

OpenAI Math Digest

CATALOGUE 046

Counterexamples to Shafarevich holomorphic convexity

Catalogue synthesis & 2 proof overviews

Independent reading guide · English edition

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AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 2 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

Counterexamples to Shafarevich holomorphic convexity

The question is whether the simply connected ordinary universal cover of every smooth connected projective complex variety is holomorphically convex. The first manuscript in official order constructs a fourfold with large fundamental group; the second constructs a surface with an infinite chain of rational curves upstairs. Both claim counterexamples in the projective setting, rather than examples only among general compact complex manifolds.

Papers in official catalogue order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 046 · 2026-10-13 checked source snapshot

#

01

Manuscript and overview

A projective fourfold with large fundamental group and non-Stein universal cover →

Short name: A projective fourfold with large fundamental group and non-Stein universal cover

October 5, 2026 · 41 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

A smooth connected projective complex fourfold X contains a simple abelian surface A , with large fundamental group and image $\pi_1(A) \cong \mathbb{Z}$. Its universal cover has no positive-dimensional compact analytic subvariety and is not Stein.

[Theorem 1.1 · p. 1](#) ↗

Role and available overview

Local covers and hyperbolic path tests preserve a cyclic image through projective realization; a final complete intersection gives largeness. The lifted closed surface has nonzero H_3 , obstructing Steinness.

[Read the proof overview](#) →

#

02

Manuscript and overview

A surface counterexample to Shafarevich holomorphic convexity →

Short name: A surface counterexample to Shafarevich holomorphic convexity

September 23, 2026 · 26 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

A smooth connected projective complex surface X contains a connected nodal curve Z_0 with rational components and infinite fundamental-group image. A singleton in the universal cover has noncompact holomorphic hull.

[Theorem 1.1 · p. 2](#) ↗

Role and available overview

A weighted group presentation ensures infinitude, while finite detection of local boundary groups preserves it through projective compactification. An infinite chain of rational curves obstructs holomorphic convexity.

[Read the proof overview](#) →

References on the arrows

- **[OpenAI26a]** OpenAI, A projective fourfold with large fundamental group and non-Stein universal cover, October 5, 2026, 41 pages. The topology of a closed lifted surface, and the use of largeness to deduce Steinness from hypothetical holomorphic convexity.

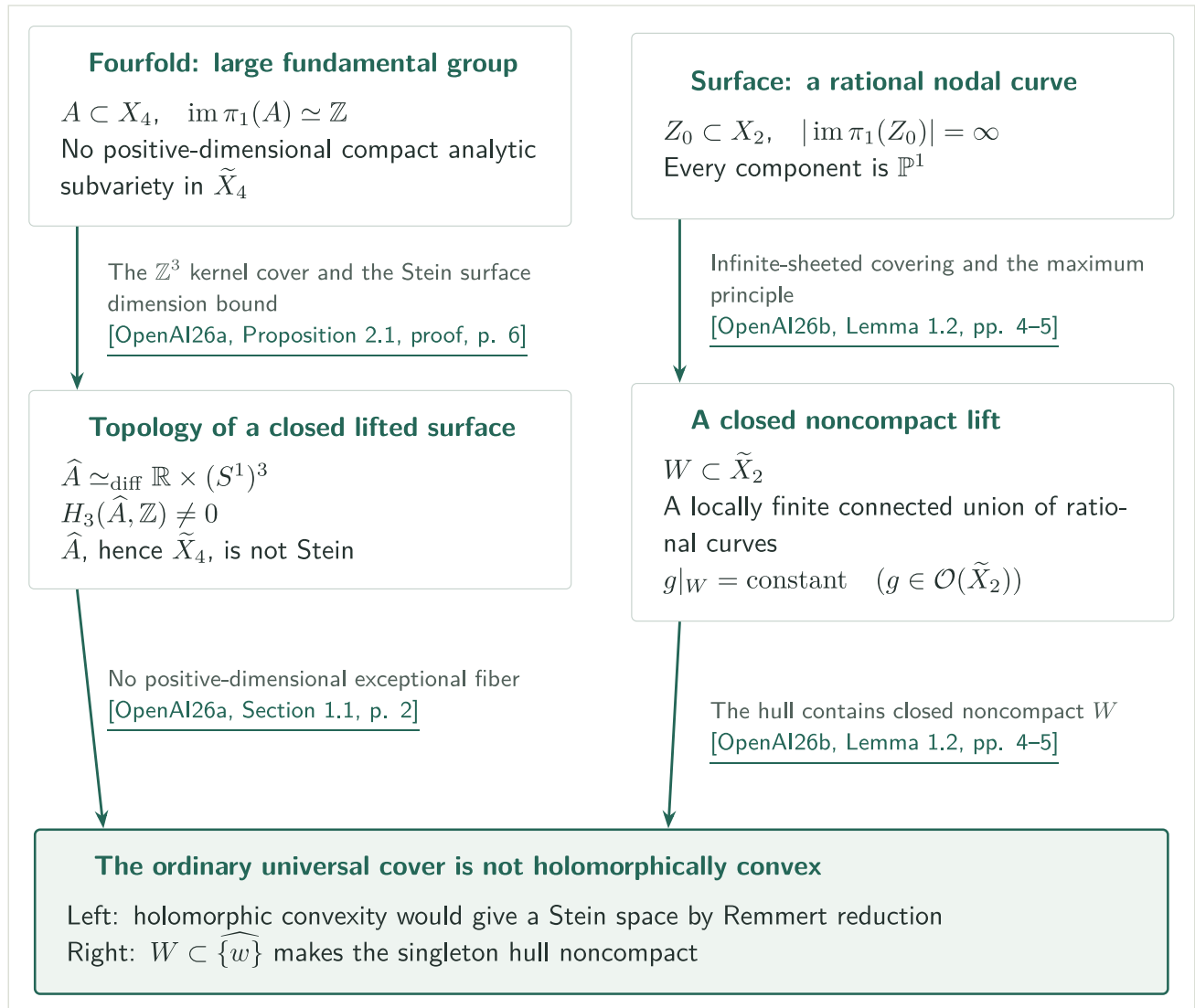
[\[OpenAI26a, Theorem 1.1, p. 1\]](#)

- **[OpenAI26b]** OpenAI, A surface counterexample to Shafarevich holomorphic convexity, September 23, 2026, 26 pages. The fundamental-group image of a whole rational nodal curve and the holomorphic-hull obstruction in its lift.

[\[OpenAI26b, Theorem 1.1, p. 2\]](#)

Two routes to the same negative conclusion

The two sides of the diagram concern separately constructed spaces; they do not represent proof dependence between the manuscripts.



The fourfold route also uses largeness; the surface route directly tests a singleton hull

[Open figure ↗](#) [TeX source ↓](#)

On the fourfold side, a lifted simple abelian surface is a closed complex surface with nonzero third homology, and hence is not Stein. The universal cover additionally has no positive-dimensional compact analytic subvariety. If it were holomorphically convex, its Remmert reduction would be an isomorphism, making it Stein. This additional condition supplies the passage to failure of holomorphic convexity. See the [fourfold proof overview](#).

[\[OpenAI26a, Proposition 2.1, proof, p. 6\]](#) [\[OpenAI26a, Section 1.1, p. 2\]](#)

On the surface side, the presence of compact rational curves alone is not the claimed obstruction. The entire connected lift is closed and noncompact, and every holomorphic function is constant on it, so it lies in the hull of any of its points. The [surface proof overview](#) explains how infinitude is retained through compactification before applying this analytic obstruction.

[OpenAI26b, Lemma 1.2, pp. 4–5]

Sources and checking scope

The official inventory is OpenAI Research Catalog, October 6, 2026, entry 046 on PDF page 6, at fixed commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a. The official subject is **Algebraic and complex geometry**. CONTENTS uses the alternative heading Shafarevich counterexamples in dimension two and with large fundamental group; the two manuscripts and their order agree.

[OpenAI Research Catalog · Catalogue 046 · p. 6](#)

We checked the authors' main statements against the passages connecting construction, preservation of the group image, and the analytic obstruction. Each article specifies checked and unresolved direct inputs. This is not independent verification of every external proof or of the complete manuscripts. No mutual citation or direct use was found within the checked passages. Inputs from other catalogues into consequences are distinguished in the surface article; uninvestigated relations are not recorded as absent.

