

OpenAI Math Digest

CATALOGUE 039

Nagata's conjecture and maximal Seshadri constants

Catalogue synthesis & 4 proof overviews

Independent reading guide · English edition

Edition: 2026-10-15

Created by: Masataka Iwai · Osaka University

AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 4 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

00 · Catalogue synthesis	3
01 · Nagata's conjecture for plane curves	11
02 · Maximal Seshadri constants on arbitrary polarized surfaces	22
03 · Maximal Multipoint Seshadri Constants in Higher Dimensions	36
04 · Maximal multipoint Seshadri constants in positive characteristic	47

Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

Nagata's conjecture and maximal Seshadri constants

These four manuscripts concern Nagata's plane-curve inequality and ordinary, equally weighted multipoint Seshadri constants of ample line bundles. The plane paper claims a strict inequality for arbitrary multiplicities at every very general tuple with $r \geq 10$. The other three claim attainment of the volume bound for **every sufficiently large integer r** on a fixed polarized variety. Each article distinguishes the authors' claims from the editorial checking scope.

Papers in this catalogue

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 039 · 2026-10-15 checked source snapshot

#

01

Manuscript and overview

Nagata's conjecture for plane curves →

Short name: Nagata's conjecture for plane curves

2026-09-23 · 24 pages

Original paper (GitHub) ↗

Principal claim

For $r \geq 10$ very general complex plane points, the strict Nagata inequality for all nonzero effective curves and arbitrary multiplicities; equal-weight constant $1/\sqrt{r}$.

Theorem 1.1 · p. 1; Corollaries 6.1–6.2 · p. 22 ↗

Role and available overview

Specialize universal equal-multiplicity systems to a normal bundle; use coefficients for squares and collision plus theta-matrix rank otherwise.

Read the proof overview →

#

02

[Manuscript and overview](#)

Maximal Seshadri constants on arbitrary polarized surfaces →

Short name: Maximal Seshadri constants on arbitrary polarized surfaces

2026-09-23 · 19 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For a fixed smooth integral complex projective surface and ample L , $\varepsilon = \sqrt{(L^2/r)}$ at very general tuples for every sufficiently large r ; extended to uncountable algebraically closed characteristic-zero fields.

[Theorem 1.1 · p. 2](#); [Corollary 5.5 · p. 18](#) ↗

Role and available overview

Node exponent bounds, compression and finite torus groups give injective jet tests and nefness against curves with unequal multiplicities.

[Read the proof overview](#) →

#

03

[Manuscript and overview](#)

Maximal Multipoint Seshadri Constants in Higher Dimensions →

Short name: Maximal Multipoint Seshadri Constants in Higher Dimensions

2026-10-05 · 20 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For a fixed smooth integral complex projective variety of dimension $n \geq 3$ and ample L , $\varepsilon = (L^n / r)^{1/n}$ at very general tuples for every sufficiently large r .

[Theorem 1.1 · pp. 1–2](#) ↗

Role and available overview

Iterate two-intercept reshaping, fill the interior by graded additivity, and obtain the lower bound from surjectivity onto full jets.

[Read the proof overview](#) →

#

04

Manuscript and overview

Maximal multipoint Seshadri constants in positive characteristic →

Short name: Maximal multipoint Seshadri constants in positive characteristic

2026-10-05 · 17 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For a fixed smooth integral projective variety of dimension $n \geq 2$ with ample L over an algebraically closed field of positive characteristic, $\varepsilon = (L^n/r)^{1/n}$ at the geometric generic tuple for all sufficiently large r , including countable fields.

[Theorem 1.1 · pp. 1–2; Remark 6.3 · p. 16](#) ↗

Role and available overview

Formal jet transfer, a node retaining two branches in characteristic two, and CRT adapt the method to positive characteristic.

[Read the proof overview →](#)

Methodological connections and reading routes

Dashed arrows show methodological inheritance and reconstruction explicitly described by the later manuscripts. They do not denote use of a predecessor's main theorem as a premise. The plane paper is shown separately because it gives a different argument covering every $r \geq 10$. Diagram keys refer to these fixed manuscripts by OpenAI.

- [OpenAI26a](#) OpenAI — Nagata's conjecture for plane curves — A separate route to the strict plane inequality and strict nefness; 2026-09-23.
- [OpenAI26b](#) OpenAI — Maximal Seshadri constants on arbitrary polarized surfaces — Surface jet injectivity from nodes, compression and finite groups; 2026-09-23.
- [OpenAI26c](#) OpenAI — Maximal Multipoint Seshadri Constants in Higher Dimensions — Generalization to graded exponent sets and full-jet surjectivity; 2026-10-05.

- [OpenAI26d](#) OpenAI — Maximal multipoint Seshadri constants in positive characteristic — Positive-characteristic reconstruction using formal substitutions and a geometric generic tuple; 2026-10-05.

[[OpenAI26a](#), Theorem 1.1; Corollary 6.2, pp. 1, 22]

[[OpenAI26b](#), Theorem 1.1; Corollary 5.5, pp. 2, 18]

[[OpenAI26c](#), Theorem 1.1, pp. 1–2]

[[OpenAI26d](#), Theorem 1.1; Remark 6.3, pp. 1–2, 16]

01 Plane curves: strict inequality for every $r \geq 10$

Nagata's conjecture for plane curves

$$d\sqrt{r} > \sum_i m_i \implies \varepsilon(\mathcal{O}(1); p_1, \dots, p_r) = 1/\sqrt{r}$$

Normal specialization, square-case coefficients, nonsquare theta matrices

This route is not used as a premise for the polarized-variety proofs below

[[OpenAI26a](#), Theorem 1.1; Corollary 6.2, pp. 1, 22]

02 Surfaces: vanishing via injective jet tests

Maximal Seshadri constants on arbitrary polarized surfaces

Node exponent bounds, compression, and finite torus groups

$$\varepsilon = \sqrt{L^2/r} \quad (\text{every sufficiently large } r)$$

Extend nodes, compression and primitive directions to labelled polynomial spaces

[[OpenAI26c](#), §1.1; Proposition 4.1, pp. 2, 9–14]

03 Higher dimensions: reconstruct for surjectivity onto full jets

Maximal Multipoint Seshadri Constants in Higher Dimensions

Two-intercept reshaping, interior filling of graded supports, interpolation

$$\varepsilon = (L^n/r)^{1/n}, \quad n \geq 3, \quad k = \mathbf{C}$$

Reprove the surface and higher-dimensional constructions using formal substitutions

[[OpenAI26d](#), §1.1; §4; §5, pp. 2, 8–14]

04 Positive characteristic: formal transfer and a geometric generic tuple

Maximal multipoint Seshadri constants in positive characteristic

$$y^2 + xy - x^3 = \xi\eta, \quad K_r = k(\overline{U_r})$$

$$\varepsilon_r = (L^n/r)^{1/n}, \quad n \geq 2, \quad \text{char } k > 0$$

Dashed arrows: methodological inheritance and reconstruction, not use of earlier maximality theorems.

Scopes of the four papers and the methodological links among surfaces, higher dimensions and positive characteristic

Open figure ↗ TeX source ↓

In the plane paper, [normal-bundle specialization](#) and [two ordered limits](#) establish strictness, while [Corollary 6.2](#) deduces strict nefness and the Seshadri constant. Eventual maximality on arbitrary surfaces should not be identified with strictness for every plane point count $r \geq 10$.

The higher-dimensional paper reconstructs the surface node, compression and primitive-direction method as [two-intercept reshaping](#). The final test changes from injectivity of jets on surfaces to surjectivity onto full jets in higher dimensions. The positive-characteristic paper further uses [formal jet transfer](#) and [a two-branch node](#), imposing all tests at the [geometric generic tuple](#). These changes accommodate characteristic two and countable algebraically closed fields.

Methodological source

Surface paper, Propositions 3.2, 3.4; Lemma 4.3

Reconstruction

Higher-dimensional paper, §1.1; Proposition 4.1

Role and checked distinction

Remaining coordinates become labels and real intercepts are reshaped; the required arguments are reproved

Methodological source

The same three surface results and higher-dimensional §§2–6

Reconstruction

Positive-characteristic paper, §1.1; §§2–5

Role and checked distinction

Formal substitutions replace analytic limits; the node becomes $y^2 + xy - x^3$. Interior filling and interpolation are also reproved

[OpenAI26b, Propositions 3.2, 3.4; Lemma 4.3, pp. 8–10, 14]

[OpenAI26c, §1.1; Proposition 4.1, pp. 2, 9–14]

[OpenAI26d, §1.1; §4; §5, pp. 2, 8–14]

Sources and checking scope

The official number and subject follow the OpenAI Research Catalog, “Algebraic and complex geometry” (032–068). Entry 039 is on overview PDF page 6. The table retains the official order in CONTENTS. The source snapshot is commit `1581d4cbdd69a0faf34b2089614fe2a814316a6a`.

[OpenAI Research Catalog · 039 · p. 6](#)

The articles check every principal conclusion against the relevant proof passages and compare direct inputs with their uses. The original statement and location in Chow’s paper remain unchecked in the plane article. Complete external proofs, all analytic estimates and independent verification of constructions in every degree were not undertaken. Connections with other catalogues have not been exhaustively investigated; uninvestigated does not mean independent. The diagram selects confirmed methodological relations and is not a dependency graph of every cited work.

Nagata's conjecture for plane curves

Specialization to a normal bundle and elliptic interpolation produce the equal-weight nef boundary

The manuscript reduces unequal multiplicities at very general plane points to an equal-multiplicity system existing at every configuration, and excludes that system. This article describes the author's statements and proof connections; it does not independently certify the complete proof.

Manuscript author: OpenAI · 2026-09-23 · 24 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A simultaneous strict inequality for all multiplicities

Let $r \geq 10$ be an integer, and let $U_r \subset (\mathbf{P}_{\mathbf{C}}^2)^r$ be the space of distinct ordered r -tuples. The author asserts the existence of a countable union $E_r \subset U_r$ of proper Zariski-closed subsets, with nonempty complement, such that the following holds for every $(p_1, \dots, p_r) \notin E_r$. For every nonzero effective plane curve C of degree $d \geq 1$ and every $m_i \in \mathbf{Z}_{\geq 0}$ satisfying $\text{mult}_{p_i} C \geq m_i$,

$$d\sqrt{r} > \sum_{i=1}^r m_i.$$

Reducible and nonreduced curves are included. One exceptional set works for every degree and multiplicity vector: the points are **very general**. This must not be replaced by a statement on one Zariski-open set. The assertion does not include $r = 9$.

[Theorem 1.1 · p. 1](#)

Corollary 6.1 — Passage to uncountable characteristic-zero fields

Let k be an uncountable algebraically closed field of characteristic zero, and let $r \geq 10$. In the space of distinct ordered r -tuples in $\mathbf{P}_k^2$, the same strict inequality, with the degree, multiplicity and curve conventions of Theorem 1.1, holds simultaneously outside a countable union of proper closed subsets with nonempty complement.

Corollary 6.1 · p. 22

Corollary 6.2 — Strict nefness, square zero and maximal multipoint Seshadri constants

Blow up the plane at a tuple covered by Theorem 1.1 or Corollary 6.1. Write H for the pullback of a line and E_i for the exceptional curves. Then

$$N_r = \sqrt{r}H - \sum_i E_i \text{ is strictly nef,} \quad N_r^2 = 0.$$

The constant here is the equal-weight multipoint constant, normalized as

$$\varepsilon(\mathcal{O}(1); p_1, \dots, p_r) = \sup\{t \geq 0 : H - t \sum_i E_i \text{ is nef}\} = \frac{1}{\sqrt{r}}.$$

It is neither a one-point constant nor a constant with separately prescribed weights. Its value is irrational for nonsquare r .

Corollary 6.2 · p. 22

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Nagata's conjecture for plane curves](#).

External inputs use the following keys.

- [StacksN](#) The Stacks Project Authors — The Stacks project — Divisors: Identify the normal bundle with $\mathcal{O}_B(k)$. Online chapter PDF and tag statement consulted 2026-10-15.
- [StacksC](#) The Stacks Project Authors — The Stacks project — Cohomology of Schemes: Compute the section-space dimension in the square case. Online chapter PDF and tag statement consulted 2026-10-15.
- [StacksT](#) The Stacks Project Authors — The Stacks project — Algebraic Curves: Choose a nontorsion degree-zero line bundle. Online chapter PDF and tag statement consulted 2026-10-15.

