

OpenAI Math Digest

CATALOGUE 034

Log abundance and effective Iitaka fibrations

Catalogue synthesis & 14 proof overviews

Independent reading guide · English edition

Edition: 2026-10-11

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AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 14 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

00 · Catalogue synthesis	3
01 · Log abundance for compact Kähler spaces under logarithmic Iitaka subadditivity	38
02 · Uniform indices for semi-log-canonical log Calabi–Yau pairs	57
03 · Conditional good minimal models for compact Kähler fourfolds	69
04 · Log abundance in characteristic zero	85
05 · Minimal metrics and interior injectivity for nef adjoints	109
06 · Fourfold nonvanishing by minimal metrics and moving jets	122
07 · Lifting sections from the reduced support of an adjoint	133
08 · Schnell fiber spaces and good canonical models	144
09 · Uniform log Iitaka fibrations and bounded moduli denominators	155
10 · Uniform Pluricanonical Iitaka Fibrations	168
11 · Relative denominators and effective systems for log Calabi-Yau fibrations	183
12 · Arithmetic Stein-degree bounds for log Calabi–Yau pairs	196
13 · Uniform effective log Iitaka fibrations for fourfolds	210
14 · Abundance after nonvanishing for compact Kähler fourfolds	219

Original-manuscript snapshot: `adc7f1241b42e322a6451854ab7e4b4c146bf78a`

Log abundance and effective Iitaka fibrations

From a first section to a good model, and from a good model to a uniform Iitaka degree. Follow the claims of 14 manuscripts and the results their arguments use.

14 manuscript summaries **23** selected connections **14** proof overviews

This is a reading map of the manuscripts' claims and uses of cited results. The arrows record selected source-to-use connections; they are not a complete dependency graph or a certification of the proofs.

All 14 manuscripts, in official order

The summaries give the principal scope, not every technical hypothesis. Follow the theorem reference for the full statement. A uniform degree here is an existence statement unless a manuscript explicitly gives a numerical value.

Catalogue 034 · Pinned source snapshot checked on 8 October 2026

#

01

Manuscript and overview

Log abundance for compact Kähler spaces under logarithmic Iitaka subadditivity

→

Short name: Kähler abundance

2026-10-04 · 148 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Under logarithmic Iitaka subadditivity (Assumption 1.1), claims semiampleness of analytically nef adjoints of compact Kähler lc pairs with effective rational boundary in every dimension.

[Theorem 1.2 · p. 3](#) ↗

Role and available overview

Uses the projective good-model argument in Log Abundance as the anchor for the fibration and simple-space cases.

[Read the proof overview](#) →

#

02

Manuscript and overview

Uniform indices for semi-log-canonical log Calabi–Yau pairs →

Short name: slc indices

2026-10-05 · 42 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For fixed dimension and a finite rational coefficient set, claims a common trivializing multiple for connected projective slc log Calabi–Yau pairs. The main theorem covers dimension at least four.

[Theorem 1.1 · p. 2](#) ↗

Role and available overview

Adds character bounds and residue gluing across the conductor to the normal lc index theorem.

[Read the proof overview](#) →

#

03

Manuscript and overview

Conditional good minimal models for compact Kähler fourfolds →

Short name: Conditional Kähler models

2026-10-05 · 138 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For globally strongly $\mathbb{Q}$ -factorial compact Kähler klt fourfold pairs with pseudo-effective adjoint, claims good minimal models under three premises: orbifold subadditivity, the specified MMP, and abundance after nonvanishing.

[Theorems 1.1–1.2 · p. 4; Assumptions 2.2–2.4 · pp. 8–9](#) ↗

Role and available overview

Supplies nonvanishing at the nef MMP endpoint, then applies Kähler fourfold abundance after nonvanishing. Its stated conditional scope is retained.

[Read the proof overview →](#)

#

04

Manuscript and overview

Log abundance in characteristic zero →

Short name: Log Abundance

2026-09-24 · 85 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Claims semiampleness for nef adjoints of projective lc rational pairs over algebraically closed characteristic-zero fields, and good models for pseudo-effective lc real pairs over $\mathbb{C}$.

