

OpenAI Math Digest

CATALOGUE 036

Numerical semiampleness and generalized minimal models

Catalogue synthesis & 4 proof overviews

Independent reading guide · English edition

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AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 4 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

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Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

Numerical semiampleness and generalized minimal models

Four manuscripts address numerical semiampleness of nef adjoints and model existence for generalized lc pairs with fixed nef b-data. A semiample numerical representative, equality of real Bott–Chern classes, and semiampleness of the original adjoint are distinct conclusions. The numerical-dimension-one paper supplies a separate construction of minimal models using surface intersections.

1. Papers in official catalogue order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 036 · 2026-10-13 checked source snapshot

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01

Manuscript and overview

Numerical semiampleness of nef adjoint classes on compact Kähler manifolds →

Short name: Numerical semiampleness of nef adjoint classes on compact Kähler manifolds

October 6, 2026 · 52 pages

Original paper (GitHub) ↗

Principal claim

For a smooth compact Kähler rational SNC klt pair, with $K+B$ pseudo-effective and M and $K+B+M$ nef, a semiample rational line represents the same real Bott–Chern class.

Theorems 1.1–1.2 · pp. 2–3 ↗

Role and available overview

Combines vanishing in algebraic dimension zero, descent of nef lines and positive parts on a projective base.

Read the proof overview →

#

02

[Manuscript and overview](#)

Numerical Semiampleness of Nef Adjoint Divisors →

Short name: Numerical Semiampleness of Nef Adjoint Divisors

October 3, 2026 · 37 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For a projective klt rational pair in characteristic zero with $K+B$ pseudo-effective, M nef on X and $K+B+M$ nef, obtains numerical equivalence to a semiample rational divisor on X .

[Theorem 1.1; Corollary 1.2 · pp. 1–2](#) ↗[Role and available overview](#)

Bigness from full nef dimension and induction for positive parts via nef reduction.

[Read the proof overview](#) →

#

03

[Manuscript and overview](#)

Minimal models and Mori fibre spaces for generalized log canonical $\mathbb{Q}$ -pairs →

Short name: Minimal models and Mori fibre spaces for generalized log canonical $\mathbb{Q}$ -pairs

September 24, 2026 · 79 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For a projective generalized lc rational pair with fixed nef rational b -data, gives a minimal model for a pseudo-effective adjoint and a Mori fibre space otherwise.

[Theorem 1.1; Corollary 1.2 · p. 2](#) ↗[Role and available overview](#)

Transfers a local phase argument and smooth relative weak Zariski decompositions to the NQC generalized MMP.

[Read the proof overview](#) →

#

04

Manuscript and overview

Minimal models in numerical dimension one →

Short name: Minimal models in numerical dimension one

September 24, 2026 · 14 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

A smooth complex projective X of dimension at least three with pseudo-effective K_X and section-growth numerical dimension one has a $\mathbb{Q}$ -factorial terminal minimal model; a separate input makes the same endpoint good.

[Theorem 1.1; Corollary 1.2 · p. 2](#) ↗

Role and available overview

A separate route to nefness using a model fixed in codimension two and two surfaces.

[Read the proof overview →](#)

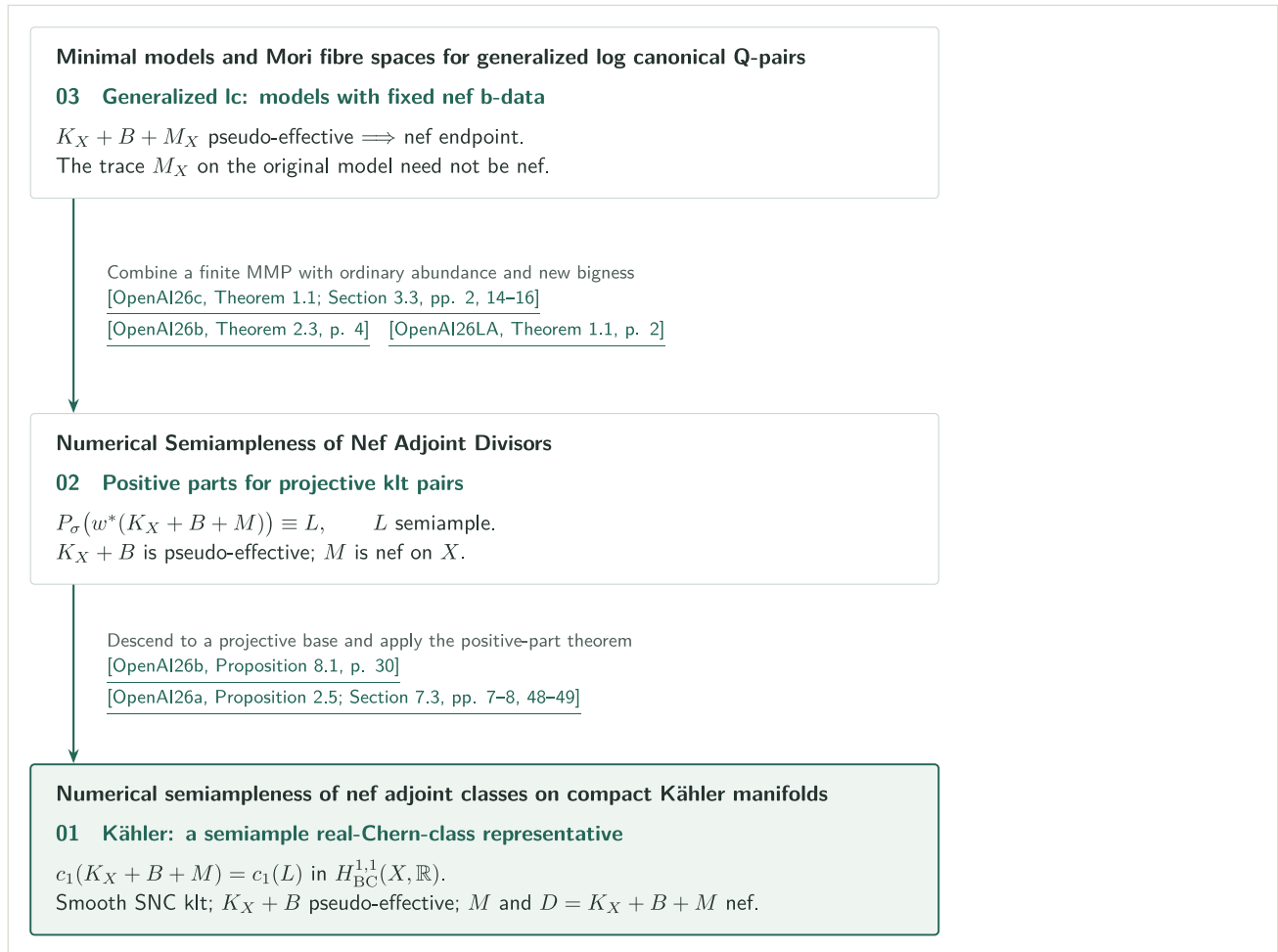
2. From model existence to numerical semiampleness

The diagram shows a chain of reductions linking three manuscripts. Downward edges are direct uses of specific results, not containment of every conclusion or verification of complete proofs. Box numbers follow the official inventory; their placement follows the dependency order.

References in the diagram

- [OpenAI26a](#) OpenAI — Numerical semiampleness of nef adjoint classes on compact Kähler manifolds. October 6, 2026.
- [OpenAI26b](#) OpenAI — Numerical Semiampleness of Nef Adjoint Divisors. October 3, 2026.
- [OpenAI26c](#) OpenAI — Minimal models and Mori fibre spaces for generalized log canonical $\mathbb{Q}$ -pairs. September 24, 2026.
- [OpenAI26d](#) OpenAI — Minimal models in numerical dimension one. September 24, 2026.
- [OpenAI26LA](#) OpenAI — Log abundance in characteristic zero. September 24, 2026.

OpenAI26c supplies model existence, OpenAI26b the projective positive part, and OpenAI26a the Kähler real-Chern-class conclusion. OpenAI26LA is an additional input for ordinary adjoints. OpenAI26d denotes the separate surface argument discussed below.



The main route from models and ordinary abundance through projective positive parts to Kähler numerical semiampleness

[Open figure ↗](#) [TeX source ↓](#)

The projective paper uses the finite MMP in the $\mathbb{Q}$ -factorial generalized klt setting to control two adjoints. It combines ordinary log abundance with its new bigness argument to construct the positive part. The Kähler paper applies this positive-part theorem to an ordinary klt pair on a **projective base** obtained after algebraic reduction. This is not an arrow running an arbitrary generalized MMP on a Kähler space.

[OpenAI26c, Theorem 1.1; Section 3.3, pp. 2, 14–16] [OpenAI26b, Theorem 2.3, p. 4]
[OpenAI26b, Proposition 8.1, p. 30] [OpenAI26a, Proposition 2.5; Section 7.3, pp. 7–8, 48–49]

Follow the projective article and the Kähler article for hypotheses and proof steps. The Kähler paper also uses ordinary Kähler log abundance (the October 6, 2026 version in 034) and Orbifold Itaka (033); their versions and distinct roles are recorded in its reference guide.

The [numerical-dimension-one](#) article fixes positive perturbations in codimension two and proves nefness using growth with a fixed twist and a surface through a negative curve. Its Theorem 1.1 does not use the generalized existence theorem above. Corollary 1.2 separately adds ordinary log abundance to make the same terminal endpoint a good minimal model.

[[OpenAI26d](#), Theorem 1.1; Corollary 1.2, p. 2] [[OpenAI26LA](#), Theorem 1.1, p. 2]

Corollary 1.2 of the [generalized lc](#) article likewise treats ordinary pairs with $M=0$, to which ordinary abundance can subsequently be added. No semiamplessness conclusion is extrapolated to nonzero nef b-data. Unlisted citation relations remain uninvestigated rather than being classified as absent.

3. Sources and checking scope

The official overview and CONTENTS are fixed at commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a. The subject is the second in official display order, “Algebraic and complex geometry”; entry 036 appears on PDF p. 5. The four titles, order, versions and page counts were checked, followed by the article statements, principal reductions, and statements and applications of direct inputs. Remaining textbook checks and unverified external proofs are described in each article. These accounts explain the manuscripts' claims and do not constitute specialist review or formal verification of the entire arguments.

[OpenAI Research Catalog · Catalog 036 · p. 5](#)

Numerical semiampleness of nef adjoint classes on compact Kähler manifolds

Kill the nef class on algebraic-reduction fibers, then pass to an ordinary adjoint on a projective base

The manuscript studies an ordinary klt adjoint $K_X + B$ plus a nef rational line bundle M on the original space. It constructs a numerical positive-part decomposition by induction on dimension and removes the negative part when the sum is nef. The output is a semiample representative of the same real Bott–Chern class; the flat part of the original line may change.

Manuscript author: OpenAI · October 6, 2026 · 52 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A semiample representative on the original space

Let X be a smooth connected compact Kähler manifold, $B \geq 0$ a rational SNC divisor with every coefficient strictly less than 1, and $M \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$. Suppose $K_X + B$ is pseudo-effective and both M and $D = K_X + B + M$ are nef. There exists a semiample rational line bundle L on X such that

$$c_1(L) = c_1(D) \quad \text{in } H_{\text{BC}}^{1,1}(X, \mathbb{R}).$$

Here M is a line bundle on X itself, rather than arbitrary nef b-divisor data living only on a higher model.

[Theorem 1.1 · p. 2](#)

Theorem 1.2 — A stronger decomposition including the negative part

Let T be a smooth connected compact Kähler manifold, $B \geq 0$ a rational SNC divisor with every coefficient strictly less than 1, and M a nef rational line bundle. If $J = K_T + B$ is pseudo-effective, there are a smooth compact Kähler modification $\mu : U \rightarrow T$, a semiample rational line bundle P , and an effective rational divisor R such that

$$\mu^*(J + M) \equiv P + R, \quad R = N(c_1(\mu^*(J + M))).$$

Moreover, $a(T) = 0$ implies $c_1(M) = 0$. Here $a(T)$ is the algebraic dimension and N is the analytic divisorial negative part. The sum $J + M$ need not be nef. Throughout, $\equiv$ means equality of real Bott–Chern classes, whereas $\sim_{\mathbb{Q}}$ means equality in $\text{Pic} \otimes \mathbb{Q}$.

Theorem 1.2 · p. 3

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Numerical semiampleness of nef adjoint classes on compact Kähler manifolds](#). External inputs use the following keys.

- [OpenAI26b](#) OpenAI — Numerical Semiampness of Nef Adjoint Divisors: Projective numerical semiampness of positive parts and a generalized klt MMP with fixed nef data. October 3, 2026 version.
- [OpenAI26K](#) OpenAI — Log abundance for compact Kähler spaces under logarithmic Iitaka subadditivity: Actual rational-line decomposition of ordinary adjoints and contraction of a known negative part. October 6, 2026.
- [OpenAI26O](#) OpenAI — Orbifold and logarithmic Iitaka subadditivity: The logarithmic Iitaka hypothesis, highest-line positivity and actual line descent to a period quotient. September 26, 2026.
- [MWWZ](#) Shin-ichi Matsumura; Juanyong Wang; Xiaojun Wu; Qimin Zhang — Beauville–Bogomolov–Yau decomposition for Kähler generalized pairs: Finite quasi-étale product decomposition for a klt pair of Calabi–Yau type with zero nef b-part. arXiv:2506.23218v2 (September 9, 2026); title-page date September 10, 2026.
- [HX](#) Christopher Hacon; Lingyao Xie — On the Kähler MMP and the transcendental base-point-free theorem: Length bound for a negative ray of an ordinary klt pair. arXiv:2607.24986v1 (July 27, 2026); title-page date July 29, 2026.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.2 — Simultaneous induction through three geometric cases

Proof target : Theorem 1.2 · p. 3 See the hypotheses and both conclusions.

Proof route. The decomposition and the vanishing assertion for $a(T) = 0$ are proved simultaneously. The projective case uses an external positive-part theorem. Algebraic dimension zero reduces to a [simple symplectic model](#). In the intermediate case, [fiberwise vanishing](#) descends M to the algebraic base, and an [ordinary klt adjoint construction](#) permits a return to the projective input. Specializing to nef D then gives Theorem 1.1.



Algebraic-dimension cases converge on the projective positive-part theorem

Open figure ↗ TeX source ↓

1. The projective and algebraic-dimension-zero cases

If $a(T) = \dim T$, the Kähler–Moishezon theorem makes T projective. The projective input supplies a rational numerically semiample positive part from pseudo-effectivity of $K_T + B$ and nefness of M on T . Separately, a generalized klt MMP with fixed nef data produces, on a common resolution,

$$u^*(K_T + B + M) \sim_{\mathbb{Q}} v^*H_m + E.$$

Here H_m is nef and $E \geq 0$ is v -exceptional. Negative-part calculus gives

$E = N(u^*(K_T + B + M))$. The trace of M on the generalized model is not assumed nef.

Proposition 2.5 · pp. 7–8

[OpenAI26b, Proposition 8.1, p. 30]

[OpenAI26b, Theorem 2.3, p. 4]

If $a(T) = 0$, Proposition 5.1 gives $M \equiv 0$. The ordinary adjoint decomposition is an actual rational-line identity $\mu^*J \sim_{\mathbb{Q}} P + R$, and its semiample part P is rationally trivial in this case. Adding M therefore leaves the negative part unchanged. The logarithmic Iitaka hypothesis required by ordinary decomposition is supplied by subadditivity with reduced SNC boundaries and $\text{Supp}(f^*D_Y) \subseteq \text{Supp}(D_X)$.

