

OpenAI Math Digest

CATALOGUE 037

The sharp ordinary-double-point volume gap

Catalogue synthesis & 2 proof overviews

Independent reading guide · English edition

Edition: 2026-10-12

Created by: Masataka Iwai · Osaka University

AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 2 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

00 · Catalogue synthesis	3
01 · The ordinary-double-point gap in every dimension	9
02 · The normalized-volume gap in dimension four	28

Original-manuscript snapshot: `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`

Catalogue 037 · Synthesis

[Read on the website ↗](#)

CATALOGUE 037

The sharp ordinary-double-point volume gap

The two papers in Catalogue 037 study normalized volumes of boundary-zero complex algebraic klt singularities. The all-dimensional paper uses the fourfold paper as an induction base and claims the upper bound $2(n-1)^n$ for singular n -dimensional germs, with equality precisely for an analytic ordinary double point (ODP).

1. The two papers in official order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 037 · 2026-10-12 checked source snapshot

#

01

Manuscript and overview

The ordinary-double-point gap in every dimension →

Short name: ODP gap in every dimension

September 24, 2026 · 57 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

A singular boundary-zero complex algebraic klt n -fold germ ($n \geq 2$) has normalized volume $\leq 2(n-1)^n$, with equality iff analytic ODP; global bounds follow for K -semistable $\mathbb{Q}$ -Fano varieties.

[Theorem 1.1; Corollary 1.2 · p. 3](#) ↗

Role and available overview

Induct from the fourfold base through stable degeneration, two stabilizer regimes, and a same-dimensional supremum estimate; recover equality at the original germ.

[Read the proof overview →](#)

#

02

Manuscript and overview

The normalized-volume gap in dimension four →

Short name: Fourfold volume gap

September 24, 2026 · 20 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

A singular boundary-zero complex algebraic klt fourfold germ has normalized volume ≤ 162 , with equality iff analytic ODP, without an isolatedness or lci assumption.

[Theorem 1.1 · p. 1](#) ↗

Role and available overview

Force embedding dimension ≤ 5 from large volume via a terminal Gorenstein threefold section and the established lci gap; supply the all-dimensional induction base.

[Read the proof overview](#) →

2. From the fourfold input to all dimensions

The official order lists the all-dimensional paper first, but the direct proof input runs in the opposite direction. The keys in the diagram refer to the following fixed manuscripts. The reference guides before the figures in each article specify the statements, roles, and versions of external inputs.

- [OpenAI26a] OpenAI — The ordinary-double-point gap in every dimension. September 24, 2026. The local gap in all dimensions, equality, and the global corollary. [原典 / Original](#).
- [OpenAI26b] OpenAI — The normalized-volume gap in dimension four. September 24, 2026. The fourfold induction base. [原典 / Original](#).

02 The four-dimensional base case**The normalized-volume gap in dimension four**

$\widehat{\text{vol}}(x, X) \leq 162$; equality precisely for an analytic ODP.

[\[OpenAI26b, Theorem 1.1, p. 1\]](#)

Use the fourfold bound and equality as the induction base

[\[OpenAI26a, Theorem 2.4, p. 9\]](#)

$n = 2, 3$ are known; a new induction step treats $n \geq 5$

01 The local gap in every dimension**The ordinary-double-point gap in every dimension**

$n \geq 2$, $\widehat{\text{vol}}(x, X) \leq 2(n-1)^n$; equality iff analytic ODP.

[\[OpenAI26a, Theorem 1.1, p. 3\]](#)

Apply the local-to-global volume comparison at a singular point

[\[OpenAI26a, §8.3, p. 53\]](#)

The comparison theorem and smooth-case inputs are listed in article 01

Global consequence in the same all-dimensional paper

Singular K -semistable $\mathbb{Q}$ -Fano varieties:

$$(-K_V)^n \leq 2 \left(\frac{n^2 - 1}{n} \right)^n < 2n^n.$$

[\[OpenAI26a, Corollary 1.2, p. 3\]](#)

The fourfold base, all-dimensional induction, and the consequence for singular Fano varieties

[Open figure ↗](#) [TeX source ↓](#)

The fourfold paper forces embedding dimension at most **5** from a large normalized volume, reduces to a hypersurface, and invokes the established lci volume theorem. The all-dimensional paper imports this fourfold bound and equality characterization as Theorem 2.4. Together with the known cases $n = 2, 3$, it starts the induction for $n \geq 5$. An intermediate comparison involving singularities of the same dimension uses the supremum M_n of their volume densities, without assuming the gap still to be proved.

[\[OpenAI26b, Theorem 1.1, p. 1\]](#) [\[OpenAI26a, Theorem 2.4, p. 9\]](#)

[\[OpenAI26a, Theorem 7.7; §8.2, pp. 48, 52\]](#)

The global volume bound for singular K -semistable $\mathbb{Q}$ -Fano varieties also requires a local-to-global comparison. The paper does not claim sharpness or classify equality for this smaller singular bound. Its separate bound $2n^n$ for $V \not\cong \mathbb{P}^n$, including equality, combines the strict singular estimate with an external theorem for smooth varieties.

[\[OpenAI26a, Corollary 1.2, p. 3\]](#) [\[OpenAI26a, §8.3, p. 53\]](#)

Source → application

Fourfold paper, Theorem 1.1 → all-dimensional paper, Theorem 2.4 and §8.2

Relation

Direct proof input

Input and place of use

The bound and analytic equality characterization for boundary-zero complex algebraic klt fourfold germs supply the $n = 4$ induction base.

Source → application

All-dimensional paper, Theorem 1.1 → its Corollary 1.2 and §8.3

Relation

Use in a consequence

Input and place of use

Apply the local bound at a singular closed point and then the local-to-global comparison to bound the anticanonical volume.

The all-dimensional statement includes the fourfold statement, but this does not mean that the fourfold proof depends on the all-dimensional paper. The [fourfold article](#) explains the hypersurface reduction; the [all-dimensional article](#) explains induction, recovery of equality, and the global corollary.

3. Sources and checking scope

The official overview calls this catalogue The sharp ordinary-double-point volume gap; CONTENTS uses The ordinary-double-point volume gap. Its field is Algebraic and complex geometry. The order and manuscripts are fixed at commit

`fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`. Both manuscripts are dated September 24, 2026, and have 57 and 20 pages, including references.

[OpenAI Research Catalog · Catalog 037 · p. 5](#)

The articles describe the authors' claims and proof strategies. Editorial checks compared the main hypotheses and conclusions, selected proof passages, and the inter-paper input and application against the sources. They include checking by someone other than each article's writer, but do not constitute expert peer review or independent verification of every proof. Full

details of the semigroup and filtration constructions, analytic estimates, and proofs of external theorems remain outside this checking scope; each article and its source record give finer limitations.

The ordinary-double-point gap in every dimension

Stable degeneration, saturation of jet semigroups, and a supremum argument in the same dimension

The manuscript propagates the ordinary-double-point (ODP) volume gap for boundary-zero complex algebraic klt singularities from dimensions at most four to every dimension. It degenerates a high-volume germ to an isolated Gorenstein K-polystable cone, then splits according to stabilizers of integral gradings. When the construction produces another singularity of the same dimension, it closes an inequality for the supremum of all singular densities instead of invoking an unavailable induction hypothesis.

Manuscript author: OpenAI · September 24, 2026 · 57 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — The ODP volume gap and equality classification

Let $n \geq 2$ and let x be a singular closed point of a normal complex algebraic n -fold X . Assume K_X is $\mathbb{Q}$ -Cartier and X is klt with zero boundary near x . Then

$$\widehat{\text{vol}}(x, X) \leq 2(n-1)^n,$$

with equality if and only if there is a complex analytic isomorphism

$$(X, x)^{\text{an}} \simeq (\{z_0^2 + \cdots + z_n^2 = 0\} \subset \mathbb{C}^{n+1}, 0).$$

No isolatedness, complete-intersection or smoothability assumption is made.

For $R = \mathcal{O}_{X,x}$ and $\mathfrak{a}_t(v) = \{f : v(f) \geq t\}$, the normalization is

$$\text{vol}(v) = \lim_{t \rightarrow \infty} \frac{n! \ell(R/\mathfrak{a}_t(v))}{t^n}, \quad \widehat{\text{vol}}(x, X) = \inf_{v \text{ centered exactly at } x} A_X(v)^n \text{vol}(v).$$

Here A_X is log discrepancy: the exceptional divisor of the ordinary blowup of a smooth n -fold point has discrepancy n . A valuation of infinite discrepancy contributes $+\infty$. Write $d(x, X) = \widehat{\text{vol}}(x, X)/n^n$ and $p_n = 2((n-1)/n)^n$.

Theorem 1.1 · p. 3 §2.1; (2.1) · pp. 6–7

Corollary 1.2 — Global volumes of K-semistable Fano varieties

Let V be an n -dimensional K-semistable $\mathbb{Q}$ -Fano variety, $n \geq 2$. If V is singular, then

$$(-K_V)^n \leq 2 \left(\frac{n^2 - 1}{n} \right)^n < 2n^n.$$

If $V \not\cong \mathbb{P}^n$, then $(-K_V)^n \leq 2n^n$, with equality precisely for a smooth quadric $Q^n \subset \mathbb{P}^{n+1}$ or $\mathbb{P}^1 \times \mathbb{P}^{n-1}$. These descriptions coincide when $n = 2$. **The smaller singular bound is not asserted to be sharp and is not accompanied by a classification of its equality cases.**

Corollary 1.2 · p. 3

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to ODP gap in every dimension. External inputs use the following keys.