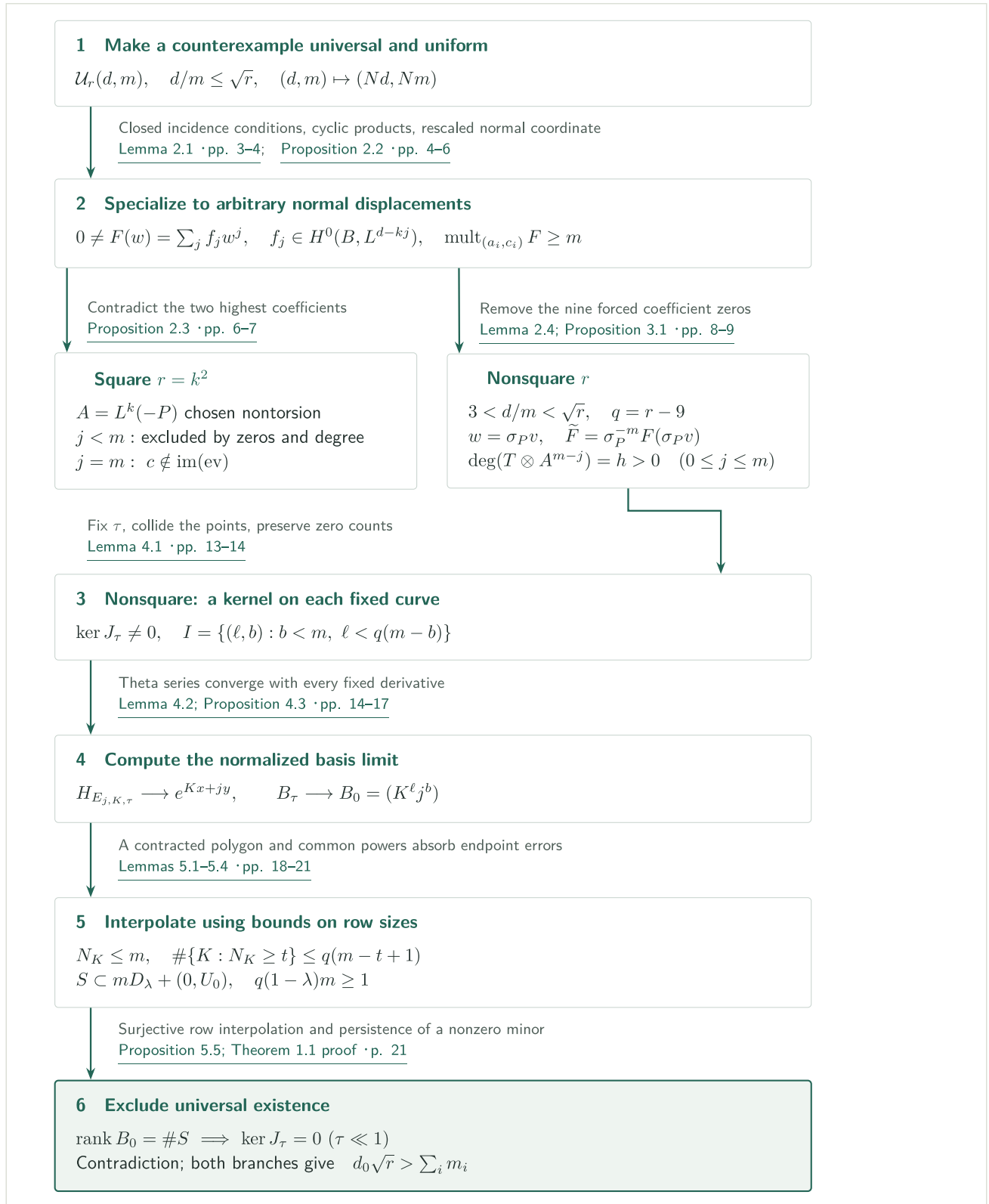
- [Chow49](#) W.-L. Chow — On Compact Complex Analytic Varieties: Algebraicity of the analytic image in Lemma 3.3; original theorem statement not checked. American Journal of Mathematics 71 (1949), 893–914; original theorem location unverified.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — Contradict universal existence using two limits

Proof target : [Theorem 1.1](#) · p. 1

Proof route. Express existence of a counterexample by closed incidence loci and take cyclic products to equalize multiplicities. For square r , at most two coefficients yield the contradiction. For nonsquare r , collide the remaining $r - 9$ points on the normal bundle of a cubic to obtain a nonzero kernel, then prove column independence of a numerical matrix obtained by degenerating the elliptic curve. The diagram collects these branches and the final contradiction.



The square and nonsquare branches of Theorem 1.1 and the limiting matrix contradiction

Open figure ↗ TeX source ↓

1. A countable exceptional set, cyclic products and normal polynomials

For a fixed degree e and multiplicity vector $\mathbf{n}$, the locus $A_{e,\mathbf{n}}$ admitting a nonzero form is closed: it is defined over $\mathbf{Q}$ by the full-column-size minors of the Taylor-condition matrix. Remove all proper such loci to form E_r . A counterexample outside E_r therefore has an incidence locus equal to all of U_r . Multiplying forms associated with the r cyclic permutations gives $d = rd_0$, $m = \sum_i m_i > 0$, and

$\mathcal{U}_r(d, m)$: every distinct-point configuration admits a nonzero degree- d form of multiplicity $\geq m$ at every point,

with $d/m \leq \sqrt{r}$. We write $\mathcal{U}_r$ for the manuscript's property $U_r(d, m)$ to distinguish it from the configuration space. The property persists under every common multiple (Nd, Nm) .

For a smooth degree- k plane curve B , put $L = \mathcal{O}_B(1)$; its normal bundle is $M = L^k$. Move points towards B and rescale the normal direction. For arbitrary distinct $a_i \in B$ and arbitrary normal vectors $c_i \in M_{a_i}$, this produces

$$0 \neq F(w) = \sum_{j=0}^{\lfloor d/k \rfloor} f_j w^j, \quad f_j \in H^0(B, L^{d-kj}),$$

of multiplicity at least m at each (a_i, c_i) . Taking the lowest power of the motion parameter leaves distinct powers of w , ensuring nonvanishing. Multiplicity is preserved by extending the local derivative conditions, centered at the moving points, to parameter zero. The proof contradicts universal existence at a special configuration; it does not assume a reverse deformation from special to general points.

Lemma 2.1 · pp. 3–4

Proposition 2.2 · pp. 4–6

[StacksN, Tag 0B3P, Lemma 15.2, p. 26]

2. Exclude equality as well in the square case

Let $r = k^2$ with $k \geq 4$. Choose $P = \sum a_i$ so that $A = M(-P)$ is nontorsion, and choose the normal-vector tuple c outside the image of $H^0(B, M) \rightarrow \bigoplus_i M_{a_i}$. These choices use countability of torsion and $h^0(B, M) = (k^2 + 3k)/2 < r$.

If j is the highest exponent of F , then $d \leq km$ implies $j \leq m$. When $j < m$, the required zeros of the leading coefficient either exceed its degree or, at the boundary $d = km$, produce a nonzero section of the nontrivial degree-zero bundle A^{m-j} . Both are impossible. The remaining case has $j = m$, $d = km$, and a nonzero constant leading coefficient f_m . Each fiber restriction equals $f_m(w - c_i)^m$, so the next coefficient satisfies $(f_{m-1}(a_i)) = -mf_m(c_i)$. This puts the chosen c in the evaluation image, again a contradiction. This second step excludes the endpoint equality.

Proposition 2.3 · pp. 6–7

[StacksC, Tag 01XT, Lemma 8.1, p. 16]

[StacksT, Tag 0C1Z, Lemma 17.1, p. 36]

3. Nonsquare case: collision on a fixed curve, then a numerical matrix limit

The rationality of d/m gives $d/m < \sqrt{r}$. Setting nine normal displacements to zero on a cubic forces the coefficient of degree $j < m$ to vanish along $(m-j)P$. Lemma 2.4 excludes $d \leq 3m$. Put $A = L^3(-P)$ and $T = L^{d-3m}$. The substitution $w = \sigma_P v$ and division $\tilde{F} = \sigma_P^{-m} F(\sigma_P v)$ place the first $m+1$ coefficient blocks in bundles $T \otimes A^{m-j}$ of the same positive degree $h = 3(d-3m)$. For $j > m$, use $\sigma_P^{j-m} H^0(B, L^{d-3j})$; their direct sum defines W_τ .

Take $B = X_\tau = \mathbf{C}^*/\tau^{\mathbf{Z}}$, $0 < \tau < 1$, with the manuscript's theta-bundle multipliers. Degree-three sections separate points and tangent directions, giving an analytic embedding; Lemma 3.3 invokes Chow's theorem to make its image an algebraic plane cubic. The location and statement in Chow's original paper remain unchecked here.

First fix τ . Collide the remaining $q = r - 9$ points along $y = 0$ in local coordinates (x, y) . After b derivatives in y , there are zeros of order at least $m - b$ at each point. A normalized finite-dimensional subsequence has a nonzero limit; Rouché's theorem preserves the total number of zeros and gives

$$J_\tau : W_\tau \longrightarrow \mathbf{C}^I, \quad I = \{(\ell, b) : 0 \leq b < m, 0 \leq \ell < q(m-b)\}, \quad \ker J_\tau \neq 0.$$

Next normalize the residue-class theta bases. Their local expressions converge to e^{Kx+jy} together with every fixed derivative, so the matrices converge to $B_0 = (K^\ell j^b)$. No continuous choice of kernel vectors across τ is used.

Lemma 2.4; Proposition 3.1 · pp. 8–9

Lemmas 3.2–3.3; §3.3 · pp. 10–12

Lemma 4.1 · pp. 13–14

Lemma 4.2; Proposition 4.3 · pp. 14–17

4. Row interpolation yields column independence and excludes every counterexample

The finite exponent set $S = \{(j, K)\}$ is defined in (4.4), including the degree-zero block $(d/3, 0)$ when $3 \mid d$. Put $a = 3(\sqrt{r} - 3)$, $\rho = 3(d/m - 3)$, and choose positive rational Δ with $\rho + \Delta\rho/9 < a$. Taking $\lambda < 1$ sufficiently close to one places S in a contracted polygon $mD_\lambda + (0, U_0)$. Its maximum horizontal width is $m\lambda$, and the interval of heights with width at least s has length $q(m\lambda - s)$. The identity $2a + a^2/9 = q$ is decisive.

First take a common multiple of (d, m) so that $q(1 - \lambda)m \geq 1$. This absorbs the lattice endpoint error and gives

$$N_K := \#\{j : (j, K) \in S\} \leq m, \quad \#\{K : N_K \geq t\} \leq q(m - t + 1) \quad (1 \leq t \leq m).$$

Interpolate the largest rows first, multiplying each correction by $\prod_{a < i} (K - K_a)$ to preserve earlier rows. Every correction still satisfies $\ell < q(m - b)$. Hence the allowed span of $K^\ell J^b$ maps onto all functions on S , and B_0 has full column rank. A nonzero maximal minor persists for sufficiently small positive τ , making J_τ injective and contradicting its nonzero kernel.

The common multiple leaves d/m , Δ and λ unchanged and is taken once, before both limits. Exclusion of $\mathcal{U}_r(d, m)$ in both square and nonsquare cases returns the strict inequality for every degree and every unequal multiplicity vector outside the original single E_r .

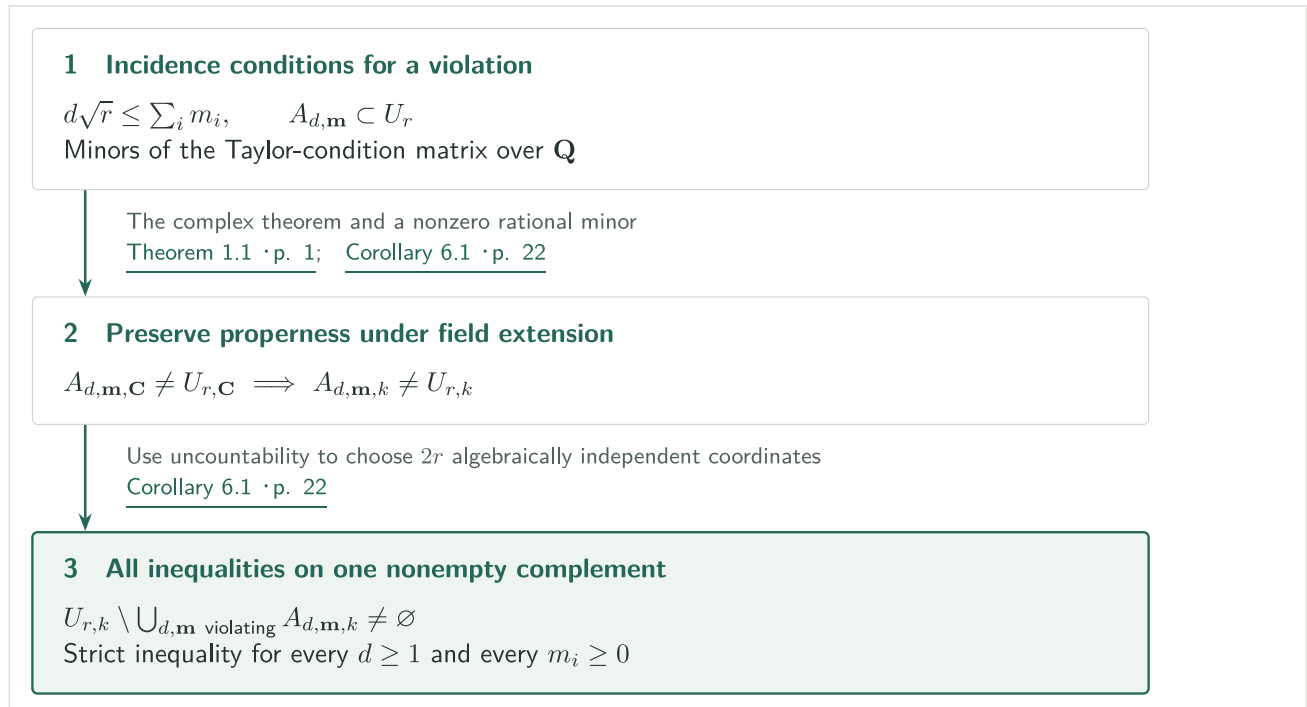
Lemmas 5.1–5.4 · pp. 18–21

Proposition 5.5; Theorem 1.1 proof · p. 21

3. Proof of Corollary 6.1 — Transfer rational closed conditions

Proof target : Corollary 6.1 · p. 22

Transfer all quantifiers of Theorem 1.1 to the fields in Corollary 6.1 above.