Theorem 2.3; Proposition 2.4 · pp. 6–7

[OpenAI26K, Definition 2.1; Theorem 2.13, pp. 6, 13]

[OpenAI26O, Corollary 6.2, pp. 40–41]

2. Keep the ordinary program over the projective algebraic base

For $0 < a(T) < n$, resolve the algebraic reduction as $h : T \rightarrow W$. Proposition 6.4 and Corollary 6.5 give $M \equiv h^*N_W$ with N_W nef, after further modifications. Choose an ample Cartier divisor A_W and an integer $b > 2n$, and decompose $J + bh^*A_W$ as an ordinary klt adjoint. The ordinary program contracting its known negative part preserves the actual semiample positive line.

Preservation of the map to W needs a separate argument. A $(J_i + bC_i)$ -negative ray is J_i -negative and is generated by a rational curve with $0 < -J_i \cdot \Gamma \leq 2n$. Since $C_i = h_i^*A_W$ is Cartier, $C_i \cdot \Gamma > 0$ would imply

$$(J_i + bC_i) \cdot \Gamma \geq -2n + b > 0,$$

a contradiction. Thus $C_i \cdot \Gamma = 0$, and connected-fiber descent preserves the actual morphism h_i . At the endpoint (Y, Δ) , the line $J_Y + bh_Y^*A_W$ is semiample.

Proposition 7.2 · pp. 47–48

[OpenAI26K, Proposition 3.8, pp. 23–24]

[HX, Theorem 1.3, p. 2]

3. Solve on the projective base and pull back the entire negative part

Combine the semiample map with h_Y and take the Stein factorization $f : Y \rightarrow Z$, $j : Z \rightarrow W$. The space Z is normal projective, and $a(T) = \dim W \leq \dim Z \leq a(Y) = a(T)$ gives positive relative dimension for f . Subtracting the ample base term yields $J_Y \sim_{\mathbb{Q}} f^*H$. Ordinary nonvanishing and exceptional comparisons make H pseudo-effective. Proposition 4.1 gives $H \sim_{\mathbb{Q}} K_Z + \Delta_Z$, with (Z, Δ_Z) klt and $\Delta_Z \geq 0$. The projective input thus applies to $H + j^*N_W$.

Pull its decomposition to a common Kähler resolution V , obtaining a semiample class represented by $g^*P_{Z'}$ and an effective divisor g^*E . Every component of g^*E maps into codimension at least two on the final projective model Z_m , but $V \rightarrow Z_m$ need not be birational. Lemma 7.1 uses mixed Hodge–Riemann relations to establish $N(r^*H_m + g^*E) = g^*E$. Adding the effective exceptional comparison F of the ordinary program gives

$$q^*(J_1 + M_1) \equiv g^*P_{Z'} + g^*E + F, \quad N(q^*(J_1 + M_1)) = g^*E + F.$$

Lemma 2.2 removes the effective exceptional errors introduced by resolution. This yields the semiample rational positive class, rationality of the remainder, and its identification with the entire negative part.

Section 7.3 · pp. 48–49

Lemma 7.1 · p. 46

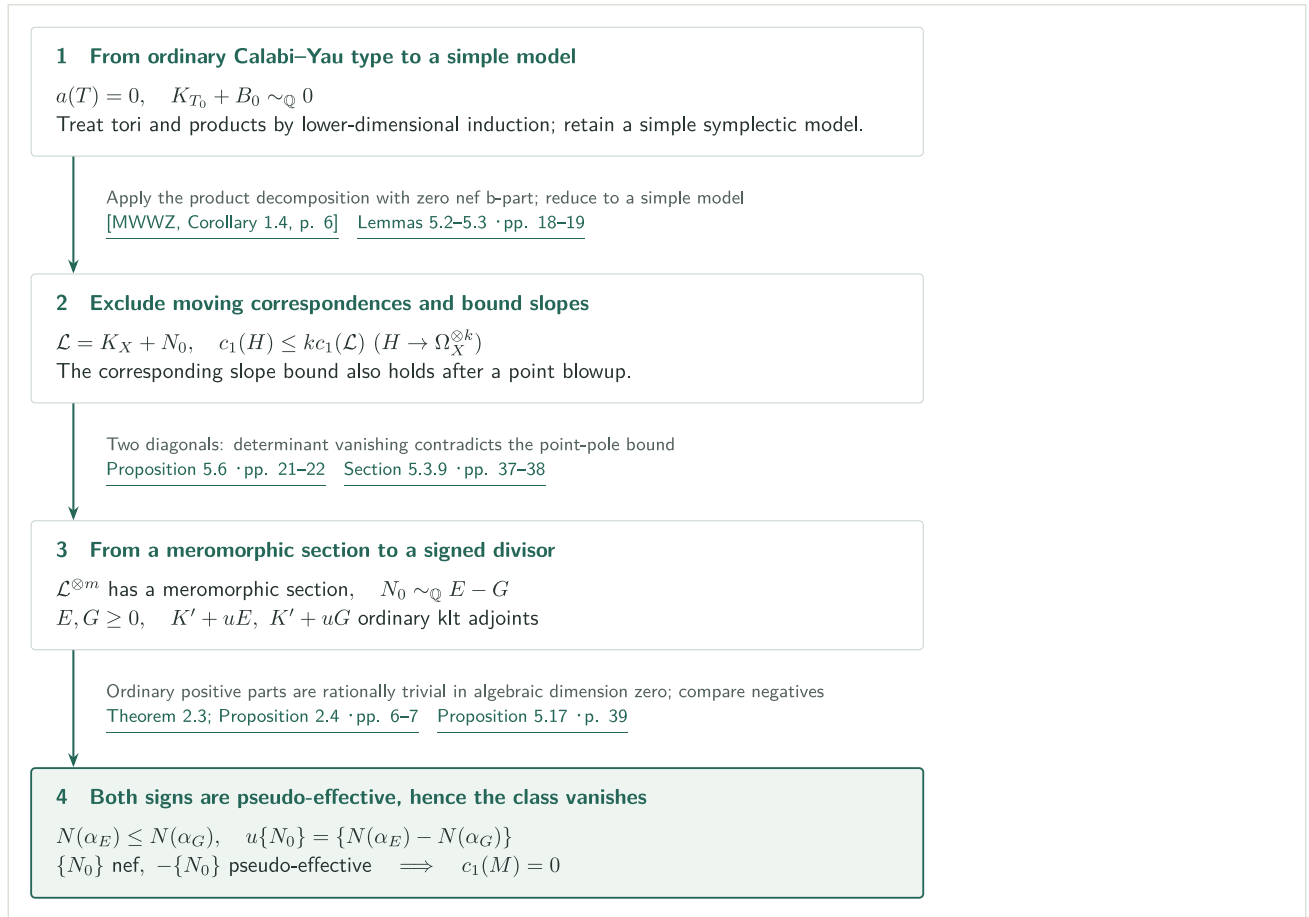
3. Proof of Proposition 5.1 — Why the nef class vanishes in algebraic dimension zero

Proof target : Proposition 5.1 · p. 18

Proposition 5.1 — The algebraic-dimension-zero induction step

Let $n \geq 1$ and assume the algebraic-dimension-zero assertion of Theorem 1.2 in every dimension below n . Let T be a smooth connected compact Kähler n -fold with $a(T) = 0$, and $B \geq 0$ a rational SNC divisor with coefficients strictly less than 1. If $K_T + B$ is pseudo-effective, every nef rational line bundle M satisfies $c_1(M) = 0$.

Proposition 5.1 · p. 18



Product decomposition and meromorphic nonvanishing eliminate the nef class

[Open figure ↗](#) [TeX source ↓](#)

1. From an ordinary Calabi–Yau-type pair to a simple model

Ordinary decomposition and contraction of its negative part produce $K_{T_0} + B_0 \sim_{\mathbb{Q}} 0$. Apply the MWWZ product decomposition to this ordinary klt pair with zero nef b-part. Rationally connected factors and higher-dimensional strict Calabi–Yau factors on the finite quasi-étale cover are incompatible with $a = 0$, leaving tori and symplectic factors. For multiple factors, lower-dimensional induction on the fibers and mixed intersection numbers kill the nef class. For one symplectic factor, Campana's minimal-fibration result reduces to a simple model with a generically nondegenerate holomorphic two-form. Complete proofs of these ancillary inputs are outside this article's checking scope.

[Lemmas 5.2–5.3 · pp. 18–19](#)

[\[MWWZ, Corollary 1.4, p. 6\]](#)

2. Extract a meromorphic section from cotangent slopes

If a finite cover has positive irregularity, the Albanese map and the torus intersection argument suffice. In the remaining case every smooth generically finite cover has irregularity zero. A family of finite correspondences sweeping $X \times X$ would produce a nonzero one-form on a fixed finite cover by contracting the two-form in a varying direction. Thus the only positive-dimensional proper subvarieties through a very general point are factor slices.

Let N_0 be an integral multiple of M and put $\mathcal{L} = K_X + N_0$. The top wedge of the symplectic form makes K_X effective. Lemma 5.5 bounds line subsheaves of $\Omega_X^{\otimes k}$ by $c_1(H) \leq kc_1(\mathcal{L})$ and supplies the corresponding bounds after a point blowup. Proposition 5.6 uses two diagonals, projective-bundle volume, and determinants of cotangent tensors to produce a nonzero **meromorphic** section of a positive power of $\mathcal{L}$. In the final step, diagonal vanishing becomes a point pole, and bounded point-pole estimates contradict growing determinant orders. This is distinct from holomorphic nonvanishing.

Proposition 5.6 · pp. 21–22

Section 5.3.9 · pp. 37–38

3. Split a signed divisor into two ordinary adjoints

Divide the meromorphic section by a canonical section and resolve to obtain $N_0 \sim_{\mathbb{Q}} E - G$ with $E, G \geq 0$. Choose a small rational $u > 0$ so that both $K' + uE$ and $K' + uG$ are ordinary klt adjoints. They are pseudo-effective, and algebraic dimension zero makes the semiample parts of their ordinary decompositions trivial. On a common model write their classes as α_E, α_G . Since $\alpha_E - \alpha_G = u\{N_0\}$ is nef,

$$N(\alpha_E) \leq N(\alpha_G), \quad u\{N_0\} = \{N(\alpha_E) - N(\alpha_G)\}.$$

Hence $-\{N_0\}$ is also pseudo-effective. Pairing the nef class $\{N_0\}$ and its negative with a Kähler power gives zero mass, so $\{N_0\} = 0$. Injectivity of pullback under generically finite maps and modifications returns the conclusion to T .

Proposition 5.17 · p. 39

4. Proof of Proposition 6.4 — The order in which flat twists are chosen

Proof target : Proposition 6.4 · pp. 44–45

Proposition 6.4 — Vanishing on algebraic-reduction fibers

Assume Theorem 1.2 in every dimension below n . Let T be a smooth connected compact Kähler n -fold with $0 < a(T) < n$, and let $h : T \rightarrow W$ be a holomorphic algebraic reduction with smooth projective base and connected fibers. Let $B \geq 0$ be a rational SNC divisor with coefficients strictly less than 1, let $J = K_T + B$ be pseudo-effective, and let M be a nef rational line bundle. Then $c_1(M|_F) = 0$ on a general smooth fiber F .

Proposition 6.4 · pp. 44–45

1 Choose global data before choosing a fiber

$h : T \rightarrow W$, $(f_i : X_i \rightarrow Z_i)_{i \in I}$, $M_i^* \equiv M$
Relative divisor cycles give countably many maps and fixed flat twists.

Coherent base change and ample twisting; countability allows a simultaneous choice
[Lemmas 6.1–6.3 · pp. 40–43](#)

2 Impose simultaneous constraints on a very general fiber

$F = T_w$, $\kappa(F, (J + M_i^*)|_F) \leq 0$ (all i)
Ratios of lifted sections are constant on fibers by algebraic reduction.

Apply lower-dimensional decomposition and monotonicity of the negative part
[Proposition 6.4 · pp. 44–45](#) [Lemma 3.3 · p. 11](#)

3 A nonzero nef class gives a positive-dimensional semiample base

$M|_F \not\equiv 0 \implies \hat{F} \xrightarrow{g} Q$, $\dim Q > 0$
Descend the detected flat difference to Q ; a flat twist preserves the big base line.

Lift sections using the effective fixed part; descend a positive current to the base
[Corollary 6.5 · p. 45](#) [Proposition 3.1; Corollary 3.2 · pp. 8–10](#)

4 The Iitaka-dimension contradiction yields numerical descent

$\kappa(F, J_F + M_i^*|_F) > 0$ is impossible; $c_1(M|_F) = 0$
 $p^*M \equiv (h')^*N$, N a nef rational line on W'

Countably many global twists lead to fiberwise vanishing and numerical descent

[Open figure ↗](#) [TeX source ↓](#)

1. Fix countably many twists on the total space first

Ordinary nonvanishing fixes $J \sim_{\mathbb{Q}} E \geq 0$. Countably many components of relative divisor-cycle spaces produce global maps $f_i : X_i \rightarrow Z_i$ detecting the semiample maps on fibers. For indices where p_i^*M is numerically trivial on a general f_i -fiber, numerical descent and birational invariance of Pic^0 fix one global representative $M_i^* \equiv M$ on T . Its pullback is an actual rational pullback from the intermediate base.

Only then choose w . Lemma 6.1 applied simultaneously to the countably many $J + M_i^*$ gives $\kappa(F, (J + M_i^*)|_F) \leq 0$ for every i , where $F = T_w$. Coherent base change and an ample twist on the projective base lift fiber sections globally; a nonconstant ratio on the fiber would contradict the algebraic reduction.

Lemmas 6.1–6.3 · pp. 40–43

Proposition 3.1; Corollary 3.2 · pp. 8–10

2. A positive-dimensional semiample base gives a contradiction

Suppose $M|_F \not\equiv 0$. Inductive decomposition and monotonicity of the negative part give

$$\rho^*(J_F + M|_F) \equiv P_F + R_F, \quad P_F \equiv (\rho^*E_F - R_F) + \rho^*(M|_F).$$

Pairing the effective and nef classes on the right with a Kähler power shows that the map $g : \widehat{F} \rightarrow Q$ defined by P_F has $\dim Q > 0$. The class of M vanishes on its general fiber, and cycle detection identifies a previously chosen index i . Using the rational triviality of M_i^* on that fiber, Lemma 3.3 descends the remaining flat difference to a resolution of Q . A flat twist preserves bigness of the base line, giving

$$\kappa(F, J_F + M_i^*|_F) > 0,$$

contrary to the preceding step. Local constancy in $R^2h_*\mathbb{R}$ then gives vanishing on a general smooth fiber.

Proposition 6.4 · pp. 44–45

Lemma 3.3 · p. 11

3. Descend a positive current and recover a rational nef line

Corollary 6.5 feeds this vanishing into Proposition 3.1. After flattening, the vertical and mixed blocks of a positive current vanish, so it descends over the smooth locus. The intersection matrix of fiber components and exceptional negativity remove the remaining vertical divisors. Injectivity of $H^2(W', \mathbb{Q}) \rightarrow H^2(T', \mathbb{Q})$ makes the descended class rational; Lefschetz (1, 1) and the curve criterion on the projective base produce a nef rational line N . The conclusion at this stage is $p^*M \equiv (h')^*N$, not linear descent of the original M .

Proposition 3.1; Corollary 3.2 · pp. 8–10

Corollary 6.5 · p. 45

5. Proof of Proposition 4.1 — From period rank to an ordinary klt adjoint

Proof target : Proposition 4.1 · p. 11

Proposition 4.1 — An adjoint presentation on the projective base

Let (Y, D) be an effective rational klt pair on a normal compact Kähler space, and let $f : Y \rightarrow Z$ be a surjective morphism with connected fibers onto a normal projective variety, with $\dim Y > \dim Z$. If a rational Cartier line bundle H on Z satisfies $K_Y + D \sim_{\mathbb{Q}} f^*H$, there is an effective rational divisor D_Z such that (Z, D_Z) is klt and $H \sim_{\mathbb{Q}} K_Z + D_Z$.