A projective fourfold with large fundamental group and non-Stein universal cover

Create an infinite cyclic image of a simple abelian surface, then use its rank-three lattice cover as an obstruction to Steinness.

The manuscript claims a counterexample to the Shafarevich conjecture that the topological universal cover $\tilde{X}$ of a smooth connected projective complex variety X is holomorphically convex. Its construction reaches projective dimension four and also imposes a large fundamental group. The ambient variety is projective; the argument is not confined to general compact complex or nonprojective Kähler manifolds. This article explains the manuscript's construction and reasoning, without certifying the correctness of its complete proof.

Manuscript author: OpenAI · October 5, 2026 · 41 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A projective fourfold with large fundamental group

There exist a smooth connected projective complex fourfold X and a simple abelian surface $A \subset X$ such that:

1. X has large fundamental group: for every positive-dimensional closed irreducible subvariety $Z \subset X$, the image $\text{im}(\pi_1(Z^\nu) \rightarrow \pi_1(X))$ of the normalization is infinite.
2. $\text{im}(\pi_1(A) \rightarrow \pi_1(X)) \simeq \mathbf{Z}$.

The universal cover $\tilde{X}$ contains no positive-dimensional compact complex-analytic subvariety and is not Stein.

[Theorem 1.1 · p. 1](#)

Together, the last two conclusions also rule out holomorphic convexity. The Remmert reduction of a holomorphically convex complex manifold is a proper holomorphic map to a Stein space with connected compact analytic fibers. In the absence of positive-dimensional compact analytic subspaces, every fiber is a point; the reduction is an isomorphism, and the original manifold is Stein. The manuscript makes this additional argument in §1.1. Non-Steinness alone is not being used to infer failure of holomorphic convexity.

Section 1.1 · p. 2

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A projective fourfold with large fundamental group and non-Stein universal cover](#). External inputs use the following keys.

- [Zar99](#) Yuri G. Zarhin — Hyperelliptic Jacobians without Complex Multiplication: Simplicity of a genus-two Jacobian with characteristic-zero Galois group S_6 . [arXiv:math/9909052v2](#), September 12, 1999 (preprint of the 2000 article).
- [LP25](#) Claudio Llosa Isenrich and Pierre Py — Groups with exotic finiteness properties from complex Morse theory: Formulation of Stover–Toledo virtual cyclic ramified covers; a double cover for each label. *Journal of Topology* 18 (2025), e70013, author-hosted published PDF.
- [HL85](#) Helmut A. Hamm and Lê Dũng Tráng — Lefschetz theorems on quasi-projective varieties: Preserve the fundamental group under general hyperplane sections of the ordinary locus. *Bulletin de la Société Mathématique de France* 113 (1985), 123–142; NUMDAM scan.
- [Mil63](#) John Milnor — Morse Theory: Andreotti–Frankel CW dimension bound for Stein manifolds. *Annals of Mathematics Studies* 51 (1963); Stanford-hosted facing-page scan.
- [EKPR09](#) Philippe Eyssidieux, Ludmil Katzarkov, Tony Pantev and Mohan Ramachandran — Linear Shafarevich Conjecture: Formulation of Remmert reduction; holomorphic convexity implies Steinness under largeness. [arXiv:0904.0693v1](#), April 4, 2009 (preprint of the 2012 article).

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1 · p. 1](#)

Proof route. The statement is Theorem 1.1 above. First construct a map from a point blowup Y of a simple abelian surface to an orbifold $\mathcal{S}$, and prove that its fundamental-group image is infinite cyclic. Preserve the actual surface homomorphism while passing to an ordinary smooth

projective ambient variety and removing the surface blowups. A complete intersection containing the fixed surface then enforces largeness, while a surface cover inside the universal cover supplies the topological obstruction.



Overview linking construction, cyclic image, projective realization and the covering obstruction

[Open figure ↗](#) [TeX source ↓](#)

1. Separate geometric construction from the group calculation

An arrangement on an arithmetic complex-hyperbolic fourfold gives a smooth projective variety V after two sign covers and branched fillings. An order-four action exchanges the factors of its distinguished fiber $C \times C$. Resolving the coarse quotient only near that fiber gives $\mathcal{S}$. Since $\mathrm{Sym}^2 C$ is a one-point blowup of $A = \mathrm{Jac}(C)$, resolving a rational map to the distinguished fiber produces $Y \rightarrow \mathcal{S}$. Meridian relations give the cyclic upper bound; path readings in local covers give the infinite-order lower bound.

Sections 4.4–4.6; Proposition 4.3 · pp. 17–19

2. Connect projective realization, largeness and the analytic obstruction

Feed the embedding $A \hookrightarrow S'$ from Theorem 7.3 into Proposition 2.1. Largeness follows by distinguishing subvarieties that move in an auxiliary abelian variety from those contained in the fixed surface. Meanwhile, the cover A_K corresponding to the kernel $K \simeq \mathbf{Z}^3$ occurs as a closed complex submanifold of $\tilde{X}$. Its nonzero H_3 obstructs Steinness. This A_K is different from the universal cover $\mathbf{C}^2$ of A .

Theorem 7.3 · p. 40; Proof of Theorem 1.1 · p. 41; Proposition 2.1 · pp. 4–6

3. Proof of Theorem 6.1 — Local labels preserve the infinite cyclic image

Proof target : Theorem 6.1 · p. 28

Theorem 6.1 — Infinite cyclic image of the distinguished surface

For a sufficiently deep arithmetic level in Proposition 4.3, $A = \mathrm{Jac}(C)$ is a simple abelian surface, $Y \rightarrow A$ is an iterated point blowup, and $\mathcal{S}$ is a smooth connected effective proper orbifold with projective coarse space. The map $f : Y \rightarrow \mathcal{S}$, whose image lies in the ordinary locus, satisfies

$$\mathrm{im}(f_* : \pi_1(Y) \rightarrow \pi_1(\mathcal{S})) \simeq \mathbf{Z}.$$

Here an orbifold fundamental group means the fundamental group of its classifying space.

Theorem 6.1 · p. 28

1 Six branch points and two pencils
 $g(C) = 2, \quad A = \text{Jac}(C) \text{ is simple}$

 Double sign covers for each label $\longrightarrow V$, fiber $C \times C$

Use branching parity, the order-four action
and resolution of the rational map
Sections 4.4–4.6; Proposition 4.3 · pp. 17–19

2 A surface map in the ordinary locus
 $Y \rightarrow A, \quad f : Y \rightarrow \mathcal{S}$

Resolve the coarse symmetry quotient at the distinguished fiber

Kill parity-zero exceptional meridians;
compare with common basing
Lemmas 6.2–6.3 · pp. 29–30

3 Cyclic upper bound
 $kh_i = 1 \ (i = 1, \dots, 4) \implies h_1 = \dots = h_4 = r$
 $h_5 = h_6 = b, \quad \text{im } \pi_1(Y) \subset \langle rb \rangle$

Add a separate lower bound:
cyclicity alone allows finite image
Propositions 5.4–5.5 · pp. 23–24 Lemma 3.1 · p. 7

4 Read endpoints in local covers

Buffered kernel compatibility + uniform hyperbolicity

 Central: $\ell_z + \ell_w$; outer: $\ker \ell_z \cap \ker \ell_w$

Each power is one edge;
transfer orbifold nullhomotopies to local relations
Lemmas 6.4–6.6 · pp. 32–34
Section 6.5 · pp. 34–35

5 Detect every nonzero power
 $\text{diam}(\text{pr}_{\mathbb{B}} e^n) < u, \quad n\ell(e) \neq 0 \ (n \neq 0)$
 $e^n \neq 1 \implies \text{im } f_* \simeq \mathbf{Z}$

Combine sign covers, the cyclic upper bound and the local-path lower bound

Open figure ↗ TeX source ↓

1. Realize six branch points and their local arrangement projectively

Choose four real algebraic branch points with $|\lambda_i| < 0.01$, and two with $\lambda_5 < -4$, $\lambda_6 > 4$. Weak approximation prescribes both these root locations and factorization types at three primes, yielding a degree-six polynomial with Galois group S_6 . Zarhin's result gives $\text{End}(A) = \mathbf{Z}$, hence simplicity. The application uses characteristic zero, irreducible degree six, and Galois group S_6 .