- [Gra15](#) Robin Graham (lecturer); Josh Swanson (initial draft); Debbie Matthews (editor) — (Compact) Lie Groups and Representation Theory Lecture Notes: Irreducibility of the graded harmonic-polynomial representations in the ODP minimum. University of Washington lecture notes, 2015-05-29.
- [XZ25](#) Chenyang Xu; Ziquan Zhuang — Stable degenerations of singularities: Minimizers, finite generation and degeneration of klt germs to K-semistable and K-polystable cones. arXiv:2205.10915v2, 2025-01-17.
- [XZ21](#) Chenyang Xu; Ziquan Zhuang — Uniqueness of the minimizer of the normalized volume function: Uniqueness, automorphism invariance and the finite degree formula. arXiv:2005.08303v1, 2020-05-17.
- [LWX21](#) Chi Li; Xiaowei Wang; Chenyang Xu — Algebraicity of the metric tangent cones and equivariant K-stability: Special degeneration from a K-semistable to a K-polystable cone. arXiv:1805.03393v2, 2021-01-08.
- [BL21](#) Harold Blum; Yuchen Liu — The normalized volume of a singularity is lower semicontinuous: Lower semicontinuity used to exclude singular puncture points by induction. arXiv:1802.09658v2, 2019-08-15; manuscript date 2019-08-16.
- [LX19](#) Yuchen Liu; Chenyang Xu — K-stability of cubic threefolds: The threefold bound 16 and equality, and the smooth upper bound in all dimensions. arXiv:1706.01933v3, 2019-01-26.
- [LZ25](#) Chi Li; Zhengyi Zhou — Minimal log discrepancy and orbifold curves: Orbifold rational curves with at most one stack point in the small-stabilizer branch. arXiv:2502.11847v1, 2025-02-17.

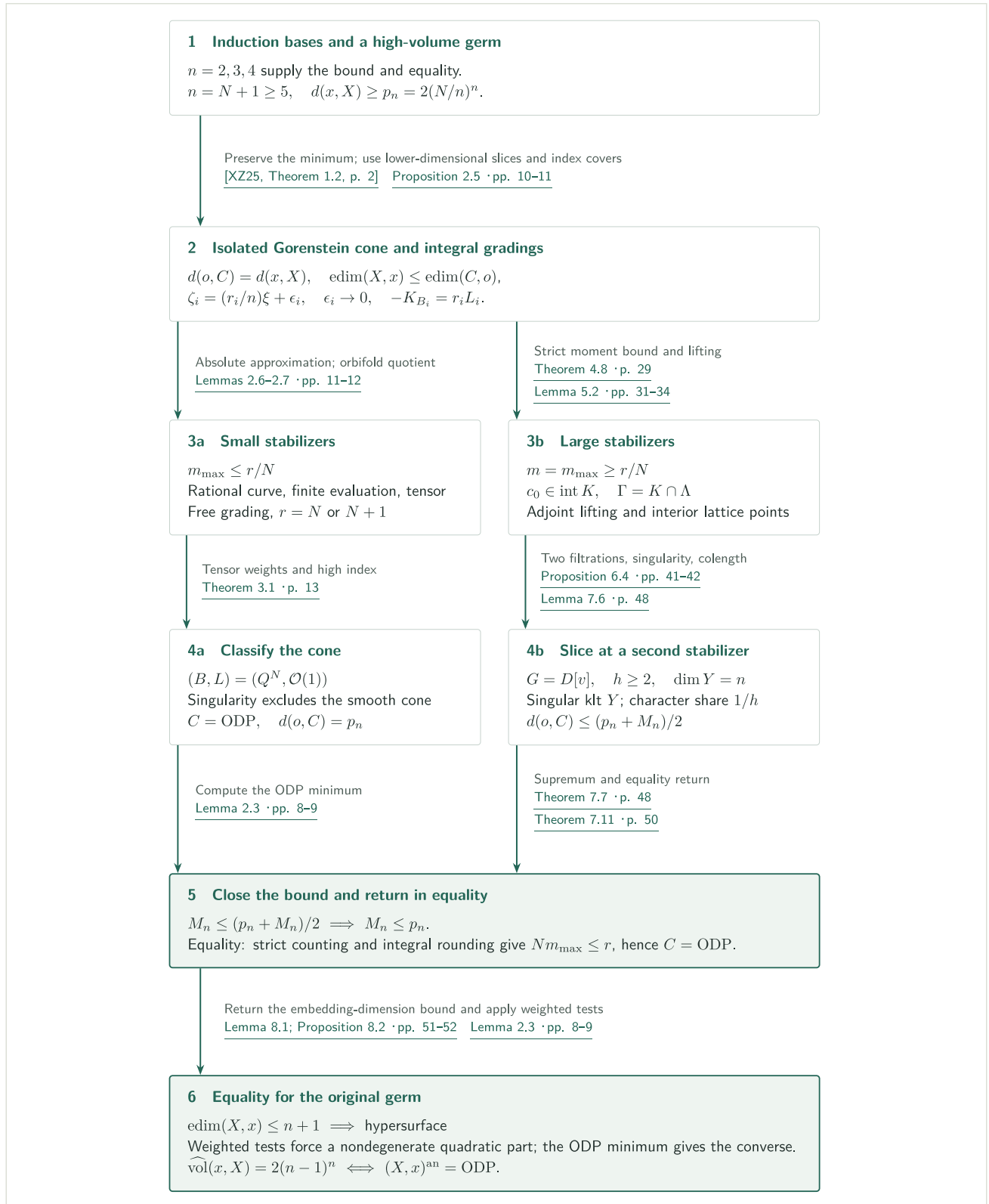
- [AOV11](#) Dan Abramovich; Martin Olsson; Angelo Vistoli — Twisted stable maps to tame Artin stacks: Properness of twisted stable maps in the finite evaluation argument. Author-hosted manuscript, twisted-tame-10-01-2009.pdf; published J. Algebraic Geometry 20 (2011).
- [CS19](#) Tristan C. Collins; Gábor Székelyhidi — Sasaki–Einstein metrics and K-stability: Ricci-flat metrics on isolated K-polystable Fano cones and the stability convention. Geometry & Topology 23 (2019), 1339–1413; publisher PDF with final back cover.
- [ADK08](#) Carolina Araujo; Stéphane Druel; Sándor J. Kovács — Cohomological characterizations of projective spaces and hyperquadrics: High-index classification after the grading is free and L is an actual ample line bundle. arXiv:0707.4310v2, 2007-10-09; manuscript date 2007-07-29.
- [How01](#) Jason A. Howald — Multiplier ideals of monomial ideals: Newton-polyhedron interior criterion for monomials on smooth atlases. arXiv:math/0003232v1, 2000-03-31.
- [Fuj23](#) Osamu Fujino — Cone and contraction theorem for projective morphisms between complex analytic spaces: Nef and big vanishing for a coarse klt pair and an integral $\mathbb{Q}$ -Cartier Weil divisor. Version 0.00, 2023-10-09; author PDF.
- [Sta78](#) Richard P. Stanley — Hilbert functions of graded algebras: Canonical modules of normal monoids through interior lattice points and the Gorenstein criterion. Advances in Mathematics 28 (1978), 57–83; author-hosted scan.
- [EM04](#) Lawrence Ein; Mircea Mustața — Inversion of adjunction for local complete intersection varieties: Inversion of adjunction in the weighted hypersurface estimate. arXiv:math/0301164v2, 2003-07-30.
- [Liu18](#) Yuchen Liu — The volume of singular Kähler–Einstein Fano varieties: Equivalence of K/Ding semistability and the local-to-global volume comparison. arXiv:1605.01034v5, 2018-05-23.
- [LM26](#) Chi Li; Minghao Miao — On the volume of K-semistable Fano manifolds: The smooth K-semistable Fano bound and equality candidates away from projective space. arXiv:2506.17420v3, 2026-05-20; manuscript title-page date 2026-05-22.
- [Ber16](#) Robert J. Berman — K-polystability of $\mathbb{Q}$ -Fano varieties admitting Kähler–Einstein metrics: K-polystability from the KE metrics on the global equality examples. arXiv:1205.6214v3, 2015-06-09.
- [Liu22](#) Yuchen Liu — K-stability of cubic fourfolds: Transfer of normalized volume under isomorphic completed local rings. arXiv:2007.14320v2, 2022-01-10.
- [OpenAI26b](#) OpenAI — The normalized-volume gap in dimension four: The unrestricted boundary-zero klt fourfold bound 162 and analytic equality as the induction base. September 24, 2026.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1](#) · p. 3 See the statement and normalization.

Proof route. Dimension two uses quotient singularities, dimension three an established gap theorem, and dimension four the companion in this catalogue. The remaining step is $n = N + 1 \geq 5$ and starts with a cone of density at least p_n . Small stabilizers classify the cone; large stabilizers lead through the [saturated semigroup](#) to a [slice estimate in the same dimension](#). Equality needs a further return from the cone to the original germ.



Cone reduction, the stabilizer split, and recovery of the original equality germ

Open figure ↗ TeX source ↓

1. Preserve volume and obtain an isolated Gorenstein cone

Stable degeneration and its K-polystable refinement preserve the minimizing valuation's value.

Lifting generators and upper semicontinuity in the flat degeneration also give

$\text{edim}(X, x) \leq \text{edim}(C, o)$, so the final cone is singular. If a puncture point P were singular, an

étale product of an $(n - 1)$ -dimensional transverse slice with a smooth curve would give

$d(P, C) \leq p_{n-1}$ by induction and Lemma 2.1. Lower semicontinuity gives $d(o, C) \leq d(P, C)$,

contradicting $p_{n-1} < p_n$. Applying the finite degree formula to the canonical index- q cover gives

$d(o, C) \leq 1/q$; since $p_n > 1/2$, the index is one.