Rational incidence conditions and nonemptiness in Corollary 6.1

Open figure ↗ TeX source ↓

1. Proper incidence loci and a nonempty complement

For every violating $(d, \mathbf{m})$, the incidence locus is defined by rational minors. Properness over $\mathbf{C}$ implies properness over $\mathbf{Q}$, and a nonzero rational minor stays nonzero under characteristic-zero field extension. Removing these countably many loci yields all inequalities simultaneously.

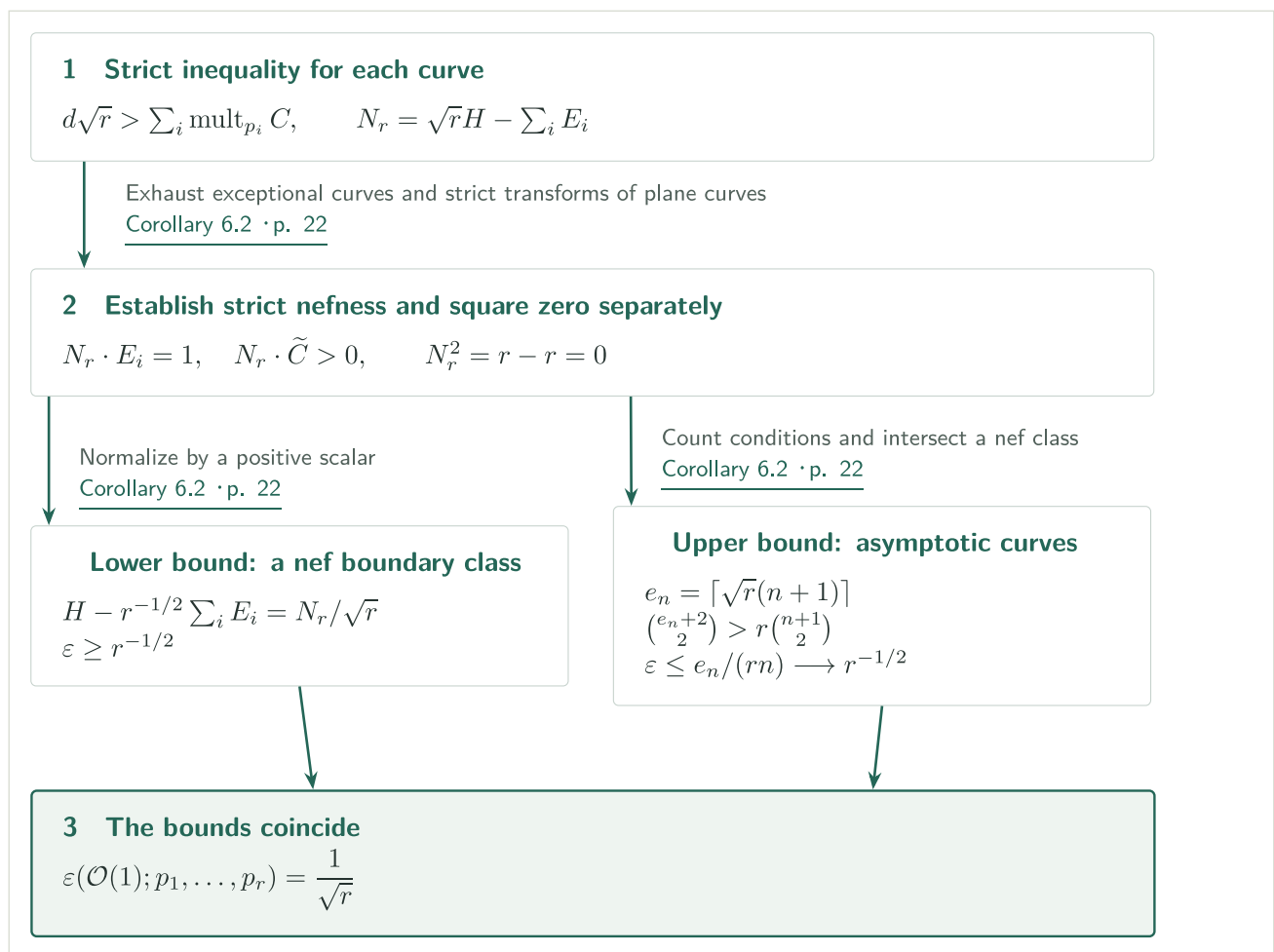
Uncountability of k permits $2r$ coordinates algebraically independent over $\mathbf{Q}$; they avoid every excluded locus and every diagonal. No embedding $k \hookrightarrow \mathbf{C}$ is required.

Corollary 6.1 · p. 22

4. Proof of Corollary 6.2 — Combine curvewise positivity with an asymptotic upper bound

Proof target : Corollary 6.2 · p. 22

For the three conclusions of Corollary 6.2, separate the lower bound obtained from the strict inequality from the upper bound obtained by counting conditions.



Strict nefness, square zero and matching bounds in Corollary 6.2

Open figure ↗ TeX source ↓

1. Intersections with every irreducible curve and the self-intersection

An irreducible curve on the blowup is exceptional or the strict transform of a plane curve. In the former case $N_r \cdot E_i = 1$; in the latter, $N_r \cdot \tilde{C} = d\sqrt{r} - \sum_i \text{mult}_{p_i} C > 0$. Thus N_r is strictly nef. Separately, $H^2 = 1$, $H \cdot E_i = 0$ and $E_i \cdot E_j = -\delta_{ij}$ give $N_r^2 = r - r = 0$. Nefness is not inferred from this numerical identity alone.

Corollary 6.2 · p. 22

2. The normalized lower bound and the interpolation-count upper bound

Nefness of $N_r/\sqrt{r} = H - r^{-1/2} \sum_i E_i$ gives $\varepsilon \geq r^{-1/2}$. For the reverse bound, set $e_n = \lceil \sqrt{r}(n+1) \rceil$. Since $\binom{e_n+2}{2} > r \binom{n+1}{2}$, there is a nonzero degree- e_n form D of multiplicity at least n at every point. If $H - t \sum_i E_i$ is nef, then

$$0 \leq e_n - t \sum_i \text{mult}_{p_i} D \leq e_n - trn,$$

so $t \leq e_n/(rn) \rightarrow 1/\sqrt{r}$. This upper bound follows from the condition count for any configuration; the difficulty at very general points is the matching lower bound. Positive intersection with each curve is compatible with an asymptotic threshold equal to the boundary value.

Corollary 6.2 · p. 22

5. Which papers supply which steps

Source and result

[StacksN, Tag 0B3P, Lemma 15.2, p. 26]

Used at

Proposition 2.2, p. 4

Role and required hypotheses

Identify the normal bundle of a smooth plane curve as an effective Cartier divisor

Checking scope

Statement and application compared

Source and result

[StacksC, Tag 01XT, Lemma 8.1, p. 16]

Used at

Proposition 2.3, p. 7

Role and required hypotheses

$H^1(\mathbf{P}^2, \mathcal{O}) = 0$ and the restriction sequence

Checking scope

Statement and application compared

Source and result

[StacksT, Tag 0C1Z, Lemma 17.1, p. 36]

Used at

Proposition 2.3, p. 6; Lemma 2.4, p. 8

Role and required hypotheses

Finiteness of fixed-order torsion for a smooth projective curve in characteristic zero

Checking scope

Statement and application compared

Source and result

Chow49

Used at

Lemma 3.3, p. 11

Role and required hypotheses

Algebraicity of the closed image of a compact analytic embedding

Checking scope

Use and hypotheses compared; original statement and location unchecked

Dumnicki's evaluation matrices, Evain's collisions and Rosengren's theta conventions are methodological background explicitly identified by the manuscript. The particular row interpolation, collision conditions and normalized convergence used here are proved within the manuscript, so no unchecked direct-dependency arrows are assigned to those references. Exhaustive connections to other catalogues have not been investigated.

6. Return to the sources

- Lemma 2.1 · pp. 3–4 — A common exceptional set for all degrees and multiplicities, and symmetrization.
- Proposition 2.2 · pp. 4–6 — Nonvanishing and multiplicity in the normal limit.
- Proposition 2.3 · pp. 6–7 — Two steps excluding equality in the square case.
- Lemma 4.1 · pp. 13–14 — Collide points only after fixing the elliptic curve.
- Lemma 4.2; Proposition 4.3 · pp. 14–17 — Basis normalization and the numerical matrix limit.
- Lemmas 5.1–5.4 · pp. 18–21 — Control the row distribution, not merely total dimension.
- Corollary 6.2 · p. 22 — Separate strict nefness, square zero, and the two bounds for the constant.

The editorial check compared the hypotheses and every principal conclusion with the statements and the listed proof passages in §§2–6. Particular attention was paid to arbitrary multiplicities, the square endpoint, the order of the fixed-curve collision and the subsequent matrix limit, lattice endpoint errors, and the separate Seshadri bounds. The three Stacks statements and their applications were compared. Chow's original theorem statement, external proofs, all background literature, and a complete independent verification of the analytic estimates remain outside this check.

§§2–6 · pp. 3–22

Maximal Seshadri constants on arbitrary polarized surfaces

From a nodal curve to finite interpolation, and from homogeneous vanishing to a nef boundary

The manuscript claims that, on every smooth integral complex projective surface with an ample line bundle, the multipoint Seshadri constant at very general points reaches its volume bound once the number of points is sufficiently large. A fixed nodal curve constrains initial exponents; compression moves them into a triangle whose shape can be tuned to an interpolation test on a finite group. Riemann–Roch then converts the exclusion of equal vanishing orders into the exclusion of curves with arbitrary multiplicities.

Manuscript author: OpenAI · 2026-09-23 · 19 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Eventual maximality on every polarized surface

Let S be a smooth integral complex projective surface and L an ample line bundle. Put $H = L^2$ and let

$$U_r = \{(p_1, \dots, p_r) \in S^r : p_i \neq p_j \ (i \neq j)\}$$

be the space of ordered tuples of distinct points. The manuscript asserts that there is an integer $r_0 = r_0(S, L)$ such that, **for every integer** $r \geq r_0$, there is a countable union $Z_r \subset U_r$ of proper Zariski-closed subsets, with nonempty complement, and for every $\mathbf{p} \in U_r \setminus Z_r$ the class

$$D_0 = \pi^* L - \sqrt{\frac{L^2}{r}} \sum_{i=1}^r E_i$$

is nef. Here $\pi : Y \rightarrow S$ is the blow-up at the specified points and E_i are its exceptional curves. Consequently,

$$\varepsilon(S, L; \mathbf{p}) = \sqrt{\frac{L^2}{r}}.$$

Theorem 1.1 · p. 2

The normalization is the **multipoint constant with equal exceptional coefficients**:

$$\varepsilon(S, L; \mathbf{p}) = \inf_C \frac{L \cdot C}{\sum_i \text{mult}_{p_i} C} = \sup\{\lambda \geq 0 : \pi^* L - \lambda \sum_i E_i \text{ is nef}\}.$$

The infimum runs over integral curves meeting at least one point. The denominator is not divided by r , and no unequal exceptional weights are introduced. The multiplicities of a given curve need not be equal. “Very general” means the complement of the countable union above; the theorem does not assert that a single Zariski-open subset satisfies all jet conditions simultaneously.

§1; equation (1.1) · pp. 1–2

The result concerns the **qualitative Nagata–Biran conjecture**, with a threshold depending on the polarized surface. The strict nonhomogeneous plane inequality for every $r \geq 10$ belongs to the separate manuscript cited as reference [14]. It does not follow merely from the unspecified threshold and non-strict inequality here. The proof of this surface theorem does not use that plane theorem.

§§1–1.1 · pp. 2–3

Corollary 5.5 — Uncountable algebraically closed fields of characteristic zero

Let K be an uncountable algebraically closed field of characteristic zero, let S/K be a smooth integral projective surface, and let L be ample. There is an integer $r_0 = r_0(S, L)$ such that, for every $r \geq r_0$, outside a countable union of proper Zariski-closed subsets of U_r , with nonempty complement,

$$\varepsilon(S, L; p_1, \dots, p_r) = \sqrt{\frac{L^2}{r}}.$$

Both characteristic zero and uncountability are hypotheses. The corollary does not assert the same nonemptiness for rational points over a countable algebraically closed field.

Corollary 5.5 · p. 18

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Maximal Seshadri constants on arbitrary polarized surfaces](#). External inputs use the following keys.