[Zar99, Theorem 2.1, p. 2]

In each of two coordinate pairs, take six slope hyperplanes and two coupling hyperplanes. Each coupling meets only the four red slopes. Proposition 4.1 realizes exactly this finite pattern in arbitrarily large bounded regions of a deep congruence ball quotient. Apply the Stover–Toledo virtual ramified-cover theorem with $d = 2$ **separately to each label divisor**, whose components are smooth, embedded and pairwise disjoint. Then take a common finite cover and form two parity characters. The theorem is not applied directly to the intersecting union of all divisors.

Proposition 4.1 · pp. 14–15; Theorem 4.2; Section 4.3 · p. 16 [LP25, Theorem 4.2, p. 8]

[LP25, Theorem 4.2, p. 8]

Blow up arrangement centers and assign an exceptional meridian the parity of the number of components through its center. After separating branch components, normalization gives a smooth V with distinguished fiber $C \times C$. The order-four action is locally $(p, q; u, v) \mapsto (q, p; v, -u)$. Resolve the coarse quotient explicitly along this fiber, retaining quotient stacks at other fixed loci. This distinction ensures that the surface map lands in the ordinary locus.

Sections 4.4–4.6; Proposition 4.3 · pp. 17–19

2. Red and blue meridians give only a cyclic upper bound

The six-point orbifold pencil has group

$$Q = \langle h_1, \dots, h_6 \mid h_i^2 = 1, h_1 \cdots h_6 = 1 \rangle.$$

Its even-word subgroup is $\pi_1(C)$. Blowing up the intersection of a coupling meridian k with a red meridian h_i introduces an exceptional divisor of parity zero. Filling therefore imposes $kh_i = 1$, identifying the four red meridians with one involution r . The sphere relation identifies the two blue meridians with an involution b , and the even-word image consists of powers of rb . The common-basepoint push in Lemma 6.2 transfers this calculation to the image of $\pi_1(Y)$. At this stage that cyclic image could still be finite.

Lemmas 6.2–6.3 · pp. 29–30

3. Detect each nonzero power as a single edge

Dihedral labels of local arrangement covers supply translations ℓ_z, ℓ_w . Central charts use the symmetry-invariant sum $\ell_z + \ell_w$; outer charts use the intersection of the two kernels. Proposition 5.4 gives equality of kernel tests on buffered overlaps, and Proposition 5.5 gives uniform hyperbolicity of the associated path graphs. Lemma 3.1 requires compatibility with prefixes, reversals and edge lifting, invariance of endpoint-equality tests after a short prefix, and uniform hyperbolicity. Lemma 6.4 verifies these hypotheses for the construction.

Lemma 3.1 · p. 7; Proposition 5.4 · pp. 23–24; Proposition 5.5 · p. 24; Lemma 6.4 · pp. 32–33

An edge is a path whose projection to the ball has diameter $< u$, with no bound on length. Thus a loop e with nonzero translation can be pushed near the distinguished fiber so that every nonzero power e^n is still one edge, with read endpoints differing by $n\ell(e) \neq 0$. The symmetry acts freely in the ordinary open set, so this difference survives the central quotient chart. Lemma 6.6 compares orbifold nullhomotopies with the local short-loop relations, using general position and transverse-disk meridian relations. The local criterion therefore proves $e^n \neq 1$ for every $n \neq 0$. Together with the cyclic upper bound, this gives image $\mathbf{Z}$. The local labels are not asserted to extend to a global dihedral representation of $\pi_1(\mathcal{S})$.

Lemma 6.6; Sections 6.4–6.5 · pp. 33–35

4. Proof of Theorem 7.3 — Preserve the surface homomorphism in projective realization

Proof target : Theorem 7.3 · p. 40

Theorem 7.3 — Embed the simple abelian surface itself

There exist a simple abelian surface A , a smooth connected projective complex variety S' , and a closed embedding $A \hookrightarrow S'$ such that

$$\mathrm{im}(\pi_1(A) \rightarrow \pi_1(S')) \simeq \mathbf{Z}.$$

Theorem 7.3 · p. 40

1 Start with a map to an orbifold

$$Y \xrightarrow{f} S, \quad \text{im } f_* \simeq \mathbf{Z}$$

Point blowups $Y \rightarrow A$; $f(Y)$ lies in the ordinary locus

Transport the Lefschetz isomorphism
by Ehresmann deformation

[Lemma 7.1 · pp. 35–36](#)

[\[HL85, Theorem 1.1.3\(ii\), print p. 126 / PDF p. 5\]](#)

2 Cut in the ordinary locus of a projective bundle

$$Y \hookrightarrow S \xrightarrow{h} S, \quad \pi_1(S) \xrightarrow{\sim} \pi_1(S)$$

Smooth projective fivefold; $h \circ i = f$

Use a surface-preserving small resolution
and repeated blowups

[Lemma 7.2 · pp. 36–40](#)

3 Adjust the normal bundle of one exceptional curve

$$E \subset Y_1 \subset T^3, \quad N_{E/T} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$$

$$G \subset \widehat{T}, \quad G \simeq \mathbf{P}^1 \times \mathbf{P}^1$$

Contract by high multiples;
verify smoothness in local coordinates

[Lemma 7.2 · pp. 36–40](#)

4 Contract the other ruling projectively

$$H|_G = \mathcal{O}(\alpha, \beta), \quad \beta > \alpha > 0$$

$$(H + \alpha G)|_G = \mathcal{O}(0, \beta - \alpha)$$

Repeat in reverse order of the point blowups

[Theorem 7.3 · p. 40](#)

5 Embed the original surface

$$A \hookrightarrow S', \quad \text{im}(\pi_1(A) \rightarrow \pi_1(S')) \simeq \mathbf{Z}$$

Preserve the ambient group and surface homomorphism at every step

Pass from an orbifold to a projective variety and remove the surface point blowups

[Open figure ↗](#) [TeX source ↓](#)

1. Replace the orbifold ambient space by an ordinary projective one

Lemma 7.1 applies to a smooth connected effective proper Deligne–Mumford stack T with projective coarse space, and a map from a smooth projective surface Y into its ordinary locus. It constructs $Y \hookrightarrow S \xrightarrow{h} T$ with $h_* : \pi_1(S) \simeq \pi_1(T)$, where S may be a smooth projective fivefold. A

projective bundle built from many copies of the tangent and trivial bundles makes the nonordinary locus have large codimension, using faithful stabilizer representations. A complete intersection can then retain the graph of Y while avoiding that locus.

For the fundamental group, the manuscript first applies Hamm–Lê's quasi-projective Lefschetz theorem to unconstrained general sections. A path in the parameter space of smooth complete intersections avoiding the boundary gives a smooth proper family; Ehresmann transports the isomorphism to the section constrained to contain Y . The constrained section is not simply assumed to be a general hyperplane.

Lemma 7.1 · pp. 35–36 [HL85, Theorem 1.1.3(ii), print p. 126 / PDF p. 5]
[HL85, Theorem 1.1.3(ii), print p. 126 / PDF p. 5]

2. Change the ambient variety to remove the surface exceptional curves

Lemma 7.2 realizes the point blowdown of a (-1) -curve on an embedded smooth projective surface inside a new smooth projective ambient variety. It preserves both the ambient fundamental group and the **actual homomorphism from the surface**. The manuscript first constructs a threefold containing the surface, uses small resolutions of isolated nodes that leave the surface unchanged, and adjusts the exceptional curve's normal bundle to $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Its blowup has exceptional divisor $G \simeq \mathbf{P}^1 \times \mathbf{P}^1$, which is contracted along the other ruling.

The global projective step constructs an ample bundle with $H|_G = \mathcal{O}(\alpha, \beta)$ and $\beta > \alpha > 0$. Then $D = H + \alpha G$ restricts to $\mathcal{O}(0, \beta - \alpha)$, and sections of a high multiple produce the desired contraction. Local coordinates verify smoothness of the target and the surface image. Repeating in reverse order of the point blowups embeds A itself, preserving the homomorphism of Theorem 6.1 and proving Theorem 7.3.