Proposition 4.1 · p. 11

1 An actual adjoint pullback and its moduli line

$$K_Y + D \sim_{\mathbb{Q}} f^*H, \quad L_S = b^*H - K_S - \Delta_S$$

Root cover of the effective klt total space: parabolic extension of the highest Hodge eigenline.

Use the integral period quotient and rational-line descent including the boundary
[\[OpenAI26O, Lemma 4.1; Lemma 4.3; Proposition 4.5, pp. 21–26\]](#)
[Lemmas 4.3–4.4 · pp. 13–15](#)

2 Period input with a real polarization

$$L_{S'} \sim_{\mathbb{Q}} p^*P, \quad P \text{ nef on projective } Q$$

$$\nu(L_{S'}) = \text{rank}(d\Phi_{\text{line}}), \quad \dim Q \leq \text{rank}(d\Phi)$$

Effectivity bounds poles, so vector fields in the line kernel lift holomorphically
[Lemma 4.5 · pp. 15–16](#)

3 Match the line and full period ranks

$$r = \nu(P) \leq \dim Q \leq \text{rank}(d\Phi) = r$$

If $\dim Q > 0$, then P is big; over a point, $L_{S'} \sim_{\mathbb{Q}} 0$.

Kodaira decomposition and Bertini; original effectivity bounds thresholds by 1
[Section 4.4 · pp. 16–17](#)

4 Replace the moduli line by an effective boundary of small coefficients

$$E_S \sim_{\mathbb{Q}} L_S, \quad (S, \Delta_S + E_S) \text{ sub-klt}$$

$$D_Z = b_*(\Delta_S + E_S) \geq 0, \quad H \sim_{\mathbb{Q}} K_Z + D_Z, \quad (Z, D_Z) \text{ klt}$$

The highest-line rank and period quotient produce an effective klt boundary

Open figure ↗ TeX source ↓

1. The discriminant and the highest Hodge eigenline

On a base resolution $b : S \rightarrow Z$, log canonical thresholds define Δ_S and $L_S = b^*H - K_S - \Delta_S$. A root cover of a relative pluriform supplies the highest Hodge eigenline, whose parabolic extension is identified with L_S . The total space need not be projective. Its real polarization need not be rational, but the ambient cohomology retains an integral lattice for the period input. Negative coefficients may occur on the crepant resolution, while effectivity of the original D is essential in the next step.

Lemmas 4.3–4.4 · pp. 13–15

[OpenAI260, Lemma 4.1; Lemma 4.3; Proposition 4.5, pp. 21–26]

2. The eigenline detects the entire period rank

A base vector field in the kernel of the eigenline period differential lifts meromorphically to the root cover by contraction with the highest form. Averaging under the root group and using the strict pole bound from the original effective boundary shows that the descended field is holomorphic and tangent to the boundary. Lift its flow through the functorial resolution and root cover: the Kodaira–Spencer class of the entire family vanishes too. Thus the line and full period differentials have the same kernel and rank.

Combine this with actual line descent $L_{S'} \sim_{\mathbb{Q}} p^*P$ to a period quotient $p : S' \rightarrow Q$:

$$r = \nu(L_{S'}) = \nu(P) \leq \dim Q \leq \text{rank}(d\Phi) = r.$$

If $\dim Q > 0$, the line P is nef and big; if Q is a point, $L_{S'}$ is rationally trivial.

Lemma 4.5 · pp. 15–16

Lemma 4.2 · p. 12

3. Replace the big moduli line by an effective boundary

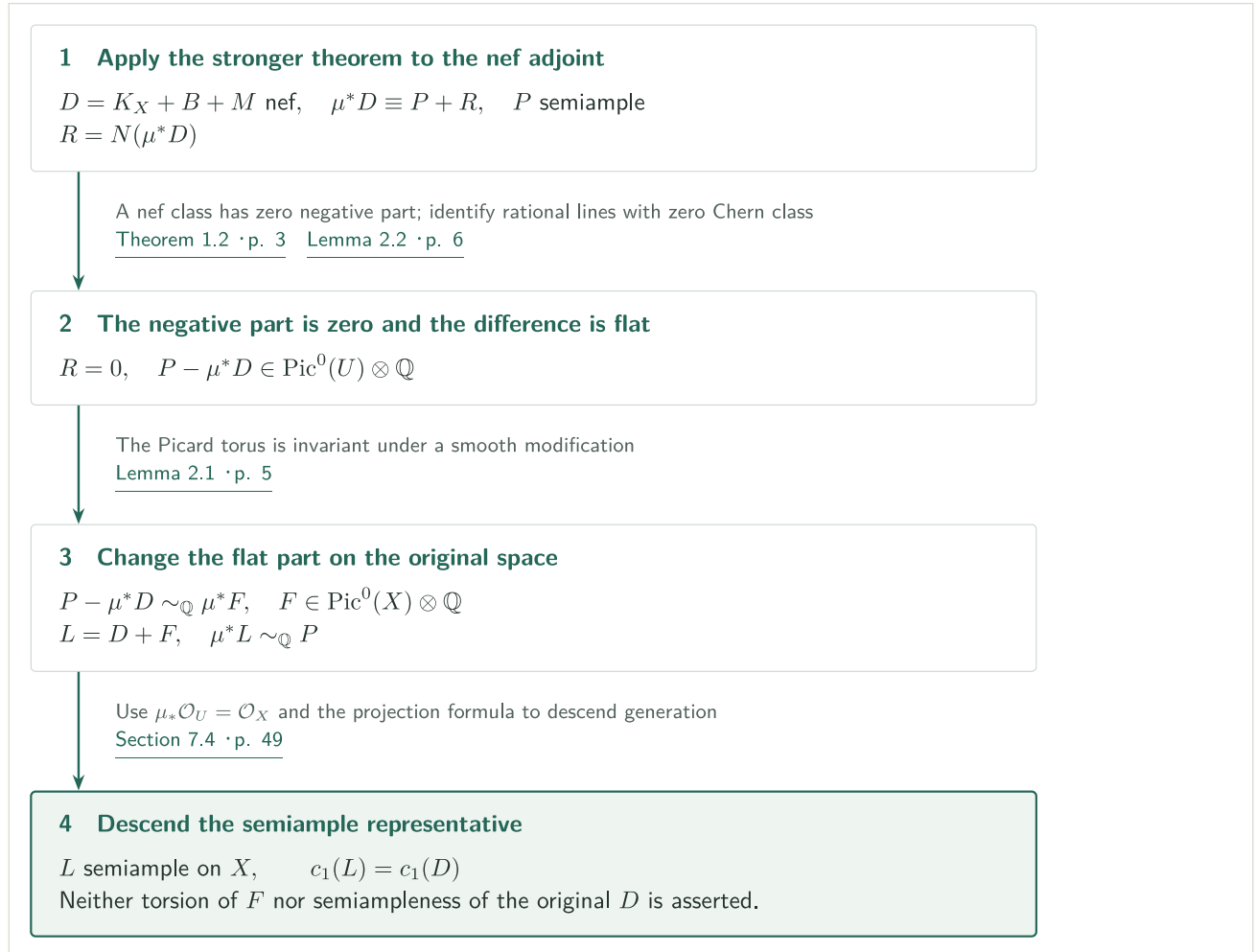
Write $P \sim_{\mathbb{Q}} A + E$ and, for small $\varepsilon > 0$, $P \sim_{\mathbb{Q}} A_\varepsilon + \varepsilon E$ with A_ε ample. A general member of a sufficiently high free system with small coefficient gives $E_S \geq 0$, $E_S \sim_{\mathbb{Q}} L_S$, and a sub-klt pair $(S, \Delta_S + E_S)$. In the point case take $E_S = 0$. Thresholds over prime divisors of the original base are at most 1 because $D \geq 0$, so $D_Z = b_*(\Delta_S + E_S)$ is effective.

Push down the principal-divisor equality to obtain $H \sim_{\mathbb{Q}} K_Z + D_Z$. Comparing its pullback with the identity upstairs gives a crepant equality, and sub-klt upstairs implies klt for (Z, D_Z) . A merely numerical canonical bundle formula would not supply the required rational-line identity.

Section 4.4 · pp. 16–17

6. Proof of Theorem 1.1 — Change the flat part and return to the original space

Proof target : Theorem 1.1 · p. 2 See the main statement.



Vanishing of the negative part and Picard-torus descent of a semiample representative

[Open figure ↗](#) [TeX source ↓](#)

1. Nefness removes the negative part

Apply Theorem 1.2 to D . Since μ^*D is nef, $R = N(\mu^*D) = 0$, and the difference between the semiample line P and μ^*D has zero real Chern class.

[Theorem 1.2 · p. 3](#)

[Lemma 2.2 · p. 6](#)

2. The flat part is allowed to change

Lemma 2.1 places this difference in $\text{Pic}^0(U) \otimes \mathbb{Q}$ and identifies $\text{Pic}^0(X) \simeq \text{Pic}^0(U)$ under a smooth modification. Thus choose $F \in \text{Pic}^0(X) \otimes \mathbb{Q}$ with

$$P - \mu^* D \sim_{\mathbb{Q}} \mu^* F, \quad L = D + F, \quad \mu^* L \sim_{\mathbb{Q}} P.$$

The equality $\mu_* \mathcal{O}_U = \mathcal{O}_X$ and the projection formula descend global generation to L . We have $c_1(L) = c_1(D)$, but F is not proved torsion. A nontorsion degree-zero line on an elliptic curve already explains why semiample-ness of the original D cannot be the conclusion.

[Lemma 2.1 · p. 5](#)

[Section 7.4 · p. 49](#)

7. Which papers supply which steps

Source and result

[\[OpenAI26b, Proposition 8.1, p. 30\]](#)

Used at

Proposition 2.5; completion of induction

Role and required hypotheses

An ordinary projective klt pair, pseudo-effective ordinary adjoint, and nef term on the original model give a numerically semiample positive part

Checking scope

Source statement and application matched

Source and result

[\[OpenAI26b, Theorem 2.3, p. 4\]](#)

Used at

Proposition 2.5

Role and required hypotheses

Nef endpoint and exceptional comparison for a generalized klt program with fixed nef b-data

Checking scope

Statement matched; not termination of every flip sequence

Source and result

[OpenAI26K, Definition 2.1; Theorem 2.13, pp. 6, 13]; [OpenAI26K, Proposition 3.8, pp. 23–24]

Used at

Theorem 2.3; Proposition 2.4; ordinary program

Role and required hypotheses

Actual rational-line decomposition and ordinary contraction of a known negative part

Checking scope

October 6 version and application matched; distinct from the October 4 version of the existing catalog034 article

Source and result

[OpenAI26O, Corollary 6.2, pp. 40–41]

Used at

Theorem 2.3

Role and required hypotheses

Logarithmic subadditivity supplying Kähler abundance Assumption 1.1

Checking scope

Support, boundary and fiber hypotheses matched

Source and result

[OpenAI26O, Lemma 4.1; Lemma 4.3; Proposition 4.5, pp. 21–26]

Used at

Lemma 4.2; projective base

Role and required hypotheses

Real polarization, ambient integral lattice, parabolic highest line, actual rational-line descent

Checking scope

Statements and uses matched; full external constructions not verified

Source and result

[MWWZ, Corollary 1.4, p. 6]

Used at

Lemma 5.3; algebraic dimension zero

Role and required hypotheses

Product decomposition of an ordinary klt Calabi–Yau-type pair with zero nef b-part

Checking scope

Input setting and specialization matched

Source and result

[HX, Theorem 1.3, p. 2]

Used at

Proposition 7.2

Role and required hypotheses

Zero nef b-part, length at most $2n$, and Cartier integrality

Checking scope

Statement and degree inequality matched

8. Return to the sources

- [Theorem 1.2 · p. 3](#) — The two conclusions in the simultaneous induction and the meaning of numerical equivalence.
- [Proposition 5.6 · pp. 21–22](#) — The core construction of a meromorphic section from cotangent-slope bounds.
- [Proposition 6.4 · pp. 44–45](#) — Why global flat twists are fixed before the fiber is chosen.
- [Proposition 4.1 · p. 11](#) — Hypotheses needed to obtain an ordinary klt adjoint on the projective base.
- [Section 7.3 · pp. 48–49](#) — Pullback of the projective input and identification of the whole negative part.
- [Section 7.4 · p. 49](#) — Descent of a semiample representative to the original space.

The reading covered all conclusions of the principal results, the ordinary and projective input hypotheses, numerical descent, the order of choosing countably many global flat twists, period-rank comparison, and the final negative-part and semiample-representative descent. The specified input statements in the table were compared with their applications. This records the author's proof route; it is not an independent verification of every proof.

The intermediate direct-image estimates and determinant calculations in Proposition 5.6, pp. 23–36, were not fully audited. Root-cover extension, rational period-factor constructions and their compatibility on all models, and every original statement behind the ancillary Campana, Ou, cycle-space, mixed Hodge–Riemann, current-support and negativity inputs remain incompletely checked. Unrecorded inter-catalog relations are uninvestigated, rather than absent.

Numerical Semiampleness of Nef Adjoint Divisors

From full nef dimension to bigness, positive parts, and numerical descent

The manuscript claims a semiample numerical representative on the original variety for an ordinary klt adjoint with a nef summand. The central positivity input concerns nef divisors of full nef dimension on Calabi–Yau pairs. Moving jets and a Frobenius comparison supply that input; nef reduction and induction on positive parts then produce the conclusion.

Manuscript author: OpenAI · October 3, 2026 · 37 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Numerical semiampleness of a nef adjoint

Let k be an algebraically closed field of characteristic zero and (X, B) a projective klt $\mathbb{Q}$ -pair. Assume that $T = K_X + B$ is pseudo-effective and that M is a nef $\mathbb{Q}$ -Cartier divisor on X itself. If $D = T + M$ is nef, there is a semiample $\mathbb{Q}$ -Cartier divisor L on the same X such that

$$D \equiv L.$$

Here B is an effective rational boundary and $K_X + B$ is $\mathbb{Q}$ -Cartier.

[Theorem 1.1 · p. 1](#)

Corollary 1.2 — Nef divisors on klt Calabi–Yau pairs

Let (X, B) be a projective klt $\mathbb{Q}$ -pair over an algebraically closed field of characteristic zero. If $K_X + B \equiv 0$, every nef $\mathbb{Q}$ -Cartier divisor on X is numerically semiample.

[Corollary 1.2 · p. 2](#)

The conclusion is neither $D \sim_{\mathbb{Q}} L$ nor semiampleness of D itself. A nontorsion degree-zero line bundle on an elliptic curve is numerically trivial and has no sections in any positive power. The ordinary-pair hypotheses also differ from the generalized-pair data used during the MMP: the original M determines a fixed nef b -divisor, whose trace on a subsequent model need not be nef.

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Numerical Semiampleness of Nef Adjoint Divisors](#). External inputs use the following keys.