[Theorem 1.1 · p. 2](#); [Theorem 11.1 · p. 73](#) ↗

Role and available overview

A central qualitative input for models and nonvanishing. Some manuscripts instead use only its independent tools in §§7–9.

[Read the proof overview](#) →

#

05

Manuscript and overview

Minimal metrics and interior injectivity for nef adjoints →

Short name: Minimal metrics

2026-09-27 · 21 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Claims zero Lelong numbers for minimal metrics on nef projective complex klt adjoints, and H^1 injectivity for SNC interior boundaries with zero-Lelong semipositive endpoint metrics.

[Theorems 1.1–1.2 · pp. 1–2](#) ↗

Role and available overview

Provides the analytic input for lifting sections from the whole reduced boundary in fourfold nonvanishing.

[Read the proof overview](#) →

#

06

Manuscript and overview

Fourfold nonvanishing by minimal metrics and moving jets →

Short name: Fourfold nonvanishing

2026-09-27 · 38 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Claims canonical nonvanishing for smooth projective complex fourfolds with pseudo-effective canonical divisor; Hashizume's reduction gives sections for nef lc rational adjoints in dimension at most four.

[Theorem 1.1 / Corollary 1.2 · p. 2](#) ↗

Role and available overview

The first-section step, using metrics, interior injectivity, and Log Abundance's finite-data Frobenius theorem.

[Read the proof overview](#) →

#

07

Manuscript and overview

Lifting sections from the reduced support of an adjoint →

Short name: Supported lifting

2026-09-27 · 40 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For projective $\mathbb{Q}$ -factorial dlt pairs, claims positive Iitaka dimension from a nonzero effective adjoint multiple supported in the floor and semiample on its entire reduced support. Deduces abundance after nonvanishing through dimension four.

[Theorems 1.1–1.3 / Corollary 1.4 · p. 2](#) ↗

Role and available overview

The step from an existing section to semiampleness. Fourfold nonvanishing enters Corollary 1.4, separately from the proof of Theorem 1.2.

[Read the proof overview →](#)

#

08

Manuscript and overview

Schnell fiber spaces and good canonical models →

Short name: Schnell

2026-09-24 · 11 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Claims a twisted nonvanishing inequality for smooth projective fibre spaces with general fibre of Kodaira dimension zero, with Campana–Peternell consequences. See the overview for the exact hypotheses.

[Theorem 1.1 / Corollaries 1.2–1.3 · pp. 1–2](#) ↗

Role and available overview

Uses the good canonical model supplied by Log Abundance to obtain a contradiction between mixed-intersection signs.

[Read the proof overview →](#)

#

09

[Manuscript and overview](#)**Uniform log Iitaka fibrations and bounded moduli denominators** →

Short name: Log Iitaka

2026-10-04 · 32 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For fixed dimension at least five and a finite rational coefficient set, claims that one rounded complete system generates the full Iitaka field of every projective lc pair with nonnegative Iitaka dimension.

[Theorem 1.1 · p. 2; Theorem 4.2 · pp. 7–8](#) ↗[Role and available overview](#)

Uses Log Abundance, normal lc indices, and arithmetic Stein degrees to control moduli denominators in the presence of horizontal boundary.

[Read the proof overview](#) →

#

10

Manuscript and overview

Uniform Pluricanonical Iitaka Fibrations →

Short name: Uniform Iitaka

2026-10-03 · 44 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Claims a dimension-only pluricanonical Iitaka degree for smooth projective varieties with nonnegative Kodaira dimension, and uniform normal lc log Calabi–Yau indices for rational DCC coefficient sets.

[Theorems 1.1–1.2 · p. 2](#) ↗

Role and available overview

Inducts simultaneously on Iitaka degrees and normal lc indices. The index theorem feeds the Log Iitaka and slc-index manuscripts.

[Read the proof overview](#) →

#

11

Manuscript and overview

Relative denominators and effective systems for log Calabi-Yau fibrations →

Short name: Relative denominators

2026-09-27 · 23 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For projective lc log Calabi–Yau fibrations of total dimension at most four with finite rational coefficients, claims an exact pullback presentation and a uniform b-Cartier multiple of the moduli b-divisor.

[Theorem 1.1 · p. 2; Proposition 6.1 · p. 15](#) ↗

Role and available overview

For a big base divisor, supplies a uniform complete system generating the base field; a relative input to effective fourfold Iitaka fibrations.

