraic and complex geometry, 050 · p. 7

The source is the September 24, 2026 version, 18 pages. Checking covers the main theorem, the proof route in §§2–6, and statements and uses of direct external inputs. It does not constitute independent verification of complete proofs, recursive verification of external literature, or expert peer review. The individual article records the detailed scope.

§§2–6 · pp. 4–16

Ample rank-two bundles on the quadric surface without Griffiths-positive metrics

Ampleness from two quotients, and an invariant-measure obstruction to a quadratic limit

The manuscript constructs a rank-two bundle on the quadric surface by identifications along three graphs. The same identifications produce sections of symmetric powers on the algebraic side and an obstruction to a common invariant measure on the analytic side. This article describes the manuscript's claims and proof route; it does not certify the complete proof.

Manuscript author: OpenAI · September 24, 2026 · 18 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Ampleness for every power and no strictly positive metric for large powers

Set $S = \mathbf{P}^1 \times \mathbf{P}^1$ and $P = \mathcal{O}_S(1, 1)$, using the convention that $\mathbf{P}(E)$ parametrizes one-dimensional quotients. For a positive integer m , let

$$f_m([z_0 : z_1], [w_0 : w_1]) = ([z_0^m : z_1^m], [w_0^m : w_1^m]), \quad \deg f_m = m^2.$$

There exists an algebraic vector bundle G of rank two on S such that, for every positive integer m ,

$$E_m = f_m^* G \otimes P$$

is ample. Moreover, there is an integer $m_0 \geq 1$ such that, for every $m \geq m_0$, E_m admits no smooth strictly Griffiths-positive Hermitian metric. Thus no smooth Hermitian metric has Chern curvature positive on every pair consisting of a nonzero tangent vector and a nonzero fiber vector.

Theorem 1.1 · p. 2

The formulation contradicted here is the modern assertion that every ample vector bundle on a smooth complex projective variety itself admits a smooth strictly Griffiths-positive Hermitian metric. The manuscript does not assert the nonexistence of semipositive metrics. It supplies no effective value of m_0 and does not settle the small values of m . Positivity after determinant twists or symmetric powers, and Finsler metrics not required to be Hermitian quadratics, concern different conclusions.

§1, Historical context and related results · pp. 1–3

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Ample rank-two bundles on the quadric surface without Griffiths-positive metrics](#). External inputs use the following keys.