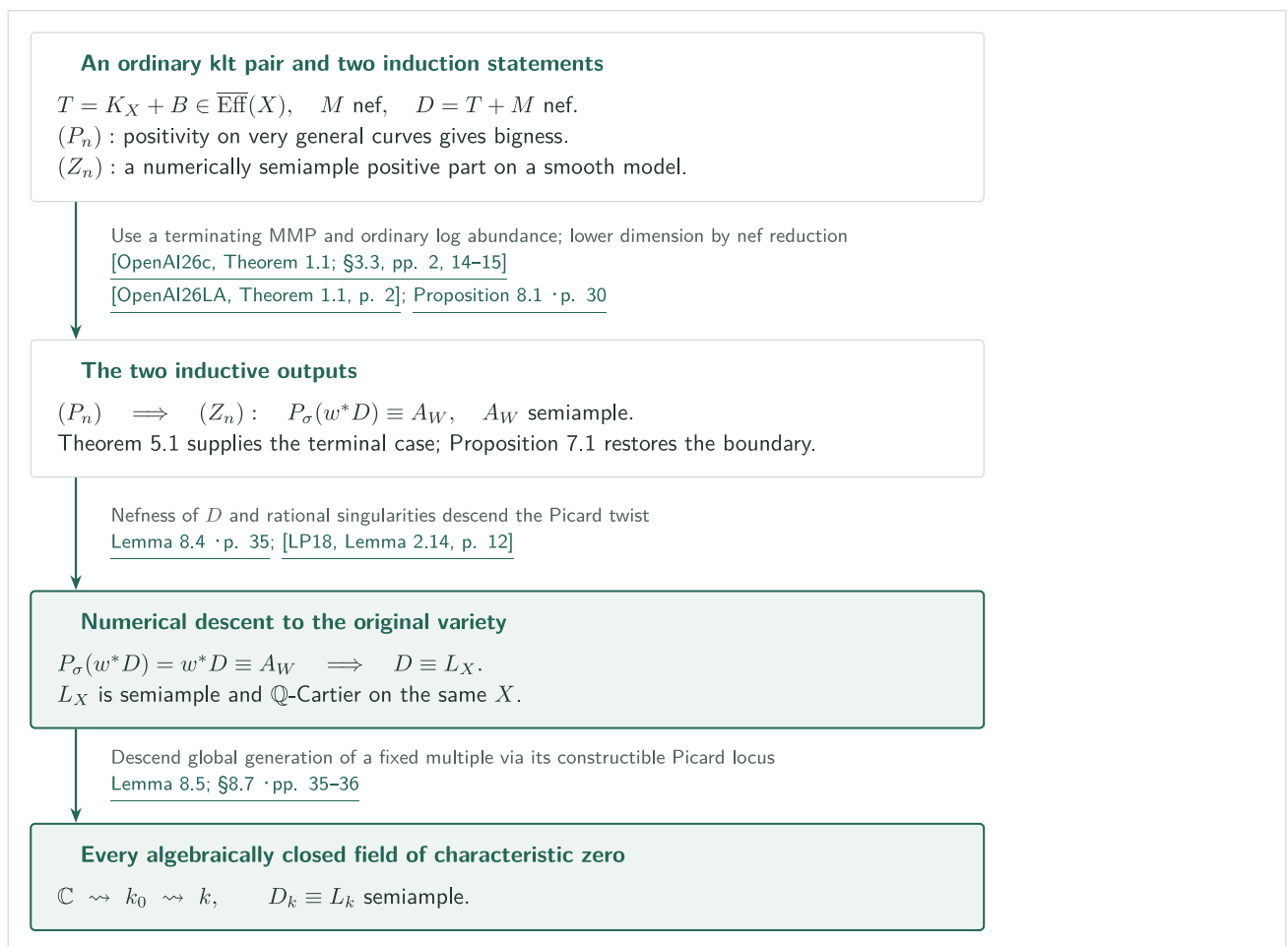
- [OpenAI26LA](#) OpenAI — Log abundance in characteristic zero: Ordinary lc abundance, moving kernels and finite-data Frobenius comparison. September 24, 2026; fixed commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a.
- [OpenAI26c](#) OpenAI — Minimal models and Mori fibre spaces for generalized log canonical $\mathbb{Q}$ -pairs: A terminating choice of MMP retaining fixed nef b -divisor data. September 24, 2026.
- [LP18](#) Vladimir Lazić; Thomas Peternell — On Generalised Abundance, I: Positive-part comparisons and numerical descent to the base and original variety. arXiv:1808.00438v2, 6 September 2018.
- [BZ15](#) Caucher Birkar; De-Qi Zhang — Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs: Uniform birationality for index-one nef data and reduced boundary bounds curve degrees. arXiv:1410.0938v2, 24 December 2015.
- [Amb03](#) Florin Ambro — The moduli b -divisor of an lc-trivial fibration: Replace a $\mathbb{Q}$ -linear pullback ordinary klt adjoint by an effective klt boundary on the base. arXiv:math/0308143v1, 14 August 2003.
- [HX11](#) Christopher D. Hacon; Chenyang Xu — Existence of log canonical closures: Obtain a relative good model from relative finite generation and good models on very general fibres. arXiv:1105.1169v3, stamp 28 June 2012; manuscript date July 24, 2020.
- [Fuj09b](#) Osamu Fujino — Birkar–Cascini–Hacon–McKernan: Finite generation of relative log canonical rings for rational klt pairs. Author notes, version 1.02, 28 November 2009.
- [CP15](#) Frédéric Campana; Mihai Păun — Orbifold generic semi-positivity: an application to families of canonically polarized manifolds: Generic semipositivity of the cotangent bundle on a smooth resolution with pseudo-effective K . Ann. Inst. Fourier 65 (2015), 835–861; published PDF.
- [BDPP13](#) Sébastien Boucksom; Jean-Pierre Demailly; Mihai Păun; Thomas Peternell — The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension: Separate a non-pseudo-effective divisor by a strongly movable curve. arXiv:math/0405285v1.
- [BCHM10](#) Caucher Birkar; Paolo Cascini; Christopher D. Hacon; James McKernan — Existence of minimal models for varieties of log general type: Crepant $\mathbb{Q}$ -factorial terminalization of the canonical index-one cover. arXiv:math/0610203v2, 14 August 2008; PDF compilation date 26 November 2024.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — Two induction statements and descent to X

Proof target : [Theorem 1.1 · p. 1](#) See the [statement](#).

Proof route. Work first over $\mathbb{C}$ with simultaneous induction. Statement (P_n) says that if, for some $a \in \mathbb{Q}_{>0}$, $T + aM$ has positive degree on every integral curve through a very general point, then $T + cM$ is big for every $c \in \mathbb{Q}_{>0}$. Statement (Z_n) drops this positivity condition and asserts that $P_\sigma(w^*(T + M))$ is rational and numerically semiample on a smooth birational model. The [terminal case](#), [boundary reduction](#), and [positive-part argument](#) are detailed below. Nefness of the original D is used at the end to identify its pullback with its positive part.



From the two induction statements to numerical semiamplessness on the original variety

[Open figure ↗](#) [TeX source ↓](#)

1. Preserve an ordinary semiample adjoint while making a generalized adjoint nef

Choose a terminating ordinary T -MMP and apply log abundance to its nef endpoint U_0 . Next choose an MMP for $U_0 + s_0 N_0$ with sufficiently small $s_0 > 0$. If dU_0 is free Cartier, the extremal-ray length bound and the discrete inequality $U_i \cdot C \geq 1/d$ force every step of this second program to be U_i -trivial. Thus the ordinary adjoint remains semiample while the generalized adjoint becomes nef.

The two effective exceptional pullback comparisons transport the positivity on very general curves to the endpoint. When $\kappa(T) > 0$, the fibres of the ordinary semiample fibration admit the lower-dimensional (P_j) . When $\kappa(T) = 0$, one has $U \sim_{\mathbb{Q}} 0$ and reduces to full nef dimension on a Calabi–Yau pair. The input is the **existence of a terminating choice**, not termination of every sequence of flips.

Lemma 3.2 · pp. 6–7

Propositions 3.3–3.4 · p. 8

[OpenAI26c, Theorem 1.1; §3.3, pp. 2, 14–15]

[OpenAI26LA, Theorem 1.1, p. 2]

2. Obtain a numerically semiample positive part and descend to X

Statement (P_n) and the lower-dimensional (Z_j) yield Proposition 8.1. Since the main theorem assumes D nef, $P_\sigma(w^*D) = w^*D$ on every smooth birational model. Choose a semiample $A_W \equiv w^*D$. Clear denominators and the torsion in numerically trivial line bundles modulo Pic^0 to obtain

$$\mathcal{O}_W(rA_W) \simeq w^*\mathcal{O}_X(rD) \otimes P_W, \quad P_W \in \text{Pic}^0(W).$$

Klt singularities are rational, so pullback identifies $\text{Pic}^0(X)$ with $\text{Pic}^0(W)$ and $P_W = w^*P_X$. The right side is now the pullback of a line bundle on X . Using $w_*\mathcal{O}_W = \mathcal{O}_X$ and the projection formula descends global generation. The resulting divisor twists rD by P_X , so the conclusion remains numerical equivalence.

Proposition 8.1 · p. 30

Lemma 8.4 · p. 35

[LP18, Lemma 2.14, p. 12]

3. Pass to every algebraically closed field of characteristic zero

Descend the variety, boundary and Cartier data to a finitely generated field, and embed its algebraic closure k_0 into $\mathbb{C}$. The complex conclusion gives a fixed $r > 0$ and $P \in \text{Pic}^0(X_{\mathbb{C}})$ for which $\mathcal{O}(rD) \otimes P$ is globally generated. In a Poincaré family over the Picard variety, global generation for this fixed r defines a constructible set of parameters. Its nonemptiness is reflected by algebraically closed field extensions. A k_0 -point therefore gives the desired representative, which can be extended to the original k . The argument with very general points itself need not be transferred to every ground field.

Lemma 8.5; §8.7 · pp. 35–36

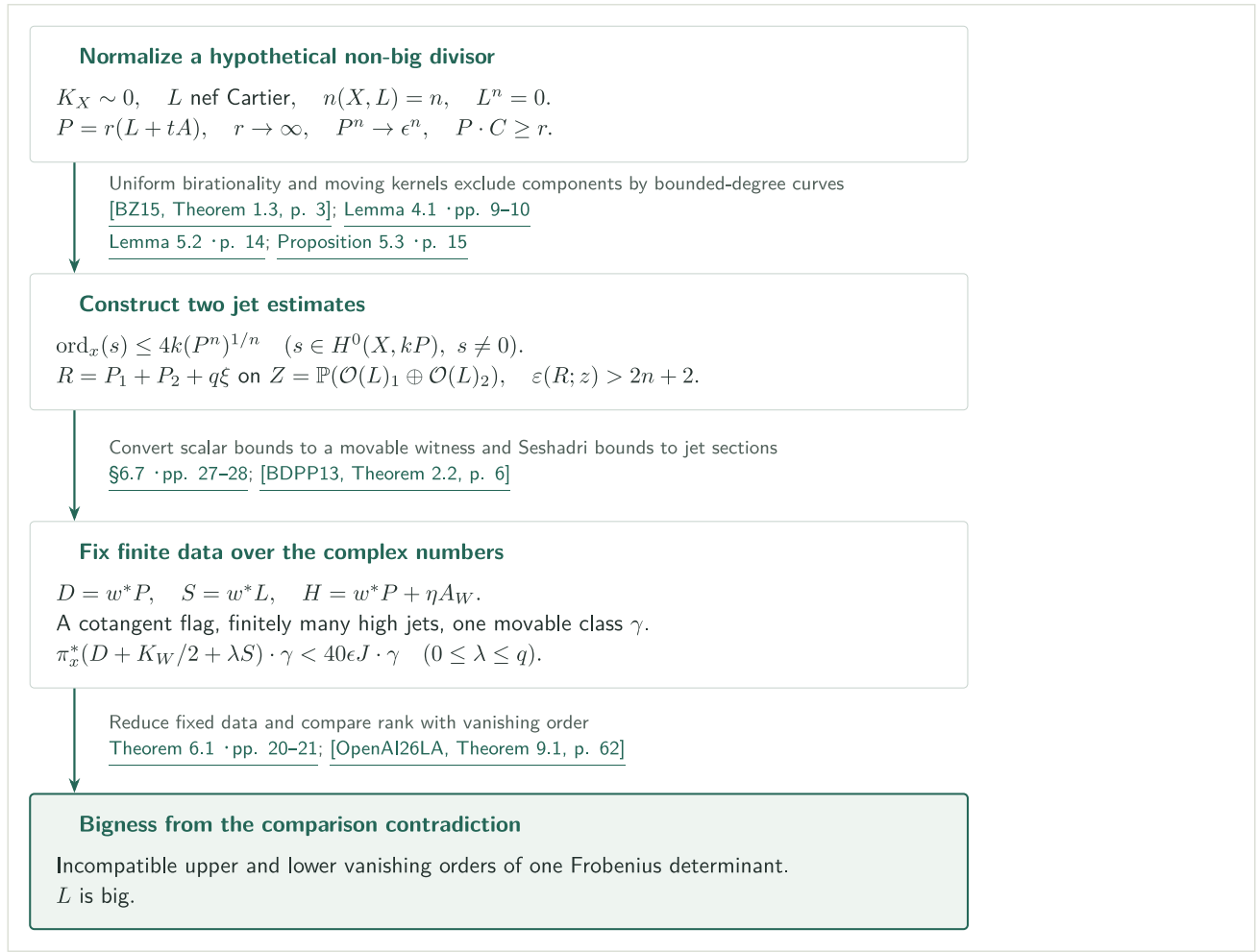
3. Proof of Theorem 5.1 — Small scalar jets and large two-slot jets

Proof target : Theorem 5.1 · p. 13

Theorem 5.1 — The terminal case with trivial canonical bundle

Let $n \geq 2$ and assume (P_j) for $j < n$. Let X be a projective $\mathbb{Q}$ -factorial terminal complex variety of dimension n with $K_X \sim 0$. If L is nef Cartier and $n(X, L) = n$, then L is big.

Theorem 5.1 · p. 13



A hypothetical non-big divisor of full nef dimension yields incompatible jet conditions

[Open figure ↗](#) [TeX source ↓](#)

1. Compare positive integral degrees with uniformly bounded curve degrees

Suppose L is not big and fix an ample Cartier divisor A . As $t \downarrow 0$, normalize

$$P = r(L + tA), \quad r \longrightarrow \infty, \quad P^n \longrightarrow \epsilon^n.$$

Every curve through a very general point has $L \cdot C \geq 1$, hence $P \cdot C \geq r$. In the opposite direction, Lemma 4.1 constructs curves of uniformly bounded degree when the testing volume and adjoint intersection ratio are bounded. Its Birkar–Zhang input uses a reduced SNC boundary, Cartier index one for the nef summand, and a big adjoint. The lower-dimensional (P_j) supply this bigness when needed.

[Lemma 4.1 · pp. 9–10](#)

[\[BZ15, Theorem 1.3, p. 3\]](#)

2. Exclude moving base components to obtain the two estimates

On the scalar side, an excessively vanishing section produces a moving base component. Its volume and canonical-intersection estimates feed into Lemma 4.1, giving $\text{ord}_x(s) \leq 4k(P^n)^{1/n}$ for every nonzero $s \in H^0(X, kP)$ at a very general x .

The two-slot construction uses

$$Z = \mathbb{P}(\mathcal{O}_X(L)_1 \oplus \mathcal{O}_X(L)_2) \longrightarrow X \times X, \quad R = P_1 + P_2 + q\xi.$$

Fix q before making t small. The proof excludes positive-dimensional slice fibres, lower-dimensional components and sweeping branch divisors, then treats finite correspondences dominant over both factors. Non-sweeping branch images reduce the possibilities to finitely many Stein covers **for each fixed t** . A product-dominant family of birational graphs between fixed covers leads to a model birational to an abelian variety; the bounded-degree obstruction then excludes the last case. Consequently there is a point outside both axes with $\varepsilon(R; z) > 2n + 2$. The finite lists of covers themselves are not asserted to be uniform in t .

Lemma 5.2 · p. 14

Proposition 5.3 · p. 15

§§5.4–5.6 · pp. 16–20

[OpenAI26LA, Lemmas 7.1–7.5, pp. 45–52]

3. Fix all complex data before applying the Frobenius comparison

On a resolution $w : W \rightarrow X$, set $D = w^*P$ and $S = w^*L$, and perturb w^*P slightly to an ample H . Pseudo-effectivity of K_W supplies generic semipositivity of the cotangent bundle and the flag required by the comparison. The two-slot Seshadri bound produces finitely many sections generating jets through order $(2n + 2)k_0$. The scalar bound implies that $\pi_x^*(2D + E) - 40\epsilon J$ on the blowup of a point is not pseudo-effective. Duality with movable curves supplies one fixed strongly movable class witnessing all the weight inequalities.