Lemma 7.2 · pp. 36–40

5. Proof of Proposition 2.1 — Largeness and failure of holomorphic convexity

Proof target : Proposition 2.1 · p. 4

Proposition 2.1 — From a cyclic-image embedding to the main theorem

Let A be a simple abelian surface and $A \hookrightarrow S'$ a closed embedding in a smooth connected projective complex variety. Assume $\text{im}(\pi_1(A) \rightarrow \pi_1(S')) \simeq \mathbf{Z}$. Then there exist a smooth connected projective fourfold X with large fundamental group and a closed embedding $A \hookrightarrow X$ with infinite cyclic fundamental-group image, such that $\tilde{X}$ is not Stein.

Proposition 2.1 · p. 4

1 A general complete intersection through the surface

$F = A \times \{0\} \subset X \subset S' \times B$, $\dim B = 6$, $\dim X = 4$
 $\pi_1(X) \simeq \pi_1(S') \times \pi_1(B)$

Four-point interpolation forces
 positive-dimensional fibers into F
[Lemma 2.3 · pp. 4–5](#)

[Proof of Proposition 2.1 · p. 5](#)

2 Infinite image for positive-dimensional subvarieties

Nonconstant $Z \rightarrow B$: compactness and universal cover $\mathbf{C}^6$
 For $Z \subset F$: simplicity of A and its infinite cyclic image

Use the equivalence for largeness and covering classification
[Lemma 2.2 · p. 4](#)

[Proof of Proposition 2.1 · pp. 5–6](#)

3 Largeness and the covering component

No positive-dimensional compact analytic subvariety in $\tilde{X}$
 $K = \ker(\pi_1(A) \rightarrow \pi_1(X)) \simeq \mathbf{Z}^3$, $A_K = \mathbf{C}^2/K \subset \tilde{X}$

A Stein surface has CW dimension at most two;
 apply to the closed submanifold
[Proof of Proposition 2.1 · p. 6](#)

[\[Mil63, Theorem 7.2, print p. 39 / PDF p. 22\]](#)

4 Topological obstruction on a closed complex surface

$A_K \simeq_{\text{diff}} (S^1)^3 \times \mathbf{R}$, $H_3(A_K, \mathbf{Z}) = \mathbf{Z}$
 Non-Stein $A_K \implies$ non-Stein $\tilde{X}$

Use both non-Steinness and absence of
 positive-dimensional compact analytic subsets
[\[EKPR09, Introduction, p. 2\]](#)

5 Rule out holomorphic convexity

If holomorphically convex, all Remmert fibers are points
 Then $\tilde{X}$ is Stein: contradiction

A complete intersection yields largeness and a rank-three lattice-cover obstruction

[Open figure ↗](#) [TeX source ↓](#)

1. A fourfold complete intersection retaining the fixed surface

Put $d = \dim S'$, take an abelian variety B of dimension six, and set $W = S' \times B$, $F = A \times \{0\}$. For a sufficiently high power of a very ample line bundle, let X be the common zero locus of $c = d + 2$ general sections of $\mathcal{I}_F \otimes L^m$. The rank-drop locus of their normal derivatives along F has codimension three, greater than $\dim F = 2$, so the successive intersections are smooth. Lefschetz gives

$$\pi_1(X) \simeq \pi_1(S') \times \pi_1(B),$$

with the image from A preserved.

Uniform interpolation in Lemma 2.3 shows that requiring all c equations to vanish at four distinct points of $W \setminus F$ in one B -fiber imposes $4c = 4d + 8$ conditions. The quadruple parameter space has dimension $4d + 6$, so a general intersection has at most three points outside F in each fiber. Every positive-dimensional fiber component is therefore contained in F .

[Lemma 2.3 · pp. 4–5](#); [Proof of Proposition 2.1 · p. 5](#)

2. Largeness for every positive-dimensional subvariety

If $Z \subset X$ moves in B , the image of $\pi_1(Z^\nu)$ in $\pi_1(B)$ is infinite. A finite image would vanish because the target is torsion-free, so the map would lift to $\mathbf{C}^6$ and be constant by compactness. If the projection is constant, the fiber property gives $Z \subset F$. For $Z = F$, the given cyclic image is infinite. For a curve with normalization D , the nonzero map $\text{Jac}(D) \rightarrow A$ is surjective by simplicity. Its integral lattice image has finite index in $\pi_1(A) = \mathbf{Z}^4$, hence remains infinite in the infinite cyclic quotient.

This proves largeness. Lemma 2.2 equates largeness with the absence of positive-dimensional compact analytic subvarieties in $\tilde{X}$. In the converse direction, the compact analytic subvariety has algebraic image in X , and the resulting finite surjection of normalizations has finite-index image on fundamental groups.

[Lemma 2.2 · p. 4](#); [Proof of Proposition 2.1 · pp. 5–6](#)

3. Third homology of the surface cover obstructs Steinness

Here $\tilde{X} \rightarrow X$ is the simply connected topological universal cover equipped with the lifted complex structure. For $K = \ker(\pi_1(A) \rightarrow \pi_1(X)) \simeq \mathbf{Z}^3$, a connected component above A is

$$A_K = \mathbf{C}^2 / K \simeq_{\text{diff}} (S^1)^3 \times \mathbf{R}, \quad H_3(A_K, \mathbf{Z}) \simeq \mathbf{Z}.$$

Covering-space classification identifies this component through the kernel, and locally separated components in covering charts make A_K a closed complex submanifold of $\tilde{X}$. The Andreotti–Frankel bound says that a Stein surface has the homotopy type of a CW complex of real dimension at most two. Thus A_K is not Stein. Since a closed complex submanifold of a Stein manifold is Stein, $\tilde{X}$ cannot be Stein either.

Proof of Proposition 2.1 · p. 6 [Mil63, Theorem 7.2, print p. 39 / PDF p. 22]

[Mil63, Theorem 7.2, print p. 39 / PDF p. 22]

Finally combine the Remmert reduction described in §1 with the absence of positive-dimensional compact analytic subvarieties. Then $\tilde{X}$ is not holomorphically convex. Largeness is the extra input that turns the topological obstruction on A_K into an obstruction to holomorphic convexity.

[EKPR09, Introduction, p. 2]

6. External inputs and their uses

Input

Zarhin, Theorem 2.1

Used at

§4.1, p.13

Content and hypotheses

Characteristic zero, degree six and Galois group S_6 yield a simple Jacobian

Checking scope

Statement in arXiv v2 and application checked; complete input proof not verified

Input

Stover–Toledo's ramified-cover theorem, as formulated in Llosa Isenrich–Py Theorem 4.2

Used at

§4.3, p.16

Content and hypotheses

Virtual double branched covers of simplest-type congruence arithmetic ball quotients along disjoint embedded totally geodesic divisors

Checking scope

LP25 p.8 and application to each label checked; Stover–Toledo's proof not verified

Input

Hamm–Lê, quasi-projective Lefschetz theorem

Used at

Lemma 7.1, pp.35–36

Content and hypotheses

Preserve the fundamental group by general sections of the ordinary locus, then deform to constrained sections

Checking scope

Application paragraph checked; external pagination recorded in the sources

Input

Andreotti–Frankel dimension bound, as formulated in Milnor Theorem 7.2

Used at

Proposition 2.1, p.6

Content and hypotheses

Vanishing of H_3 for a Stein surface

Checking scope

Statement and application to A_K checked; Morse-theoretic proof outside scope

Input

Remmert reduction, recalled in the introduction of Eyssidieux–Katzarkov–Pantev–Ramachandran

Used at

§1.1, p.2

Content and hypotheses

Holomorphic convexity and a Stein reduction with compact analytic fibers

Checking scope

Formulation and the deduction using largeness checked

[Zar99, Theorem 2.1, p. 2] [LP25, Theorem 4.2, p. 8] [HL85, Theorem 1.1.3(ii), print p. 126 / PDF p. 5]
 [Mil63, Theorem 7.2, print p. 39 / PDF p. 22] [EKPR09, Introduction, p. 2]

The external proofs of the Borel–Harish-Chandra arithmetic lattice theorem, Baily–Borel projectivity, standard hyperbolic estimates, Serre vanishing, Bertini, numerical ampleness and general position for stacks were not recursively audited. The overview records their uses and roles within the manuscript; their theorem statements in the external originals were also not checked in this work. No proof-dependency arrow from the other surface counterexample in this catalog is recorded.