Proposition 2.5 · pp. 10–11 [XZ25, Theorem 1.2, p. 2] [LWX21, Theorem 1.2, p. 2]

[BL21, Theorem 33(1), p. 23] [XZ21, Theorem 1.3, p. 1] [LX19, Theorem 1.3(1), p. 3]

[OpenAI26b, Theorem 1.1, p. 1]

2. Small stabilizers force a free grading and high-index classification

Normalize $A_C(\xi) = n$ and approximate by primitive positive integral gradings $\zeta_i = (r_i/n)\xi + \epsilon_i$

with **absolute error** $\epsilon_i \rightarrow 0$. The smooth orbifold $B_i = [C^\circ/\mathbb{G}_m]$ has a faithful weight-one ample

line bundle L_i and $-K_{B_i} = r_i L_i$. If $m_{\max} \leq r/N$, the Li-Zhou curve theorem supplies a

minimal-degree map $\mathbb{P}(1, m) \rightarrow B$. Unless the deformation character count directly forces

freeness, its evaluation map is finite and surjective.

The symmetric product of infinitesimal translations over all sheets gives a nonzero rank- b tensor

of integral weight $-bm$. A Bochner argument for the Ricci-flat cone metric bounds its Reeb

weight by $\tau \geq -b$; estimates in fixed ring generators control the character error independently of

b . Thus the integer $nm - r$ is at most $o(1)$. Rounding and faithfulness of the tangent action force

all stabilizers to be trivial and $r = N$ or $N + 1$. High-index classification of the resulting smooth

polarized Fano variety identifies its section ring with the quadric ring or a polynomial ring.

Singularity of C excludes the latter.

Lemmas 2.6–2.7 · pp. 11–12 Theorem 3.1; proof · pp. 13–19 [LZ25, Proposition 2.6, p. 6]

[AOV11, Theorem 4.3, p. 17] [CS19, Theorem 1.1, print p. 1340 / PDF p. 2]

[ADK08, Theorems 1.1 and 1.3, pp. 1–2]

3. Large stabilizers lead to an inequality for the supremum

For $m = m_{\max} \geq r/N$, record original degree and transverse Taylor initial in a semigroup Γ .

Moment estimates from near-minimality and an exponential expectation inequality put

$c_0 = (r, 1, \dots, 1)$ inside its real cone K . Adjoint lifting then yields $\Gamma = K \cap \Lambda$ and

$\text{int } K \cap \Lambda = c_0 + \Gamma$. The Theorem 5.3 section explains this exact integral strengthening.

Two filtrations produce a central ring $G = D[v]$ with Hilbert series $(1 - z)^{-n}$. A point with a different stabilizer μ_h , $h \geq 2$, gives a singular klt slice Y of **the same dimension n** . The share $1/h$ of each character in leading colength yields $d(o, C) \leq (p_n + M_n)/2$, where M_n is the supremum of all singular n -dimensional klt densities. Applying this to germs approaching M_n proves $M_n \leq p_n$. The original m and the new h belong to different actions.

Theorem 4.8 · p. 29 Theorem 5.3 · pp. 34–36 Proposition 6.4; Corollary 6.6 · pp. 41–43
Theorem 7.7 · p. 48

4. Classify equality on the cone and return to the germ

When $d(o, C) = p_n$, every inequality in the slice estimate is an equality. Thus $h = 2$ and, after normalizing $u(\lambda_Y) = 1$, one has $A_Y(u) = N$ and $\text{vol}(u) = 2$. The restricted valuation w satisfies $A_C(w) = 2N$ and $\text{vol}(w) = 2^{-N}$ and computes the minimum; uniqueness gives $w = (2N/n) \text{wt}_\xi$. If the distinguished section $v \in S_m$ had $w(v) > 2$, a positive leading-order loss in the low-filtration quotient would make the volume estimate strict. Hence $w(v) \leq 2$. Absolute approximation gives $Nm - r \leq o(1)$, so integrality returns the grading to $Nm \leq r$, the small-stabilizer branch. Theorem 7.11 therefore identifies C as the ODP cone.

The original singular germ has embedding dimension at most $n + 1$, hence exactly $n + 1$, and is a hypersurface. Lemma 8.1 supplies the weighted bound

$$\widehat{\text{vol}}(x, X) \leq \frac{d_F(\sum_{i=0}^n w_i - d_F)^n}{\prod_{i=0}^n w_i},$$

where $F = 0$ is a local equation, $w_i > 0$ are rational weights, and d_F is its least weight. Weights all equal to one exclude multiplicity $q > 2$. If $q = 2$ but the quadratic part omits a direction, giving that direction weight $1 - \varepsilon$ yields the strict bound $2(n - 1 - \varepsilon)^n / (1 - \varepsilon) < 2(n - 1)^n$. Equality forces a nondegenerate quadratic part, and the holomorphic Morse lemma gives an analytic ODP. An ODP degeneration alone would not establish this last assertion.

Lemmas 7.8–7.10; Theorem 7.11 · pp. 49–50 [XZ21, Theorem 1.1; Corollary 1.2, p. 1]
Lemma 8.1; Proposition 8.2 · pp. 51–52 [EM04, Theorem 1.1, p. 1] Proof of Theorem 1.1 · p. 52

5. Sharpness requires the infimum at an ODP

For the converse, Lemma 2.3 uses uniqueness to make a quadric cone's minimizer invariant under cone scaling and $\text{SO}(n + 1, \mathbb{C})$. Irreducibility of each graded harmonic-polynomial representation forces it to be a positive multiple of the ordinary degree valuation. That valuation has discrepancy $n - 1$ and volume two, so the **infimum over all valuations** is $2(n - 1)^n$. This is stronger than testing only the ordinary blowup to obtain an upper bound. Completion invariance transfers the value to an analytic ODP, establishing sharpness and the reverse equality implication.

3. Theorem 5.3 — From the canonical shift to exact saturation

Proof target : Theorem 5.3 · pp. 34–36

Theorem 5.3 — The saturated jet semigroup

Fix the orbifold jet data of §§4–5: a smooth effective proper complex N -orbifold B with projective coarse space, an ample line bundle L with $-K_B = rL$, and a point with stabilizer μ_m . The fibre character of L is one and transverse coordinate characters are b_i . Let Γ record the total-degree–lexicographic Taylor initials (j, γ) of sections of $H^0(B, jL)$, and put

$$\Lambda = \{(j, \gamma) \in \mathbb{Z} \times \mathbb{Z}^N : j \equiv \sum b_i \gamma_i \pmod{m}\}, \quad K = \overline{\text{cone}(\Gamma)}, \quad c_0 = (r, 1, \dots, 1).$$

The cone is full-dimensional, has bounded slices of bounded degree, contains the positive constant ray, projects onto the exponent orthant, and satisfies the jet realization properties of §4. If $c_0 \in \text{int } K$, then K is rational polyhedral, every facet's primitive nonnegative support form $\eta \in \Lambda^*$ has $\eta(c_0) = 1$, and

$$\Gamma = K \cap \Lambda, \quad \text{int } K \cap \Lambda = c_0 + \Gamma.$$

In particular, Γ is finitely generated and saturated and generates the group Λ . In the large-stabilizer branch, Theorem 4.8 supplies the interiority hypothesis for every sufficiently far grading.

Theorem 5.3 · pp. 34–36

1 Taylor semigroup retaining characters

$\Gamma \subset \Lambda = \{(j, \gamma) : j \equiv \sum b_i \gamma_i \pmod{m}\},$
 $K = \text{cone}(\Gamma), \quad c_0 = (r, 1, \dots, 1) \in \Lambda.$

Moments contradict a support
[Lemmas 4.2–4.3 · pp. 22–23](#)
[Theorem 4.8 · p. 29](#)

Multiplier ideals and vanishing
[\[How01, Main Theorem, p. 2\]](#)
[Lemma 5.2 · pp. 31–34](#)

2a Numerical interiority

$H \geq 2/s \geq 2, \quad C_* < 2$
 Near-minimality and exponential bound
 $c_0 \in \text{int } K$

An interior ball bounds support norms
[Theorem 5.3 · pp. 34–36](#)

2b Independent adjoint lifting

$q \in \Lambda, \quad q + c_0 \in \text{int } K$
 $\implies q \in \Gamma$
 Lift the prescribed jet on smooth stacks

Lattice approximation and lifting give a contradiction
[Theorem 5.3 · pp. 34–36](#)

3 Force primitive facet supports to height one

$\eta \in \Lambda^*, \quad \eta(c_0) = 1.$
 At differentiability points, an irrational support ray or height at least two would give $q \notin K$ but $q + c_0 \in \text{int } K$.
 A ball around c_0 bounds support norms, giving finitely many supports.