[Read the proof overview →](#)

#

12

Manuscript and overview

Arithmetic Stein-degree bounds for log Calabi–Yau pairs →

Short name: Stein degree

2026-09-25 · 15 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

For projective lc log Calabi–Yau pairs over a characteristic-zero field k with $H^0(\mathcal{O}_X)=k$, claims a bound depending on dimension and t for the constant-field degree of boundary components of coefficient at least t .

[Theorem 1.1 · p. 1](#) ↗

Role and available overview

Controls finite field extensions in Stein factorization when adjunction or residue passes to a boundary component.

[Read the proof overview →](#)

#

13

Manuscript and overview

Uniform effective log Itaka fibrations for fourfolds →

Short name: Fourfold Itaka

2026-09-26 · 52 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Claims a uniform canonical index for projective klt Calabi–Yau fourfolds and a uniform effective Itaka degree for pseudo-effective lc fourfold adjoints with finite rational coefficients.

[Theorems 1.1–1.2 · pp. 1–2](#) ↗

Role and available overview

Combines Log Abundance nonvanishing, semiamplessness from Supported lifting, and relative denominator bounds.

[Read the proof overview](#) →

#

14

Manuscript and overview

Abundance after nonvanishing for compact Kähler fourfolds →

Short name: Kähler after nonvanishing

2026-09-27 · 92 pages

[Original paper \(GitHub\) ↗](#)

Principal claim

Claims semiampleness for analytically nef adjoints of compact Kähler klt fourfolds with effective rational boundary and nonvanishing, without projectivity or $\mathbb{Q}$ -factoriality.

[Theorem 1.1 · p. 3 ↗](#)

Role and available overview

Handles the Kähler fourfold step after nonvanishing, proving its analytic boundary arguments within the manuscript.

[Read the proof overview →](#)

Three strands of the argument

These are reading groups for this guide. The numbered manuscript inventory above retains the official catalogue order. Short names used in diagrams are paired with the full titles there.

01

From nonvanishing to abundance ↓

Obtain a first section and a good model.

Log Abundance supplies an all-dimensional qualitative input. In dimension four, metrics and nonvanishing meet lifting and abundance after nonvanishing. Schnell applies good models.

- 04 Log Abundance
- 05 Minimal metrics
- 06 Fourfold nonvanishing
- 07 Supported lifting

- 08 Schnell

02

From indices to uniform Iitaka degrees ↓

Control the degree by dimension and coefficients.

Combines arithmetic Stein degrees, normal lc indices, and moduli denominators. The slc index problem adds descent of trivializations across the conductor.

- 02 slc indices
- 09 Log Iitaka
- 10 Uniform Iitaka
- 11 Relative denominators
- 12 Stein degree
- 13 Fourfold Iitaka

03

Compact Kähler spaces ↓

Keep the nonprojective hypotheses visible.

Separates conditional all-dimensional lc abundance from fourfold klt models and abundance after nonvanishing. Similar titles alone do not create dependency edges.