7. Return to the sources

- [Proposition 2.1 and proof](#) · pp. 4–6; [Section 1.1](#) · p. 2 — Conjecture, covering component and the consequence for holomorphic convexity
- [Sections 4.4–4.6](#); [Proposition 4.3](#) · pp. 17–19 — Construct the distinguished fiber and surface map into the ordinary locus
- [Theorem 6.1](#); [Sections 6.1–6.5](#) · pp. 28–35 — Separate the cyclic upper bound from the infinite-order lower bound
- [Lemmas 7.1–7.2](#); [Theorem 7.3](#) · pp. 35–40 — Projective realization and surface blowdown preserving the homomorphism

The reading covered all conclusions of the main result, the covering and topological obstruction in §2, the surface and orbifold construction in §4, the cyclic upper bound, detection of all nonzero powers and comparison with orbifold relations in §6, and projective realization and surface blowdown in §7. The reductions explained here were compared with those passages. The complete constant estimates for the local path criterion in §3, and all finite-model kernel-compatibility and uniform-hyperbolicity arguments in §5, were not fully checked. These are essential internal inputs to the infinite-order argument; this overview does not independently validate them. External checking centers on the statements and applications listed above. Recursive proof verification, formal verification and expert peer review have not been performed.

A surface counterexample to Shafarevich holomorphic convexity

Preserving a weighted pro-2 quotient on a projective surface, then lifting an infinite string of rational curves

The manuscript addresses the usual smooth-projective formulation of the Shafarevich conjecture: the universal cover of every smooth connected projective complex variety is holomorphically convex. The author claims a counterexample in complex dimension two. Finite covers first produce a smooth projective surface; its ordinary simply connected universal cover is taken only afterwards. The argument does not identify an arbitrary non-Stein cover as a counterexample.

Manuscript author: OpenAI · September 23, 2026 · 26 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A projective surface with a noncompact singleton holomorphic hull upstairs

There exist a smooth connected projective complex surface X and a connected nodal curve $Z_0 \subset X$ whose irreducible components are smooth rational curves, such that

$$|\mathrm{im}(\pi_1(Z_0) \longrightarrow \pi_1(X))| = \infty.$$

In the simply connected universal cover $\tilde{X}$, a connected component above Z_0 is a closed, connected, noncompact, locally finite union of compact rational curves. The holomorphic hull of any one of its points is noncompact. In particular, $\tilde{X}$ is not holomorphically convex.

[Theorem 1.1 · p. 2](#)

For a nonempty compact subset K of a complex manifold V , the hull is

$$\hat{K}_V = \{v \in V : |g(v)| \leq \sup_K |g| \text{ for every } g \in \mathcal{O}(V)\}.$$

A projective surface is also a compact Kähler manifold, so a valid construction also contradicts the universal assertion for that larger class. The constructed objects are nevertheless smooth projective surfaces: the manuscript is not a classification of covers of arbitrary compact complex manifolds, nor a claim of nonconvexity for every Kähler manifold.

§1 · p. 2

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [A surface counterexample to Shafarevich holomorphic convexity](#). External inputs use the following keys.

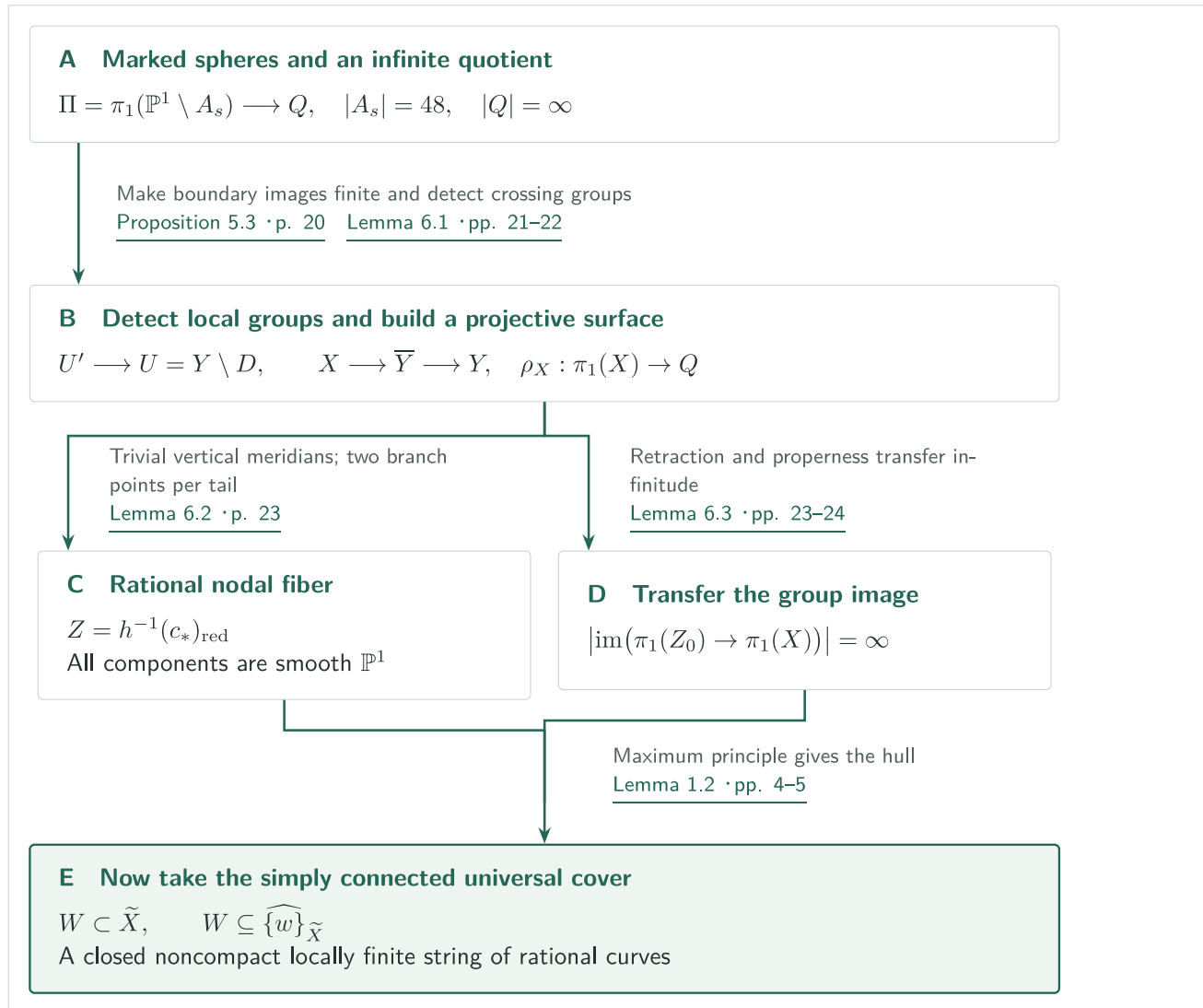
- [Groth03](#) Alexander Grothendieck et al. — Revêtements étales et groupe fondamental (SGA 1): Algebraization of nonproper finite étale covers and uniqueness of finite normal extensions. 2003 revised edition; arXiv:math/0206203v2. Main printed pagination of this edition is used, not the old marginal pagination..
- [EH02](#) Santiago Encinas; Herwig Hauser — Strong resolution of singularities in characteristic zero: Strong resolution preserving the smooth open locus. *Commentarii Mathematici Helvetici* 77 (2002), 821–845; publisher PDF.
- [Loj64](#) Stanisław Łojasiewicz — Triangulation of semi-analytic sets: Triangulation compatible with the special fiber. *Annali della Scuola Normale Superiore di Pisa, Ser. 3*, 18 (1964), 449–474; Numdam PDF with cover page.
- [Stacks26a](#) The Stacks Project Authors — The Stacks Project — Relative normalization: Lemma 29.54.15: Finiteness of relative normalization under the Nagata and finite-type hypotheses. HTML tag 03GR, accessed 2026-10-13; unpaginated.
- [Stacks26b](#) The Stacks Project Authors — The Stacks Project — Finite morphisms are projective: Lemma 29.45.16: Projectivity of finite morphisms. HTML tag 0B3I, accessed 2026-10-13; unpaginated.
- [Stacks26c](#) The Stacks Project Authors — The Stacks Project — Embedded resolution: Lemma 54.15.6: Boundary resolution by point blowups of a regular surface. HTML tag 0BIC, accessed 2026-10-13; unpaginated.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — From the construction to a holomorphic hull

Proof target : [Theorem 1.1 · p. 2](#) See the statement.