Integral facet values and translation; apply the canonical-module formula
[Theorem 5.3 · pp. 34–36](#)
[\[Sta78, Theorem 6.7; normal-monoid reformulation, print pp. 81–82 / PDF pp. 25–26\]](#)

4 Exact saturation and the canonical shift

$\Gamma = K \cap \Lambda, \quad \text{int } K \cap \Lambda = c_0 + \Gamma.$
 Finite generation and saturation; the interior-lattice ideal is principal.
 $\mathbb{C}[\Gamma]$ is Gorenstein, $\deg \omega_{\bar{S}} = (r, N).$

Independent moment and adjoint arguments combine to give exact saturation

[Open figure ↗](#) [TeX source ↓](#)

1. Establish interiority numerically

Rescale original degree j to $D_0 = Nj/r$, so the cone is the epigraph $D_0 \geq f(x)$ with lattice density $s = r/(Nm) \leq 1$. Put $H = \int_{\mathbb{R}_{\geq 0}^N} e^{-f(x)} dx$. The volume lower bound gives $H \geq 2/s$; near-minimality of valuations mixing degree with Taylor order gives a second moment inequality. Failure of interiority would give $f(\mathbf{1}) \geq N$. A supporting linear form then allows the uniform exponential estimate of Theorem 4.4, whose constant $C_* = (3/4)e^{4/3} - e/3 < 2$ contradicts $H \geq 2$. The asymptotic semigroup count used here does not assume saturation.

Lemmas 4.2–4.3 · pp. 22–23 Theorem 4.4; proof · pp. 24–28 Theorem 4.8 · p. 29

2. Prove adjoint lifting independently of interiority

Lemma 5.2 says that $(j, \alpha) \in \Lambda$ and $(j, \alpha) + c_0 \in \text{int } K$ imply $(j, \alpha) \in \Gamma$. Choose positive integral weights agreeing with finitely many known Taylor initials and construct the weighted blowup as a smooth stack. The canonical shift cancels the exceptional cost, leaving the nef and big class $(j + r - j_0)\pi^*L$ in (5.6). Howald's monomial multiplier-ideal formula places the prescribed exceptional monomial in the required ideal; orbifold vanishing in Lemma 5.1 lifts the section. Its coarse-space reduction uses Fujino's theorem for a klt pair and an integral $\mathbb{Q}$ -Cartier Weil divisor. Retaining characters on the smooth atlases is essential for descent.

Lemmas 5.1–5.2 · pp. 30–34 [How01, Main Theorem, p. 2] [Fuj23, Theorem 3.2.4, print p. 31 / PDF p. 41]

3. Height-one supports give finite generation and the canonical module

If a supporting ray at a differentiability point were irrational, or its primitive integral form had height at least two on c_0 , lattice approximation would produce $q \notin K$ with $q + c_0 \in \text{int } K$. Adjoint lifting forbids this. A small ball around c_0 bounds the norms of height-one supports, so only finitely many integral supports occur. This proves rational polyhedrality.

For $q \in K \cap \Lambda$, translation by c_0 reaches the interior, hence lifting gives $q \in \Gamma$. Conversely, each primitive facet takes an integer value at least one on an interior lattice point, so subtracting c_0 remains in K . These are the two set equalities; finite generation follows. The interior-translation formula makes the normal semigroup ring's canonical module principal, supplying the Gorenstein property and degree (r, N) needed in Lemma 6.1. Saturation alone would not imply Gorensteinness.

Theorem 5.3 · pp. 34–36

[Sta78, Theorem 6.7; normal-monoid reformulation, print pp. 81–82 / PDF pp. 25–26]

Lemma 6.1 · pp. 37–39

4. Theorem 7.7 — Control the same-dimensional slice by a supremum

Proof target : Theorem 7.7 · p. 48

Theorem 7.7 — The inductive upper bound

In the setting of §7, let $n = N + 1 \geq 5$ and assume the local ODP volume theorem for boundary-zero klt germs in every dimension below n . Let M_n be the supremum of d over singular normal complex algebraic $\mathbb{Q}$ -Gorenstein boundary-zero klt n -fold germs. Then

$$M_n \leq p_n = 2 \left(\frac{n-1}{n} \right)^n.$$

Theorem 7.7 · p. 48

1 Two filtrations give another action

$\bar{S} = \text{gr}_T S$, $D = \bar{S}/(\bar{v})$, $G = \text{gr}_W S = D[v]$.
 $\text{Hilb}_W G(z) = (1-z)^{-n}$, nontrivial stabilizer $\mu_h, h \geq 2$.

Compare Rees canonical forms; exclude non-klt and smooth slices
[Proposition 6.4 · pp. 41–42](#) [Lemmas 7.1–7.3 · pp. 44–46](#)

2 A singular klt slice of the same dimension

$\dim Y = n$, λ_Y has character -1 , $\ell = u(\lambda_Y) > 0$.
 $A_C(w) \leq N\ell + A_Y(u)$, $\text{vol}(w) \leq \ell^{-n}$.

Uniqueness, character injections via λ_Y , and colength splitting
[\[XZ21, Theorem 1.1; Corollary 1.2, p. 1\]](#) [Lemma 7.4; Proposition 7.5 · p. 47](#)

3 Count only one character

u_* = invariant minimizer, $A' = A_Y(u_*)$, $\sigma = \text{vol}(u_*)$.
 $\dim(R/\mathfrak{a}_t)_X = \frac{\sigma}{h} \frac{t^n}{n!} + o(t^n)$,
 $\text{vol}(w) \leq \min_{0 \leq x \leq 1} \{ \ell^{-n} x^n + (1-x)^n \sigma / h \}$.

Optimize, apply Hölder and use $h \geq 2$
[Lemma 7.6 · p. 48](#)

4 Use only the definition of the supremum

$d(o, C) \leq (N/n)^n + (A'/n)^n \sigma / h$
 $= p_n/2 + d(y, Y)/h \leq (p_n + M_n)/2$.
 Since Y is singular klt, $d(y, Y) \leq M_n$.

For $d_i > p_n$, exclude the small-stabilizer branch and take the limit
[Theorem 3.1 · p. 13](#) [Theorem 7.7 · p. 48](#)

5 Close the supremum inequality

$d_i \rightarrow M_n$, $d_i \leq (p_n + M_n)/2 \implies M_n \leq p_n$.
 Choose approximating gradings separately on each cone. The supremum need not be attained.

Two filtrations, the character share $1/h$, and closure of the supremum inequality

[Open figure ↗](#) [TeX source ↓](#)

1. Construct a singular klt slice of the same dimension

The first filtration T is transverse Taylor order: $\bar{S} = \text{gr}_T S$ and $D = \bar{S}/(\bar{v})$. The second assigns $W = a + T(e_i)$ to a basis $v^a e_i$, giving $\text{gr}_W S = D[v]$. Reducedness of D uses the volume hypothesis and $N \geq 4$. If every puncture stabilizer were trivial, Hilbert series $(1 - z)^{-N}$ would force D to be a polynomial ring generated in degree one, and ultimately C to be smooth. Thus a stabilizer μ_h , $h \geq 2$, exists.

A transverse slice Y in the W -Rees total space through that point has a function λ_Y of character -1 . Comparison of canonical forms gives, for a quasi-monomial valuation u of Y , $\ell = u(\lambda_Y)$ and its restriction w to C ,

$$A_C(w) \leq N\ell + A_Y(u), \quad \text{vol}(w) \leq \ell^{-n}.$$

If Y were not klt, tests with $A_Y(u)/\ell$ tending to a nonpositive value would give $\widehat{\text{vol}}(o, C) \leq N^n$, contradicting the lower bound $2N^n$. Smoothness of Y would produce a nonzero μ_h -fixed point in the other deformation's general fibre $\mathbb{A}^N$, also a contradiction. Establishing singularity is necessary before using M_n .

Proposition 6.4; Corollary 6.6 · pp. 41–43 Lemmas 7.1–7.3 · pp. 44–46

2. Restriction to one character improves the coefficient

Uniqueness makes the minimizer u_* on Y invariant under μ_h . With $\sigma = \text{vol}(u_*)$, injections between character spaces induced by powers of λ_Y show that each character carries $1/h$ of the leading colength. Splitting the valuation quotient by the filtration, the second part injects into one character quotient, rather than the full quotient. Thus

$$\text{vol}(w) \leq \min_{0 \leq x \leq 1} \left\{ \frac{x^n}{\ell^n} + \frac{(1-x)^n \sigma}{h} \right\}.$$

Combining this with the discrepancy estimate and Hölder's inequality, and putting $A' = A_Y(u_*)$, yields

$$d(o, C) \leq \left(\frac{N}{n} \right)^n + \frac{(A'/n)^n \sigma}{h} = \frac{p_n}{2} + \frac{d(y, Y)}{h} \leq \frac{p_n + M_n}{2}.$$

No same-dimensional ODP bound has been applied to Y .