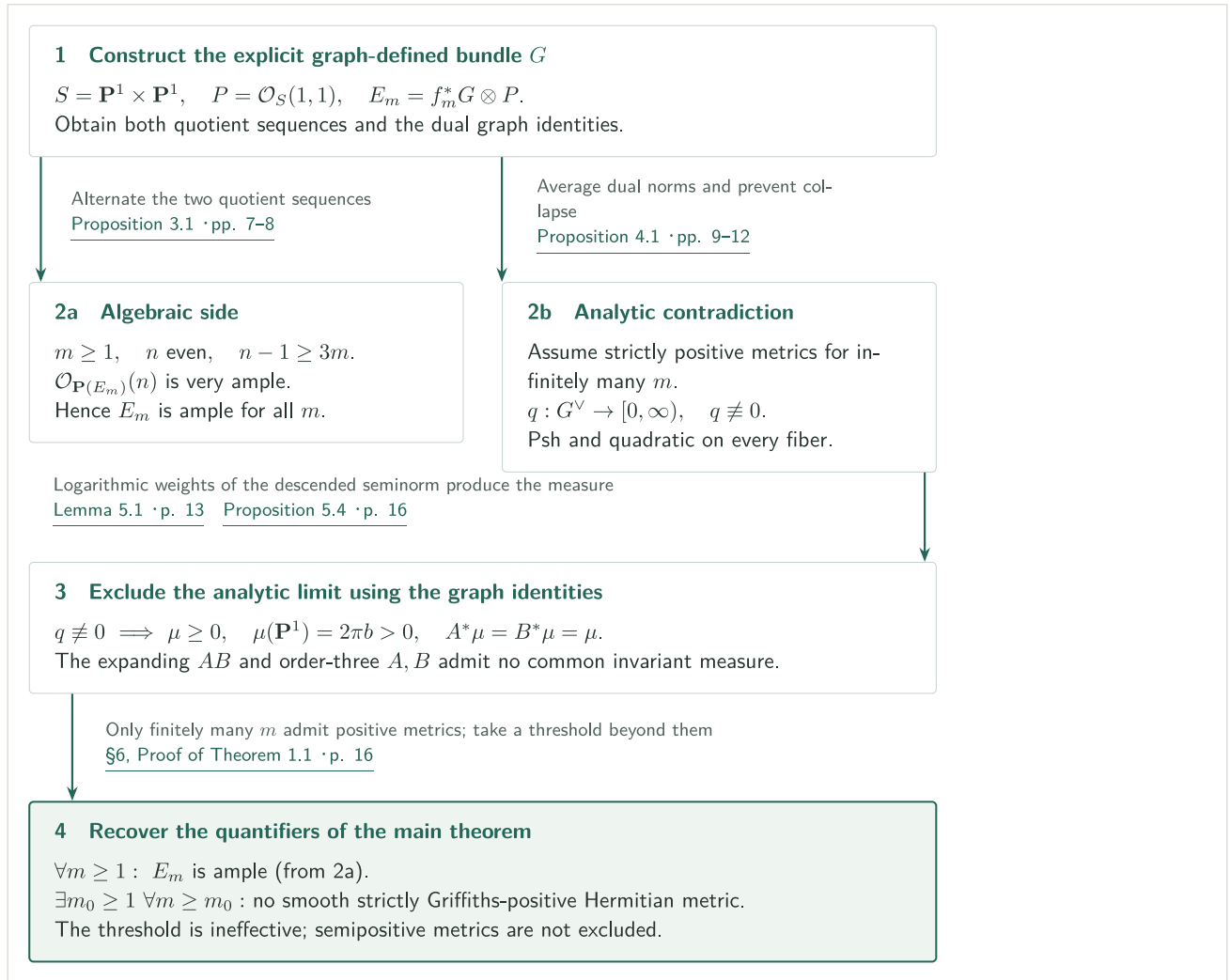
- [Dem12](#) Jean-Pierre Demailly — Complex analytic and differential geometry: Removable singularities and decreasing limits of psh functions. Version of 21 June 2012.
- [StacksCoh](#) The Stacks Project Authors — The Stacks Project — Cohomology of Schemes: Čech comparison for the standard affine cover. Version ed88ff78, compiled 14 July 2026; consulted 22 October 2026.
- [StacksVar](#) The Stacks Project Authors — The Stacks Project — Varieties: Künneth for line-bundle cohomology on the product. Version ed88ff78, compiled 14 July 2026; consulted 22 October 2026.
- [StacksCon](#) The Stacks Project Authors — The Stacks Project — Constructions of Schemes: Quotient projectivization, closed immersions, and the Segre hyperplane bundle. Version ed88ff78, compiled 14 July 2026; consulted 22 October 2026.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1](#) · p. 2

Proof route. Construct G from [three graphs](#), then alternate its two quotient sequences to obtain [ampleness for every \$m\$](#) . If positive metrics existed for infinitely many m , averaged dual norms would yield a nonzero psh function quadratic on every fiber. Applying the graph identities to this function would produce an [impossible common invariant measure](#).



The two conclusions and the limiting contradiction

[Open figure ↗](#) [TeX source ↓](#)

1. An explicit bundle and ampleness for all m

The bundle G of Proposition 2.3 has two surjections $G \rightarrow L_i$. After pullback by f_m , the product of their kernels is $\mathcal{O}_S(18m, 18m)$. Proposition 3.1 incorporates this positive product into a symmetric-power filtration and proves that $\mathcal{O}_{\mathbf{P}(E_m)}(n)$ is very ample whenever n is even and $n - 1 \geq 3m$. Every $m \geq 1$ admits such an n , giving ampleness for all m .

[Proposition 2.3 · p. 6](#)

[Proposition 3.1 · pp. 7–8](#)

2. Infinitely many positive metrics produce one limit

If positive metrics exist for infinitely many m , choose a sequence with $m \rightarrow \infty$. Proposition 4.1 gives a finite, nonnegative, nonzero psh function q on $F = G^\vee$, with $q(x, v) = H_x(v, v)$ and $H_x \geq 0$ at every $x \in S$. Rank may drop, but the function cannot vanish identically or cease to be quadratic. The pointwise statement allows the graph identities on the divisor to be evaluated directly in q .