Proof route. Starting with a family of spheres minus 48 points, construct a pro-2 group Q that remains infinite after all required boundary relations. A finite quotient detecting the local boundary groups produces the projective surface X . Rationality of its special-fiber components and an infinite fundamental-group image then meet in a curve Z_0 . Finally inspect a singleton hull in the universal cover. The details are organized as the [infinite quotient](#), [finite cover and descent](#), [special fiber](#), and [holomorphic hull](#).



The infinite quotient and rational special fiber meet before passage to the universal cover

[Open figure ↗](#) [TeX source ↓](#)

1. Distinguish the three covers

First, $B' \rightarrow B$ is a finite étale parameter cover splitting the 48 markings into sections. Its projective completion $C' \rightarrow \mathbb{P}^1$ has ramification index exactly two at the distinguished point $c_* \in C' \setminus B'$. Next, $U' \rightarrow U$ is the finite étale cover defined by a finite group P_f ; normalization

and resolution produce X . The projective morphism $X \rightarrow Y$ includes resolution and is not assumed globally finite or unramified. Only the final map $\tilde{X} \rightarrow X$ is the simply connected universal cover.

§5.2; Lemma 6.1 · pp. 18, 21–22

2. Connect every conclusion

Lemma 6.1 supplies projectivity and smoothness, Lemma 6.2 supplies the rational nodal special fiber, and Lemma 6.3 supplies a connected component Z_0 with infinite fundamental-group image. This infinitude comes from loops in the incidence graph, not from an individual $\mathbb{P}^1$. Lemma 1.2 then supplies closedness, local finiteness, noncompactness and the singleton-hull conclusion in Theorem 1.1.

Lemmas 6.2–6.3; proof of Theorem 1.1 · pp. 23–24

3. Proof of Proposition 5.3 — Keep infinitude after the geometric relations

Proof target : Proposition 5.3 · p. 20

Proposition 5.3 — Finite boundary images and dense fiber images

In the construction of §§3–5, let $U = Y \setminus D$ be the SNC boundary complement in the smooth projective compactification of the marked family. Let Q be the quotient of the weighted free pro-2 group L obtained by imposing the sign-involution relations, sufficiently deep monodromy relations, and sufficiently large powers of two of the remaining boundary meridians. Then Q is infinite, and $\rho : \pi_1(U) \rightarrow Q$ has dense image on every punctured smooth fiber. Each component meridian of D has finite-order image; every component meridian of the distinguished fiber D_* has trivial image.

Proposition 5.3 · p. 20

1 Based monodromy and weights

$$K \rightarrow \Pi, \quad p\varphi_m = \beta_m p, \quad \deg(\varphi_m - 1) \geq 1$$

Preserve degrees in a difference basis; economize via Frattini
[Proposition 4.2; Lemma 4.3; §5.1 · pp. 15–17](#)

2 48 generators and 24 involution relations

$$P(t) = (1+t)G(t), \quad G(t) = 3t(1+t)(1+t^2)^2$$

$$\text{Relation cost} \leq tG(t)$$

Count b and choose e after the finite cover and compactification
[§5.2–5.3; Lemma 5.2 · pp. 18–20](#) [Proposition 5.3 · p. 20](#)

3 Depth first, boundary powers after compactification

$$97t \frac{(4t)^J}{1-4t} < \frac{1}{100}, \quad bt^{2^e} < \frac{1}{100}$$

Weighted criterion, including convergence of the relator series
[Lemma 2.2; Lemma 5.2; Proposition 5.3 · pp. 6–7, 20](#)

4 Infinitude survives all boundary relations

$$t = \frac{49}{200}, \quad 1 - G(t) + \frac{1}{50} < -\frac{1}{125} < 0$$

$$|Q| = \infty, \quad \rho(\pi_1(F_s)) = Q$$

Control the costs of involution and monodromy relations and reserve room for boundary powers

[Open figure ↗](#) [TeX source ↓](#)

1. Control the actual sphere monodromy

Take $E = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}\omega)$ with $\omega^3 = 1 \neq \omega$, the quotient $q : E \rightarrow E/\{\pm 1\} = \mathbb{P}^1$, and the degree-16 map f induced by [4]. Here $\bar{\omega}$ denotes the automorphism of the quotient sphere induced by $[\omega]$. Over the five-punctured parameter curve B ,

$$A_s = \{z : f(z) = \bar{\omega}^j s \text{ for some } 0 \leq j \leq 2\}$$

consists of 48 points, colliding in 24 pairs near $s_* = q(e_0)$ for $0 \neq e_0 \in E[2]$. The six-punctured torus group F has rank 7; the covering subgroup K for [4] has rank 97; the 48-punctured sphere group Π has rank 47. The based surjection $p : K \rightarrow \Pi$ and identity $p\varphi_m = \beta_m p$ in Proposition 3.2 transfer the torus estimates to the **actual sphere monodromy**.

[Lemma 3.1; Proposition 3.2 · pp. 8–9](#)

Three double covers $Y_j^2 = q(z) - \bar{\omega}^j s$, with lifted basepoint sections, fix the weight-one quotient. Assigning weight two to the pair boundaries makes $\varphi_m - \text{id}$ degree-raising by Proposition 3.4. Collapsing the core and pair boundaries gives the 48-generator group L ; imposing the sign involution σ gives the pro-2 completion of the rank-24 pinched sphere group. Lemma 3.5 uses $K \twoheadrightarrow \Pi$ to deduce invariance of $\Pi \rightarrow Q$ without assuming that the pinching kernel is monodromy invariant.

Lemmas 3.3, 3.5; Proposition 3.4 · pp. 9–12

2. Economize on 24 relations, then order the choices

Successive differences under the two translations of $H = (\mathbb{Z}/4)^2$ give a basis $u_{i\ell,j}$ of weight $i + \ell + 1$. Using $R = C \oplus dC$ in the Frattini quotient, Proposition 4.2 selects 24 relations $r_g = \sigma(g)g^{-1}$ that impose the entire involution. Their degree cost is at most $tG(t)$, where

$$P(t) = 3t(1 + t + t^2 + t^3)^2 = (1 + t)G(t), \quad G(t) = 3t(1 + t)(1 + t^2)^2.$$

There are at most 4^N chosen parameter generators at depth N , each imposing relations on 97 generators of K . Degree transfer in Lemma 4.3 bounds the total cost by $97t(4t)^J/(1 - 4t)$.

Proposition 4.2; Lemma 4.3; §5.1 · pp. 15–17

Fix $t = 49/200 < 1/4$ and first choose J so that this cost is less than $1/100$. After taking the finite parameter cover and compactifying, let b be the remaining number of boundary components and choose e so that $bt^{2^e} < 1/100$. Thus there is no circular advance estimate for the number of boundary components. The manuscript's bound $G(t) > 257/250$ gives

$$1 - P(t) + tG(t) + \frac{1}{100} + \frac{1}{100} < -\frac{1}{125} < 0.$$

The weighted infinitude criterion proved as Lemma 2.2 makes Q infinite. For the infinite family of monodromy relations, bounded-degree finiteness and convergence are also checked. The meridians of D_* already vanish: each is expressed using the section's base loop and a pair boundary killed by pinching (Lemma 5.2).