Lemma 7.4; Proposition 7.5 · p. 47 Lemma 7.6 · p. 48

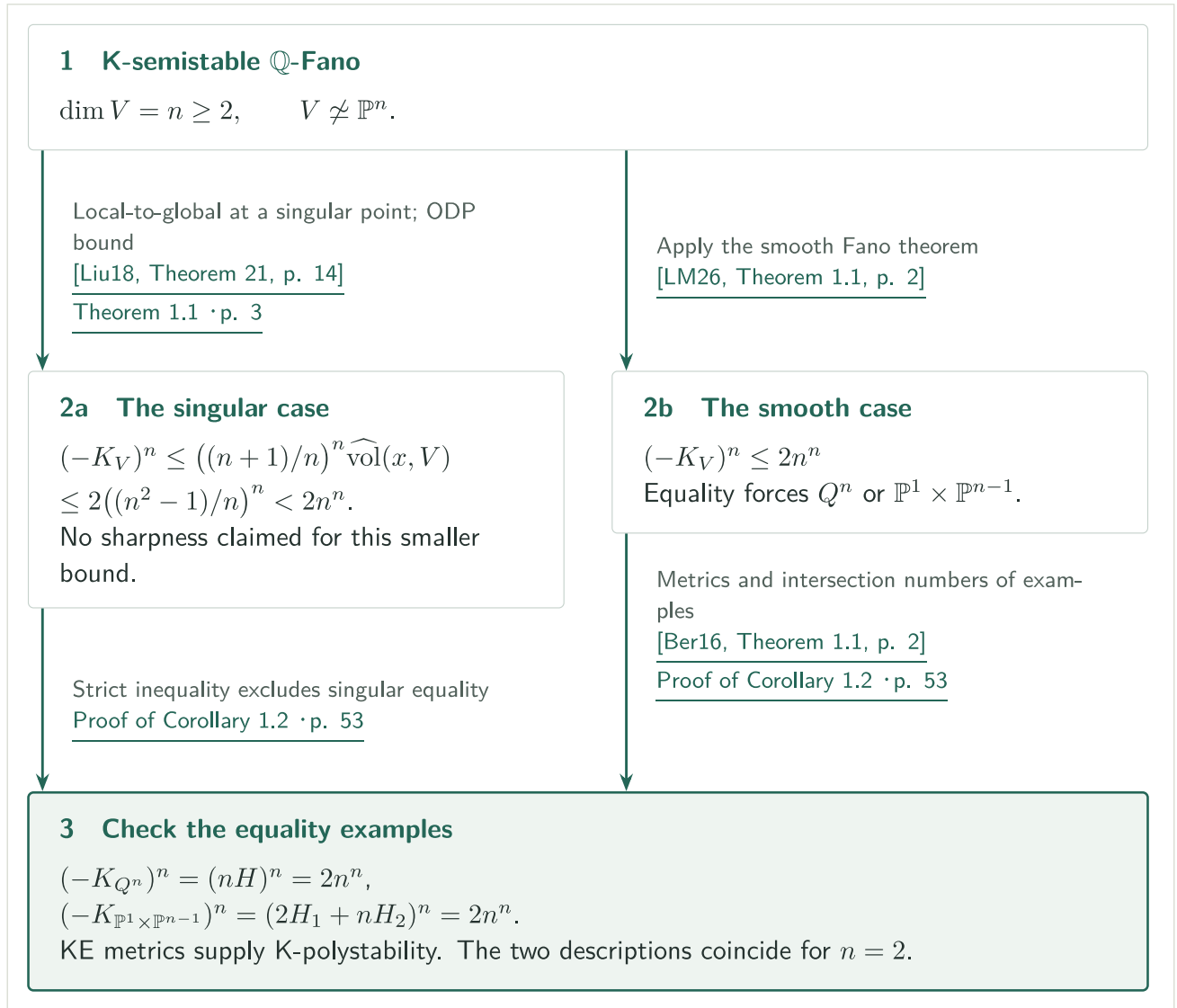
3. The supremum need not be attained

If $M_n > p_n$, choose germs with $d_i > p_n$ and $d_i \rightarrow M_n$. On each corresponding cone an infinite small-stabilizer subsequence contradicts Theorem 3.1 and the ODP minimum. A sufficiently far large-stabilizer grading therefore exists on each fixed cone. Passing to the limit in $d_i \leq (p_n + M_n)/2$ contradicts $M_n > p_n$. Approximation constants may depend on the cone; no uniform rate over all germs and no attained supremum is required.

Theorem 7.7 · p. 48

5. Corollary 1.2 — From local volumes to global bounds and examples

Proof target : Corollary 1.2 · p. 3 See the three conclusions.



Combine the singular local-to-global bound with the smooth classification

[Open figure ↗](#) [TeX source ↓](#)

1. Obtain the stricter bound for singular Fano varieties

Equivalence of K-semistability and Ding semistability and the local-to-global inequality for every centered valuation give

$$(-K_V)^n \leq \left(\frac{n+1}{n} \right)^n \widehat{\text{vol}}(x, V).$$

Apply Theorem 1.1 at a singular closed point to obtain $2((n^2-1)/n)^n$. This is strictly below $2n^n$; it does not establish sharpness or classify equality at the smaller value.

[\[Liu18, Theorem 21, p. 14\]](#) [\[Liu18, Theorem 13\(2\), p. 9\]](#) [Proof of Corollary 1.2 · p. 53](#)

2. Reduce equality at $2n^n$ to the smooth classification

For smooth $V \not\simeq \mathbb{P}^n$, apply Li–Miao. The strict singular bound excludes singular equality cases, so equality at $2n^n$ forces a smooth quadric or $\mathbb{P}^1 \times \mathbb{P}^{n-1}$. The Li–Miao input concerns smooth Fano manifolds and is not applied directly to singular V .

[LM26, Theorem 1.1, p. 2]

3. Compute equality on the two examples

The manuscript uses the homogeneous and product Kähler–Einstein metrics to ensure K-polystability of these examples. For hyperplane classes H, H_1, H_2 ,

$$(-K_{Q^n})^n = (nH)^n = 2n^n, \quad (-K_{\mathbb{P}^1 \times \mathbb{P}^{n-1}})^n = (2H_1 + nH_2)^n = 2n^n.$$

When $n = 2$, $Q^2 \simeq \mathbb{P}^1 \times \mathbb{P}^1$, so the descriptions are not distinct examples. The proof of the general metric-to-K-polystability theorem is outside this overview's checking scope.

Proof of Corollary 1.2 · p. 53 [Ber16, Theorem 1.1, p. 2]

6. Which papers supply which steps

Input

[OpenAI26b, Theorem 1.1, p. 1]

Used at

Theorem 2.4 and induction base

Role and required hypotheses

Bound 162 and analytic equality for unrestricted boundary-zero klt fourfolds

Checking scope

Both statements and pp.9–10 checked; the companion proof belongs to its own article

Input

[XZ25, Theorem 1.2, p. 2] [XZ21, Theorem 1.1; Corollary 1.2, p. 1] [XZ21, Theorem 1.3, p. 1]
[LWX21, Theorem 1.2, p. 2]

Used at

Proposition 2.5 and §7

Role and required hypotheses

Volume-preserving cone degeneration, uniqueness, finite degree

Checking scope

Statements and uses compared

Input

[LZ25, Proposition 2.6, p. 6] [AOV11, Theorem 4.3, p. 17] [CS19, Theorem 1.1, print p. 1340 /
PDF p. 2] [ADK08, Theorems 1.1 and 1.3, pp. 1–2]

Used at

Theorem 3.1

Role and required hypotheses

Orbifold curves, properness, Ricci-flat metric, high index after freeness

Checking scope

Hypotheses, statements and use sites compared; input proofs not reverified

Input

[How01, Main Theorem, p. 2] [Fuj23, Theorem 3.2.4, print p. 31 / PDF p. 41] [Sta78, Theorem
6.7; normal-monoid reformulation, print pp. 81–82 / PDF pp. 25–26]

Used at

Lemma 5.2; Lemma 6.1

Role and required hypotheses

Monomial ideals, nef and big vanishing, interior lattice points in the canonical module

Checking scope

Statements and their smooth-atlas, coarse klt and normal-monoid interfaces checked

Input

[EM04, Theorem 1.1, p. 1]

Used at

Lemma 8.1

Role and required hypotheses

Inversion of adjunction for a normal Cartier hypersurface in a smooth ambient variety

Checking scope

Theorem 1.1 and use of the weighted ideal compared

Input

[Liu18, Theorem 21, p. 14] [LM26, Theorem 1.1, p. 2]

Used at

Corollary 1.2

Role and required hypotheses

Local-to-global comparison and the smooth Fano bound with equality candidates

Checking scope

Statements and p.53 compared; inputs to a consequence of the main theorem

7. Return to the sources

- §2.1; (2.1) · pp. 6–7 — Normalization and the valuation domain.
- Proposition 2.5 · pp. 10–11 — The reduction derives isolatedness and Gorensteinness.
- Theorem 3.1; proof · pp. 13–19 — From orbifold curves through tensors to free gradings.
- Theorem 4.8 · p. 29 — Where the exponential estimate forces canonical interiority.
- Lemmas 5.1–5.2 · pp. 30–34 — Vanishing, multiplier ideals and the prescribed jet lift.
- Theorem 5.3 · pp. 34–36 — Rational polyhedrality, height one and the two lattice identities.
- Proposition 6.4; Corollary 6.6 · pp. 41–43 — Two filtrations producing the second stabilizer.
- Lemma 7.4; Proposition 7.5 · p. 47 — The comparison counts a single character rather than total length.

- Theorem 7.7 · p. 48 — The supremum argument avoiding same-dimensional induction.
- Lemmas 7.8–7.10; Theorem 7.11 · pp. 49–50 — Strict counting and integral rounding return equality to small stabilizers.
- Lemma 8.1; Proposition 8.2 · pp. 51–52 — The weighted test making the original quadratic part nondegenerate.
- Proof of Corollary 1.2 · p. 53 — Separate roles of the local-to-global bound and smooth equality examples.

This overview reports the authors' claims and proof architecture. Editorial reading compared the hypotheses and every conclusion of the theorem and corollary, the cone reduction in §2, the classification interfaces in §3, the interiority, adjoint lifting and two-filtration statements in §§4–6, and the inequalities, equality analysis and hypersurface recovery in §§7–8. The external inputs above were checked at the level of statements and applicability, without recursively checking their proofs.

Unverified scope. We have not independently verified every calculation in the exponential optimization of §4.3, all flatness, normality and canonical-module lifting details in §6, every boundary and obstruction-theoretic step for twisted stable maps in §3, the external proofs of harmonic-polynomial irreducibility and completion invariance in the ODP calculation, or the proof of the general metric-to-K-polystability theorem. Tracing the main interfaces does not certify all 57 pages of the proof.