Proposition 4.1 · pp. 9–12

3. From the measure contradiction to finitely many exceptions

Proposition 5.4 says that such a q must be zero for the constructed bundle. Hence the set of integers m admitting a positive metric is finite. Choose m_0 larger than its maximum, or $m_0 = 1$ if it is empty. The proof excludes an infinite sequence of metrics rather than computing the curvature obstruction at a specified m .

Proposition 5.4 · p. 16

§6, Proof of Theorem 1.1 · p. 16

3. Proposition 2.3 — Constructing the bundle from three graphs

Proof target : Proposition 2.3 · p. 6

Fix the following scalars and matrices:

$$a = e^{i\pi/3}, \quad s = \frac{1}{2} - \frac{i(1 + 2\sqrt{2})}{2\sqrt{3}}, \quad A = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \quad B = \begin{pmatrix} s & 1 \\ \sqrt{2}/3 & 1 - s \end{pmatrix}.$$

For $M \in \{I, A, B\}$ set $\Gamma_M = \{(x, Mx) : x \in \mathbf{P}^1\}$ and let $D = \Gamma_I \cup \Gamma_A \cup \Gamma_B$ be reduced. Set $b = 24$, $L_1 = \mathcal{O}_S(b, 0)$, and $L_2 = \mathcal{O}_S(0, b)$.

§2, equations (2.1)–(2.2) · pp. 4–5

Proposition 2.3 — Gluing, local freeness, and two quotients

The maps $M^{\otimes b}$ on Γ_M glue to an isomorphism $\alpha : L_1^{-1}|_D \simeq L_2^{-1}|_D$. With $\beta = (\alpha^\vee)^{-1}$,

$$G = \{(u_1, u_2) \in L_1 \oplus L_2 : u_2|_D = \beta(u_1|_D)\}$$

is an algebraic rank-two vector bundle. Both projections $\rho_j : G \twoheadrightarrow L_j$ give exact sequences

$$0 \longrightarrow L_i(-D) \longrightarrow G \xrightarrow{\rho_j} L_j \longrightarrow 0, \quad \{i, j\} = \{1, 2\}.$$

The dual inclusions $\iota_i = \rho_i^\vee : L_i^{-1} \hookrightarrow F = G^\vee$ span every fiber over $S \setminus D$ and satisfy $\iota_2 \alpha = \iota_1$ on D .

Proposition 2.3 · p. 6

1 Finite orders and six nodes

$\text{ord}(A) = \text{ord}(B) = 6, \quad \text{ord}(A^{-1}B) = 8, \quad b = 24.$
 $D = \Gamma_I \cup \Gamma_A \cup \Gamma_B \in |\mathcal{O}_S(3, 3)|$; no triple points.

The relative eigenvalues have 24th power one
[Lemmas 2.1–2.2 · p. 5](#)

2 Glue the compatible lifts

$Nv = \lambda Mv \implies \lambda^{24} = 1 \implies (Nv)^{\otimes 24} = (Mv)^{\otimes 24}.$
 $\alpha : L_1^{-1}|_D \xrightarrow{\sim} L_2^{-1}|_D, \quad \beta = (\alpha^\vee)^{-1}.$

Descend in the nodal local ring and lift the unit h
[Proposition 2.3 · p. 6](#)

3 A rank-two bundle with a local basis

$G = \{(u_1, u_2) : u_2|_D = \beta(u_1|_D)\}.$
 $e = (1, h), \quad f = (0, d), \quad h \in \mathcal{O}_S^\times.$

Read surjectivity and kernels in the basis, then dualize
[Proposition 2.3 · p. 6](#)

4 Two quotients and the dual graph identity

$0 \rightarrow L_i(-D) \rightarrow G \xrightarrow{\rho_j} L_j \rightarrow 0, \quad \{i, j\} = \{1, 2\}.$
 $\iota_i = \rho_i^\vee : L_i^{-1} \hookrightarrow G^\vee, \quad \iota_2\alpha = \iota_1 \text{ on } D.$
 The two lines span each fiber over $S \setminus D$.