- [SPVar](#) The Stacks Project Authors — The Stacks Project — Varieties: Smoothness of general members of a very ample system. version ed88ff78, compiled July 14, 2026; retrieved October 15, 2026.
- [SPMor](#) The Stacks Project Authors — The Stacks Project — More on Morphisms: Geometric irreducibility in Bertini. version ed88ff78, compiled July 14, 2026; retrieved October 15, 2026.
- [StacksN](#) The Stacks Project Authors — The Stacks Project — Divisors: Regular immersions and the Rees algebra of a blow-up. version ed88ff78, compiled July 14, 2026; retrieved October 15, 2026.
- [SPMod](#) The Stacks Project Authors — The Stacks Project — Sheaves of Modules: Principal parts and finite-order differential operators. version ed88ff78, compiled July 14, 2026; retrieved October 15, 2026.
- [SPChow](#) The Stacks Project Authors — The Stacks Project — Chow Homology and Chern Classes: Riemann–Roch on a smooth projective surface. version ed88ff78, compiled July 14, 2026; retrieved October 15, 2026.
- [SPDual](#) The Stacks Project Authors — The Stacks Project — Duality for Schemes: Serre duality for top cohomology. version ed88ff78, compiled July 14, 2026; retrieved October 15, 2026.
- [SPRes](#) The Stacks Project Authors — The Stacks Project — Resolution of Surfaces: Exceptional coefficients and powers of point ideals. version ed88ff78, compiled July 14, 2026; retrieved October 15, 2026.
- [SPLim](#) The Stacks Project Authors — The Stacks Project — Limits of Schemes: Descent of finite-presentation data. version ed88ff78, compiled July 14, 2026; retrieved October 15, 2026.
- [SPDes](#) The Stacks Project Authors — The Stacks Project — Descent and Algebraic Spaces: Descent of ampleness under field extension. version ed88ff78, compiled July 14, 2026; retrieved October 15, 2026.
- [StacksC](#) The Stacks Project Authors — The Stacks Project — Cohomology of Schemes: Commutation of global sections with field extension. version ed88ff78, compiled July 14, 2026; retrieved October 15, 2026.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

Stacks result numbers follow the chapter PDFs used here, with stable Tags also recorded. Checking an external statement and its application is distinguished from recursively verifying its proof.

2. Proof of Theorem 1.1 — Tuned triangles and the maximal bound

Proof target : [Theorem 1.1](#) · p. 2 See the [hypotheses](#) and all conclusions.

Proof route. A nodal curve bounds initial exponents, and diagonal compression gives a triangle of area $H/2$. For each sufficiently large r , choose its shape and an index- r lattice before choosing the jet orders. The resulting torus test works uniformly for all k, m . Two finite-matrix transfers and openness move the test to very general surface points. The lower bound supplied by Proposition 5.3 is then combined with the universal upper bound of Lemma 5.4.

1 Build exponent regions from a node

$$V \in |dL|, \quad H = L^2, \quad \nu_t(H^0(kL)) \subset kQ_t$$

$$P_t = \text{conv}\{(-b, 0), (a, 0), (0, 1/d)\}, \quad \text{area}(P_t) = H/2$$

Tune the determinant range to $\sqrt{rH}$; test each character triangle
[Proposition 4.2 · pp. 12–13](#)
[Lemma 4.3; Corollary 4.4 · pp. 14–15](#)

2 Tune the shape and group for each point count

$$r \geq r_0 : \quad t, G \text{ fixed}, \quad |G| = r, \quad w = \sqrt{H/r}$$

$$\forall k, m \in \mathbb{Z}_{>0}, \quad m > kw : \quad J_G^m|_{\mathcal{M}(kP_t)} \text{ is injective}$$

Compression and analytic transfer give witnesses; openness and Baire give a common locus
[Lemma 3.3; Proposition 3.4; Corollary 3.5 · pp. 9–11](#)
[Lemma 5.1 · p. 15](#)

3 Make all jet tests hold at very general tuples

$$U_r \setminus Z_r \neq \emptyset, \quad H^0(S, kL \otimes \bigcap_i I_{p_i}^m) = 0 \quad (m > kw)$$

Here Z_r is a countable union of proper Zariski-closed subsets

Make a negative curve produce a forbidden section; count jets separately for the upper bound
[Proposition 5.3 · pp. 16–17](#)
[Lemma 5.4 · p. 17](#)

4 Combine the nef lower bound and universal upper bound

$$D_0 = \pi^*L - w \sum_i E_i \text{ is nef} \implies \varepsilon \geq w$$

$$\varepsilon \leq w \implies \boxed{\varepsilon(S, L; \mathbf{p}) = \sqrt{L^2/r}}$$

Overview from a nodal curve through triangular diagrams, finite subgroups and very general jet tests to nefness and maximality

[Open figure ↗](#) [TeX source ↓](#)

1. Intersection at the node bounds exponents; compression transfers jets

Choose eL very ample. On the blow-up of a point, $2e\pi^*L - E$ is very ample. Choose a general member of its square meeting the exceptional curve transversely at two distinct points. Bertini supplies smoothness and irreducibility; pushing down gives an integral $V \in |dL|$, $d = 4e$, smooth away from one ordinary node. Fix local coordinates in which the two branches are $xz = 0$.

Lemma 3.1; Proposition 3.2 · pp. 6–8

[SPVar, Lemma 47.3 (Tag 0FD6), p. 108]

[SPMor, Lemma 32.3 (Tag 0G4F), p. 94]

Choose $t > T > 1$ so that $(1+t)^2 > d^2Ht$, and set $a = dHt/(1+t)$, $b = dH/(1+t)$ and $c = 1/d$. For a section of kL whose divisor does not contain V , its least weight v satisfies $v(1+1/t) \leq kdH$ by intersection with the two branches. If V occurs with coefficient j , remove it first and translate the initial exponent by (j, j) . Thus all initial exponents lie in

$$kQ_t, \quad Q_t = \text{conv}\{(0, 0), (a, 0), (c, c), (0, b)\}$$

Writing $q = \alpha - \beta$ makes diagonal slices vertical. Moving their lower endpoints to zero produces

$$P_t = \text{conv}\{(-b, 0), (a, 0), (0, c)\}, \quad \text{area}(P_t) = H/2$$

Area equality alone does not transfer jet evaluation. Proposition 3.4 uses the basis $u^q v^{\beta_q} (v-1)^h$ on each integer slice, substitutes $v = 1 + sy$, and divides by s^h to obtain a limit $u^q y^h$. Persistence of a nonzero maximal minor transfers injectivity from P_t to Q_t . The weighted analytic dilation in Proposition 2.2 then transfers it from initial monomials to the original sections. The constructed points remain distinct and may depend on k, m .

Lemma 3.3; Proposition 3.4; Corollary 3.5 · pp. 9–11

Lemma 2.1; Proposition 2.2 · pp. 4–6

2. For every sufficiently large integer r , choose one group valid for all orders

Put $w = \sqrt{H/r}$. Lemma 4.3 chooses $t > T$ and a primitive lattice vector u so that the range of $\det(u, -)$ on P_t is exactly $\sqrt{rH}$. Explicitly, put $R = \sqrt{rH}$, $W = dH$ and $a_0 = WT/(1+T)$, and choose a power of two j with $R/(4W) < j \leq R/(2W)$. For large r , the interval $(d(R-jW), d(R-ja_0))$ has length greater than two and contains an odd integer i . Then $u = (i, j)$ is primitive. The continuous parameter t realizes the unique $a \in (a_0, W)$ satisfying $i/d + ja = R$. This is the step handling every sufficiently large integer rather than only square point counts.

Lemma 4.3; Corollary 4.4 · pp. 14–15

Complete u to an integer basis (u, v_0) and set $M = \mathbb{Z}u + \mathbb{Z}(rv_0)$. Its annihilator in the torus is a group G of exactly r points. Character projections separate the exponent support into cosets of M ; take a nonzero component and express it in lattice- M coordinates. Its supporting triangle has area $(kw)^2/2$ and vertical range kw . Lemma 4.1 bounds its order at $(1, 1)$ by kw : factor the common power $(U - 1)^e$ from the horizontal rows and bound the remaining height range by $kw - e$. The coordinate map has Jacobian determinant $r \neq 0$, so it preserves local orders. A nonzero polynomial cannot therefore vanish to order $m > kw$ at every point of G .

The quantifier order is essential: first fix r_0 from (S, L) , then fix t, G for each r , and only then consider **all positive integers k, m with $m > kw$** . An asymptotic interpolation theorem for a fixed multiplicity bound would not suffice.

Proposition 4.2 · pp. 12–13

Lemma 4.1 · pp. 11–12

3. From special test points to very general configurations and a nef lower bound

Corollary 3.5 realizes each test at some tuple of r surface points. Lemma 5.1 expresses evaluation as an algebraic vector-bundle map using principal parts. Its noninjectivity locus is cut out by maximal minors, so each injective locus is nonempty and Zariski open. There are countably many pairs (k, m) ; their common locus is the complement of a countable union of proper closed subsets. The Baire property of $U_r(\mathbb{C})$ makes it nonempty. Only at this step does one tuple satisfy all orders simultaneously.

Lemma 5.1 · p. 15

[SPMod, Section 29 (Tag 0G3P), pp. 45–46]

[StacksN, Sections 22, 33 (Tags 0638, 01OF), pp. 47, 81–82]

Apply Proposition 5.3 at each such tuple to obtain nefness of D_0 , hence $\varepsilon \geq w$. The passage from excluding equal vanishing orders to excluding curves with unequal multiplicities uses the proposition's Riemann–Roch argument; it is not left as a merely numerical equivalence.

Proposition 5.3 · pp. 16–17

4. A separate universal upper bound gives equality

The upper bound is independent of the interpolation construction and holds at every tuple of distinct points. Fix $0 < \alpha < w$ and $m_k = \lceil k\alpha \rceil$. Riemann–Roch gives $h^0(kL) \geq Hk^2/2 + O(k)$, exceeding the number of jet conditions $r \binom{m_k+1}{2} = r\alpha^2 k^2/2 + O(k)$. Thus there is an effective $F_k \sim k\pi^*L - m_k \sum E_i$. If $\pi^*L - \lambda \sum E_i$ is nef, then

$$0 \leq (\pi^*L - \lambda \sum E_i) \cdot F_k = kH - \lambda r m_k.$$

First let $k \rightarrow \infty$, then $\alpha \uparrow w$, to obtain $\lambda \leq w$. Combining this with $\varepsilon \geq w$ gives $\varepsilon = w$. Computing $D_0^2 = 0$ alone would not establish the lower bound.

[Lemma 5.4 · p. 17](#)

[Proof of Theorem 1.1 · p. 17](#)

3. Proof of Proposition 5.3 — From equal multiplicities to all curves

Proof target : [Proposition 5.3 · pp. 16–17](#)

Proposition 5.3 — Homogeneous tests imply nefness

Let S be a smooth integral complex projective surface, L ample, and $p_1, \dots, p_r$ distinct points. Put $H = L^2$ and $w = \sqrt{H/r}$. Suppose that for every pair of positive integers k, m with $m > kw$, no nonzero section of $L^{\otimes k}$ vanishes to order at least m at every p_i . On the blow-up Y of the points, the class $D_0 = \pi^*L - w \sum_i E_i$ is nef.

[Proposition 5.3 · pp. 16–17](#)

Write $A = \pi^*L$ and $E = \sum_i E_i$. Then A is nef, $A^2 = H$, $A \cdot E_i = 0$ and $E_i \cdot E_j = -\delta_{ij}$.

1 Assume failure of nefness

$$D_0 = A - wE, \quad D_0^2 = 0, \quad D_0 \cdot C < 0$$

$$C \neq E_i, \quad A \cdot C > 0, \quad A = \pi^*L \text{ is nef}$$

Use $-2\delta D_0 \cdot C > 0$, then move the coefficient to $s > w$
[Proposition 5.3 · pp. 16–17](#)

2 Perturb rationally to positive square

$$D = A - sE - \delta C, \quad s > w, \quad s, \delta \in \mathbb{Q}_{>0}$$

$$D^2 > 0, \quad A \cdot D > 0$$

Use nef A to exclude $K_Y - kD$, then apply Riemann–Roch
[Lemma 5.2 · p. 16](#)
[\[SPDual, Lemma 27.1\(6\)–\(7\) \(Tag 0FVV\), p. 69\]](#)
[\[SPChow, Section 66 \(Tag 02UO\), p. 154\]](#)

3 Produce an effective multiple

$$h^2(kD) = 0, \quad h^0(kD) \geq \chi(kD) = \frac{1}{2}k^2D^2 + O(k) > 0$$

$$F \sim kD, \quad F \geq 0$$

Add the curve back; push exceptional order down to point-ideal powers
[\[SPRes, Lemma 3.4\(4\), Section 3 \(Tag 0AGP\), p. 7\]](#)
[Proposition 5.3 · pp. 16–17](#)

4 Contradict the homogeneous vanishing tests

$$F + k\delta C \sim kA - mE, \quad m = ks > kw$$

$$H^0(S, kL \otimes \bigcap_i I_{p_i}^m) \neq 0 \implies \perp \implies \boxed{D_0 \text{ is nef}}$$

A negative curve gives a rational positive-square divisor and then a forbidden section with equal vanishing orders

[Open figure ↗](#) [TeX source ↓](#)

1. Subtract a negative curve and perturb rationally beyond the boundary

If D_0 were not nef, an integral curve $C \subset Y$ would satisfy $D_0 \cdot C < 0$. Since $D_0 \cdot E_i = w > 0$, this curve is not exceptional and $A \cdot C > 0$. For a sufficiently small positive rational δ ,

$$(D_0 - \delta C)^2 = -2\delta D_0 \cdot C + \delta^2 C^2 > 0, \quad A \cdot (D_0 - \delta C) > 0.$$

By continuity, choose a rational $s > w$ sufficiently close to w so that $D = A - sE - \delta C$ still satisfies $D^2 > 0$ and $A \cdot D > 0$. Nefness of D itself is not required.