The normalized-volume gap in dimension four

From a lower volume bound to a hypersurface local ring, then to the established lci bound

The manuscript asserts the upper bound 162 for singular complex algebraic klt fourfold germs with zero boundary, together with the analytic classification of equality. Its structural contribution is that volume at least 162 forces embedding dimension at most five. Transverse jets control stabilizers on a stable-degeneration cone; separate discrepancy and maximal-ideal threshold estimates then produce a terminal Gorenstein threefold section.

Manuscript author: OpenAI · September 24, 2026 · 20 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — The sharp fourfold bound and equality

Let X be a normal complex algebraic variety and $x \in X$ a singular closed point. Assume $\dim X = 4$, K_X is $\mathbb{Q}$ -Cartier, and X is klt with zero boundary near x . Then

$$\widehat{\text{vol}}(x, X) \leq 162 = 2 \cdot 3^4.$$

Equality holds if and only if the analytic germ is isomorphic to

$$(X, x) \simeq (\{z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0\} \subset \mathbb{C}^5, 0).$$

No isolatedness, lci, quotient, or smoothability hypothesis is imposed. What follows describes the manuscript's argument; it is not an independent verification of every proof.

Theorem 1.1 · p. 1

For $\mathfrak{a}_t(v) = \{f \in \mathcal{O}_{X,x} : v(f) \geq t\}$, the normalization is

$$\mathrm{vol}(v) = \lim_{t \rightarrow \infty} \frac{4!}{t^4} \ell(\mathcal{O}_{X,x}/\mathfrak{a}_t(v)), \quad \widehat{\mathrm{vol}}(x, X) = \inf_{c_X(v)=x} A_X(v)^4 \mathrm{vol}(v).$$

Here $A_X(F) = 1 + \mathrm{coeff}_F(K_Y - \mu^* K_X)$ is the **log discrepancy**; valuations of infinite discrepancy contribute $+\infty$. The infimum over all centered real valuations equals that over prime-divisor valuations centered at x . With this convention the smooth fourfold value is 256 and the ordinary-double-point value is 162.

Section 2 · pp. 3–5

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Fourfold volume gap](#). External inputs use the following keys.

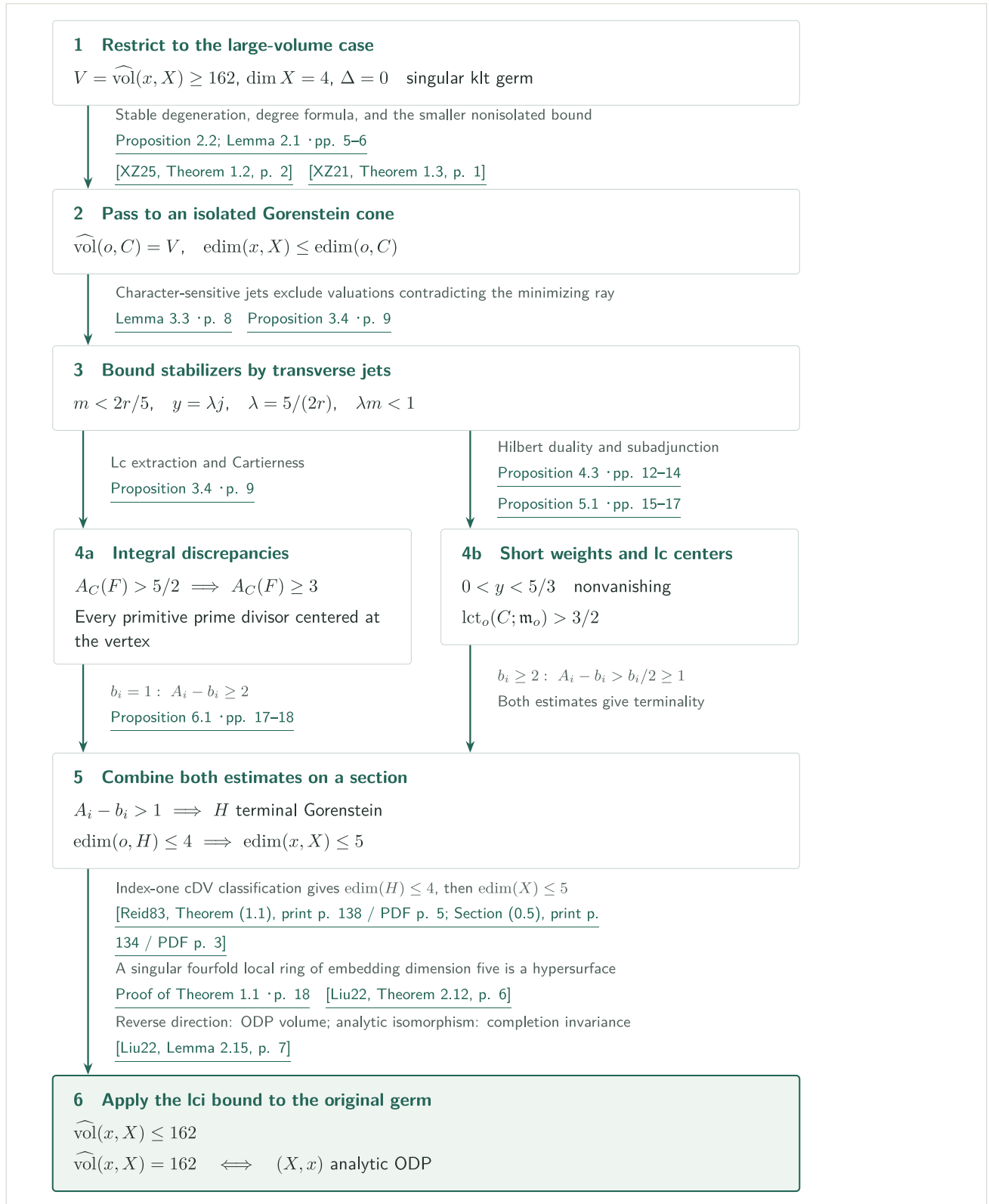
- [XZ25](#) Chenyang Xu; Ziquan Zhuang — Stable degenerations of singularities: Finite generation of klt minimizers and stable degeneration to a K-semistable cone. arXiv:2205.10915v2; published JAMS 38 (2025).
- [LX18](#) Chi Li; Chenyang Xu — Stability of valuations: higher rational rank: Preservation of volume and discrepancy and minimization over all valuations on the cone. arXiv:1707.05561v2; published Peking Mathematical Journal 1 (2018).
- [Li18](#) Chi Li — Minimizing normalized volumes of valuations: Izumi estimate for the embedding-dimension comparison. arXiv:1511.08164v4, 2017-07-18; manuscript date 2017-07-19.
- [XZ21](#) Chenyang Xu; Ziquan Zhuang — Uniqueness of the minimizer of the normalized volume function: Finite-degree formula for normalized volume under crepant finite Galois covers. arXiv:2005.08303v1, 2020-05-17; published CJM 9 (2021).
- [Liu22](#) Yuchen Liu — K-stability of cubic fourfolds: Nonisolated bound, sharp lci bound and equality, and invariance under completion isomorphism. arXiv:2007.14320v2, 2022-01-10; manuscript date 2022-01-11.
- [FG12](#) Osamu Fujino; Yoshinori Gongyo — On canonical bundle formulae and subadjunctions: Supplies a klt boundary on a minimal lc center in the affine setting. arXiv:1009.3996v1, 2010-09-21; internal version 1.17, 2010-09-12.
- [dFH09](#) Tommaso de Fernex; Christopher D. Hacon — Singularities on normal varieties: Q-factoriality of surfaces admitting a klt boundary. arXiv:0805.1767v2, 2008-10-25; published Compositio 145 (2009).
- [Reid83](#) Miles Reid — Minimal models of canonical 3-folds: cDV classification and hypersurface presentation of index-one terminal threefold points. Advanced Studies in Pure Mathematics 1 (1983), 131–180; original spread scan.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : Theorem 1.1 · p. 1

Proof route. The bound in the [statement](#) already holds when $V := \widehat{\text{vol}}(x, X) < 162$. Under $V \geq 162$, the argument constrains the local ring. After reduction to a cone, the [stabilizer estimate](#) branches into $A_C(F) \geq 3$ and the [threshold bound](#) $> 3/2$; these meet at the threefold section. One then returns to the original germ and invokes the lci theorem. The final input supplies the upper bound, necessity in the equality case, and sharpness, whose roles are distinguished below.



The fourfold structural reduction, bound, and equality

[Open figure ↗](#) [TeX source ↓](#)

1. Reduction to a cone with the same volume

Stable degeneration produces $C = \operatorname{Spec} \operatorname{gr}_v \mathcal{O}_{X,x}$ and a minimizing Reeb vector ξ_* . Finite generation, preservation of volume and log discrepancy, and minimization over all centered valuations are essential: minimization only within a fixed Reeb cone would not suffice. Lemma 2.1 lifts homogeneous generators and uses the Izumi estimate to obtain $\operatorname{edim}(x, X) \leq \operatorname{edim}(o, C)$.

[XZ25, Theorem 1.2, p. 2] [LX18, Theorem 1.3, p. 3; Lemma 4.10, p. 35] [Li18, Theorem 3.1, p. 6]

A singular point on the punctured cone would have an orbit closure making the singular locus positive-dimensional at the vertex, contradicting $V \leq 4096/27 < 162$. The finite-degree formula applies to the index-one cover of degree d and gives $dV \leq 256$, hence $d = 1$. Cohen–Macaulayness of klt singularities and graded Nakayama make C Gorenstein. For primitive integral gradings approaching the minimizing ray, write $r > 0$ and h for the discrepancy and volume of the degree valuation: $r^4 h \geq V$ and $r^4 h \rightarrow V$.