Finite-order lifts yield two quotient sequences and the graph identity

[Open figure ↗](#) [TeX source ↓](#)

1. Finite orders provide compatibility at intersections

The orders of $A, B, A^{-1}B$ in $\text{SL}_2(\mathbf{C})$ are **6, 6, 8**. Each pair of graphs meets transversely at the two distinct eigenlines of the relative matrix. There are no triple points because A and B have no common eigenline. Thus D has bidegree **(3, 3)** and six nodes. At an intersection, $Nv = \lambda Mv$ and $\lambda^{24} = 1$, so $M^{\otimes 24}$ and $N^{\otimes 24}$ agree on the actual fiber. The nodal local-ring sequence

$$0 \longrightarrow R/(xy) \longrightarrow R/(x) \oplus R/(y) \longrightarrow R/(x, y) \longrightarrow 0$$

turns this agreement into gluing on the whole divisor.

[Lemmas 2.1–2.2 · p. 5](#)

2. The local basis also proves surjectivity

Let d be a local equation of D and let the unit h lift the local coefficient of β . Then $e = (1, h)$ and $f = (0, d)$ form a basis of G , including at nodes. This basis proves that both projections are surjective and their kernels are $L_i(-D)$. Dualizing gives $\iota_2\alpha = \iota_1$, the precise relation that will identify the seminorms coming from the two rulings.

Proposition 2.3 · p. 6

4. Proposition 3.1 — Ampleness with the full range of powers and twists

Proof target : Proposition 3.1 · pp. 7–8

Proposition 3.1 — An explicit very-ampleness range

For the bundle G above, $E_m = f_m^*G \otimes P$ is ample for every positive integer m . More precisely, $\mathcal{O}_{\mathbf{P}(E_m)}(n)$ is very ample for every even integer n satisfying $n - 1 \geq 3m$.

Proposition 3.1 · pp. 7–8

Write $V = f_m^*G$, $\tilde{L}_i = f_m^*L_i$, $\tilde{D} = f_m^*D$, $K_i = \tilde{L}_i(-\tilde{D})$, and $K = K_1 \otimes K_2$.

1 Alternate the two quotient sequences

$$V = f_m^* G, \quad K_1 = \mathcal{O}_S(21m, -3m), \quad K_2 = \mathcal{O}_S(-3m, 21m). \\ K = K_1 K_2 = \mathcal{O}_S(18m, 18m), \quad \tilde{D} = f_m^* D.$$

Each pair of steps extracts the positive product K
[§3, equations \(3.1\)–\(3.5\) · pp. 7–8](#)

2 Line factors of the symmetric power

$$K^j \tilde{L}_2^{n-2j}, \quad K^j \tilde{L}_1^{n-2j}(-\tilde{D}) \quad (0 \leq j < n/2), \quad K^{n/2}.$$

Each quotient sequence locally splits even where the kernels coincide.

Offset the remaining $-3m$ by $n-1$ and use H^1 vanishing
[\[StacksCoh, Tag 01XD, Lemma 2.6, p. 3\]](#) [\[StacksVar, Tag 0BED, Lemma 29.1, p. 54\]](#)

3 Twist and lift sections

$$m \geq 1, \quad n \in 2\mathbf{Z}_{>0}, \quad n-1 \geq 3m.$$

$W = \mathrm{Sym}^n V \otimes P^{n-1}$: all line factors have nonnegative bidegrees.

Global generation and $H^1 = 0$ imply global generation of W .

Use quotient projectivization; the final P embeds the base
[§3, The projective embedding and its line bundle · p. 8](#)
[\[StacksCon, Tag 01WD, Lemma 13.6, p. 29\]](#)

4 Veronese and Segre give very ampleness

$$\mathbf{P}(V) \hookrightarrow \mathbf{P}(H^0(S, W)) \times S \hookrightarrow \mathbf{P}^N.$$

$$\mathcal{O}_{\mathbf{P}(V)}(n) \otimes \pi^* P^{n-1} \otimes \pi^* P = \mathcal{O}_{\mathbf{P}(E_m)}(n).$$

Thus E_m is ample for every $m \geq 1$.

Alternating symmetric-power filtrations produce a projective embedding

[Open figure ↗](#) [TeX source ↓](#)

1. Alternating quotients group the two negative directions

Although $K_1 = \mathcal{O}_S(21m, -3m)$ and $K_2 = \mathcal{O}_S(-3m, 21m)$, their product is $K = \mathcal{O}_S(18m, 18m)$.