Lemma 2.2; Lemma 5.2; Proposition 5.3 · pp. 6–7, 20

4. Proof of Lemma 6.1 — Detect the entire local group in a finite quotient

Proof target : Lemma 6.1 · pp. 21–22

Lemma 6.1 — Finite local detection and descent

Let Y be a smooth connected projective complex surface, D an SNC divisor and $U = Y \setminus D$. Suppose $\rho : \pi_1(U) \rightarrow Q$ has dense image in a profinite group and sends every component meridian to a finite-order element. There exists a finite quotient $Q \twoheadrightarrow P_f$ injective on the **entire image** of every sufficiently small local boundary-complement group. The connected finite cover $U' \rightarrow U$ defined by its kernel extends to a projective morphism $p : X \rightarrow Y$, where X is smooth and connected and $D_X = X \setminus U'$ is SNC. For the inclusion $i : U' \hookrightarrow X$ there is a homomorphism

$$\rho_X : \pi_1(X) \longrightarrow Q, \quad \rho_X i_* = \rho \pi_*.$$

The resolution may preserve every open subset of the normalization on which the surface is smooth and its reduced boundary is SNC.

Lemma 6.1 · pp. 21–22

1 Local boundary groups, including crossings

$A_V = \rho(\pi_1(V \setminus D))$, $\pi_1(V \setminus D) \cong \mathbb{Z}^r$, $r \leq 2$
 Each A_V is finite; finitely many conjugacy types

Profinite separation avoids every nonidentity local element
[Lemma 6.1 · pp. 21–22](#)

2 Detect every entire local image

$N \triangleleft Q$ open, $N \cap gA_Vg^{-1} = 1$, $P_f = Q/N$

Algebraize, normalize finitely, then resolve
[\[Groth03, Exposé XII, Théorème 5.1, print p. 251 / PDF p. 267\]](#)
[\[EH02, The result, print pp. 822–823 / PDF pp. 2–3\]](#)

3 Normalize and resolve the finite cover

$U' \rightarrow U$, $X \rightarrow \bar{Y} \rightarrow Y$
 Smooth connected projective X ; preserve the good open locus

Project a transverse disk; its image lies in both subgroups
[Lemma 6.1 · pp. 21–22](#)

4 Kill exceptional meridians and descend

$\rho\pi_*(\mu_E) \in N \cap gA_Vg^{-1} = 1$
 $\rho_X : \pi_1(X) \rightarrow Q$, $\rho_X i_* = \rho\pi_*$

Simultaneous detection at boundary crossings also kills the meridians of resolution exceptional curves

[Open figure ↗](#) [TeX source ↓](#)

1. Finiteness, projectivity and resolution

In a small boundary polydisk the complement group is $\mathbb{Z}^r$, $r = 1, 2$. Its image $A_V \subset Q$, generated by commuting finite-order elements, is finite; only finitely many types occur up to conjugacy. Profinite separation gives an open normal subgroup N avoiding all nonidentity elements of these groups, so $N \cap gA_Vg^{-1} = \{1\}$. Detecting only the orders of individual meridians would not suffice.

Riemann existence applies to the nonproper U and algebraizes the $P_f = Q/N$ cover as a finite étale cover. Normalizing Y in the function field of U' gives a finite, hence projective, morphism. Strong resolution in characteristic zero and point blowups resolving the boundary produce X . Preserving the already good open subset is essential for retaining the special fiber in the next section.

[\[Groth03, Exposé XII, Théorème 5.1, print p. 251 / PDF p. 267\]](#)

[EH02, The result, print pp. 822–823 / PDF pp. 2–3]

2. Why the new boundary meridians vanish

The kernel of $i_* : \pi_1(U') \rightarrow \pi_1(X)$ is normally generated by component meridians of D_X . A small disk transverse to any such component, including a resolution exceptional curve, maps into a boundary polydisk downstairs. Its meridian therefore has Q -image in some gA_Vg^{-1} . Since it is a closed loop upstairs, its image in P_f vanishes, so its Q -image also lies in N . Trivial intersection kills every new meridian, yielding ρ_X .

Lemma 6.1 · pp. 21–22

5. Proof of Lemma 6.2 and Lemma 6.3 — Transfer the infinite image to a rational special fiber

Proof target : Lemma 6.2 · p. 23

Lemma 6.2 — Every component of the distinguished fiber is rational

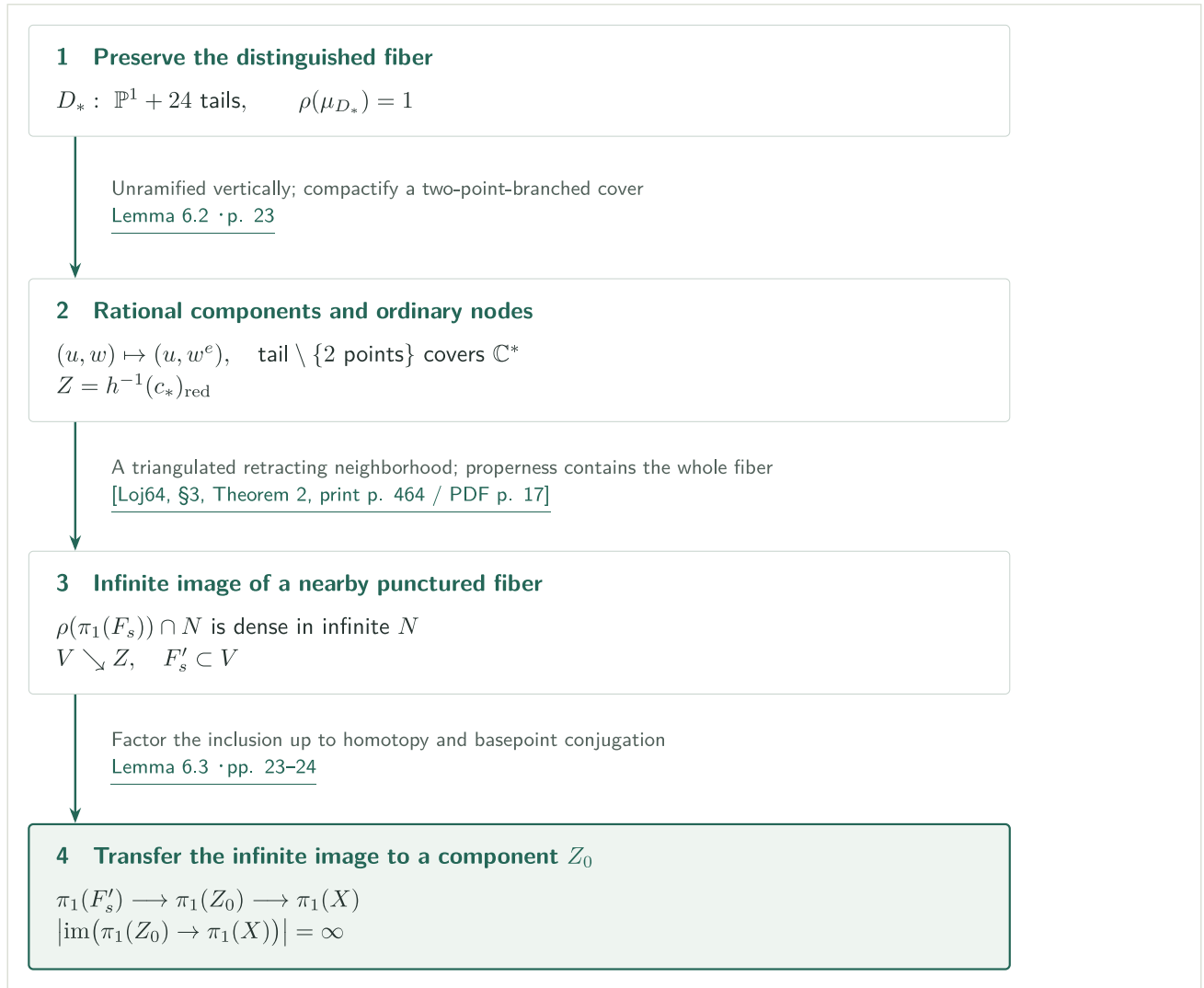
In the construction above, let $h : X \rightarrow Y \rightarrow C'$. The resolution can be chosen so that $Z = h^{-1}(c_*)_{\text{red}}$ has only smooth rational irreducible components and ordinary transverse crossings.

Lemma 6.2 · p. 23

Lemma 6.3 — Some connected component has infinite fundamental-group image

The rational nodal fiber Z of Lemma 6.2 has a connected component Z_0 for which $\text{im}(\pi_1(Z_0) \rightarrow \pi_1(X))$ is infinite.