Theorem 6.1 rules out coexistence of these fixed data by opposite bounds on the vanishing order of a Frobenius-evaluation determinant. Thus the assumption that L is not big is contradictory. The manuscript reproves the comparison theorem; this overview checks its statement and the application in §6.7. The full slope and determinant calculation in §§6.2–6.6, the external Hanamura/Blanc sources used in §5.6, and the restriction theorem producing the flag were not independently checked.

§6.7 · pp. 27–28

[CP15, Theorem 2.1, print p. 844 / PDF p. 11]

[BDPP13, Theorem 2.2, p. 6]

[OpenAI26LA, Theorem 9.1, p. 62]

4. Proof of Proposition 7.1 — Restore the boundary on a Calabi–Yau pair

Proof target : Proposition 7.1 · p. 28

Proposition 7.1 — The full-nef-dimension induction statement

Let $n \geq 2$ and assume (P_j) for $j < n$, together with the log-abundance and terminating-MMP inputs of Theorems 2.2–2.3. Let (X, B) be a projective klt $\mathbb{Q}$ -pair of dimension n over $\mathbb{C}$, and M a nef $\mathbb{Q}$ -Cartier divisor on X . Suppose $T = K_X + B$ is pseudo-effective and, for some $a \in \mathbb{Q}_{>0}$, every integral curve C through a very general point satisfies $(T + aM) \cdot C > 0$. Then $T + cM$ is big for every $c \in \mathbb{Q}_{>0}$.

Proposition 7.1 · p. 28

Reduce by the ordinary litaka dimension

$$T = K_X + B, \quad (T + aM) \cdot C > 0.$$

$\kappa(T) > 0$: induct on smaller fibres; $\kappa(T) = 0$: $K_Y + B_Y \sim_{\mathbb{Q}} 0$.

The two MMPs preserve the ordinary semiample adjoint; treat the remaining case
[Propositions 3.3–3.4 · p. 8](#); [Theorem 5.1 · p. 13](#)

[\[BCHM10, Corollary 1.4.3, p. 10\]](#)

A zero-boundary Calabi–Yau pair

$$K_X \equiv 0 \implies K_X \sim_{\mathbb{Q}} 0 \implies K_{\tilde{X}} \sim 0.$$

Use an index-one cover and crepant terminalization, then Theorem 5.1.

M big.

Positive litaka dimension or disappearance of the boundary supplies effective G
[§7 · pp. 29–30](#)

Perturb the boundary to an ordinary klt adjoint

$$T_* = K_X + (1 + \alpha)B \sim_{\mathbb{Q}} \alpha B, \quad T_* + M \text{ big.}$$

$$G \sim_{\mathbb{Q}} \alpha B + M, \quad \Delta_u = (1 - u\alpha)B + uG, \quad K_X + \Delta_u \sim_{\mathbb{Q}} uM.$$

Apply abundance to the nef ordinary klt adjoint, then use positive combinations
[\[OpenAI26LA, Theorem 1.1, p. 2\]](#); [Proposition 7.1 · p. 28](#)

Close the full-nef-dimension induction

$$M \text{ semiample, } n(X, M) = n \implies M \text{ big.}$$

$$T + cM \text{ big for every } c \in \mathbb{Q}_{>0}.$$

Close the full-nef-dimension induction using an index-one cover and a boundary perturbation

[Open figure ↗](#) [TeX source ↓](#)

1. Transfer the zero-boundary case to Theorem 5.1

The preceding reductions allow us to focus on $K_X + B \equiv 0$ and M of full nef dimension. If $B = 0$, log abundance gives $K_X \sim_{\mathbb{Q}} 0$. The index-one cover is klt and canonical; after crepant terminalization it is terminal and $\mathbb{Q}$ -factorial with $K \sim 0$. Apply Theorem 5.1 to a Cartier multiple of the pulled-back M , then descend bigness, which is detected by a dominant generically finite pullback.

[\[BCHM10, Corollary 1.4.3, p. 10\]](#)

2. Enlarge the boundary slightly and realize M as an ordinary adjoint

For $B \neq 0$, set $T_* := K_X + (1 + \alpha)B \sim_{\mathbb{Q}} \alpha B$. If $\kappa(T_*) > 0$, Proposition 3.3 gives bigness of $T_* + M$. If it is zero, the two-stage MMP has $B_Y = 0$ at its endpoint, so the zero-boundary case just established gives the same conclusion. This avoids using Proposition 7.1 before completing its proof.

Choose an effective $G \sim_{\mathbb{Q}} \alpha B + M$. For a sufficiently small positive rational u ,

$$\Delta_u = (1 - u\alpha)B + uG \geq 0, \quad K_X + \Delta_u \sim_{\mathbb{Q}} uM,$$

and (X, Δ_u) is an ordinary klt pair. Its adjoint is nef, so log abundance now makes M itself semiample. Full nef dimension forces the associated morphism to be generically finite, and hence M is big. Finally, the positive-combination argument of Proposition 3.4 gives bigness of $T + cM$ for every $c > 0$. The stronger intermediate semiampleness conclusion belongs to this specially constructed ordinary-adjoint situation.

§7 · pp. 29–30

[OpenAI26LA, Theorem 1.1, p. 2]

5. Proof of Proposition 8.1 — Nef reduction and descent of positive parts

Proof target : Proposition 8.1 · p. 30

Proposition 8.1 — Rationality and numerical semiampleness of the positive part

Let (X, B) be a projective klt $\mathbb{Q}$ -pair over $\mathbb{C}$, with $K_X + B$ pseudo-effective, and let M be a nef $\mathbb{Q}$ -Cartier divisor on X . There is a projective birational morphism $w : W \rightarrow X$ from a smooth projective variety such that

$$P_{\sigma}(w^*(K_X + B + M))$$

is rational and numerically equivalent to a semiample $\mathbb{Q}$ -divisor on W . Nefness of $K_X + B + M$ itself is not assumed here.

Proposition 8.1 · p. 30

An MMP trivial on the nef divisor

$L = m_0 M$ Cartier, $m > 2n$, $T + mL$ nef at the endpoint.
Integral L -degrees and ray lengths force every step to be L -trivial.

Take nef reduction at the endpoint and apply ordinary abundance on its fibres
[Lemma 8.3 · pp. 31–32](#); [§§8.3–8.4 · pp. 32–33](#)

Two cases according to nef dimension

$n(X, T + mL) = n : (P_n) \Rightarrow T + M$ big $\Rightarrow$ ordinary good model.

$n(X, T + mL) < n : T|_F \equiv M|_F \equiv 0, \quad K_F + B_F \sim_{\mathbb{Q}} 0.$

In the lower-dimensional case: nef descent, a relative good model, and the bundle formula
[\[LP18, Lemma 3.1, p. 13\]](#); [\[Fuj09b, Theorem 1.4, p. 2\]](#)
[\[HX11, Theorem 2.12, p. 12\]](#); [\[Amb03, Theorem 0.2, p. 2\]](#)

An ordinary adjoint and nef divisor on a smaller base

$p^*M \equiv f^*M_1, \quad K_{X_2} + B_2 \sim_{\mathbb{Q}} g^*(K_{Z_2} + B_{Z_2}).$
 $(Z_{\dim Z_2}), \quad \dim Z_2 < n, \quad (Z_2, B_{Z_2})$ klt.

Pull back nef positive parts, remove effective exceptional terms, and use numerical invariance
[Lemma 8.2 · p. 31](#); [\[LP18, Lemmas 2.4–2.5, pp. 7–8\]](#)

A rational, numerically semiample positive part

$P_{\sigma}(w^*(T + M)) \equiv A_W, \quad A_W$ semiample over a point.
Both cases return to the original model on a common resolution.

The full-nef-dimension and lower-dimensional nef-reduction cases yield the positive part

[Open figure ↗](#) [TeX source ↓](#)

1. Prepare the alternatives by an MMP trivial on the nef divisor

Make $L = m_0 M$ Cartier and choose $m > 2n$. On a $(T + mL)$ -negative ray, the inequality $L \cdot C > 0$ would imply $(T + mL) \cdot C \geq m - 2n > 0$, using integrality and the ordinary ray-length bound. Thus the chosen terminating MMP is L -trivial and also T -negative. Its endpoint has $T + mL$ nef and retains a nef transform of M .

If $T + mL$ has full nef dimension, (P_n) makes $T + M$ big. A decomposition $T + M \sim_{\mathbb{Q}} A + G$ realizes a positive rational multiple $(1 + v)(T + M)$ as an ordinary big klt adjoint. The pullback of the semiample adjoint on its good minimal model gives its positive part.

[Lemma 8.3 · pp. 31–32](#)

[§§8.3–8.4 · pp. 32–33](#)

2. Reduce to an ordinary adjoint and a nef divisor on a smaller base

If the nef dimension is smaller, a very general compact nef-reduction fibre F has $T|_F$ pseudo-effective, $M|_F$ nef, and $(T + mL)|_F \equiv 0$. These signs make both restrictions numerically trivial. Ordinary log abundance then gives $K_F + B|_F \sim_{\mathbb{Q}} 0$.

Birational descent of nef divisors produces $f : X_1 \rightarrow Z_1$ after resolution, with $p^*M \equiv f^*M_1$ and M_1 nef. This is **numerical equivalence**. Write $K_{X_1} + B_1 = p^*T + E$, where $E \geq 0$ is p -exceptional. The ordinary adjoint has Iitaka dimension zero on the general fibre. Relative finite generation of its log canonical ring and good minimal models of very general fibres give a relative good model. Its semiample fibration $g : X_2 \rightarrow Z_2$ satisfies

$$K_{X_2} + B_2 \sim_{\mathbb{Q}} g^*A_2, \quad \dim Z_2 = \dim Z_1 < n.$$

Apply Ambro's result to this **Q-linear identity for an ordinary klt adjoint**, replacing A_2 by $K_{Z_2} + B_{Z_2}$ with effective klt boundary. Pseudo-effectivity of the ordinary adjoint upstairs and sections supplied by its good model descend to show that the base adjoint is pseudo-effective. Statement $(Z_{\dim Z_2})$ therefore applies.

[LP18, Lemma 3.1, p. 13]

[Fuj09b, Theorem 1.4, p. 2]

[HX11, Theorem 2.12, p. 12]

[HX11, Corollary 2.9, p. 9]

[Amb03, Theorem 0.2, p. 2]

§8.5 · pp. 33–34

3. Use positive-part comparisons to return to the original model

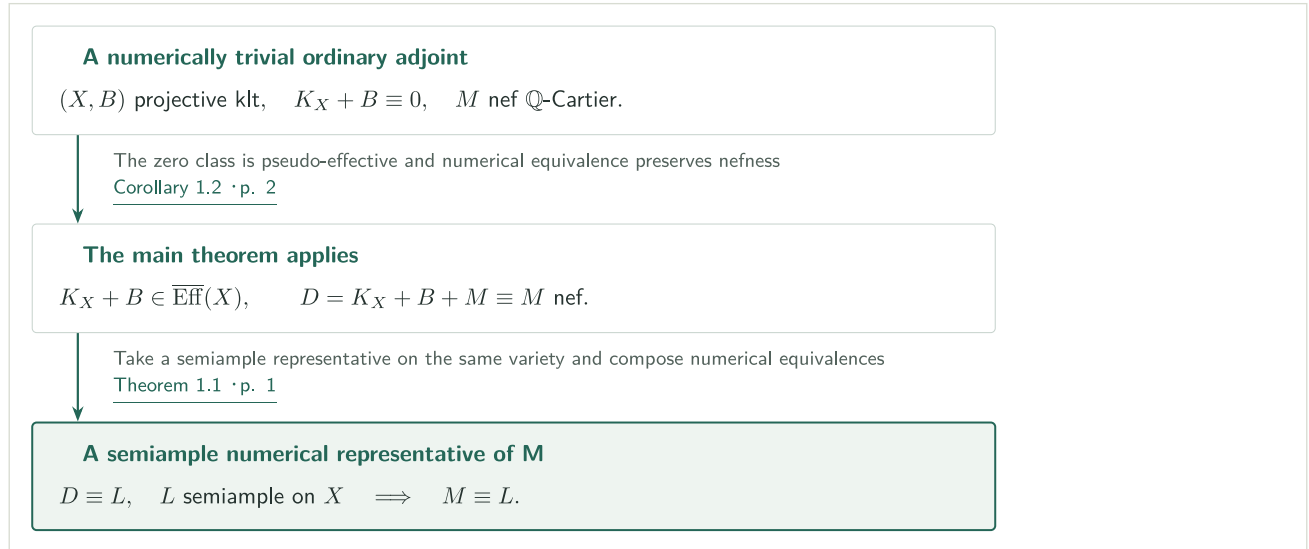
The lower-dimensional positive part is numerically semiample, hence nef. Use the pullback formula for a nef positive part, its invariance under adding effective exceptional divisors, and the numerical invariance of its rationality and numerical semiampleness. These rules allow the numerical descent of M and the Q-linear ordinary-adjoint formula to enter the same comparison. Reverse the relative MMP from X_2 and then remove the p -exceptional E , obtaining the asserted positive part of $w^*(T + M)$. This proves (Z_n) from (P_n) and the lower-dimensional (Z_j) .

Lemma 8.2 · p. 31

[LP18, Lemmas 2.4–2.5, pp. 7–8]

6. Proof of Corollary 1.2 — A numerically trivial ordinary adjoint

Proof target : Corollary 1.2 · p. 2 See the statement.



Apply the main theorem to every nef divisor on a klt Calabi–Yau pair

[Open figure ↗](#) [TeX source ↓](#)

1. Check the hypotheses of the main theorem

If $K_X + B \equiv 0$, the ordinary adjoint is pseudo-effective. For every nef M , the divisor $D = K_X + B + M \equiv M$ is nef. The semiample divisor $L \equiv D$ supplied by Theorem 1.1 is therefore numerically equivalent to M . This replaces a numerical representative; it does not produce sections of a nontorsion numerically trivial line bundle.

Corollary 1.2 · p. 2

7. Which papers supply which steps

Source

Log abundance in characteristic zero

Used at

Lemma 3.2, §7, §8

Role and hypotheses

Actual semiampleness of nef ordinary lc/klt adjoints; no direct application to a generalized nef trace

Checking scope

Theorem 1.1 and its uses checked; full source proof excluded

Source

Minimal models and Mori fibre spaces for generalized log canonical $\mathbb{Q}$ -pairs

Used at

Theorem 2.3, Lemmas 3.2 and 8.3

Role and hypotheses

A terminating choice retaining rational nef b -divisor data; preservation for $\mathbb{Q}$ -factorial generalized klt input compared with §3.3

Checking scope

Direct input within catalogue 036

Source

Birkar–Zhang and the moving-kernel results of Log abundance

Used at

Lemma 4.1, §5

Role and hypotheses

Uniform birationality for index-one data and reduced boundary; intersection bounds for moving components

Checking scope

Input statements and use checked

Source

Campana–Păun and BDPP

Used at

§6.7

Role and hypotheses

Generic semipositivity; separation of a non-pseudo-effective divisor by a movable curve

Checking scope

Statements and use checked; external flag-restriction source not checked

Source

Lazić–Peternell, Fujino, Hacon–Xu and Ambro

Used at

§8

Role and hypotheses

Numerical descent of nef divisors, relative good models, the ordinary klt canonical bundle formula, and positive-part comparisons

Checking scope

Statements, hypotheses and use checked; external proofs excluded

Log abundance in catalogue 034 and the generalized-minimal-model manuscript in catalogue 036 are direct inputs. Further dependency relations with the Kähler and numerical-dimension-one manuscripts of catalogue 036 are not inferred merely from the present reference check.