Alternating the sequences $0 \rightarrow K_i \rightarrow V \rightarrow \tilde{L}_j \rightarrow 0$ gives, for even n , a filtration of $\mathrm{Sym}^n V$ with line factors

$$K^j \tilde{L}_2^{n-2j}, \quad K^j \tilde{L}_1^{n-2j}(-\tilde{D}) \quad (0 \leq j < n/2), \quad K^{n/2}.$$

Each step uses one locally split quotient sequence, so it remains valid where the two kernels have the same fiber image. For $n = 2$, the manuscript checks local bases to retain locally free quotients at nodes and on nonreduced pullbacks of the divisor.

[§3, equations \(3.1\)–\(3.5\) · pp. 7–8](#)

2. The twist P^{n-1} makes all factors globally generated

The line factors of $W = \mathrm{Sym}^n V \otimes P^{n-1}$ have bidegrees

$$\begin{aligned}
& (18mj + n - 1, 18mj + 24m(n - 2j) + n - 1), \\
& (18mj + 24m(n - 2j) - 3m + n - 1, 18mj - 3m + n - 1), \\
& (9mn + n - 1, 9mn + n - 1).
\end{aligned}$$

The inequality $n - 1 \geq 3m$ offsets the sole remaining negative term $-3m$. A line bundle of nonnegative bidegree is globally generated and has $H^1 = 0$. The long exact sequence therefore lifts sections through successive quotients, proving that W is globally generated. The cohomology calculation uses Čech comparison for the standard affine cover of $\mathbf{P}^1$ and the Künneth formula on the product.

§3, Global generation after twisting · p. 8

3. One final factor P embeds the base

Evaluation and the relative Veronese give $\mathbf{P}(V) \hookrightarrow \mathbf{P}(H^0(S, W)) \times S$. The hyperplane bundle from the first factor pulls back to $\mathcal{O}_{\mathbf{P}(V)}(n) \otimes \pi^* P^{n-1}$. Embed S by P and compose with Segre. The resulting hyperplane pullback is

$$\mathcal{O}_{\mathbf{P}(V)}(n) \otimes \pi^* P^n = (\mathcal{O}_{\mathbf{P}(V)}(1) \otimes \pi^* P)^n = \mathcal{O}_{\mathbf{P}(E_m)}(n).$$

Separating the roles of P^{n-1} and the final P explains the passage from global generation to very ampleness.

§3, The projective embedding and its line bundle · p. 8

5. Proposition 4.1 — Removing the twist while retaining a quadratic at every point

Proof target : Proposition 4.1 · pp. 9–12

Proposition 4.1 — A necessary condition for an arbitrary bundle

Let G be any holomorphic vector bundle of positive rank on $S = \mathbf{P}^1 \times \mathbf{P}^1$, and set $F = G^\vee$. If $f_m^* G \otimes P$ admits a smooth strictly Griffiths-positive Hermitian metric for infinitely many positive integers m , there exists a finite, nonnegative, nonzero psh function $q : F \rightarrow [0, \infty)$ on the total space such that, for every $x \in S$, $q(x, v) = H_x(v, v)$ for a finite positive semidefinite Hermitian form H_x .

Proposition 4.1 · pp. 9–12

1 Trace dual norms from infinitely many positive metrics

$$m \rightarrow \infty, \quad (f_m^* G \otimes P)^\vee = f_m^* F \otimes P^{-1}.$$

$$q_{m,t} = m^{-2} \sum_{y \in f_m^{-1}(x)} \|v \otimes (f_m^* t)^{1/m}(y)\|_{m,y}^2.$$

Extend psh across ramification and obtain local bounds

[Dem12, Chapter I, Theorem 5.24, p. 45]

Proposition 4.1, equations (4.1)–(4.5) · pp. 9–10

2 Normalize and take a distributional coefficient limit

$$q_{m,gt} = |g|^{2/m} q_{m,t}, \quad |g|^{2/m} \rightarrow 1.$$

$$\max_{|v|_0=1} N_m = 1, \quad q_{m,t} = v^* H_m v, \quad H_m \rightarrow H.$$

Apply the submean inequality to a fixed vector near maxima

Proposition 4.1, equation (4.6) · p. 11

3 Prevent collapse of the weak limit

$$\int \chi c^* H_m c \geq \frac{1}{2} \text{Vol}(B(0, \varrho)) > 0.$$

$$H \not\equiv 0, \quad H \geq 0 \text{ almost everywhere.}$$

Decrease radial convolutions and polarize finitely many values

[Dem12, Chapter I, Theorem 5.4, p. 39]