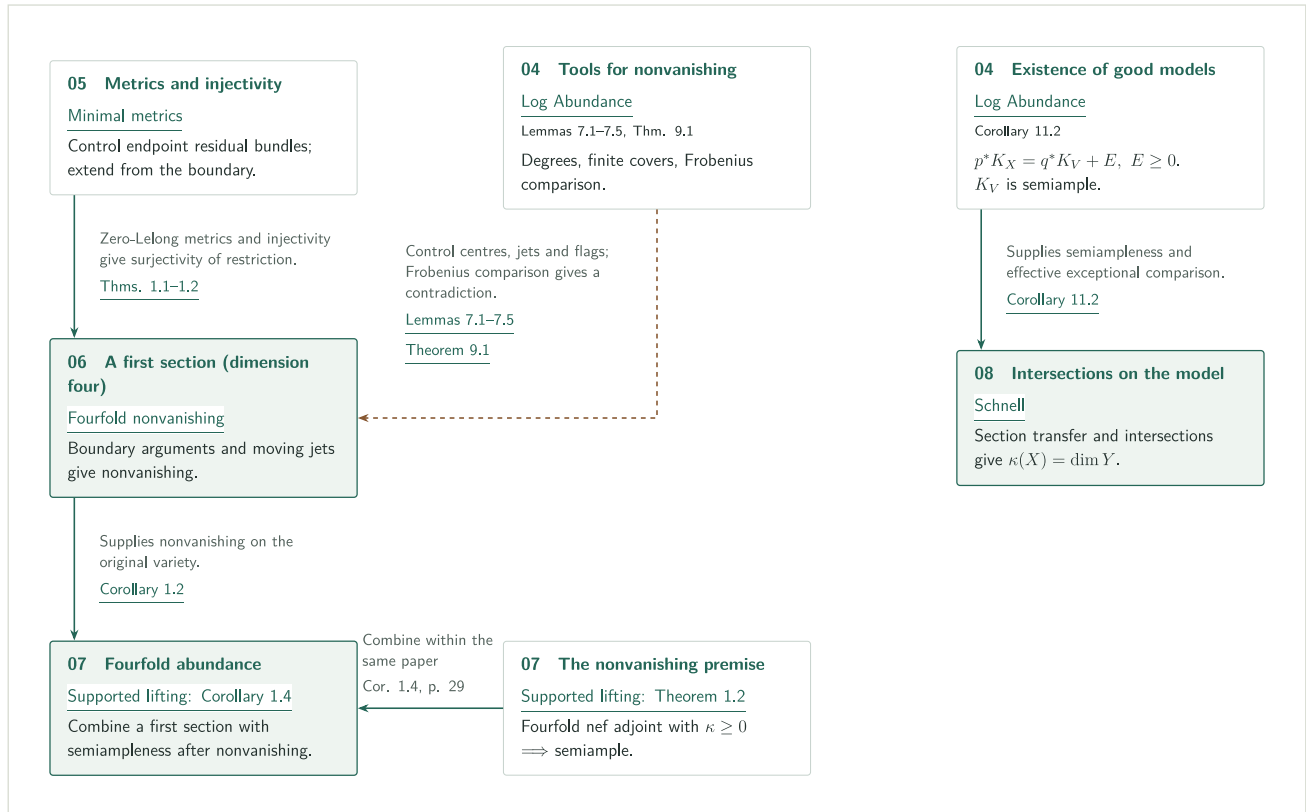
- 01 Kähler abundance
- 03 Conditional Kähler models
- 14 Kähler after nonvanishing

The connection between the strands

Log Abundance supplies qualitative good models to Schnell and to the effective-degree papers. The fourfold route combines nonvanishing, lifting and relative denominators. In the Kähler strand, projective good models become an anchor for analytic induction; its subadditivity and MMP premises remain visible.

1. Nonvanishing and abundance: from a first section to generation

Log Abundance supplies all-dimensional good models. Alongside it, the fourfold papers separate nonvanishing from lifting. Schnell is an application of the geometry of the resulting good model.



Solid arrows supply theorems or corollaries; dashed arrows supply auxiliary results. The two Log Abundance boxes denote different results in the same paper. Underlined arrow labels open the connection details; paper names open their proof overviews.

[Open figure ↗](#) [TeX source ↓](#)

1. Metrics extend boundary sections; nonzero sections need another input

Fourfold nonvanishing uses Minimal metrics, Theorems 1.1–1.2, for zero-Lelong metrics and interior injectivity on the endpoint residual bundles. These supply surjectivity of restriction. Three-dimensional semi-dlt abundance and descent separately produce a nonzero section on the whole boundary; combining the two steps gives nonvanishing upstairs.

Connection details: [Minimal metrics](#) · [Fourfold nonvanishing](#) · [Theorems 1.1–1.2](#)

[Fourfold nonvanishing](#) · [Proof overview](#) →

2. The remaining nonvanishing argument imports Log Abundance tools

The moving-jet argument imports Log Abundance, Lemmas 7.1–7.5, to control centres, degrees and covers. Theorem 9.1 then rules out coexistence of the constructed jets, flag and movable curve by Frobenius comparison. These are auxiliary inputs, rather than an application of Log Abundance’s abundance main theorem.