8. Return to the sources

- [Lemma 3.2 · pp. 6–7](#) — The two-stage MMP keeps an ordinary semiample adjoint while making a generalized adjoint nef.
- [Theorem 5.1 · p. 13](#) — Entry to the full-nef-dimension bigness argument.
- [Theorem 6.1 · pp. 20–21](#) — The fixed complex data and what is transferred to positive characteristic.
- [§7 · pp. 29–30](#) — Close P_n using an index-one cover and boundary perturbation.
- [Proposition 8.1 · p. 30](#) — The positive-part induction and its two cases.

- §8.5 · pp. 33–34 — Relative good models, canonical bundle formula and lower-dimensional positive parts.
- Lemma 8.4 · p. 35 — The Pic⁰ argument descending a numerical representative to X .
- Lemma 8.5; §8.7 · pp. 35–36 — The finite-type parameter argument for changing algebraically closed fields.

The two principal statements and their conclusions were compared with the source. Reading covered the reductions and geometric exclusions in §§3–5, the statement of Theorem 6.1 and its application in §6.7, and the proofs in §§7–8. The comparison distinguishes ordinary from generalized pairs, numerical from $\mathbb{Q}$ -linear equivalence, and a representative on the original variety from a positive part on a birational model.

The full Frobenius calculation in §§6.2–6.6 and comparison with the external Hanamura/Blanc and Mehta–Ramanathan statements remain unchecked. Input proofs have not been recursively verified. This is neither an independent certification of the complete manuscript nor expert peer review.

Minimal models and Mori fibre spaces for generalized log canonical $\mathbb{Q}$ -pairs

Fixed nef b-data, local character congruences, and relative weak Zariski decompositions

The manuscript claims existence of a minimal model or a Mori fibre space according to the sign of the adjoint of a generalized lc $\mathbb{Q}$ -pair. The proof controls characters of local valuations to terminate smooth canonical programs, constructs relative weak Zariski decompositions, and transfers to the generalized MMP. This is an account of the authors' argument, not an independent verification of all 79 pages.

Manuscript author: OpenAI · September 24, 2026 · 79 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Models preserving fixed generalized nef data

Let k be algebraically closed of characteristic zero and $(X, B + M)$ a projective generalized lc $\mathbb{Q}$ -pair. Thus X is normal, integral and projective, B is an effective rational boundary, M is the trace of a nef rational Cartier divisor on a higher projective birational model, and $D = K_X + B + M$ is $\mathbb{Q}$ -Cartier. Fix this nef b-divisor.

There exist a normal integral projective Y and a birational map $\phi : X \dashrightarrow Y$ whose inverse contracts no divisors. With $B_Y = \phi_* B$ and M_Y the trace of the same nef b-divisor, $(Y, B_Y + M_Y)$ is generalized lc and, for every prime divisor E over the common function field,

$$A_{X, B+M}(E) \leq A_{Y, B_Y+M_Y}(E).$$

The inequality is strict for prime divisors on X contracted by ϕ . Moreover:

1. If D is pseudo-effective, then $K_Y + B_Y + M_Y$ is nef.

2. Otherwise there is a projective contraction $h : Y \rightarrow T$ to a normal projective variety with connected fibres, $\dim T < \dim Y$, $\rho(Y/T) = 1$, and $-(K_Y + B_Y + M_Y)$ ample over T .

Neither X nor Y is required to be $\mathbb{Q}$ -factorial. The discrepancy is the log discrepancy, taking value $1 - b$ on a boundary component of coefficient b . The nef conclusion does not assert semiampleness or a good minimal model. The theorem does not assert termination of every arbitrarily chosen MMP.

[Theorem 1.1 · p. 2](#)

Corollary 1.2 — Ordinary lc pairs

For a projective lc $\mathbb{Q}$ -pair (X, B) over an algebraically closed field of characteristic zero, there is a birational map $\phi : X \dashrightarrow Y$ extracting no divisors, with $B_Y = \phi_* B$, such that (Y, B_Y) is lc and $A_{X,B} \leq A_{Y,B_Y}$ for all prime divisors, strictly for prime divisors on X contracted by ϕ . If $K_X + B$ is pseudo-effective, $K_Y + B_Y$ is nef. Otherwise Y has a Mori fibre contraction of relative Picard number one with relatively anti-ample adjoint.

[Corollary 1.2 · p. 2](#)

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Minimal models and Mori fibre spaces for generalized log canonical \$\mathbb{Q}\$ -pairs](#). External inputs use the following keys.

- [TX24](#) Nikolaos Tsakanikas; Lingyao Xie — Remarks on the existence of minimal models of log canonical generalized pairs: Transfer from relative NQC weak Zariski decompositions and choose a terminating MMP from X ; good models with ample boundary check relative Picard rank. *Mathematische Zeitschrift* 307 (2024), Article 20; published 26 April 2024.
- [DEL00](#) Jean-Pierre Demailly; Lawrence Ein; Robert Lazarsfeld — A subadditivity property of multiplier ideals: Global generation of twists of asymptotic multiplier ideals and subadditivity. *Prépublication de l'Institut Fourier* no. 494 (2000), author PDF, 19 pages; pagination differs from *Michigan Math. J.* publication.
- [Fuj09a](#) Osamu Fujino — Effective base point free theorem for log canonical pairs—Kollár type theorem: Fix a dimension-dependent globally generated multiple of a nef Cartier divisor on a klt pair. *Tohoku Mathematical Journal* 61 (2009), 475–481, published version.
- [BDPP13](#) Sébastien Boucksom; Jean-Pierre Demailly; Mihai Păun; Thomas Peternell — The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension: Obtain canonical pseudo-

effectivity from non-uniruledness of a smooth projective cover. arXiv:math/0405285v1, 14 May 2004; statement numbering checked in this version.

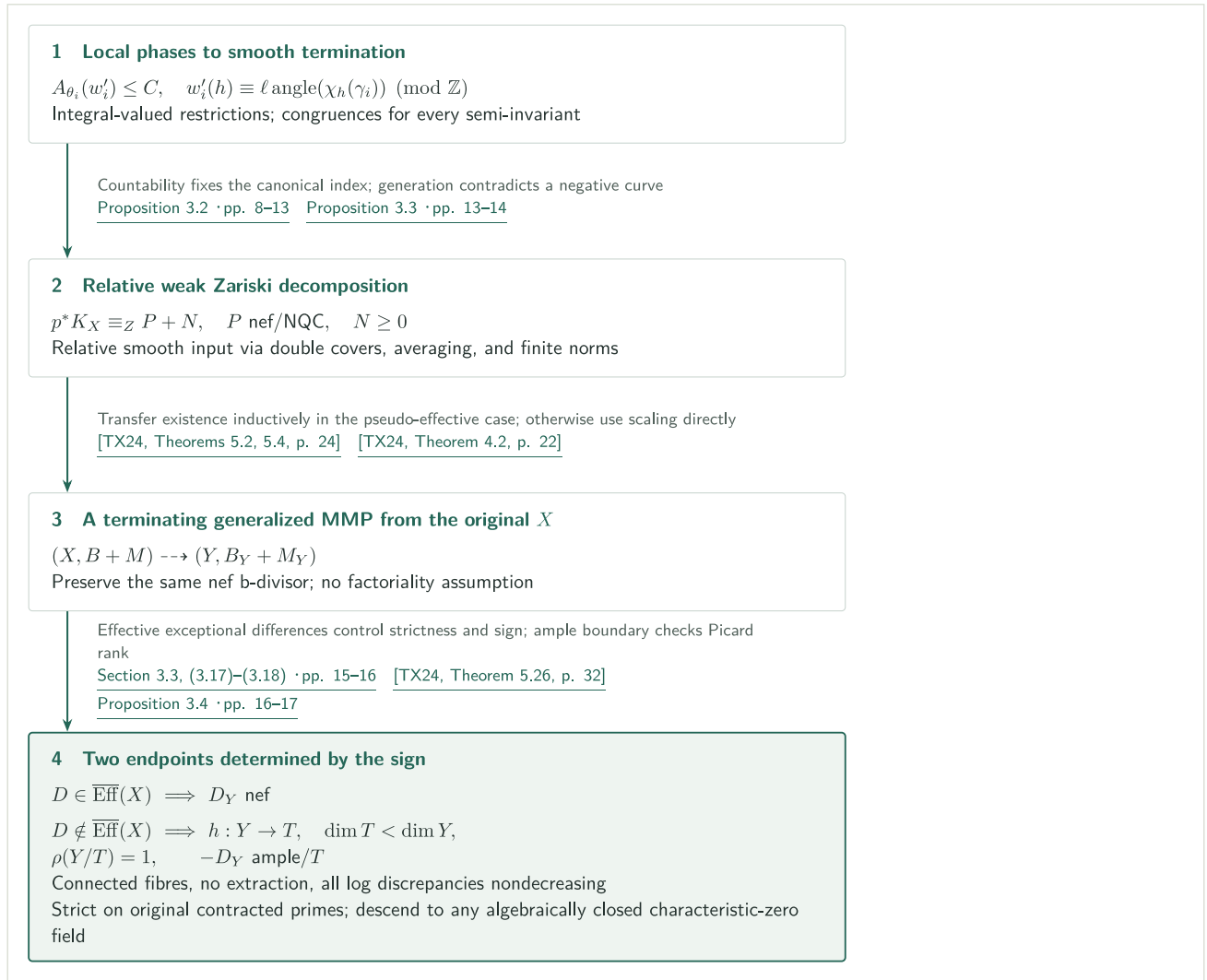
- [BCHM10](#) Caucher Birkar; Paolo Cascini; Christopher D. Hacon; James McKernan — Existence of minimal models for varieties of log general type: Finiteness of ample models for a klt boundary polytope; external proof excluded. arXiv:math/0610203v2, archive label 14 August 2008; downloaded PDF title-page date November 26, 2024 retained.
- [OpenAI26LA](#) OpenAI — Log abundance in characteristic zero: Post-theorem semiampness for ordinary pairs only; the reverse use occurs in the finite-generation proof of Corollary 12.1(i). September 24, 2026; fixed catalog commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a.
- [KM98](#) János Kollár; Shigefumi Mori — Birational geometry of algebraic varieties: Relative Kawamata–Viehweg vanishing for late flips; the application was read, but the original book passage remains unchecked. Cambridge Tracts in Mathematics 134 (1998); source passage not directly checked.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.1 — Proof overview of the main theorem

Proof target : [Theorem 1.1](#) · p. 2

Proof route. We follow every conclusion through local characters, smooth termination, relative reduction, transfer to generalized pairs, exact discrepancies and endpoints, and descent of the ground field. The first diagram gives the full route; the next expands the local-to-global termination argument.



Overall passage to a minimal model or Mori fibre space with fixed nef data

[Open figure ↗](#) [TeX source ↓](#)

1. Passing local characters to global termination

The local input, Theorem 3.1, concerns Gorenstein klt germs (V_i, o_i) of fixed dimension $n \geq 2$, cyclic groups G_i fixing o_i , semi-invariant canonical generators θ_i , and arbitrary elements $\gamma_i \in G_i$. After a subsequence it supplies divisorial valuations w'_i centred at o_i , a constant $C > 0$ and a positive integer ℓ , constant along the subsequence, such that $A_{\theta_i}(w'_i) \leq C$ and, for every nonzero rational semi-invariant h ,

$$w'_i(h) \equiv \ell \text{angle}(\chi_h(\gamma_i)) \pmod{\mathbb{Z}}.$$

The restriction to the invariant field is therefore integral-valued; these valuations need not be primitive. The construction bounds finite-window lengths using the degree-product estimate $\text{vol}(v)d_1 \cdots d_n \leq C$ for independent cone degrees. Valutive optimization and logarithmic Cartier selection in positive characteristic complete a tuple and round its characters. The germ and finite test data are fixed before increasing the characteristic; all finite recursive budgets precede the final common integer.

Theorem 3.1 · p. 8 Proposition 5.1; Lemma 6.1; Section 8 · pp. 22, 29–30, 39–44

Congruences on a completed tuple alone would not suffice. In §8.3, for any regular eigenfunction h , the functions $h^j f^\alpha$ are placed in one finite window. Their number exceeds the window length, forcing cancellation among terms of the same character and lowest valuation. The Gauss property forces two distinct powers $j \neq j'$ of h , with $1 \leq |j - j'| \leq D_{\text{fin}}$, to annihilate the congruence error. One choice $\ell = \text{lcm}(1, \dots, D_{\text{fin}})$ works for all h . Products of group translates supply invariant regular denominators for rational functions. This gives the strength needed for a ratio of canonical forms below.

Section 8.3, (8.16)–(8.20) · pp. 44–45

2. Relative reduction and transfer to the generalized MMP

Proposition 3.3 constructs an NQC weak Zariski decomposition $p^*K_X \equiv_Z P + N$ with rational Cartier parts when smooth complex quasi-projective X is projective over normal quasi-projective Z and K_X is pseudo-effective over Z . This is **relative numerical equivalence**.

Compactify the morphism and take a sufficiently positive double cover of its base, making the canonical divisor of a smooth projective cover $X^\sharp$ pseudo-effective. BDPP is used through non-uniruledness of the general fibres and of the new general-type base. Apply Proposition 3.2, average the resulting decomposition under the covering involution, and descend to the quotient, retaining $\mathbb{Q}$ -Cartierness by the finite norm argument. The base twist becomes numerically zero over Z .

Proposition 3.3 · pp. 13–14 [BDPP13, Theorem 0.2; Corollary 0.3, p. 2]

Apply TX24 Theorem 5.2 to these relative smooth decompositions, inductively on dimension, to obtain smooth minimal models. Theorem 5.4 transfers to NQC generalized lc pairs; Theorem 2.7 translates existence into the Birkar–Shokurov sense. Rational nef Cartier b-data are NQC. In the non-pseudo-effective case this existence input is unnecessary. In both cases choose a sufficiently positive ample A with small general coefficients so that $(X, B + A + M)$ is generalized lc and $D + A$ is nef. Theorem 4.2 then supplies a terminating MMP from the original X .

Section 3.3 · p. 14 [TX24, Theorems 5.2, 5.4, p. 24] [TX24, Theorem 2.7, p. 9]
[TX24, Theorem 4.2, p. 22]

3. No extraction, discrepancies, and every Mori fibre condition

On a common resolution of each birational step put $F = p^*D - q^*D^+$. It is q -exceptional and anti-nef over q , so negativity gives $F \geq 0$. A zero coefficient along a contracted prime would allow a contracted curve through its general point to have both nonnegative intersection by effectivity and negative intersection from D . The same nef b-divisor occurs on both pullbacks, so

its contribution cancels: coefficients of F are precisely increases in log discrepancy. Composition gives nondecrease everywhere and strict increase on original contracted primes. Smallness of each positive model excludes extraction by the inverse; generalized lc is preserved.

At a fibre-type endpoint, connectedness, dimension drop and relative anti-ampleness follow from the extremal contraction. Relative Picard number one requires a further check without factoriality. For a Cartier divisor L of degree zero on the contracted ray, add a small general effective boundary $A \sim_{\mathbb{Q}} L + h^*H - D_Y$. Its adjoint is nef, so the MMP in TX24 Theorem 5.26 has no negative step and that adjoint is semiample over T on Y itself. A relatively numerically trivial semiample divisor contracts every connected fibre to a point. Thus L is numerically pulled back from T , and the annihilator of the contracted ray is exactly $h^*N^1(T)_{\mathbb{R}}$, of codimension one. This auxiliary semiampleness does not assert semiampleness of the original D_Y .

Section 3.3, (3.17)–(3.18) · pp. 15–16 [TX24, Theorem 5.26, p. 32]

Effective exceptional pullback differences preserve pseudo-effectivity in both directions. A nef endpoint forces the original D to be pseudo-effective; a pseudo-effective D cannot end in an anti-ample restriction to a general positive-dimensional fibre. This identifies the two alternatives in Theorem 1.1.