Lemma 6.3 · pp. 23–24



Combine rationality of the special fiber with transfer of the infinite image through a neighborhood

[Open figure ↗](#) [TeX source ↓](#)

1. Branching remains only at the two markings on each tail

After the double base change, each collision has equation $w^2 = u^2$ and one blowup separates its two sections. The downstairs fiber D_* has an unmarked main $\mathbb{P}^1$ and 24 tails. Since all vertical meridians vanish, the normalization cover is locally an isomorphism at smooth unmarked points and vertical nodes. At a horizontal–vertical crossing, if the horizontal meridian has order e , its local normal extension is

$$(u, w) \longmapsto (u, v) = (u, w^e).$$

Analytification preserves finiteness and normality, and uniqueness of finite normal extension identifies this model with the algebraic normalization. The pair is already smooth and SNC here, so resolution leaves it unchanged. A component above the main sphere is an unramified cover of degree one. A component above a tail is the compactification of a finite cover of $\mathbb{C}^*$, branched only at the two markings; it is therefore $\mathbb{P}^1$.

[Groth03, Exposé XII, Propositions 2.1(vi), 3.2(vi), 5.3, print pp. 242, 245, 254 / PDF pp. 258, 261, 270]

2. Properness places the entire nearby fiber in a retracting neighborhood

The punctured smooth fiber $F_s \subset U$ has dense image in Q , hence surjects onto P_f . Its inverse image $F'_s \subset U'$ is consequently connected, with Q -image $\rho(\pi_1(F_s)) \cap N$, dense in the infinite open subgroup N . The descent identity then makes its image in $\pi_1(X)$ infinite.

Compatibly triangulate the compact real analytic manifold X and closed semianalytic subset Z . After barycentric subdivision, Z is a full subcomplex. The locus V where the sum of the coordinates at its vertices is positive strongly deformation retracts to Z by normalizing those coordinates. Properness of h makes $h(X \setminus V)$ closed and disjoint from c_* , so the **whole fiber** over a sufficiently nearby point lies in V . The inclusion of the connected F'_s then factors through one component Z_0 up to homotopy. Thus the fundamental-group homomorphism factors up to basepoint conjugation, transferring its infinite image.

[Loj64, §3, Theorem 2, print p. 464 / PDF p. 17]

Lemma 6.3 · pp. 23–24

Although the incidence graph of D_* is a tree, a degree- d cover of a tail meets d different degree-one lifts of the main sphere. The upstairs graph need not be a tree. The manuscript does not prescribe that graph arbitrarily; its neighborhood argument establishes the required infinite image.

6. Proof of Lemma 1.2 — Inspect the singleton hull directly

Proof target : Lemma 1.2 · pp. 4–5

Lemma 1.2 — A closed infinite string of rational curves obstructs holomorphic convexity

Let X be a connected complex manifold and $Z \subset X$ a compact connected nodal curve with smooth rational irreducible components. If $\pi_1(Z) \rightarrow \pi_1(X)$ has infinite image, every connected component W above Z in the simply connected universal cover $\tilde{X}$ is closed and noncompact, and is a locally finite union of compact rational curves. Every function in $\mathcal{O}(\tilde{X})$ is constant on W . Consequently $\widehat{\{w\}}_{\tilde{X}}$ is noncompact for every $w \in W$.

Lemma 1.2 · pp. 4–5

1 Lift to the simply connected universal cover

$\tilde{X} \rightarrow X, \quad W \subset \pi^{-1}(Z_0)$
 $\pi_1(Z_0)/\ker(\pi_1(Z_0) \rightarrow \pi_1(X))$ is infinite

Covering kernel, $\pi_1(\mathbb{P}^1) = 0$, and two branches at each node
[Lemma 1.2 · pp. 4–5](#)

2 A closed noncompact string of rational curves

$W = \bigcup C_\alpha, \quad C_\alpha \cong \mathbb{P}^1$
 Locally finite; an infinite closed discrete fiber

Maximum principle; constants agree at intersections
[Lemma 1.2 · pp. 4–5](#)

3 Every holomorphic function is constant on W

$g|_{C_\alpha}$ constant, W connected $\implies g|_W$ constant

Apply the definition of the holomorphic hull directly
[Lemma 1.2 · pp. 4–5](#)

4 A singleton hull is noncompact

$W \subseteq \widehat{\{w\}}_{\tilde{X}} \quad (w \in W)$
 A compact hull cannot contain the closed noncompact W

An infinite-sheeted cover and the maximum principle put a closed noncompact set inside a singleton hull

[Open figure ↗](#) [TeX source ↓](#)

1. Closedness, local finiteness and noncompactness of the lift

The cover $W \rightarrow Z$ corresponds to $\ker(\pi_1(Z) \rightarrow \pi_1(X))$, hence has infinitely many sheets. Its closed discrete infinite fiber rules out compactness. As a connected component of a closed inverse image, W is closed. Each rational component is simply connected, so its lifts are isomorphic compact copies of $\mathbb{P}^1$. Small evenly covered neighborhoods meeting at most two branches at a node establish local finiteness of the lifted components.

2. The closed infinite string lies in a singleton hull

By the maximum principle, $g \in \mathcal{O}(\tilde{X})$ is constant on each compact component. Connectedness and local finiteness make these constants agree along paths between components. Thus, for any $w \in W$,

$$W \subseteq \widehat{\{w\}}_{\tilde{X}}.$$

If the hull were compact, its closed subset W would be compact, a contradiction. This directly obstructs holomorphic convexity; merely exhibiting a compact curve and concluding non-Steinness would not establish the theorem.

Lemma 1.2 · pp. 4–5

7. Which external results supply which steps

Source

Groth03, Exposé XII, Théorème 5.1

Used at

§5.2, Lemma 6.1

Content and hypotheses

Comparison of finite étale and analytic covers over finite-type complex schemes; properness is unnecessary

Checking scope

Statement and both applications checked

Source

Groth03, Exposé XII, Propositions 2.1(vi), 3.2(vi), 5.3

Used at

Lemma 6.2

Content and hypotheses

Analytification of normality and finiteness; uniqueness of finite normal extensions from a dense open subset

Checking scope

Statements and use for the local model checked

Source

EH02, “The result”; Stacks26a–c, Tags 03GR, 0B3I, 0BIC

Used at

Lemma 6.1

Content and hypotheses

Finite relative normalization, projectivity of finite morphisms, resolution preserving the smooth open locus and boundary resolution

Checking scope

Statements and applications checked; complete external proofs not verified

Source

Loj64, §3, Theorem 2

Used at

Lemma 6.3

Content and hypotheses

Triangulation of a real analytic manifold compatible with a locally finite semianalytic collection; compactness of X gives finiteness

Checking scope

Statement and use for the retracting neighborhood checked

The weighted Golod–Shafarevich method has cited predecessors, but this article uses the criterion proved internally as Lemma 2.2. It does not claim independent reconstruction of all antecedent theorems. The special-Kähler abelianity and semialgebraic-universal-cover manuscripts cited in §1.2 supply further comparisons of the resulting example, not inputs to its construction. Those consequences lie outside this article’s review of the main construction.

8. Return to the sources

- [Theorem 1.1 · p. 2](#) — Exact setting and every conclusion
- [Proposition 4.2; Lemma 4.3; §5.1 · pp. 15–17](#) — Weighted relation costs and actual sphere monodromy
- [Proposition 5.3 · p. 20](#) — Infinitude after all boundary relations

- [Lemma 6.1 · pp. 21–22](#) — Detection of entire local groups and exceptional meridians
- [Lemmas 6.2–6.3; proof of Theorem 1.1 · pp. 23–24](#) — Combine rationality and infinite image to finish the theorem
- [Lemma 1.2 · pp. 4–5](#) — Why a singleton hull contains a closed noncompact subset

The editorial reading covers Theorem 1.1, Lemma 1.2, and the main construction, degree bounds, finite local detection and special-fiber transfer in §§3–6, together with the statements and applications of the external inputs listed above. This is an account of the author's claims, not completed independent verification of every proof or specialist peer review. Independent reconstruction of all Frattini linear calculations, all monodromy calculations and the complete completed-algebra proof of Lemma 2.2, as well as the historical comparisons in §1.1 and additional consequences in §1.2, remains unperformed. These limits are distinct from the connections actually checked.