Proposition 4.1, equations (4.7)–(4.8); Remark 4.2 · pp. 11–12

4 A psh representative quadratic at every point

$$q_\delta = v^*(H * \rho_\delta)v \downarrow q_U.$$

Polarize at each point and glue by uniqueness of psh representatives.

$$q : F \rightarrow [0, \infty), \quad q \not\equiv 0, \quad q(x, v) = H_x(v, v) \quad (\forall x \in S).$$

Tracing across ramification, preventing collapse, and recovering every fiber

[Open figure ↗](#) [TeX source ↓](#)

1. Tracing over the power map removes the twisting transition

The squared dual norm is psh on the total space of $f_m^* F \otimes P^{-1}$. For a local frame t of P^{-1} , use $f_m^* P^{-1} \simeq P^{-m}$ to define

$$q_{m,t}(x, v) = \frac{1}{m^2} \sum_{y \in f_m^{-1}(x)} \|v \otimes (f_m^* t)^{1/m}(y)\|_{m,y}^2.$$

The sum counts multiplicities, and different root choices differ by a unit-modulus constant, leaving the squared norm unchanged. Off the branch locus it is a sum of psh pullbacks by holomorphic inverse branches; across that locus it is continuous and locally bounded. The inverse image of the branch locus in the total space is closed and pluripolar, so the removable-singularities theorem applies. A frame change $t' = gt$ gives $q_{m,t'} = |g|^{2/m} q_{m,t}$, whose extra factor tends uniformly to one on compact subsets.

Proposition 4.1, equations (4.1)–(4.5) · pp. 9–10

[Dem12, Chapter I, Theorem 5.24, p. 45]

2. Convert a normalized maximum into nonvanishing of the limit

Using background metrics, normalize the global quadratic $N_m = q_{m,t}|t|_0^{-2/m}$ to have maximum one on the unit sphere bundle. Its local coefficient matrices H_m are uniformly bounded; a subsequence converges distributionally to $H \in L_{\text{loc}}^\infty$. Let maximizing points (x_m, v_m) converge and replace the nearby fiber vectors by a fixed vector c . The submean inequality for the psh function $c^* H_m c$ gives a fixed nonnegative test function χ with

$$\int \chi c^* H_m c \geq \frac{1}{2} \text{Vol}(B(0, \varrho)) > 0.$$

Passing this estimate to the distributional limit proves $H \neq 0$. This integral bound supplies the noncollapse that normalization of the maximum alone would not establish.

Proposition 4.1, equation (4.6) · p. 11

3. Recover a psh representative usable on the divisor

Convolve the coefficients in the base variables with a radial kernel, setting $q_\delta(x, v) = v^*(H * \rho_\delta)(x)v$. These are smooth psh functions and decrease as $\delta \downarrow 0$. Their local bounds give a finite psh limit q_U . Polarization recovers all matrix entries from the limiting values at the finite list $e_j, e_j + e_k, e_j + ie_k$, proving quadraticity at every point. On overlaps the local psh representatives agree almost everywhere, and the submean inequality together with upper semicontinuity gives equality everywhere.

Proposition 4.1, equations (4.7)–(4.8); Remark 4.2 · pp. 11–12

[Dem12, Chapter I, Theorem 5.4, p. 39]

The representatives thus glue to a global q . The graph identities are evaluated in this single everywhere-defined function, rather than in a raw pointwise limit or a limit of restrictions to the divisor. Remark 4.2 explicitly makes no assertion of convergence of those restrictions.

Proposition 4.1, equations (4.7)–(4.8); Remark 4.2 · pp. 11–12

6. Proposition 5.4 — No common invariant measure

Proof target : Proposition 5.4 · p. 16

Proposition 5.4 — Every quadratic psh function on this bundle vanishes

For G from Proposition 2.3 and $F = G^\vee$, every finite nonnegative psh function on the total space of F whose restriction to each fiber is a Hermitian quadratic form is identically zero.