Connection details: [Log Abundance](#) → [Fourfold nonvanishing](#) · [Lemmas 7.1–7.5](#)
[Log Abundance](#) → [Fourfold nonvanishing](#) · [Theorem 9.1](#)

[Fourfold nonvanishing](#) · [Proof overview](#) → [Log Abundance](#) · [Proof overview](#) →

3. Combine nonvanishing with abundance after nonvanishing

Supported lifting, Theorem 1.2, proves fourfold semiample assuming nonnegative Kodaira dimension. Fourfold nonvanishing, Corollary 1.2, supplies the first section, giving Supported lifting, Corollary 1.4. A separate input to its conditional Theorem 1.3 is FN Lemma 8.1: rational linear algebra converting an effective real presentation to a rational one. The two uses must be distinguished.

Connection details: [Fourfold nonvanishing](#) → [Supported lifting](#) · [Corollary 1.2](#)
[Fourfold nonvanishing](#) → [Supported lifting](#) · [Lemma 8.1](#)

[Supported lifting](#) · [Proof overview](#) →

4. Once a good model is supplied, Schnell finishes by intersection theory

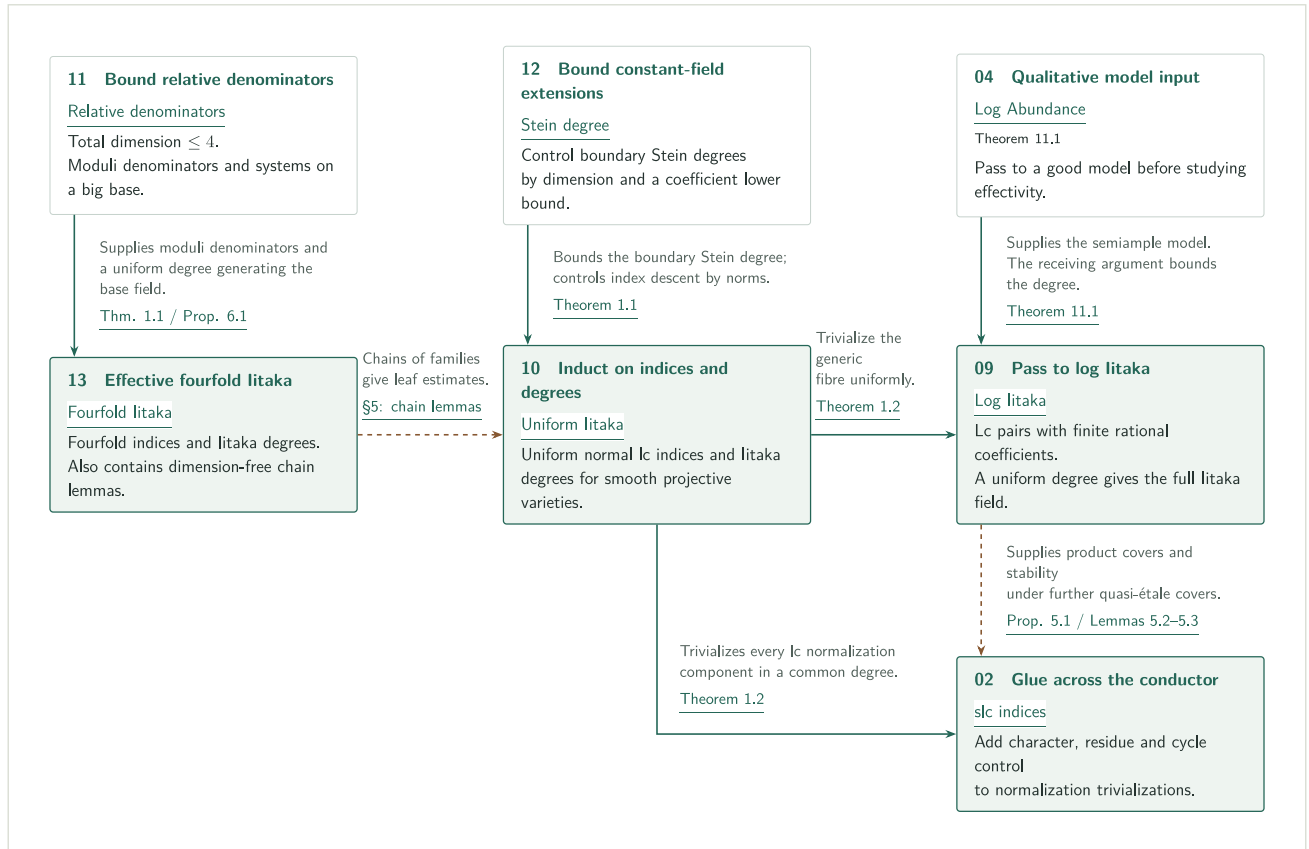
Log Abundance, Corollary 11.2, supplies a model with semiample canonical divisor and an effective exceptional comparison. Schnell transfers sections to bound the Kodaira dimension, then uses the mixed intersections of Lemma 2.3 for the remaining inequality. Model existence is the imported input; the subsequent intersection argument is carried out in Schnell.