[Proposition 5.3 · pp. 16–17](#)

2. Serre duality and Riemann–Roch produce an effective multiple

For sufficiently large k clearing the denominators, $A \cdot (K_Y - kD) < 0$. Since A is nef, $K_Y - kD$ cannot be linearly equivalent to an effective divisor. Serre duality gives $h^2(kD) = 0$, and

$$h^0(kD) \geq \chi(\mathcal{O}_Y(kD)) = \frac{k^2 D^2 - kD \cdot K_Y}{2} + \chi(\mathcal{O}_Y) > 0.$$

This is Lemma 5.2. The cited Stacks statement is Grothendieck–Riemann–Roch, specialized here to a line bundle on a smooth projective surface.

Lemma 5.2 · p. 16

[SPDual, Lemma 27.1(6)–(7) (Tag 0FVV), p. 69]

[SPChow, Section 66 (Tag 02UO), p. 154]

3. Add the curve back and contradict the homogeneous tests

Take an effective $F \sim kD$, with k divisible by the denominators of s, δ . Then

$$F + k\delta C \sim kA - mE, \quad m = ks > kw.$$

The blow-up identity $\pi_* \mathcal{O}_Y(-mE) = \bigcap_i I_{p_i}^m$, together with the projection formula, gives a nonzero section of kL vanishing to order at least m at every point, contradicting the hypothesis. Hence D_0 is nef. The original multiplicities of C may be unequal: effectivity and adding back the curve produce precisely the homogeneous test needed for the contradiction.

[SPRes, Lemma 3.4(4), Section 3 (Tag 0AGP), p. 7]

Proposition 5.3 · pp. 16–17

4. Proof of Corollary 5.5 — Descend the failure loci

Proof target : Corollary 5.5 · p. 18 See the hypotheses and conclusion.

Proof route. Descend (S, L) to a subfield $F \subset K$ finitely generated over $\mathbb{Q}$, and use an embedding $F \hookrightarrow \mathbb{C}$ to obtain a threshold. Base-change the noninjectivity loci of jet evaluation rather than transferring the analytically constructed complex points.

1 Descend to a finitely generated field

$$F \subset K, \quad F/\mathbb{Q} \text{ finitely generated}, \quad F \hookrightarrow \mathbb{C}$$

$$(S_0, L_0)/F \longmapsto (S_{0,\mathbb{C}}, L_{0,\mathbb{C}}), \quad r_0$$

The complex nef boundary excludes kernels; base-change sections and maximal minors

[Theorem 1.1 · p. 2](#)

[\[StacksC, Lemma 5.2 \(Tag 02KH\), p. 10\]](#)

[Corollary 5.5 · p. 18](#)

2 Preserve properness of every jet failure locus

$$B_{a,m} \subset U_{r,0}, \quad m > a\sqrt{H/r}$$

$$B_{a,m,\mathbb{C}} \neq U_{r,\mathbb{C}} \implies B_{a,m,K} \neq U_{r,K}$$

Uncountability supplies $2r$ independent elements; algebraic closedness extends the embedding

[Corollary 5.5 · p. 18](#)

3 Find a point in the common complement

$$F(U_{r,0}) \hookrightarrow K, \quad Z_r = \bigcup_{a,m:m > a\sqrt{H/r}} B_{a,m,K}$$

$$U_{r,K} \setminus Z_r \neq \emptyset$$

All jet tests give the lower bound; surface Riemann–Roch gives the upper bound

[Proposition 5.3 · pp. 16–17](#)

[Lemma 5.4 · p. 17](#)

4 Apply algebraic nefness and the upper bound

$$\pi^* L - \sqrt{H/r} \sum_i E_i \text{ is nef,} \quad \boxed{\varepsilon = \sqrt{L^2/r}}$$

Descent to a finitely generated field, proper failure loci, and a nonempty complement over the uncountable field

[Open figure ↗](#) [TeX source ↓](#)

1. Complex maximality makes every failure locus proper

Descend the projective equations and finite gluing data of L . Choose $(S_0, L_0)/F$ retaining ampleness, embed F into $\mathbb{C}$, and fix r_0 from Theorem 1.1 there. Intersection number H is unchanged by field extension. For each $r \geq r_0$ and $m > aw$, define the noninjectivity locus $B_{a,m}$ on $U_{r,0}$ by maximal minors of jet evaluation. Flat base change for global sections allows comparison of the same matrix condition over the different fields.

[\[SPLim, Lemma 10.1 \(Tag 01ZM\), p. 25\]](#)

[\[SPDes, Lemma 13.1 \(Tag 0D3C\), pp. 21–22\]](#)

[\[StacksC, Lemma 5.2 \(Tag 02KH\), p. 10\]](#)

At a complex tuple where $A - wE$ is nef, a nonzero section in the evaluation kernel would give an effective class $aA - mE$ with negative intersection $aH - rmw < 0$ against that nef class. Hence $B_{a,m}$ is proper over $\mathbb{C}$, and therefore over F and over K . The same r_0 works for every (a, m) .

Corollary 5.5 · p. 18

2. Uncountability supplies the common locus; the algebraic argument finishes

The field F is countable, whereas K is uncountable. Choose $2r$ elements of K algebraically independent over F , then use algebraic closedness to extend to an embedding $F(U_{r,0}) \hookrightarrow K$.

The resulting K -point maps to the generic point of $U_{r,0}$ and avoids every $B_{a,m}$. Thus

$Z_r = \bigcup_{a,m:m > aw} B_{a,m,K}$ has nonempty complement. Every tuple in that complement satisfies all homogeneous jet tests. The algebraic arguments of Lemma 5.2, Proposition 5.3 and Lemma 5.4 apply over K and give the nef lower bound together with the matching upper bound.

Corollary 5.5 · p. 18

5. Which references supply which steps

Source and result

SPVar Lemma 47.3; SPMor Lemma 32.3

Used at

Lemma 3.1, p. 7

Role and required hypotheses

Smooth integral general member of a very ample system on a smooth integral surface.

Transversality at two exceptional points is secured by the manuscript's surjective restriction map.

Checking scope

Statements and application conditions checked; external proofs excluded.

Source and result

StacksN §§22, 33; SPMoD §29

Used at

Lemma 3.1, p. 7; Lemma 5.1, p. 15

Role and required hypotheses

Rees-algebra blow-up, regular diagonal and principal parts. Openness itself follows from the manuscript's maximal-minor argument.

Checking scope

Source constructions and the manuscript's finite-jet explanation checked.

Source and result

SPDual Lemma 27.1; SPChow §66

Used at

Lemma 5.2, p. 16; Lemma 5.4, p. 17

Role and required hypotheses

Top-cohomology vanishing and a quadratic Euler characteristic. Nefness is imposed on A , not D ; use $D^2, A \cdot D > 0$.

Checking scope

Statements, specialization and use checked.

Source and result

SPRes Lemma 3.4(4)

Used at

Proposition 5.3, p. 17

Role and required hypotheses

On a point blow-up of a two-dimensional regular local ring, exceptional order becomes an ideal power.

Checking scope

Statement and application checked.

Source and result

SPLim Lemma 10.1; SPDes Lemma 13.1; StacksC Lemma 5.2

Used at

Corollary 5.5, p. 18

Role and required hypotheses

Finite-presentation descent, descent of ampleness, and flat base change. Uncountability supplies a point in the common complement.

Checking scope

Statements and application checked.

The plane Nagata manuscript is a comparison for the threshold and strict inequality, not a direct input to Theorem 1.1 here. Dumnicki, Ciliberto–Farnik–Küronya–Lozovanu–Roé–Shramov, and Gundlach are discussed as methodological background; the required transfers, exponent bounds and triangle multiplicity estimate are proved within this manuscript. Their background proofs were not examined in this check. Unrecorded connections to other catalogues remain uninvestigated.

§§1–1.1 · pp. 2–3

6. Return to the sources

- Lemma 3.1; Proposition 3.2 · pp. 6–8 — How intersection at the node bounds initial exponents.
- Lemma 3.3; Proposition 3.4; Corollary 3.5 · pp. 9–11 — Transfer diagonal compression through finite jet matrices.
- Proposition 4.2 · pp. 12–13 — Interpolation from an index- r lattice and character projection.
- Lemma 4.3; Corollary 4.4 · pp. 14–15 — Choose the threshold before all jet orders.
- Proposition 5.3 · pp. 16–17 — Pass from equal vanishing orders to all curve inequalities.
- Lemma 5.4 · p. 17 — Check the universal upper bound separately from the lower bound.
- Corollary 5.5 · p. 18 — Extension in characteristic zero and the role of uncountability.

The editorial check covered the hypotheses and all conclusions of the principal results, the indicated proof passages in §§2–5, and the statements and applications of the direct Stacks inputs. Particular attention was given to fixing r_0 before jet orders, openness in passing from

special points to very general configurations, effectivity in reducing unequal curve multiplicities to homogeneous tests, and the separate lower and upper bounds. This is an account of the author's argument and a check of its connections, not independent certification of the entire proof or recursive verification of all external inputs. An optimal r_0 , exact values for small point counts, arbitrary weighted constants, and positive-characteristic extensions are not asserted in this article.

§§2–5 · pp. 4–18

Maximal Multipoint Seshadri Constants in Higher Dimensions

Real-intercept reshaping and graded additivity yield jet separation at the volume bound

The manuscript claims exact maximality of the ordinary multipoint Seshadri constant on a fixed smooth complex projective variety for every sufficiently large integer point count. Its mechanism reshapes initial exponents into an elongated simplex without losing volume, fills its interior using additivity, and obtains surjectivity onto full jets. This article presents the manuscript's argument and checks its connections; it does not certify the complete proof.

Manuscript author: OpenAI · 2026-10-05 · 20 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Maximality at very general tuples

Let X be a smooth integral complex projective variety of dimension $n \geq 3$, let L be an ample line bundle, and put $V = L^n$. Write

$$U_r(X) = \{(p_1, \dots, p_r) \in X^r : p_i \neq p_j \ (i \neq j)\}.$$

The convention throughout is

$$\varepsilon(X, L; \mathbf{p}) = \inf_C \frac{L \cdot C}{\sum_{i=1}^r \text{mult}_{p_i} C},$$

where $C \subset X$ ranges over integral curves meeting at least one marked point and the multiplicity is zero outside C . This is the ordinary, equally weighted multipoint constant defined by curves. The denominator is not divided by r , and no higher-cycle invariant is substituted.

The manuscript asserts the existence of an integer $r_0 = r_0(X, L) \geq 1$ such that **for every integer $r \geq r_0$** there is a countable union Z_r of proper Zariski-closed subsets of $U_r(X)$, with nonempty complement, and

$$\varepsilon(X, L; \mathbf{p}) = \left(\frac{V}{r} \right)^{1/n} \quad (\mathbf{p} \in U_r(X) \setminus Z_r).$$

Equivalently, for the blow-up $\pi : Y \rightarrow X$ at such a tuple and its exceptional divisors E_i , the real divisor class

$$D_w = \pi^* L - w \sum_{i=1}^r E_i, \quad w = (V/r)^{1/n}$$

is nef and has $D_w^n = 0$. “Very general” here imposes countably many conditions; it is not replaced by a single nonempty Zariski-open set. The threshold depends on the fixed polarized variety, and equality holds at each point count beyond it, not merely asymptotically as $r \rightarrow \infty$.

Theorem 1.1 · pp. 1–2

This is the qualitative Nagata–Biran–Szemberg assertion in dimension $n \geq 3$. The theorem does not add a claim about every plane point count $r \geq 10$, surfaces, arbitrary characteristic, or weighted constants.

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Maximal Multipoint Seshadri Constants in Higher Dimensions](#). External inputs use the following keys.

- [OpenAI26b](#) OpenAI — Maximal Seshadri constants on arbitrary polarized surfaces: Methodological antecedent for node bounds, diagonal compression and primitive directions; the adaptation is proved here, without using surface maximality. September 23, 2026.