Proposition 2.2; Lemma 2.1 · pp. 5–6 [XZ21, Theorem 1.3, p. 1] [Liu22, Corollary 2.18, p. 9]

2. The two estimates meet at a terminal section

Proposition 3.4 gives $A_C(F) \geq 3$ for every primitive prime divisor centered at the vertex. Proposition 5.1 gives $\operatorname{lct}_o(C; \mathfrak{m}_o) > 3/2$. On a log resolution of the maximal ideal, write $\mathfrak{m}_o \mathcal{O}_Y = \mathcal{O}_Y(-\sum b_i E_i)$ and $A_i = A_C(E_i)$. The ordinary discrepancy along a general section H is $A_i - 1 - b_i$. As a Cartier section of a Gorenstein Cohen–Macaulay fourfold, H is Gorenstein and S_2 ; Bertini gives smoothness away from the vertex, hence R_1 and normality.

If $b_i = 1$, then $A_i - b_i \geq 2$; if $b_i \geq 2$, then $A_i - b_i > b_i/2 \geq 1$. Every coefficient is positive, so H is a terminal Gorenstein threefold germ. The threshold estimate alone is not being used to infer terminality. Reid's index-one classification makes a singular H an isolated cDV point, hence an analytic hypersurface. Including the smooth case, $\operatorname{edim}(o, H) \leq 4$, and therefore

$$\operatorname{edim}(x, X) \leq \operatorname{edim}(o, C) \leq \operatorname{edim}(o, H) + 1 \leq 5.$$

Proposition 6.1 · pp. 17–18

[Reid83, Theorem (1.1), print p. 138 / PDF p. 5; Section (0.5), print p. 134 / PDF p. 3]

3. Upper bound, necessity for equality, and sharpness

A singular normal fourfold germ with embedding dimension at most five has local ring equal to a five-dimensional regular local ring modulo a height-one prime. A regular local ring is a UFD, so this ideal is principal. Thus the original X is an algebraic local hypersurface, hence lci; showing only that the degeneration cone is a hypersurface would not suffice.

Liu's lci theorem now yields $V \leq 2(4 - 1)^4 = 162$. Its equality classification says that $V = 162$ forces the original germ to be an ODP. Conversely, one needs the actual ODP volume to be 162: merely exhibiting one valuation of normalized volume 162 would not establish the infimum. For the exceptional divisor E of the ordinary blow-up of the ODP vertex, $A_X(E) = 3$ and $\text{vol}(E) = 2$, giving the candidate value $3^4 \cdot 2 = 162$. The manuscript invokes the established lci equality theorem and this ODP calculation at this point, and uses invariance under isomorphism of completed local rings to pass to the analytic characterization. This also establishes sharpness of the constant.

Proof of Theorem 1.1 · p. 18 [Liu22, Theorem 2.12, p. 6] [Liu22, Lemma 2.15, p. 7]

3. Proposition 3.4 — Transverse jets control stabilizers and discrepancies

Proof target : Proposition 3.4 · p. 9

Proposition 3.4 — Stabilizers on a large-volume cone

Take the cone C of Proposition 2.2 with $V \geq 162$ and primitive positive integral gradings approaching its minimizing ray; let r be the degree valuation's discrepancy. For all sufficiently close gradings, every point of $C \setminus \{o\}$ has stabilizer order $m < 2r/5$.

Consequently every primitive prime divisor F centered at o has $A_C(F) \geq 3$.

Proposition 3.4 · p. 9

1 A test valuation on a finite slice

$$(u, z_1, z_2, z_3) : (1, \alpha, \alpha, \alpha), \quad A_C(v_\alpha) = r + 3\alpha$$

The cyclic action on the smooth slice controls valuation-ideal lengths

[Lemma 3.1 · p. 7](#) [Lemma 3.3 · p. 8](#)

2 Count jets in one character

$$\dim S_j \sim h j^3 / 6, \quad \dim(\text{jets in one character}) \sim \ell^3 / (6m)$$

$$\text{vol}(v_\alpha) \leq \frac{h}{(1 + \alpha(mh)^{1/3})^3}$$

Use minimality over all centered valuations and set $\alpha = r\delta$

[Proposition 3.4 · p. 9](#)

3 A large stabilizer would lower volume

$$m/r \geq 2/5 \implies Q = mhr^3 \geq 324/5 > 64$$

$$\frac{V}{r^4 h} \leq \frac{(1 + 3\delta)^4}{(1 + \delta Q^{1/3})^3} \leq \kappa < 1$$

Choose a fixed δ using derivative $12 - 3Q^{1/3} < 0$ at zero

[Proposition 2.2; Lemma 2.1 · pp. 5–6](#) [Proposition 3.4 · p. 9](#)

4 Contradict approximation to the minimizing ray

$$r^4 h \longrightarrow V \quad \text{forces} \quad m < 2r/5$$

Use lc of $(\tilde{C}, E)$ and local Cartierness of mE

[Lemma 3.1 · p. 7](#) [Proposition 3.4 · p. 9](#)

5 Use Cartierness for an integral lower bound

$$A_C(F) \geq r \text{ ord}_F(E) \geq r/m > 5/2$$

$$A_C(F) \in \mathbb{Z} \implies A_C(F) \geq 3$$

Character-sensitive transverse jets and the discrepancy bound

[Open figure ↗](#) [TeX source ↓](#)

1. Test valuations against the minimum over all valuations

The grading extraction in Lemma 3.1 has $K_{\tilde{C}} = \pi^* K_C + (r - 1)E$ and an lc pair $(\tilde{C}, E)$. On the smooth slice cover with stabilizer μ_m , assign weights $(1, \alpha, \alpha, \alpha)$ to (u, z_1, z_2, z_3) and restrict the resulting valuation to the function field. For rational $\alpha > 0$, this v_α is proportional to a divisorial valuation and $A_C(v_\alpha) = r + 3\alpha$.

The image of functions of degree j is bounded by the minimum of $\dim S_j$ and the dimension of slice jets in the matching character. The latter has leading term $\ell^3/(6m)$: forgetting the character and hence m would lose the required estimate. Integrating over degrees gives

$$V \leq (r + 3\alpha)^4 \frac{h}{(1 + \alpha(mh)^{1/3})^3}.$$

[Lemma 3.1 · p. 7](#) [Lemma 3.3 · p. 8](#)

2. Contradiction to approximate minimality, then integrality

If $m/r \geq 2/5$, then $Q = mhr^3 \geq 324/5 > 64$. Set $\alpha = r\delta$ and divide the bound by r^4h . The derivative at zero of $(1 + 3\delta)^4/(1 + \delta Q^{1/3})^3$ is $12 - 3Q^{1/3} < 0$. A fixed small positive rational δ gives a ratio uniformly below one, contradicting $V/(r^4h) \rightarrow 1$. The cutoff limit is taken separately for each valuation, so uniformity of the jet-error constants in the grading is unnecessary.

Local Cartierness of mE gives $\text{ord}_F(E) \geq 1/m$. The lc pair then yields $A_C(F) \geq r \text{ord}_F(E) > 5/2$. Since K_C is Cartier, this is an integer, hence at least three. Rescaling to $y = \lambda j$, $\lambda = 5/(2r)$, fixes the canonical weight at $5/2$ and gives $\lambda m < 1$ at every nonvertex point.

[Proposition 3.4 · p. 9](#)

4. Proposition 5.1 — Low-weight functions and minimal lc centers raise the threshold

Proof target : [Proposition 5.1 · pp. 15–17](#)

Proposition 5.1 — The maximal-ideal threshold

Let C be the fourfold cone of Proposition 2.2, with vertex $\mathfrak{o}$. Fix a grading satisfying Proposition 3.4 and rescale its weights so that the canonical weight is $5/2$. Then

$$\text{lct}_{\mathfrak{o}}(C; \mathfrak{m}_{\mathfrak{o}}) > \frac{3}{2}.$$

The threshold refers to log canonicity in a neighborhood of $\mathfrak{o}$. For the maximal ideal it is the infimum of $A_C(F)/\text{ord}_F(\mathfrak{m}_{\mathfrak{o}})$ over prime divisors centered at $\mathfrak{o}$.