Proposition 5.4 · p. 16

1 Assume a nonzero quadratic psh function

$$q : G^\vee \rightarrow [0, \infty), \quad q(x, v) = H_x(v, v), \quad H_x \geq 0.$$

$$\iota_i : L_i^{-1} \hookrightarrow G^\vee, \quad \iota_2 \alpha = \iota_1 \text{ on } D.$$

Use the maximum principle on the other $\mathbf{P}^1$ and semidefiniteness
[Lemma 5.1 · p. 13](#)

2 Descend along rulings and identify on three graphs

$$a_\bullet : \mathcal{O}_{\mathbf{P}^1}(-b) \rightarrow [0, \infty), \quad a_\bullet \neq 0.$$

$$a_\bullet(Mx, M^{\otimes b}v) = a_\bullet(x, v) \quad (M = A, B).$$

Use logarithmic integrability, a reference metric, and isometries
[Lemma 5.2 · pp. 14–15](#)

3 A positive curvature measure including zeros

$$a_\bullet(z, we) = u_e(z)|w|^2, \quad \log u_e \in L^1_{\text{loc}}.$$

$$\mu = i\partial\bar{\partial} \log u_e \geq 0, \quad \mu(\mathbf{P}^1) = 2\pi b > 0, \quad A^*\mu = B^*\mu = \mu.$$

Half-open annuli have equal mass; finiteness concentrates it at two fixed points
[Lemma 5.3 · p. 16](#) [Lemmas 2.1–2.2 · p. 5](#)

4 The action on the support is impossible

$AB : z \mapsto \lambda z, \quad |\lambda| > 1 \implies \text{Supp } \mu \subset \{0, \infty\}.$
 Projective order three $\implies$ each fixes every point of the support.
 There is no common eigenline. Therefore $q = 0$.

The rulings produce a measure of positive mass, excluded by the dynamics

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1. Combine two rulings and three graphs into one seminorm

Suppose $q \neq 0$. In $q \circ \iota_i$, hold the i th coordinate and the fiber vector fixed. The resulting finite subharmonic function on the complementary $\mathbf{P}^1$ is constant. Each ruling therefore yields a psh squared seminorm a_i on $\mathcal{O}_{\mathbf{P}^1}(-b)$. The identity along Γ_I gives $a_1 = a_2 =: a_\bullet$; those along Γ_A, Γ_B give

$$a_\bullet(Mx, M^{\otimes b}v) = a_\bullet(x, v), \quad M = A, B.$$

If $a_\bullet = 0$, the two spanning lines over $S \setminus D$ and Cauchy–Schwarz for positive semidefinite forms force $q = 0$ there. The submean inequality extends the vanishing to the total space. Thus $a_\bullet$ is nonzero.

Lemma 5.1 · p. 13

2. Include the zeros when computing curvature mass

In a local frame write $a_\bullet(z, we(z)) = u_e(z)|w|^2$. Total-space plurisubharmonicity gives $u_\delta(u_\delta)_{z\bar{z}} - |(u_\delta)_z|^2 \geq 0$ for a positive smoothing, so $\log u_\delta$ is subharmonic. Nontriviality and the submean inequality bound its negative part in L^1 ; the manuscript thereby establishes $\log u_e \in L^1_{\text{loc}}$, including at zeros. Consequently

$$\mu = i\partial\bar{\partial}\log u_e \geq 0$$

is a global Radon measure. Frame changes add harmonic terms, and the lifts of A, B are isometries, so $A^*\mu = B^*\mu = \mu$. Subtracting the reference logarithmic weight of $(1 + |z|^2)^b$ on $\mathcal{O}(-b)$ gives a globally integrable potential whose $i\partial\bar{\partial}$ has integral zero. Hence

$$\mu(\mathbf{P}^1) = \int_{\mathbf{C}} \frac{b i dz \wedge d\bar{z}}{(1 + |z|^2)^2} = 2\pi b > 0.$$

Retaining zeros and atoms is essential to preserving the nonzero mass.

Lemma 5.2 · pp. 14–15

3. An expanding product is incompatible with order three

The eigenvalues of $T = AB$ have different moduli; in projective coordinates it acts as $z \mapsto \lambda z$ with $|\lambda| = R > 1$. The disjoint half-open annuli $C_n = \{R^n \leq |z| < R^{n+1}\}$ have equal measure, so finiteness forces each to have measure zero. The nonempty support of μ is therefore contained in the two fixed points of T . Both A and B preserve this support. Since their projective orders are three, their permutations of a set of at most two points are trivial. Every support point would be fixed by both transformations, contradicting the absence of a common eigenline in Lemma 2.1. This excludes $q \neq 0$.