4. Descent of the ground field

Proposition 3.4 descends the pair, nef data and a log resolution carrying them to a countable algebraically closed $F \subset k$, embeds F into $\mathbb{C}$, and takes the finite complex MMP. Spread and specialize the line bundle and generating sections defining each contraction. Equality of contracted curves identifies its Stein target. Recover the positive model as

$\mathrm{Proj}_T \bigoplus_{m \geq 0} f_* \mathcal{O}_S(mbD)$ for suitable b . The argument checks descent of divisorial sheaves, finite generation and relative Proj before extending to k . Nefness, relative Picard number, relative ampleness and effective pullback differences persist, giving all conclusions.

Proposition 3.4 · pp. 16–17

3. Proposition 3.2 — Smooth scaling termination

Proof target : Proposition 3.2 · pp. 8–13

Proposition 3.2 — Termination assuming the local phase theorem

Assume Theorem 3.1. Let X be smooth projective complex with pseudo-effective K_X , and choose an effective ample rational H such that (X, H) is klt and $K_X + H$ is ample.

Proposition 3.2 asserts termination of this scaling MMP and a decomposition

$p^*K_X = P + N$ on a smooth birational model $p : U \rightarrow X$, with P nef rational Cartier and $N \geq 0$ rational Cartier.

Proposition 3.2 · pp. 8–13

1 Assume an infinite smooth scaling MMP

$$\lambda_i \rightarrow 0, \quad \mathcal{J}_s(t) = \mathcal{J}(X, s\|K_X + tH\|) \rightarrow \mathcal{J}_s$$

A common globally generated twist stabilizes the ideals

Subadditivity and base orders give two-sided bounds
[\[DEL00, Corollary 1.5; Theorem 1.8; Variants 2.4–2.5, pp. 7–8, 12\]](#)
[Section 3.1, \(3.4\)–\(3.8\) · pp. 9–10](#)

2 Uniform error for all valuations

$$u(\mathcal{J}_s)/s \leq E_i(u) \leq \sigma_u \leq (u(\mathcal{J}_s) + A_X(u))/s$$

$$\Delta_i(u) \leq A_X(u)/s$$

Integral values and bounded discrepancy give a fixed countable set of finite limits

Apply the all-semi-invariant congruence to a ratio of canonical forms
[Theorem 3.1 · p. 8](#) [Section 3.1, \(3.9\)–\(3.11\) · pp. 11–12](#)

3 One common Cartier index

RK_{X_i} Cartier for every i

Arbitrary character angles α on index-one covers cannot all lie in a countable set

Late-flip vanishing; fix the dimension-dependent generation multiple before regularity
[Section 3.1, \(3.12\)–\(3.14\) · pp. 12–13](#)
[\[Fuj09a, Theorem 1.1; Remark 1.2, print p. 475 / PDF p. 1\]](#)

4 Generation contradicts a negative curve

$$R^k f_{i*} \mathcal{O}_{X_i}(mK_{X_i}) = 0 \quad (k > 0, i \gg_m 0)$$

$$RK_{X_i} + af_i^* J \text{ globally generated, } RK_{X_i} \cdot C < 0$$

Terminate at a nef endpoint: $p^*K_X = P + N$, P nef, $N \geq 0$

Uniform multiplier-ideal estimates bound canonical indices and terminate the MMP

[Open figure ↗](#) [TeX source ↓](#)

1. Uniform ideal estimates

Dimensions zero and one are immediate. In dimension at least two, suppose the MMP is infinite. Only finitely many divisorial contractions occur, leaving a tail of flips. If the positive scaling numbers do not tend to zero, finiteness of ample models for a fixed rational polytope of klt boundaries gives a contradiction. The common ample part and klt hypothesis needed for BCHM are obtained by small rational perturbations on one log resolution.

[BCHM10, Corollary 1.1.5, p. 7]

Put $A_i(u) = A_{X_i}(u)$, $E_i(u) = A_i(u) - A_X(u)$ and $\Delta_i(u) = A_{i+1}(u) - A_i(u)$. A sufficiently positive twist makes the asymptotic multiplier ideals $\mathcal{J}_s(t)$ of $L_t = K_X + tH$ uniformly globally generated as $t \downarrow 0$. Their section spaces stabilize inside a fixed finite-dimensional space, so the ideals stabilize to $\mathcal{J}_s$. This uses generation, not a descending chain condition for ideals. Subadditivity and valuation estimates give, for i sufficiently large depending on s , and every divisorial u ,

$$\frac{u(\mathcal{J}_s)}{s} \leq E_i(u) \leq \sigma_u \leq \frac{u(\mathcal{J}_s) + A_X(u)}{s}, \quad 0 \leq \Delta_i(u) \leq \frac{A_X(u)}{s}.$$

Section 3.1, (3.4)–(3.8) · pp. 9–10 [DEL00, Corollary 1.5; Theorem 1.8; Variants 2.4–2.5, pp. 7–8, 12]

2. Countability and canonical index

For integral-valued valuations with bounded $A_X(u)$, finite limits of $E_i(u)$ lie in one fixed countable set. The manuscript descends X and the countably many ideals to a countable algebraically closed field and takes a diagonal limiting valuation. Repeatedly blowing up a centre of codimension at least two raises the Jacobian value by at least one, contradicting bounded discrepancy. The limit is thus trivial or divisorial.

If canonical indices are unbounded, cyclic index-one covers allow the characters of canonical generators to approach any $\alpha \in \mathbb{R}/\mathbb{Z}$. Restricting the w'_i of Theorem 3.1 downstairs gives integral-valued valuations with bounded discrepancy. Applying the congruence to a ratio of canonical forms forces a limiting discrepancy change to have residue $\ell\alpha$. The union, over positive integers, of inverse images of a countable set under multiplication remains countable and cannot contain every α . Thus one $R > 0$ makes every RK_{X_i} Cartier.

Section 3.1, (3.9)–(3.11) · pp. 11–12

3. A fixed Cartier index contradicts a negative curve

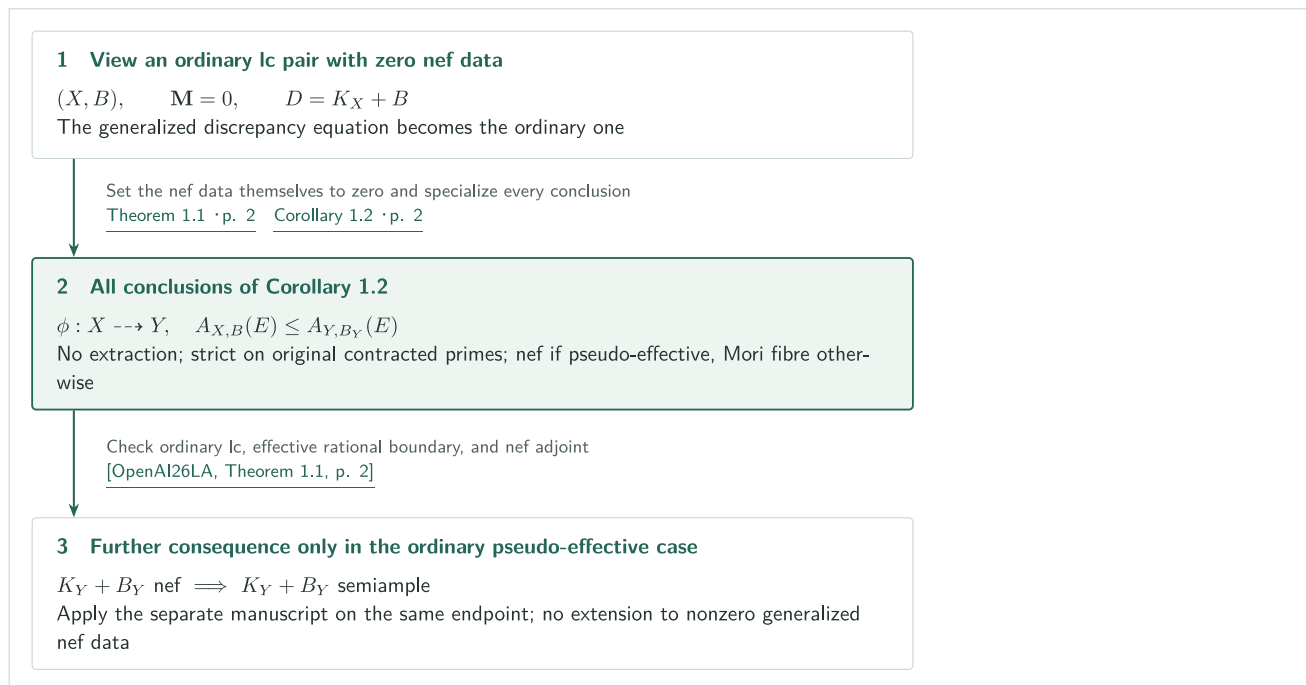
For a late flip $f_i : X_i \rightarrow Z_i$, the Δ_i estimate and relative Kawamata–Viehweg vanishing imply $R^k f_{i*} \mathcal{O}_{X_i}(mK_{X_i}) = 0$ for $k > 0$ and each fixed $m > 0$. On a common resolution the coefficient to be rounded is $A_i - 1 - (m - 1)\Delta_i > -1$, identifying the required pushforward with $\mathcal{O}(mK_{X_i})$. No stage uniform in all m is asserted.

Choose J sufficiently ample on Z_i so that $D_i = -RK_{X_i} + f_i^*J$ and $D_i - K_{X_i}$ are ample. Fujino's effective base point free theorem, with $a = 1$, gives a dimension-dependent N for which ND_i is globally generated. Fix N first and then choose a stage admitting vanishing for the finite list $m = R + jNR$, $1 \leq j \leq n$. Regularity makes $RK_{X_i} + af_i^*J$ globally generated, although it has negative degree on an f_i -contracted curve. This contradiction proves termination. Pseudo-effectivity excludes a fibre-type endpoint.

Section 3.1, (3.12)–(3.14) · pp. 12–13 [Fuj09a, Theorem 1.1; Remark 1.2, print p. 475 / PDF p. 1]

4. Corollary 1.2 — Specializing to zero nef data

Proof target : Corollary 1.2 · p. 2



Zero nef b-data give the ordinary lc pair and its two possible endpoints

[Open figure ↗](#) [TeX source ↓](#)

1. Ordinary discrepancies and endpoints

Adding the fixed zero nef b-divisor to an ordinary lc pair makes the generalized crepant discrepancy equation the ordinary one. Every conclusion of Theorem 1.1 therefore remains: no extraction, nondecrease and strict increase of discrepancies, a nef endpoint in the pseudo-effective case, and a Mori fibre space otherwise. The operation sets the nef data themselves to zero; it does not choose a different numerically equivalent divisor.

[Corollary 1.2 · p. 2](#)

2. Separating the link with Log abundance

Only in the ordinary pseudo-effective case, Theorem 1.1 of the separate Log abundance manuscript makes $K_Y + B_Y$ semiample on the same endpoint. This is a further consequence after the present main theorem. It gives no semiampleness assertion for nonzero generalized nef data.

In the other direction, the proof of Corollary 12.1(i) in Log abundance uses the present Corollary 1.2. The effective exceptional difference on a common resolution identifies sections in every rounded degree, leading to finite generation of the lc log canonical ring. This edge must be distinguished from an input to the main abundance theorem itself.

[OpenAI26LA, Theorem 1.1, p. 2] [OpenAI26LA, Corollary 12.1(i), proof and (67), p. 76]

5. Direct dependencies and their use

Input

[BCHM10, Corollary 1.1.5, p. 7]

Used at

Start of termination

Role and hypotheses

Finitely many ample models of a klt boundary polytope with a common ample part force scaling to zero

Checking scope

Statement compared with the perturbation on p.9; external proof excluded

Input

[DEL00, Corollary 1.5; Theorem 1.8; Variants 2.4–2.5, pp. 7–8, 12]

Used at

Uniform estimates

Role and hypotheses

Uniform generation of twists and subadditivity for smooth X and big $K_X + tH$

Checking scope

Author manuscript pp.7–8,12 compared with pp.9–10

Input

[Fuj09a, Theorem 1.1; Remark 1.2, print p. 475 / PDF p. 1]

Used at

Termination contradiction

Role and hypotheses

Dimension-dependent generation for klt X_i , nef Cartier D_i , and ample $D_i - K_{X_i}$

Checking scope

Print p.475/PDF p.1 compared with p.12

Input

[BDPP13, Theorem 0.2; Corollary 0.3, p. 2]

Used at

Relative reduction

Role and hypotheses

Non-uniruledness and canonical pseudo-effectivity for smooth projective varieties

Checking scope

arXiv v1 p.2 compared with pp.13–14

Input

[TX24, Theorem 2.7, p. 9; Theorem 4.2, p. 22; Theorems 5.2, 5.4, p. 24]

Used at

Generalized transfer

Role and hypotheses

Relative smooth input, NQC, generalized lc, and a suitable scaling ample divisor

Checking scope

Published pp.9,22,24 compared with p.14

Input

[TX24, Theorem 5.26, p. 32]

Used at

Relative Picard number

Role and hypotheses

Relative semiampleness after adding effective relatively ample boundary to obtain a nef adjoint

Checking scope

Published p.32 compared with p.15

Input

[OpenAI26LA, Theorem 1.1, p. 2]

Used at

Ordinary further consequence

Role and hypotheses

Ordinary lc, effective rational boundary and nef $\mathbb{Q}$ -Cartier adjoint

Checking scope

Statement p.2 compared with p.2; proof excluded

6. Return to the sources

- Theorem 1.1 · p. 2 — Hypotheses, all discrepancy conditions, and both endpoints.
- Theorem 3.1 · p. 8 — The local congruence for every semi-invariant.
- Proposition 3.2 · pp. 8–13 — Ideal stabilization, countability, index bound and termination contradiction.
- Proposition 3.3 · pp. 13–14 — Relative numerical decomposition by double covers, averaging and norms.
- Section 3.3, (3.17)–(3.18) · pp. 15–16 — Fixed-nef-data differences, strictness and nonfactorial relative Picard rank.
- Proposition 3.4 · pp. 16–17 — Ground-field descent of the finite MMP and relative Proj.
- Section 8.3, (8.16)–(8.20) · pp. 44–45 — Extension from tuple congruences to all rational eigenfunctions.

- Theorems 9.1, 10.1; Section 13.4 · pp. 45, 50, 77 — Entry to the deeper uniform estimates left unverified in this account.

We read Theorem 1.1, Corollary 1.2 and §3 (pp.8–17), including the reductions, discrepancies, Mori fibre endpoint and ground-field descent, and compared the statements of the direct inputs listed above with their sources. For the local input we read Theorem 3.1, the degree-product and window statements, and §8.3 extending to all semi-invariants. The full selection and continuation arguments of §§4–8 and the uniform lifting, mixing, trait comparison and lc-incidence proofs of §§9–13 remain unverified. In particular, Theorem 10.1→Theorem 9.1→Proposition 5.2→Proposition 5.1 is recorded at the level of roles and calls; its entire chain has not been verified.

The manuscript's application conditions for relative Kawamata–Viehweg vanishing were checked, but the corresponding Kollár–Mori passage was not directly compared in this pass. External normalized-volume, bounded-complement and b-semiampleness inputs in the local argument also remain unchecked. This work does not recursively verify external proofs, certify the whole manuscript, or constitute expert peer review or formal verification. The article and diagrams are AI-generated; important claims should be checked in the original sources.

Minimal models in numerical dimension one

Stabilization of perturbed MMPs in codimension two and two surface tests for nefness

The manuscript constructs a minimal model by finitely many canonical-negative operations when the numerical dimension, measured by section growth with a fixed ample twist, is one. Finite MMPs for each positive perturbation do not by themselves terminate their concatenation at coefficient zero. The argument fixes one model and detects nefness of its canonical divisor using surface intersections and orders of vanishing.