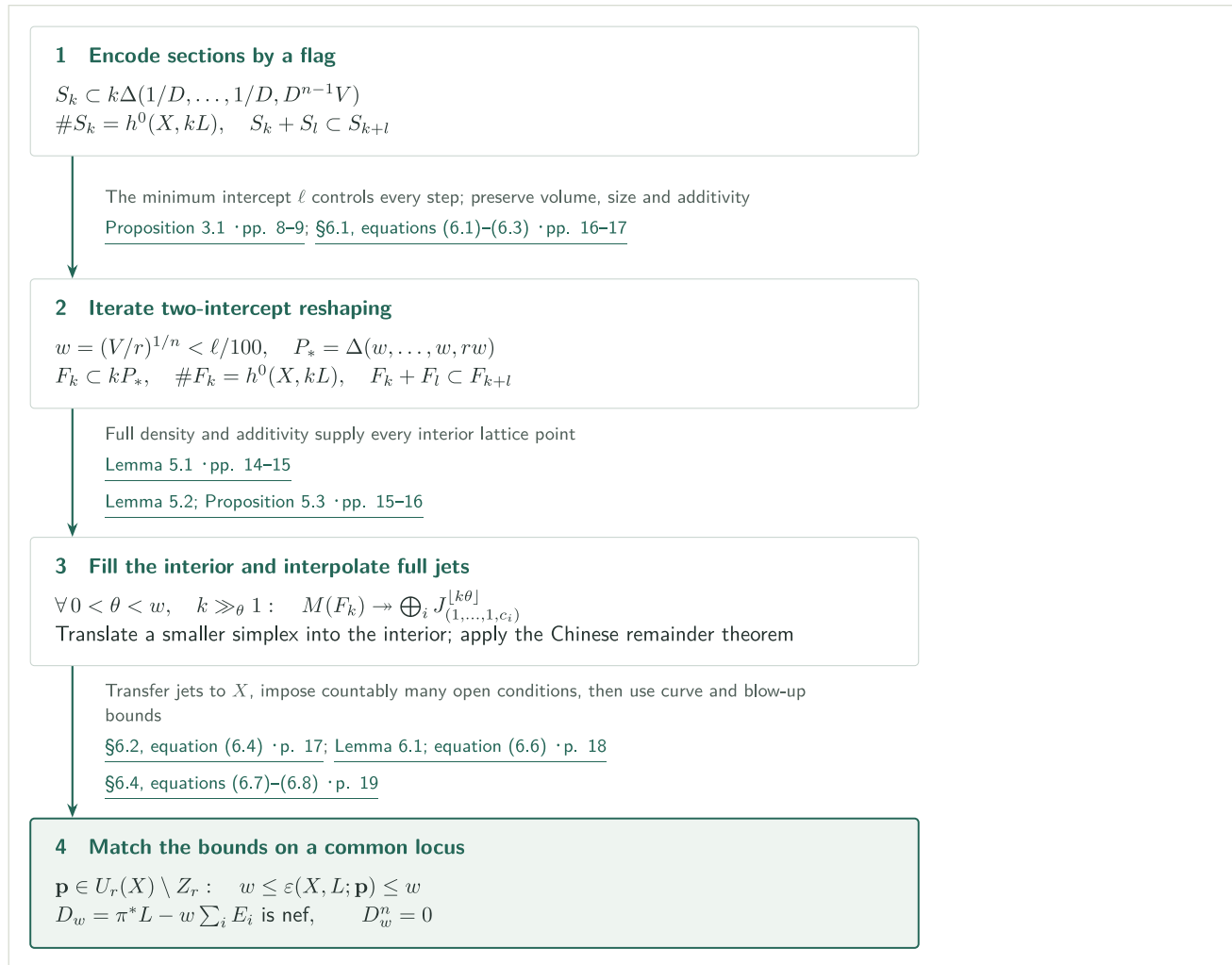
Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

The external paper listed above is a methodological antecedent. The finite-degree jet transfers, compression and interior filling used in the main argument are proved in this manuscript; surface maximality is not a premise.

2. Proof of Theorem 1.1 — From exponent sets to full jets

Proof target : Theorem 1.1 · pp. 1–2 See the hypotheses and conclusions.

Proof route. A complete-intersection flag supplies an additive exponent family. Repeating two-intercept reshaping $n - 1$ times gives an elongated simplex. Additivity and asymptotically full density supply its interior lattice points, and one-variable Hermite interpolation separates full jets. The next section explains reshaping; Section 4 explains the common very-general locus and the matching bounds. Write $\Delta(a_1, \dots, a_n) = \{x \in \mathbb{R}_{\geq 0}^n : \sum x_i/a_i \leq 1\}$ and let $M(F)$ denote the Laurent monomial space with exponents in F .



Overall route from additive supports to interpolation and Seshadri equality

Open figure ↗ TeX source ↓

1. A flag supplies the initial simplex of the correct volume

Choose D with DL very ample and a smooth integral complete-intersection flag cut out by $n - 1$ general sections. Divide sections successively along the flag and bound the final vanishing order by the degree on its last curve. A lexicographic initial exponent α then satisfies

$$D \sum_{i < n} \alpha_i + \frac{\alpha_n}{D^{n-1}V} \leq k.$$

Thus $S_k \subset k\Delta(1/D, \dots, 1/D, D^{n-1}V)$ and $\#S_k = h^0(X, kL) = Vk^n/n! + O(k^{n-1})$, while multiplicativity of least terms gives $S_k + S_l \subset S_{k+l}$. At each fixed degree, choose positive weights exposing the finitely many least terms and shrink local coordinates. Persistence of a nonzero maximal jet minor transfers surjectivity from $M(S_k)$ to $H^0(X, kL)$. The weights may depend on the degree, but the flag, order and additive support family remain fixed.

Proposition 3.1 · pp. 8–9

2. One threshold works for every subsequent point count

Let $\ell > 0$ be the smallest intercept of the initial simplex. Before fixing r , choose r_0 so that $w = (V/r)^{1/n} < \ell/100$ for every $r \geq r_0$. Proposition 4.1 replaces (A, B) by $(w, AB/w)$; carry the large new intercept into the next pair. Every input pair has $A, B \geq \ell$, so $w \max(3/A, 2/B) \leq 3w/\ell < 1/16$ and the proposition remains applicable. After $n - 1$ replacements, the product of intercepts is still V , and

$$P_* = \Delta(w, \dots, w, rw), \quad F_k \subset kP_* \cap \mathbb{Z}^n, \quad F_k + F_l \subset F_{k+l}, \quad \#F_k = h^0(X, kL).$$

No restriction to square point counts or another subsequence occurs.

§6.1, equations (6.1)–(6.3) · pp. 16–17

3. Turn density into prescribed exponents and interpolate

Since $\text{vol}(P_*) = V/n!$, only $o(k^n)$ lattice points are missing. Density alone does not guarantee a particular exponent. In Lemma 5.1, each $y \in kK \cap \mathbb{Z}^n$, for a fixed compact K inside the simplex, has $\gg k^n$ decompositions $y = y_1 + y_2$ into degrees $k_1 = \lfloor k/2 \rfloor$ and $k_2 = k - k_1$. The holes exclude only $o(k^n)$ candidates. Some pair lies in $F_{k_1} \times F_{k_2}$, and additivity puts y in F_k . No boundary saturation is asserted.

Lemma 5.1 · pp. 14–15

Given $0 < \theta < w$, choose $\theta/w < \lambda < 1$ and translate λP_* into the interior. For large k , its lattice points lie in F_k . The corresponding monomial translation acts by a unit on jet algebras at torus points. With $m = \lfloor k\theta \rfloor$, the set

$$E_{r,m} = \{(\beta, b) \in \mathbb{Z}_{\geq 0}^{n-1} \times \mathbb{Z}_{\geq 0} : |\beta| + b/r < m + 1\}$$

lies in $k\lambda P_*$. Choose distinct $c_i \in \mathbb{C}^*$. For each coefficient of $(x' - 1)^\alpha$, apply the Chinese remainder theorem modulo $(Y - c_i)^{m+1-|\alpha|}$. This prescribes arbitrary full jets through degree m at $(1, \dots, 1, c_i)$. Proposition 5.3 then gives surjectivity, which transfers backward through every reshaping step to some distinct tuple on X . That tuple may depend on θ and k .

Lemma 5.2; Proposition 5.3 · pp. 15–16

3. Proof of Proposition 4.1 — Reshape two real intercepts

Proof target : Proposition 4.1 · pp. 9–10

Proposition 4.1 — Two-intercept reshaping

Let $n \geq 2$ and let $A, B, a_3, \dots, a_n > 0$ be real. Suppose finite sets

$S_k \subset k\Delta(A, B, a_3, \dots, a_n) \cap \mathbb{Z}^n$ satisfy $S_k + S_l \subset S_{k+l}$ for all $k, l \geq 1$. Put $H = AB$ and $d = \max(3/A, 2/B)$. For every real $w > 0$ with $dw < 1/16$, there exist finite sets S'_k such that

$$S'_k \subset k\Delta(w, H/w, a_3, \dots, a_n) \cap \mathbb{Z}^n, \quad \#S'_k = \#S_k, \quad S'_k + S'_l \subset S'_{k+l}.$$

For every $k \geq 1, m \geq 0, r \geq 1$, if $M(S'_k)$ surjects onto jets through degree m at some distinct torus r -tuple, then $M(S_k)$ does so at some distinct torus r -tuple. The tuples need not coincide.

Proposition 4.1 · pp. 9–10

Fixing the remaining exponents as a label γ , the two-coordinate bound is $\kappa\Delta(A, B)$, where $\kappa = k - \sum_{i \geq 3} \gamma_i/a_i \geq 0$. Every operation retains this label and commutes with real scaling by κ .

1 Retain the other coordinates as labels

$$\kappa = k - \sum_{i \geq 3} \gamma_i / a_i, \quad f \in M(\kappa \Delta(A, B))$$

$$g = y^2 - x^2(1+x) = \xi\eta, \quad f = g^j f_0, \quad H = AB$$

Compare normalization degree with branch orders, then restore the factor g^j

[§4, equations \(4.3\)–\(4.13\) · pp. 10–11](#)

2 Bound least terms by the two branch orders

$$\nu_t(f) \in \kappa Q, \quad Q = \text{conv}\{0, (a, 0), (c, c), (0, b)\}$$

$$a + b = W = dH, \quad c = 1/d, \quad \text{area}(Q) = H/2$$

Shear and compress; choose a power of two j and odd i with $i/d + ja = R$

[§4, equations \(4.14\)–\(4.19\) · pp. 12–14](#)

3 Tune the width in a primitive direction

$$P = \text{conv}\{(-b, 0), (a, 0), (0, c)\}, \quad R = H/w$$

$$u = (i, j) \text{ is primitive, } \text{range}_P \det(u, \cdot) = R$$

Define both compressions by vanishing orders; transfer jets using a nonzero finite-degree minor

[Lemmas 2.2–2.7 · pp. 5–7; Proof of Proposition 4.1 · pp. 10–14](#)

4 Recover the width from area and compress twice

$$h_{\max} = H/R = w, \quad S'_k \subset k\Delta(w, H/w, a_3, \dots, a_n)$$

$$\#S'_k = \#S_k, \quad S'_k + S'_l \subset S'_{k+l}$$

Full-jet surjectivity after reshaping $\implies$ surjectivity before it

Two-intercept reshaping through a nodal bound and a primitive lattice direction

[Open figure ↗](#) [TeX source ↓](#)

1. A node gives a quadrilateral of equal area

Use $g = y^2 - x^2(1+x)$ and local coordinates with $g = \xi\eta$. Write $f = g^j f_0$. On the normalization $x = v^2 - 1, y = v(v^2 - 1)$, the residual polynomial has degree at most $(\kappa - jd)W$, where $W = \max(2A, 3B) = dH$. Its orders at the two branches $v = \pm 1$ show that its least $(1, t)$ -weight μ satisfies $(1 + 1/t)\mu \leq (\kappa - jd)W$. Restoring the least exponent (j, j) of g^j puts the least exponent of f in

$$\kappa Q, \quad Q = \text{conv}\{(0, 0), (a, 0), (c, c), (0, b)\}, \quad a = \frac{Wt}{1+t}, \quad b = \frac{W}{1+t}, \quad c = 1/d.$$

For $t > T = d^2 H$, this quadrilateral has area $H/2$. Least-term bases in each label and compatibility of products preserve both cardinality and additivity across degrees.

[§4, equations \(4.3\)–\(4.13\) · pp. 10–11](#)

2. Tune a primitive direction to the exact required width

Apply the shear $(\alpha, \beta) \mapsto (\alpha - \beta, \beta)$ and compress by vanishing orders in the second coordinate. The result lies in $P = \text{conv}\{(-b, 0), (a, 0), (0, c)\}$. Put $R = H/w$ and choose j as the largest power of two at most $R/(2W)$. The inequality $dw < 1/16$ makes the interval $(d(R - jW), d(R - ja_0))$, with $a_0 = WT/(1 + T)$, longer than 2. Choose an odd integer i in it and then $t > T$ with $i/d + ja = R$. The vector $u = (i, j)$ is primitive, and the range of $\det(u, \cdot)$ on P is exactly R .

Extend u to an integral basis. Horizontal slice lengths form a tent, whose area $H/2$ and height range R force its maximum width to be $H/R = w$. Horizontal and then vertical compression yield $\Delta(w, R)$. No rationality of the intercepts is required. Compression does not merely translate lattice points: it replaces each row of Laurent polynomials by its possible vanishing orders at $z = 1$.

§4, equations (4.14)–(4.19) · pp. 12–14

3. Separate fixed supports from degree-dependent jet transfers

The initial order, integral changes and exact vanishing orders defining compression are fixed in all degrees and preserve multiplication of labelled spaces. Small analytic scaling parameters and weights exposing tie-breaks may instead depend on the finite degree and test tuple. Convergence of derivatives and persistence of a nonzero finite minor transfer surjectivity backward for full jets, including the unchanged $n - 2$ variables. This distinction preserves both the graded structure and the interpolation implications under iteration.

Lemmas 2.2–2.7 · pp. 5–7

4. Proof of Theorem 1.1 — A very-general locus and matching bounds

Proof target : Theorem 1.1 · pp. 1–2 This completes the equality, nefness and zero top intersection in the main theorem.