Lemma 5.3 · p. 16

7. Which papers supply which steps

The direct inputs are the two psh theorems from Dem12 for the analytic limit, and the Stacks Project results on Čech comparison, Künneth, projective bundles, and embeddings for the algebraic argument. Their statements and the applications below were compared; the complete external proofs were not verified.

Input

[Dem12, Chapter I, Theorem 5.24, p. 45]

Application and hypotheses

Proposition 4.1, p.10: the inverse image of the branch locus is closed pluripolar, and the average is continuous and locally bounded

Role

Extend the psh average across ramification

Input

[Dem12, Chapter I, Theorem 5.4, p. 39]

Application and hypotheses

Proposition 4.1, p.12: the convolved quadratics decrease to a finite, locally bounded limit

Role

Obtain an everywhere-defined psh representative

Input

[StacksCoh, Tag 01XD, Lemma 2.6, p. 3]

Application and hypotheses

Proposition 3.1, p.8: the standard affine cover and its intersections; quasi-coherent line bundles

Role

Identify Čech and sheaf cohomology; vanishing itself comes from the calculation on $\mathbf{P}^1$

Input

[StacksVar, Tag 0BED, Lemma 29.1, p. 54]

Application and hypotheses

Proposition 3.1, p.8: $\mathbf{P}^1$ is quasi-compact and separated

Role

Deduce $H^1(\mathcal{O}(c, t)) = 0$ on the product

Input

[StacksCon, Tag 01OA, §21, pp. 45–46]

Application and hypotheses

Proposition 3.1, p.8: evaluation is surjective

Role

Match quotient projectivization with the tautological line

Input

[StacksCon, Tag 01N1, Lemma 11.5, p. 22]

Application and hypotheses

Proposition 3.1, p.8: surjections of symmetric algebras generated in degree one

Role

Identify the closed immersion and hyperplane pullback

Input

[StacksCon, Tag 01WD, Lemma 13.6, p. 29]

Application and hypotheses

Proposition 3.1, p.8: P embeds S

Role

Add the final P through Segre to obtain very ampleness

The manuscript supplies the needed arguments for graph gluing and local freeness, the alternating symmetric-power filtration, psh dual norms, smoothing and polarization, integrability of logarithmic weights, and the dynamical measure obstruction. References to Maruyama, Hartshorne, Berndtsson–Păun, Raufi, Păun–Takayama, and Furstenberg serve as methodological background or comparison; their presence is not converted into direct-dependency arrows through unread theorems. The main results and proofs of the other counterexamples discussed in the introduction are outside this article's checking scope.

8. Return to the sources

- [Theorem 1.1 · p. 2](#) — The quantifiers and the strict-positivity conclusion

- §2, equations (2.1)–(2.2) · pp. 4–5 — The explicit matrices and the choice $b=24$
- Proposition 2.3 · p. 6 — Gluing at nodes, the local basis, and two quotients
- Proposition 3.1 · pp. 7–8 — The symmetric-power filtration and the very-ampleness range
- Proposition 4.1 · pp. 9–12 — Noncollapse and a psh representative quadratic at every point
- Lemma 5.2 · pp. 14–15 — Logarithmic integrability including zeros and mass $2\pi b$
- Lemma 5.3 · p. 16 — Annuli and the order-three argument excluding invariant measures
- §6, Proof of Theorem 1.1 · p. 16 — The final passage from finitely many exceptions to m_0

The reading covers Theorem 1.1 and the proof text of §§2–6 in the September 24, 2026 version, particularly gluing at nodes, symmetric-power factors, noncollapse and the everywhere-defined quadratic representative, the mass calculation including zeros, and the conclusion of finitely many exceptions. The stated Dem12 and Stacks inputs and their application hypotheses were also compared. This is a check of the proof route and direct inputs, not independent verification of every calculation and complete proof, recursive verification of external proofs, or expert peer review. Effective values of m_0 , metrics for individual small m , nonexistence of semipositive metrics, and priority or equivalence with other counterexamples remain outside the scope.