Manuscript author: OpenAI · September 24, 2026 · 14 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — A minimal model in numerical dimension one

Let X be a smooth connected complex projective variety of dimension $n \geq 3$. Assume that K_X is pseudo-effective and $\kappa_\sigma(X, K_X) = 1$. The convention in (1.1), for a Cartier divisor D , is

$$\kappa_\sigma(X, D) = \max \left\{ k \in \{0, \dots, n\} : \exists A \text{ ample Cartier, } \limsup_{m \rightarrow \infty} \frac{h^0(X, mD + A)}{m^k} > 0 \right\}.$$

The value is $-\infty$ if the set is empty; A is fixed before taking the limit. This convention is not replaced by an intersection definition for K_X , whose nefness is not yet known.

There exist a normal projective $\mathbb{Q}$ -factorial terminal variety Y and a finite sequence $\varphi : X \dashrightarrow Y$ of K -negative divisorial contractions and flips such that K_Y is nef. The inverse map contracts no prime divisor. On a common smooth projective resolution $p : W \rightarrow X$, $q : W \rightarrow Y$, compatible canonical divisors satisfy

$$p^*K_X = q^*K_Y + E, \quad E \geq 0, \quad E \text{ is a } q\text{-exceptional rational divisor,}$$

and $\text{Supp } E$ contains the strict transform of every prime divisor of X contracted by φ .

Theorem 1.1; equation (1.1) · pp. 1–2

Corollary 1.2 — The same endpoint is a good minimal model

Under the hypotheses of Theorem 1.1, the same $\mathbb{Q}$ -factorial terminal endpoint Y has semiample K_Y , and hence is a good minimal model of X . This is a separate consequence using log abundance. The fixed-twist growth argument of Theorem 1.1 alone does not assert semiampleness of K_Y or nonvanishing of K_X .

Corollary 1.2 · p. 2

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Minimal models in numerical dimension one](#). External inputs use the following keys.

- [BCHM10](#) Caucher Birkar; Paolo Cascini; Christopher D. Hacon; James McKernan — Existence of minimal models for varieties of log general type: Finite scaling MMP for klt pairs with big boundary. [arXiv:math/0610203v2](#) (2008-08-14); downloaded 81-page PDF bears 2024-11-26 compilation date.
- [Laz10](#) Robert Lazarsfeld — A short course on multiplier ideals: Nadel vanishing for restriction with a fixed twist. Author-hosted 37-page PDF, [arXiv:0901.0651v1](#) (2009-01-06); cited publication 2010.
- [OpenAI26LA](#) OpenAI — Log abundance in characteristic zero: Ordinary lc log abundance added only for the corollary. September 24, 2026; commit 1581d4cbdd69a0faf34b2089614fe2a814316a6a.
- [KM98](#) János Kollár; Shigefumi Mori — Birational Geometry of Algebraic Varieties: MMP, base point free and negativity; original book result/page checks remain open. Cambridge Tracts in Mathematics 134, 1998; book passages not independently located.
- [Har77](#) Robin Hartshorne — Algebraic Geometry: Surface Riemann–Roch and Hodge index; manuscript uses read, book checks remain open. Graduate Texts in Mathematics 52, 1977; book passages not independently located.
- [Ful98](#) William Fulton — Intersection Theory: Cycle intersections and Borel–Moore localization; manuscript uses read, book checks remain open. Second edition, 1998; book passages not independently located.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — One fixed model and two surfaces

Proof target : Theorem 1.1; equation (1.1) · pp. 1–2

Proof route. See Theorem 1.1 for the statement. Proposition 2.1 fixes $X \dashrightarrow Y$ so that sufficiently divisible multiples of $M_j = K_Y + t_j B_Y$, $t_j \rightarrow 0$, are generated outside sets of codimension at least three. A first, very general surface converts positive square into quadratic growth with one fixed twist. The hypothesis $\kappa_\sigma = 1$ forces that square to vanish. A second surface detects any negative curve, contradicting the zero-square limit and proving nefness.

1 Fix perturbations in codimension two

$X \dashrightarrow Y$, $M_j = K_Y + t_j B_Y$, $t_j \rightarrow 0$.
 $|d_j M_j|$ is generated on $Y \setminus Z_j$, $\text{codim } Z_j \geq 3$.

Nadel vanishing and Koszul give surjective restriction
[Lemma 3.1 · pp. 8–10](#) [\[Laz10, Theorem 2.4, p. 11\]](#)

2 First surface: growth with one fixed twist

$L = K_Y$, $h^0(\hat{Y}, m\pi^*L + P) \geq \frac{m^2}{2}L^2H^{n-2} + O(m)$.
 S and P stay fixed while $j = j(m)$ varies.

One ample A_X excludes positive quadratic limsup
[Section 4; equation \(4.1\) · pp. 13–14](#)

3 Transfer sections to the original X

$u^*K_X \geq r^*\pi^*L$, $\kappa_\sigma(X, K_X) = 1$
 $\implies L^2 \cdot H^{n-2} = 0$.

Jets and Hodge index on a fixed point and exceptional curve
[Lemmas 3.2–3.3; Proposition 3.4 · pp. 10–13](#)

4 Second surface: a negative curve is impossible

$L \cdot C_0 < 0 \implies \text{coeff}_e F_j \geq ck_j$
 $\implies M_j^2 \cdot [T] \geq \epsilon > 0$, $M_j^2 \cdot [T] \rightarrow 0$.

Combine nefness with discrepancies of the finite prefix
[Proposition 2.1\(iii\); Lemma 2.2 · pp. 4–5, 8](#)

Minimal model and canonical comparison

K_Y nef, $p^*K_X = q^*K_Y + E$, $E \geq 0$, $q_*E = 0$.
Finite K -negative sequence; no extraction; $E > 0$ on contracted divisors.

The perturbed MMP, fixed-twist growth and exclusion of negative curves

[Open figure ↗](#) [TeX source ↓](#)

1. Stabilize positive perturbations in codimension two

Choose an effective ample rational divisor B such that (X, B) is klt and $K_X + B$ is ample. Put $t_j = 2^{-j}$. Starting from the preceding nef endpoint, run the $(K + t_{j+1}B)$ -MMP with scaling of $(t_j - t_{j+1})B$. The transformed boundary remains big and effective; ampleness need not persist. BCHM supplies a finite scaling MMP. Bigness of the adjoint excludes a Mori fibre space, and the base point free theorem applies to the nef and big klt adjoint at each endpoint.

[BCHM10, Corollary 1.4.2, p. 10]

On a scaling ray, $(K + t_{j+1}B) \cdot R < 0$ and $(K + t_{j+1}B + \lambda C) \cdot R = 0$ imply $B \cdot R > 0$ and $K \cdot R < 0$. For a small contraction, the relative base point free theorem descends $K - b(K + t_{j+1}B) \sim_{\mathbb{Q}} h^*A$. With $b > 0$, relative ampleness of the flipped adjoint gives relative ampleness of the flipped canonical divisor. Every step is therefore a canonical MMP step.

Lemma 2.3 does not assert termination of the concatenation. Finiteness of divisorial contractions and strict increase of discrepancies first remove codimension-two components of the flipped loci. A decrease in the dimension of the finite-dimensional space spanned by algebraic $(n - 2)$ -cycle classes then handles the flipping loci. Choose one endpoint Y after this finite prefix. Each later finite composition is an isomorphism outside sets of codimension at least three on both sides. Normality transfers sections, giving generation of $|d_j M_j|$ on $Y \setminus Z_j$. Both d_j and Z_j may depend on j .

Proposition 2.1; Lemmas 2.2–2.3 · pp. 4–8

2. Force zero square while keeping the ample twist fixed

Put $L = K_Y$. Termination makes Y smooth in codimension two. Choose a very general complete intersection surface $S \subset Y$ avoiding every Z_j and $\text{Sing } Y$ simultaneously. Each $M_j|_S$ is semiample, so the limit $L|_S$ is nef and $L^2 \cdot H^{n-2} \geq 0$.

On a resolution $\pi : \widehat{Y} \rightarrow Y$, put $r = n - 2$, $A = \pi^*H$, and fix $P = K_{\widehat{Y}} + rA + A_0$ with A_0 ample Cartier. For each m , first choose $j = j(m)$ with mt_j sufficiently small, and then choose a **multiple k of d_j with $k > m$** . A general $D_m \in |k\pi^*M_j|$ gives $R_m = (m/k)D_m$ whose multiplier ideal is trivial near S . The difference $m\pi^*L + P - iA - K_{\widehat{Y}} - R_m$ is ample for $0 \leq i \leq r$. Nadel vanishing and the Koszul complex give surjectivity of restriction to S .

[Laz10, Theorem 2.4, p. 11]

Surface Riemann–Roch then gives, along multiples of one integer d with dL Cartier,

$$h^0(\widehat{Y}, m\pi^*L + P) \geq \frac{m^2}{2} L^2 \cdot H^{n-2} + O(m).$$

The twist P and the coefficients of the error term are independent of m, j, k . On a common resolution, the effective difference $u^*K_X - r^*\pi^*L \geq 0$ transfers sections to X . The fixed Cartier divisor $P_X = u_*r^*P$ is bounded above by one ample divisor A_X . A positive square would yield $h^0(X, mK_X + A_X) \geq cm^2$, contrary to (1.1). Hence $L^2 \cdot H^{n-2} = 0$. No uniform $O(m)$ upper bound is inferred from $\kappa_\sigma = 1$.

Lemma 3.1 · pp. 8–10 Section 4; equation (4.1) · pp. 13–14

3. Use a surface through a negative curve to establish nefness

Suppose $L \cdot C_0 < 0$. Fix a surface component T of a different complete intersection through a suitable point $x \in C_0$. The original very general S can miss C_0 , which necessitates this second surface. The squares of M_j on all components are nonnegative, and their multiplicity-weighted sum is $M_j^2 \cdot H^{n-2} \rightarrow 0$. Thus $M_j^2 \cdot [T] \rightarrow 0$.

On a resolution, fix a curve through a point z above x that dominates C_0 . The ambient jet estimate in Lemma 3.2 forces every pulled-back section of $k_j M_j$ to vanish to order at least ck_j at z , with $c > 0$ fixed. Blow up once on a fixed resolution of T . Its exceptional curve e occurs in the fixed divisor F_j with $\text{coeff}_e F_j \geq ck_j$. The Hodge index theorem gives a uniform bound $F_j^2 \leq -\epsilon k_j^2$. Nonnegative square of the residual moving system yields

$$0 \leq (k_j f^*(M_j|_T) - F_j)^2 = k_j^2 (M_j^2 \cdot [T]) + F_j^2 \implies M_j^2 \cdot [T] \geq \epsilon,$$

contradicting the zero limit. The construction also covers curves contained in the singular locus. This is the application of Proposition 3.4 to $L = K_Y$.

Lemmas 3.2–3.3; Proposition 3.4 · pp. 10–13

4. Recover all birational conclusions from the finite prefix

Projectivity, $\mathbb{Q}$ -factoriality, terminality, and non-extraction of divisors come from the finite canonical prefix in Proposition 2.1. Sum the effective canonical differences of Lemma 2.2 on a common resolution to obtain $p^*K_X = q^*K_Y + E$. Components dominating prime divisors of Y have coefficient zero, so E is q -exceptional. Every prime divisor on X contracted by the map changes from log discrepancy one to a value greater than one on terminal Y . Its strict transform therefore has positive coefficient in E . This accounts for the remaining conclusions of Theorem 1.1.

Proposition 2.1 (iii); Lemma 2.2 · pp. 4–5, 8

3. Proof of Corollary 1.2 — Add ordinary log abundance

Proof target : Corollary 1.2 · p. 2

Corollary 1.2 applies an additional input to the same constructed endpoint.

The same endpoint of Theorem 1.1

Y is projective, $\mathbb{Q}$ -factorial and terminal.

$(Y, 0)$ lc, K_Y nef and $\mathbb{Q}$ -Cartier.

Apply log abundance to the ordinary lc pair with boundary zero

[\[OpenAI26LA, Theorem 1.1, p. 2\]](#) [Corollary 1.2 · p. 2](#)

Corollary 1.2: good minimal model

$\exists m > 0 : mK_Y$ is Cartier and globally generated.

Semiamplessness of K_Y itself, not just a numerical replacement.

Ordinary log abundance upgrades the terminal nef endpoint to a good minimal model

[Open figure ↗](#) [TeX source ↓](#)

1. From a nef to a semiample canonical divisor

The variety Y is normal projective terminal, and K_Y is nef and $\mathbb{Q}$ -Cartier. Thus the ordinary pair $(Y, 0)$ is lc with effective rational boundary zero. Log abundance in characteristic zero makes a **positive Cartier multiple of K_Y itself** globally generated; it does not merely supply a numerically equivalent divisor. The map and canonical comparison constructed in Theorem 1.1 are retained, and the same endpoint is a good minimal model.

[\[OpenAI26LA, Theorem 1.1, p. 2\]](#) [Corollary 1.2 · p. 2](#)

4. Which papers supply which steps

Input

BCHM scaling MMP with big boundary

Used at

Proposition 2.1, p. 7

Role and hypotheses

klt, big effective boundary, and a nef adjoint before scaling; each positive stage is finite

Checking scope

Corollary 1.4.2, its preceding termination statement, and the application checked

Input

Lazarsfeld's Nadel vanishing

Used at

Lemma 3.1, p. 9

Role and hypotheses

Smooth resolution, effective R_m , ample difference; surjective restriction with a fixed twist

Checking scope

Theorem 2.4 and its use in (3.7) checked

Input

Kollár–Mori MMP, base point free and negativity results

Used at

§§2.1–2.2, pp. 5–8

Role and hypotheses

Canonical negativity, relative descent, singularities and effective differences

Checking scope

Uses in the manuscript read; individual statements and page locations in the book remain unchecked

Input

Surface Riemann–Roch, Hodge index and cycle intersections

Used at

Lemmas 3.1, 3.3; Proposition 3.4

Role and hypotheses

Quadratic growth and uniformly negative exceptional fixed parts on fixed surfaces

Checking scope

Calculations and hypotheses in the manuscript read; page-level checks against Hartshorne/Fulton remain open

Input

Log abundance in characteristic zero (034)

Used at

Corollary 1.2, p. 2

Role and hypotheses

Makes the nef adjoint of the ordinary lc pair $(Y, 0)$ semiample

Checking scope

Statement of Theorem 1.1 and application checked; its proof is outside scope

AHK's four-dimensional flip argument is a methodological precedent for Lemma 2.3, not an imported termination theorem in arbitrary dimension. Theorem 1.1 here also does not take the generalized lc manuscript in 036 as an input. Containment in a related existence theorem does not create an unchecked direct-dependency arrow.

5. Return to the sources

- Theorem 1.1; equation (1.1) · pp. 1–2 — The section-growth convention and all conclusions
- Proposition 2.1; Lemmas 2.2–2.3 · pp. 4–8 — From positive perturbations to a model fixed in codimension two
- Lemma 3.1 · pp. 8–10 — The fixed twist and the choice of $j(m)$
- Lemmas 3.2–3.3; Proposition 3.4 · pp. 10–13 — Excluding negative curves on a fixed surface
- Section 4; equation (4.1) · pp. 13–14 — Section transfer to X and use of numerical dimension
- Corollary 1.2 · p. 2 — Where log abundance is added

Theorem 1.1, Corollary 1.2 and the proof text of §§2–4 were read, including where the model, twist and exceptional curve are fixed. Statements and applications of BCHM, Nadel vanishing and log abundance were checked. The classical textbook passages listed above, proofs of external inputs, and independent verification of the entire argument remain outside this checking scope. These are the manuscript's claims, not a certification by specialist review or formal verification.