1 Nonempty open conditions in each degree

$$O_{j,k} = \{\mathbf{p} : H^0(X, kL) \rightarrow \bigoplus_i J_{p_i}^{[k\theta_j]}(kL)\}$$

$$0 < \theta_j \uparrow w, \quad k \geq k_j$$

Local freeness of principal parts gives openness; the complex Baire property gives nonemptiness

§6.2, equation (6.4) · p. 17

2 Choose a common very-general tuple

$$Z_r = \bigcup_{j \geq 1} \bigcup_{k \geq k_j} (U_r(X) \setminus O_{j,k})$$

$$U_r(X) \setminus Z_r \neq \emptyset$$

Choose a section nonzero on the curve; bound local intersections by Hilbert–Samuel multiplicity

Lemma 6.1; equation (6.6) · p. 18

3 Bound intersections with every integral curve

$$kL \cdot C \geq [k\theta_j] \sum_i \text{mult}_{p_i} C$$

$$\varepsilon \geq \theta_j \quad (j \geq 1), \quad \theta_j \uparrow w$$

Check strict transforms and exceptional curves; zero top intersection alone does not imply nefness

§6.4, equations (6.7)–(6.8) · p. 19

4 Use the independent upper bound and obtain nefness

$$0 \leq t < \varepsilon : \quad D_t \text{ nef}, \quad 0 \leq D_t^n = V - rt^n$$

$$\varepsilon \leq w \implies \varepsilon = w, \quad D_w \text{ nef}, \quad D_w^n = 0$$

Countably many jet conditions and ordinary curve multiplicities give matching bounds

Open figure ↗ TeX source ↓

1. From special test tuples to a common very-general locus

For fixed integers k, m , surjectivity onto jets is Zariski open: smoothness makes principal parts locally free, and maximal minors define the surjectivity locus. Choose $\theta_j \uparrow w$. For each j , all $k \geq k_j$ yield nonempty open sets $O_{j,k} \subset U_r(X)$. Remove $Z_r = \bigcup_j \bigcup_{k \geq k_j} (U_r(X) \setminus O_{j,k})$. The complex space $U_r(X)(\mathbb{C})$ is locally compact Hausdorff and hence Baire; each proper algebraic closed subset has empty complex interior. Thus the complement is nonempty. The constructed special torus tuples are not themselves called general: nonzero rank conditions and a countable intersection provide that passage.

§6.2, equation (6.4) · p. 17

2. Full jets give the lower bound with ordinary multiplicities

Lemma 6.1 deduces $A \cdot C \geq m \sum_i \text{mult}_{p_i} C$ from separation of full jets through degree $m \geq 1$ by A . Lift a nonzero element of $\mathfrak{m}_C^m / \mathfrak{m}_C^{m+1}$ at one marked point and prescribe zero jets at the others. The resulting section does not vanish identically on C . Even when C is singular, for nonzero $f \in \mathfrak{m}_C^m$,

$$q \operatorname{length}(\mathcal{O}_C/(f)) = \operatorname{length}(\mathcal{O}_C/(f^q)) \geq mq e(\mathcal{O}_C) + O(1)$$

bounds the local intersection by m times the ordinary Hilbert–Samuel multiplicity. Take $A = kL$ and $m = \lfloor k\theta_j \rfloor$. First letting $k \rightarrow \infty$ and then $j \rightarrow \infty$ gives $\varepsilon \geq w$.

Lemma 6.1; equation (6.6) · p. 18

3. The blow-up upper bound and the nef-class conclusion

For any distinct tuple, put $D_t = \pi^*L - t \sum E_i$. Curves inside E_i have nonnegative intersection because $D_t|_{E_i} = t\mathcal{O}_{\mathbb{P}^{n-1}}(1)$. Every other curve Γ is the strict transform of an integral curve $C \subset X$, and

$$D_t \cdot \Gamma = L \cdot C - t \sum_i \operatorname{mult}_{p_i} C.$$

The manuscript explains $E_i \cdot \Gamma = \operatorname{mult}_{p_i} C$, including for singular C , using the blow-up of its maximal ideal and the Hilbert–Samuel multiplicity of the exceptional fibre. Hence D_t is nef for $0 \leq t < \varepsilon$, and

$$0 \leq D_t^n = V - rt^n$$

gives the universal upper bound $\varepsilon \leq w$. Combine it with Step 2 for equality. Substituting the lower bound into every curve intersection proves that D_w is nef, and the intersection computation gives $D_w^n = 0$. Conversely, nefness of D_w gives $\varepsilon \geq w$ on strict transforms; the upper bound then gives equality. Zero top self-intersection alone is not used to infer nefness.

§6.4, equations (6.7)–(6.8) · p. 19

5. Which papers supply which steps

The surface paper is a methodological antecedent, not a direct input through its maximality theorem. Its final test uses jet **injectivity** on finite torus subgroups, whereas this manuscript reshapes polynomial spaces to obtain full-dimensional jet **surjectivity**. Section 1.1 explicitly states that all arguments needed for the adaptation are proved here.

§1.1; §4 · pp. 2, 9

Source and result

Surface paper, Proposition 3.2

Location in this manuscript

Proposition 4.1, pp. 10–11

Relation and role

Generalization of method: two-branch node bounds for labelled polynomial spaces

Checking scope

Antecedent statement and adaptation proof compared

Source and result

Surface paper, Proposition 3.4

Location in this manuscript

Proposition 4.1, p. 12

Relation and role

Generalization of method: diagonal slice compression

Checking scope

Injectivity distinguished from surjectivity

Source and result

Surface paper, Lemma 4.3

Location in this manuscript

Proposition 4.1, pp. 12–14

Relation and role

Generalization of method: primitive directions tuned continuously to a real width

Checking scope

Threshold and choice of direction compared

[OpenAI26b, Proposition 3.2, p. 8] · [OpenAI26b, Proposition 3.4, pp. 9–10] ·
[OpenAI26b, Lemma 4.3, p. 14]

Nagata, Biran, Küchle and Roé–Ross are cited for background and are not placed on proof-dependency arrows here. Demailly, Ito, Okounkov-body and semigroup theory, and Dumnicki provide context for the methods; the connections explained above are followed through the manuscript's own lemmas. Complete proofs of these external historical results and unlisted cross-catalogue connections remain uninvestigated.

6. Return to the sources

- [Theorem 1.1 · pp. 1–2](#) — Quantifiers, ground field, ordinary multipoint constant and nef-class conclusion.
- [Proof of Proposition 4.1 · pp. 10–14](#) — The node, compression and primitive-direction mechanism with real intercepts.
- [Lemma 5.1 · pp. 14–15](#) — Why additivity across degrees upgrades density to interior saturation.
- [Lemma 5.2; Proposition 5.3 · pp. 15–16](#) — Full-jet interpolation in the elongated simplex and translation into the interior.
- [§6.2, equation \(6.4\) · p. 17](#) — From existence in each degree to a common very-general tuple.
- [Lemma 6.1; equation \(6.6\) · p. 18](#) — The lower bound using ordinary multiplicities, including singular curves.
- [§6.4, equations \(6.7\)–\(6.8\) · p. 19](#) — The blow-up upper bound and the nefness equivalence.

Editorial checking covered the full conclusions and quantifiers of Theorem 1.1, the principal proof passages in Sections 2–6, and the three surface-paper statements cited above together with their uses as methodological antecedents. It followed the complete-intersection flag, reshaping, interior filling, interpolation, countable common locus, ordinary curve multiplicities and blow-up bounds. Recursive verification of external historical proofs, specialist review and formal verification were not performed. The authors' theorem claim is distinct from this limited comparison.

Maximal multipoint Seshadri constants in positive characteristic

Formal jet transfer and lattice filling attain the volume bound at a geometric generic tuple

The manuscript claims that the ordinary, equally weighted multipoint Seshadri constant on a smooth polarized projective variety in positive characteristic is maximal for every sufficiently large point count. It turns the section ring into graded exponent sets, reshapes two intercepts using a node, and constructs the lower bound by interpolation. The tuple is geometric generic; no choice of very general closed points over a countable field is required.

Manuscript author: OpenAI · 2026-10-05 · 17 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Maximality at the geometric generic tuple

Let k be algebraically closed of characteristic $p > 0$, let X/k be smooth, integral and projective of dimension $n \geq 3$, and let L be ample. Let $U_r \subset X^r$ be the configuration space of ordered distinct points, put $K_r = \overline{k(U_r)}$, and let $p_1, \dots, p_r \in X(K_r)$ be the points defined by its geometric generic point. The constant is

$$\varepsilon_r(X, L) = \inf_C \frac{L_{K_r} \cdot C}{\sum_{i=1}^r \text{mult}_{p_i} C},$$

where C ranges over integral curves meeting at least one marked point; multiplicity outside C is zero. All weights are one. There is no division by r in the denominator and no replacement of curves by higher-dimensional cycles.

The authors assert that an integer $r_0 = r_0(X, L)$ exists such that every integer $r \geq r_0$ satisfies

$$\varepsilon_r(X, L) = \left(\frac{L^n}{r} \right)^{1/n}.$$

Remark 6.3 gives the same conclusion for $n = 2$. This does not assert maximality for every point count below r_0 , or for arbitrary tuples of k -rational points.

Theorem 1.1 · pp. 1–2

Remark 6.3 · p. 16

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Maximal multipoint Seshadri constants in positive characteristic](#). External inputs use the following keys.

- [Stacks0H90](#) The Stacks Project Authors — The Stacks Project — Lemma 10.133.9 (Tag 0H90): Principal parts via the diagonal ideal. Used at §2 p4, Lemma 2.4 pp5–6. HTML, accessed 2026-10-15; unpaginated.
- [Stacks0H8Z](#) The Stacks Project Authors — The Stacks Project — Lemma 10.133.8 (Tag 0H8Z): Base change for principal parts. Used at §2 p4, Proposition 5.3 p14. HTML, accessed 2026-10-15; unpaginated.
- [Stacks0FD6](#) The Stacks Project Authors — The Stacks Project — Lemma 33.47.3 (Tag 0FD6): Smoothness of general hyperplane sections of an embedding. Used at Proposition 3.1 p7. HTML, accessed 2026-10-15; unpaginated.
- [Stacks0G4G](#) The Stacks Project Authors — The Stacks Project — Remark 37.32.4 (Tag 0G4G): Smooth irreducible hyperplane sections in dimension at least two. Used at Proposition 3.1 p7. HTML, accessed 2026-10-15; unpaginated.
- [Stacks0AZV](#) The Stacks Project Authors — The Stacks Project — Definition 43.15.1 (Tag 0AZV): Hilbert–Samuel multiplicity as the leading coefficient of length. Used at Lemma 6.1 p15; Lemma 6.2 p16. HTML, accessed 2026-10-15; unpaginated.
- [OpenAI26b](#) OpenAI — Maximal Seshadri constants on arbitrary polarized surfaces: Methodological background; required statements are reproved here rather than importing its main theorem. 2026-09-23.
- [OpenAI26c](#) OpenAI — Maximal Multipoint Seshadri Constants in Higher Dimensions: Methodological background; required statements are reproved here rather than importing its main theorem. 2026-10-05.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

The Stacks references are unpaginated HTML statements identified by their tags and result names. The diagrams cite statements reproved in this manuscript. The surface and higher-dimensional companions are methodological sources; their maximality theorems are not premises of this proof.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : Theorem 1.1 · pp. 1–2

Proof route. The lower bound in Theorem 1.1 passes through a flag, two-intercept reshaping, and interior filling and interpolation. Put $V = L^n$ and $w = (V/r)^{1/n}$. Intersection theory on the blow-up supplies a separate upper bound.

1 From a flag to graded exponents

$$S_N \subset N\Delta(1/D, \dots, 1/D, D^{n-1}V)$$

$$\#S_N = h^0(X, NL) = VN^n/n! + o(N^n)$$

Use the flag bound and reshape two intercepts $n - 1$ times
[Proposition 3.1 · pp. 7–8](#) [Proposition 4.1 · pp. 8–11](#)

2 Preserve intercept product and cardinality

$$F_N \subset N\Delta(w, \dots, w, rw), \quad w = (V/r)^{1/n}$$

Keep gradedness and backward jet transfer

Density and additivity, CRT, and a nonempty open locus
[Lemma 5.1 · p. 12](#) [Proposition 5.3 · pp. 13–14](#)

3 Interior filling gives jet separation

$$0 < \theta < w, \quad m = \lfloor N\theta \rfloor$$

$$H^0(X_{K_r}, NL_{K_r}) \twoheadrightarrow \bigoplus_i J_{p_i}^m(NL_{K_r})$$

Local lengths give the lower bound; the blow-up gives the upper bound
[Lemma 6.1 · p. 15](#) [Lemma 6.2 · pp. 15–16](#)

4 Match the lower and upper bounds

$$\varepsilon_r \geq \lim_{\theta \uparrow w} \lim_{N \rightarrow \infty} \frac{\lfloor N\theta \rfloor}{N} = w$$

$$D_s \text{ nef} \implies 0 \leq D_s^n = V - rs^n \implies \varepsilon_r \leq w$$

$$\varepsilon_r = w, \quad D_w \text{ nef}, \quad D_w^n = 0$$

Exponent sets yield jets and the matching lower and volume upper bounds

[Open figure ↗](#) [TeX source ↓](#)

1. Reshape intercepts while keeping one graded structure

Choose D with DL very ample and a smooth complete-intersection flag. At each Bertini step the variety being cut has dimension at least two, and the morphism induced by DL is an embedding. Lexicographically least exponents in adapted coordinates give

$$S_N \subset N\Delta(1/D, \dots, 1/D, D^{n-1}V), \quad S_N + S_M \subset S_{N+M}, \quad \#S_N = h^0(X, NL) = \frac{V}{n!}N^n + o(N^n).$$

Here $\Delta(a_1, \dots, a_n) = \{x_i \geq 0 : \sum_i x_i/a_i \leq 1\}$. The containment follows by successively dividing by flag equations and then bounding the final vanishing order by the degree of a zero divisor on the terminal curve. Apply Proposition 4.1 $n - 1$ times, preserving the product of intercepts and the cardinality in each degree, to reach $\Delta(w, \dots, w, rw)$. At every stage, jet separation after transformation transfers back to the preceding space.