[Proposition 5.1 · pp. 15–17](#)

1 Find short-weight functions

$$\lambda m < 1, \quad \text{wt}(\omega_C) = 5/2, \quad D = 5/3$$

$$\bigoplus_{0 < y < D} S_y \neq 0$$

After the Hilbert frequency gap and sign tests, average general equal-weight members

[Lemma 4.1 · pp. 10–11](#) [Proposition 4.3 · pp. 12–14](#)

[Proposition 5.1, boundary and lc centers · pp. 15–16](#)

2 A small threshold produces an lc boundary

$$\text{lct}_o(C; \mathfrak{m}_o) \leq 3/2 \quad \text{assumed}$$

$$(C, G) \text{ lc}, \quad \text{wt}(G) = td_0 < 5/2$$

Boundary weight $< 5/2$ excludes the vertex; apply affine subadjunction

[Proposition 5.1, boundary and lc centers · pp. 15–16](#) [\[FG12, Theorem 7.2, p. 11\]](#)

3 Pass to a positive-dimensional minimal lc center

$$1 \leq \dim Z \leq 3, \quad Z \text{ normal, CM}, \quad T = S/I_Z$$

$$T_y = 0 \quad (0 < y < D), \quad \lambda m < 1$$

Curve normality / surface $\mathbb{Q}$ -factoriality / adjunction on the threefold center

[Proposition 5.1, center dimensions · pp. 16–17](#)

[\[dFH09, Proposition 7.2, p. 16; Corollary 7.15, p. 18\]](#)

4 Verify nonvanishing hypotheses on the center

$$\dim Z = 1 : T = \mathbb{C}[z], \quad 0 < \text{wt}(z) < 1$$

$$\dim Z = 2 : \omega_T^{[q]} \hookrightarrow T \quad \text{send a positive-weight generator to 1}$$

$$\dim Z = 3 : \omega_T^{[q]} \hookrightarrow T, \quad \deg = b - 5q/2 < 0$$

In dimensions two and three, canonical-module positivity gives short-weight nonvanishing

[Lemma 4.2 · pp. 11–12](#) [Proposition 4.3 · pp. 12–14](#)

5 Low-weight vanishing contradicts nonvanishing

$$\exists y \in (0, D) : T_y \neq 0$$

$$\text{lct}_o(C; \mathfrak{m}_o) > \frac{3}{2}$$

Vanishing of low-weight functions contradicts nonvanishing on a minimal lc center

[Open figure ↗](#) [TeX source ↓](#)

1. A Hilbert-series frequency gap forces short weights

For a normal Cohen–Macaulay graded domain T , use the natural weights on its canonical module to form

$$\mu_T = \sum_y \dim T_y \delta_y + (-1)^{\dim T - 1} \sum_y \dim(\omega_T)_y \delta_{-y}.$$

Lemma 4.1 uses $\lambda m < 1$ to place all nonzero Hilbert-series pole frequencies outside $[-1, 1]$. A Schwartz test F with Fourier transform supported in $(-1, 1)$ then satisfies

$\int F d\mu_T = \int F(y)P_T(y) dy$, where P_T encodes the polar part of $H_T(s) = \sum_y \dim T_y e^{-sy}$ at $s = 0$. The manuscript derives $H_{\omega_T}(z) = (-1)^{\dim T} H_T(z^{-1})$ by dualizing a graded free resolution; we follow that internal calculation. In this identity $H_T(z)$ is the integer-graded Hilbert series, with $z = e^{-\lambda s}$.

For the four-dimensional ring S , canonical weight $5/2$ makes the density $a(x^3 + cx)$ after shifting to $x = y + 5/4$. The manuscript multiplies squares of band-limited functions by sign-changing polynomials. After deriving $c \geq -1$ and $a \geq 432/625$, it shows that the absence of function weights in $(0, D)$, $D = 5/3$, gives contradictory integral bounds. This is Proposition 4.3(a). Its two- and three-dimensional version (b) has an **additional hypothesis**: an injection $\omega_T^{[q]} \hookrightarrow T$ must lower weights by a positive amount. Lemma 4.2 then gives positive canonical-section weights and positivity of the next-to-leading coefficient of P_T .

Lemma 4.1 · pp. 10–11 Lemma 4.2 · pp. 11–12 Proposition 4.3 · pp. 12–14

2. A small threshold creates an lc center where all low weights vanish

Assume $\text{lct}_o(C; \mathfrak{m}_o) \leq 3/2$. Choose a basis f_i of the homogeneous functions with $0 < y < D$ and put $d_0 = \max \text{wt}(f_i) < D$. Power them to a common weight B , obtaining an ideal $\mathfrak{b}$. Its threshold $t = \text{lct}_o(C; \mathfrak{b}^{d_0/B})$ satisfies $0 < t \leq 3/2$ and $td_0 < 5/2$. Averaging sufficiently many general members constructs an effective homogeneous boundary G of total weight td_0 . The pair (C, G) is lc, and every f_i vanishes on every lc center. The argument does not assume that $\mathfrak{b}$ is maximal-ideal primary.

On the grading extraction, $A_{C,G}(F) > 0$ for every divisor centered exactly at o : the strict inequality $td_0 < 5/2$ makes the exceptional coefficient difference positive. Thus o itself is not an lc center. A minimal center Z has $1 \leq \dim Z \leq 3$. Affine subadjunction applies because C is normal affine, G is an effective rational boundary, $K_C + G$ is $\mathbb{Q}$ -Cartier, and the pair is lc. The resulting klt pair (Z, Δ_Z) makes Z normal with rational singularities, hence Cohen–Macaulay.

Proposition 5.1, boundary and lc centers · pp. 15–16 [FG12, Theorem 7.2, p. 11]

3. Check the nonvanishing hypothesis in each center dimension

Give $T = S/I_Z$ the induced grading. Its stabilizers equal those of the same points in C , so $\lambda m < 1$ remains valid even if the action is ineffective. However, T_y is an image of S_y , and all f_i vanish on Z , so $T_y = 0$ for $0 < y < D$.

If $\dim Z = 1$, the normal graded curve has ring $\mathbb{C}[z]$, and the stabilizer bound forces the weight of z below one, a contradiction. If $\dim Z = 2$, a surface admitting a klt boundary is $\mathbb{Q}$ -factorial; removing the effective boundary makes Z itself $\mathbb{Q}$ -Gorenstein klt. A homogeneous generator of a reflexive canonical power has positive weight. Sending it to 1 gives the required weight-lowering injection.

If $\dim Z = 3$, then $U = Z \setminus \{o\}$ is a Cartier divisor in smooth $C \setminus \{o\}$. Choose a nonzero low-weight function f , put $b = \text{wt}(f) < D$ and $q = \text{ord}_Z(f) \geq 1$. Equivariant adjunction and multiplication by f give $\omega_U^{\otimes q} \rightarrow \mathcal{O}_U$ of degree $b - q(5/2) < 0$. Reflexive extension across the vertex yields $\omega_T^{[q]} \hookrightarrow T$. No global $\mathbb{Q}$ -Gorenstein hypothesis on Z is required. In dimensions two and three, Proposition 4.3(b) now contradicts low-weight vanishing, proving the strict threshold bound.

Proposition 5.1, center dimensions · pp. 16–17 [dFH09, Proposition 7.2, p. 16; Corollary 7.15, p. 18]

5. Which papers supply which steps

Input

[XZ25, Theorem 1.2, p. 2] [LX18, Theorem 1.3, p. 3; Lemma 4.10, p. 35] [Li18, Theorem 3.1, p. 6]

Used at

Proposition 2.2, Lemma 2.1

Role and hypotheses

Finite generation for a klt minimizer, preservation and minimization of volume on the cone, embedding-dimension comparison with the original local ring

Checking scope

Stated results in the specified versions and their uses on pp.4–6 checked

Input

[XZ21, Theorem 1.3, p. 1] [Liu22, Corollary 2.18, p. 9]

Used at

Proposition 2.2

Role and hypotheses

Degree formula for crepant finite Galois covers, smaller bound at nonisolated points, $dV \leq 256$

Checking scope

Hypotheses of the theorem and corollary compared with p.6

Input

[FG12, Theorem 7.2, p. 11] [dFH09, Proposition 7.2, p. 16; Corollary 7.15, p. 18]

Used at

Proposition 5.1

Role and hypotheses

A klt boundary on a minimal affine lc center, $\mathbb{Q}$ -factoriality in dimension two

Checking scope

Statements compared with their uses on pp.16–17

Input

[Reid83, Theorem (1.1), print p. 138 / PDF p. 5; Section (0.5), print p. 134 / PDF p. 3]

Used at

Proposition 6.1

Role and hypotheses

The terminal Gorenstein threefold section is a cDV hypersurface

Checking scope

Theorem (1.1) and §(0.5) read in the original spread scan and compared with p.18

Input

[Liu22, Theorem 2.12, p. 6] [Liu22, Lemma 2.15, p. 7]

Used at

End of Theorem 1.1

Role and hypotheses

Upper bound, equality classification, and analytic invariance once the original germ is lci

Checking scope

Theorem 2.12 and Lemma 2.15 compared with p.18

The relationship with the every-dimensional manuscript is recorded in the catalogue guide. No arrow asserting that this paper depends on that manuscript is introduced. Methodological comparisons with sphere packing and orbifold Riemann–Roch are distinct from the direct inputs above.

6. Return to the sources

- [Theorem 1.1 · p. 1](#) — Hypotheses, zero boundary, dimension, and equality
- [Proposition 2.2; Lemma 2.1 · pp. 5–6](#) — Volume preservation and embedding-dimension comparison with the original ring
- [Lemma 3.3 · p. 8](#) — Where character-sensitive jets detect stabilizer order
- [Proposition 4.3 · pp. 12–14](#) — Hilbert duality and sign tests for short-weight nonvanishing
- [Proposition 5.1 · pp. 15–17](#) — The lc boundary and canonical modules in each center dimension
- [Proposition 6.1 · pp. 17–18](#) — Combining the two estimates to obtain a terminal section
- [Proof of Theorem 1.1 · p. 18](#) — The final lci input for the bound and equality

We checked the statement and volume convention, Proposition 2.2, the estimate from Lemma 3.3 to Proposition 3.4, the sign and weight reductions in §4, all three center dimensions in Proposition 5.1, and the section argument and application of the lci theorem in §6 against the proof text. For external inputs, the scope is the statements and applications in the table, without recursive verification of their proofs.

Remaining unchecked are the full local algebra of the slice/extraction construction in Lemma 3.1, an independent recomputation of all Fourier, approximation, and moment calculations in §4, and independent verification of every sheaf identification involving natural gradings and reflexive extension on three-dimensional centers. The manuscript's explanations of these connections were read; completion of this overview does not certify the whole manuscript.