Proposition 3.1 · pp. 7–8

Proposition 4.1 · pp. 8–11

2. A lower bound from jets and a separate volume upper bound

For each fixed $0 < \theta < w$, Proposition 5.3 supplies jets through order $\lfloor N\theta \rfloor$ at the geometric generic tuple for every sufficiently large N . In Lemma 6.1 one prescribes a nonzero class in $\mathfrak{m}^m/\mathfrak{m}^{m+1}$ of one curve local ring, and zero jets at the other points. The resulting section vanishes to order at least m at every marked point while remaining nonzero on the curve. The local inequality $\text{length}(R/(f)) \geq me(R)$ then gives, including for singular integral curves,

$$NL_{K_r} \cdot C \geq \lfloor N\theta \rfloor \sum_i \text{mult}_{p_i} C.$$

Let $N \rightarrow \infty$ and then $\theta \uparrow w$ to obtain $\varepsilon_r \geq w$.

Separately, for rational $0 \leq s < \varepsilon_r$, consider $D_s = \pi^*L_{K_r} - s \sum E_i$ on the blow-up. Curves inside E_i are handled by $D_s|_{E_i} = s\mathcal{O}_{\mathbf{P}^{n-1}}(1)$; other curves by $E_i \cdot \tilde{C} = \text{mult}_{p_i} C$. Thus D_s is nef and $0 \leq D_s^n = V - rs^n$, giving $\varepsilon_r \leq w$. The upper bound alone does not establish maximality: it must meet the lower bound constructed from jets.

Proposition 5.3 · pp. 13–14

Lemma 6.1 · p. 15

Lemma 6.2 · pp. 15–16

3. The boundary class and dimension two

Closedness of the nef cone and equality of the bounds imply that D_w is nef with $D_w^n = 0$. This connection uses the curve definition and the blow-up intersection calculation, rather than only a numerical correspondence. Remark 6.3 observes that the flag, reshaping, interpolation and intersection arguments all apply when $n = 2$, with only one reshaping step. Strict positivity on every curve is not asserted here.

Proof of Theorem 1.1; Remark 6.3 · p. 16

3. Proposition 4.1 — Two-intercept reshaping through a node

Proof target : Proposition 4.1 · pp. 8–11

Proposition 4.1 — Preserve cardinality, products and jet transfer

Let $n \geq 2$, let $A, B, a_3, \dots, a_n > 0$, and let a graded family of finite sets $S_N \subset \mathbf{Z}^n$ satisfy $S_N \subset N\Delta(A, B, a_3, \dots, a_n)$. Put $H = AB$ and $d = \max\{3/A, 2/B\}$. For every $w > 0$ with $dw < 1/16$, there is a graded family S'_N such that

$$S'_N \subset N\Delta(w, H/w, a_3, \dots, a_n), \quad \#S'_N = \#S_N,$$

and the transformation has backward jet transfer in every degree. This means that separation of order- m jets by $M(S'_N)$ at some r distinct torus points implies the same separation by $M(S_N)$ at some tuple. Here $M(S)$ is the span of Laurent monomials with exponents in S . For $n = 2$ there are no remaining intercepts.

Proposition 4.1 · pp. 8–11

1 Keep two nodal branches even in characteristic two

$$g = y^2 + xy - x^3 = \xi\eta, \quad x = v(v+1), \quad y = v^2(v+1)$$

Bound the sum of orders at $v = 0, -1$ by the degree

Use the two branch valuations and formal substitutions
[Lemma 4.2 · p. 9](#) [Lemmas 2.2–2.4 · pp. 4–6](#)

2 Put least exponents in a fixed-area quadrilateral

$$\nu_t(f) \in \kappa Q, \quad \text{area}(Q) = H/2$$

Realize finite initial terms by integral weights and transfer jet rank back

Use $dw < 1/16$ to fix t and the primitive direction in every degree
[Lemma 4.3 · pp. 10–11](#)

3 Prescribe height in a primitive direction

$$u = (i, j) \in \mathbb{Z}^2, \quad \gcd(i, j) = 1, \quad q_2 - q_0 = H/w$$

An integral basis change after shear and compression

Area gives the maximal slice w ; compressions preserve counts and jet transfer
[Lemmas 2.5–2.6 · p. 6](#) [Proposition 4.1 · pp. 8–11](#)

4 Horizontal and vertical compression reshape intercepts

$$S'_N \subset N\Delta(w, H/w, a_3, \dots, a_n), \quad \#S'_N = \#S_N$$

Obtain interval bounds, without yet claiming filling of intermediate integers

Two distinct branches even in characteristic two, followed by lattice compression

[Open figure ↗](#) [TeX source ↓](#)

1. Bound exponents using the two branches

Use $g = y^2 + xy - x^3$. Hensel lifting the simple roots $0, -1$ of $Z^2 + Z - x$ gives formal coordinates with $g = \xi\eta$. The roots remain distinct when $p = 2$, since $-1 = 1 \neq 0$. Under $x = v(v+1)$, $y = v^2(v+1)$, the sum of the orders at $v = 0, -1$ of a factor not divisible by g is bounded by its degree. Consequently its least exponent for the order $(\alpha + t\beta, \beta)$ lies in a quadrilateral Q of area $H/2$.

Formal coordinates alone do not transfer interpolation. Lemma 2.3 chooses common integral weights realizing the finitely many initial terms in each degree. Lemma 2.4 substitutes $z_a = s^{\lambda_a}(Q_a + \delta_a)$ into truncated jet rings. After column normalization, the evaluation matrix specializes at $s = 0$ to the monomial matrix. A nonzero maximal minor remains nonzero over $k((s))$. Setting $\delta = 0$ defines actual $k((s))$ -points, and the substitution factors through their order- m jet algebras. Surjectivity of the matrix and equality of dimensions make these algebra maps

isomorphisms. Repeated centers would contradict surjectivity, so separation occurs at distinct points and the corresponding open locus is nonempty. The auxiliary weights may depend on the degree, but the formal coordinates and monomial order defining the graded family stay fixed.

Lemma 4.2 · p. 9

Lemmas 2.2–2.4 · pp. 4–6

2. Fix the primitive direction and compressions in every degree

A shear and compression turn Q into a triangle of the same area. Lemma 4.3 chooses t and a primitive vector $u = (i, j)$ so that the height range under $z \mapsto \det(u, z)$ is exactly H/w . Choosing j as a power of two and i as an odd integer in a suitable open interval provides both primitivity and the required real intercept. After an integral basis change, horizontal and vertical compressions give the bounding triangle $\Delta(w, H/w)$. The other coordinates remain labels.

Compression does not fill every integer in a slice. In characteristic p , the orders of $\langle 1, z^p \rangle$ at $z = 1$ are $0, p$, leaving gaps. Lemma 2.6 preserves the interval bound, cardinality, gradedness and backward transfer. Filling every interior lattice point requires the separate argument in the next section.

Lemma 4.3 · pp. 10–11

Lemmas 2.5–2.6 · p. 6

4. Proposition 5.3 — From interior filling to geometric generic jets

Proof target : Proposition 5.3 · pp. 13–14

Proposition 5.3 — Jets at the geometric generic tuple

Let k be algebraically closed of positive characteristic, let X/k be smooth, integral and projective of dimension $n \geq 2$, let L be ample, and put $V = L^n$. There is $r_0(X, L)$ such that for each integer $r \geq r_0$ and real $0 < \theta < (V/r)^{1/n}$ there is $N_0(r, \theta)$ with the following property. For every $N \geq N_0$, $H^0(X, NL)$ separates jets through order $\lfloor N\theta \rfloor$ at some r distinct k -points, possibly depending on N . After base change to K_r , the same separation holds at the fixed geometric generic tuple $p_1, \dots, p_r$.

Proposition 5.3 · pp. 13–14

1 Density of graded supports

$$F_N \subset NP_*, \quad \#F_N = \text{vol}(P_*)N^n + o(N^n)$$

$$F_N + F_M \subset F_{N+M}, \quad P_* = \Delta(w, \dots, w, rw)$$

Decompose into nearly equal degrees and use additivity
[Lemma 5.1 · p. 12](#)

2 All lattice points in a compact interior region

$$K \Subset \text{int}(P_*) \implies NK \cap \mathbf{Z}^n \subset F_N$$

$o(N^n)$ missing points cannot exclude cN^n decompositions

Contract and translate to include E_m ; apply CRT to interpolate jets
[Lemma 5.2 · p. 13](#) [Proposition 5.3 · pp. 13–14](#)

3 Multipoint interpolation in an elongated simplex

$$E_m = \{(\beta, b) : |\beta| + b/r < m + 1\}, \quad m = \lfloor N\theta \rfloor$$

$$q_i = (1, \dots, 1, c_i), \quad c_i \neq c_j, \quad c_i \neq 0$$

Backward jet transfer, openness of evaluation rank, and base change
[Lemmas 2.2–2.4 · pp. 4–6](#) [Proposition 5.3 · pp. 13–14](#)

4 Pass to the same geometric generic tuple

$$\exists (x_1, \dots, x_r) \text{ with jets} \implies U_{N,m} \subset U_r \text{ nonempty open}$$

$$\text{Spec } K_r \longrightarrow U_{N,m}, \quad K_r = \overline{k(U_r)}$$

No closed point in a countable intersection is chosen

All interior lattice points, Chinese remainder interpolation and openness

[Open figure ↗](#) [TeX source ↓](#)

1. Turn density into the presence of individual lattice points

Let ℓ be the smallest initial intercept and choose r_0 so that $w < \ell/100$. At each reshaping step the processed intercepts A, B are at least ℓ , hence $w \max(3/A, 2/B) < 1/16$. The final family F_N lies in $P_* = \Delta(w, \dots, w, rw)$ and has $\#F_N = \text{vol}(P_*)N^n + o(N^n)$.

Lemma 5.1 combines this density with additivity. Each lattice point y in a fixed compact interior region has at least cN^n decompositions $y = y_1 + y_2$ in nearly equal degrees. Missing lattice points exclude only $o(N^n)$ decompositions. One remains with both summands in the family, so $y \in F_N$. Density alone is not being identified with pointwise filling.

[Lemma 5.1 · p. 12](#)

2. Interpolate in the elongated simplex and transfer to a nonempty open set

For $m = \lfloor N\theta \rfloor$, set

$$E_m = \{(\beta, b) \in \mathbf{Z}_{\geq 0}^{n-1} \times \mathbf{Z}_{\geq 0} : |\beta| + b/r < m + 1\}.$$

Choose distinct nonzero $c_i \in k$ and express arbitrary jets at $q_i = (1, \dots, 1, c_i)$ as a sum of terms $(x' - 1)^\alpha h_\alpha(Y)$. The Chinese remainder theorem supplies h_α with $\deg h_\alpha < r(m - |\alpha| + 1)$, so all monomials lie in E_m . Neither factorials nor ordinary derivatives are required.

Choose $\theta/w < \lambda < 1$ and a small positive vector v such that $v + \lambda P_* \subset \text{int } P_*$. An integral translate puts E_m inside the filled region; multiplication by a Laurent monomial is a unit on the torus and preserves jet separation. Backward transfer returns to some tuple on X . For fixed r, θ, N , surjectivity of evaluation into the principal-parts bundles is an open rank condition on U_r .

Lemma 5.2 · p. 13

Proposition 5.3 · pp. 13–14

3. Impose every condition at the same geometric generic tuple

The generic point of integral U_r belongs to each nonempty open locus, and base change for principal parts respects jet evaluation. Thus the same K_r -tuple satisfies all required separations. No k -point needs to be chosen in a countable intersection of open sets, so countable algebraically closed fields are included. This conclusion must not be replaced by a statement for arbitrary general closed points over k .

Proposition 5.3 · pp. 13–14

5. Which papers supply which steps

Source

Stacks 0FD6 / 0G4G (reference guide above)

Used at

Proposition 3.1, pp.7–8

Role and hypotheses

An embedding and dimension at least two give a smooth integral flag

Checking scope

Statements and application conditions checked

Source

Stacks 0H90 / 0H8Z

Used at

§2 p.4; Proposition 5.3 p.14

Role and hypotheses

Principal parts, base change and the open evaluation condition

Checking scope

Statements and use checked; local freeness also uses smoothness and the symmetric-power filtration stated in the manuscript

Source

Stacks 0AZV

Used at

Lemma 6.1 p.15

Role and hypotheses

Multiplicity as the leading coefficient of local length

Checking scope

Definition and the manuscript's local-length calculation checked

Source

Surface and higher-dimensional companions

Used at

§1.1 p.2; §4 p.8

Role and hypotheses

Methodological sources for nodes, graded supports and reshaping

Checking scope

No direct dependence on their main theorems; the required results are reproved here

6. Return to the sources

- [Lemmas 2.2–2.4](#) · pp. 4–6 — Transfer from formal substitutions to jet separation.
- [Proposition 4.1](#) · pp. 8–11 — Node, primitive lattice direction, and compressions.
- [Proposition 5.3](#) · pp. 13–14 — From density to all interior lattice points, then to geometric generic jets.
- [Proof of Theorem 1.1; Remark 6.3](#) · p. 16 — Matching bounds and the dimension-two extension.

The editorial reading covered the proof passages in §§2–6: formal transfer, the two nodal branches, primitive lattice directions, interior filling, interpolation, passage to the generic tuple, and matching bounds. The Stacks statements and their application conditions were checked. This is not a recursive verification of their proofs, expert peer review, or formal verification. A complete independent verification of the construction in all degrees and an exhaustive investigation of connections with other catalogues were not performed. The authors' maximality claim remains distinct from this editorial checking scope.