Connection details: [Log Abundance](#) → [Schnell](#) · [Corollary 11.2](#)

[Schnell](#) · [Proof overview](#) →

2. Effectivity: from existence of sections to a uniform degree

Good models alone do not bound the degree uniformly. The arguments separately control generic-fibre indices, moduli denominators on the base and degrees of finite covers. The diagram shows the six papers addressing these tasks and an input from Log Abundance.



The dashed arrow from Fourfold litaka to Uniform litaka uses chain lemmas stated in arbitrary dimension, not the fourfold main theorem. The text below adds the other applications of Log Abundance and Supported lifting.

[Open figure ↗](#) [TeX source ↓](#)

1. Separate qualitative good models from uniform denominators

In Fourfold litaka, Log Abundance supplies nonvanishing from pseudo-effectivity and Supported lifting makes the nef adjoint semiample after nonvanishing. Relative denominators, Theorem 1.1 and Proposition 6.1, control relative denominators and complete systems on a big base uniformly. The qualitative step supplies the fibration; the effective step captures its function field in a uniform degree.

Connection details: Log Abundance → Fourfold litaka · Theorems 1.1, 11.1 / Lemma 2.2

Supported lifting → Fourfold litaka · Theorem 1.2

Relative denominators → Fourfold litaka · Theorem 1.1 / Proposition 6.1

Fourfold litaka · Proof overview → Relative denominators · Proof overview →

2. Induct on normal lc indices and litaka degrees together

Uniform litaka combines Log Abundance good models with Stein degree, Theorem 1.1, bounding constant-field extensions of boundary components and hence norm descent. Arbitrary-dimensional chain lemmas from Fourfold litaka lead from covering families to estimates on leaves. The output is a uniform complete system generating the full litaka field, not just an image of the right dimension.

Connection details: [Log Abundance](#) → [Uniform litaka](#) · [Theorem 11.1](#)

[Stein degree](#) → [Uniform litaka](#) · [Theorem 1.1](#)

[Fourfold litaka](#) → [Uniform litaka](#) · [Lemma 5.1](#) / [Corollary 5.2](#) / [Lemma 5.4](#)

[Uniform litaka](#) · [Proof overview](#) → [Stein degree](#) · [Proof overview](#) →

3. Use normal indices on the generic fibre to obtain log litaka effectivity

Log litaka applies Uniform litaka, Theorem 1.2, to the generic fibre, fixing a uniform trivializing multiple and an exact pullback presentation. Log Abundance supplies the semiample model, while Stein degree controls finite extensions in the residue induction. The finite rational coefficient set of the log litaka main theorem should be distinguished from the rational DCC coefficient scope of the normal lc index theorem.

Connection details: [Uniform litaka](#) → [Log litaka](#) · [Theorem 1.2](#) [Log Abundance](#) → [Log litaka](#) · [Theorem 11.1](#)

[Stein degree](#) → [Log litaka](#) · [Theorem 1.1](#)

[Log litaka](#) · [Proof overview](#) →

4. For slc pairs, glue normalization trivializations across the conductor

slc indices applies the normal lc index theorem on normalization components, but descent to the original slc pair needs more. Product covers and stability under further covers from Log litaka, the rational-image lemma from Uniform litaka, and Log Abundance good models enter the character bounds. Residues and cycles must then be made compatible. The slc paper itself supplies the final conductor gluing.

Connection details: [Uniform litaka](#) → [slc indices](#) · [Theorem 1.2](#)

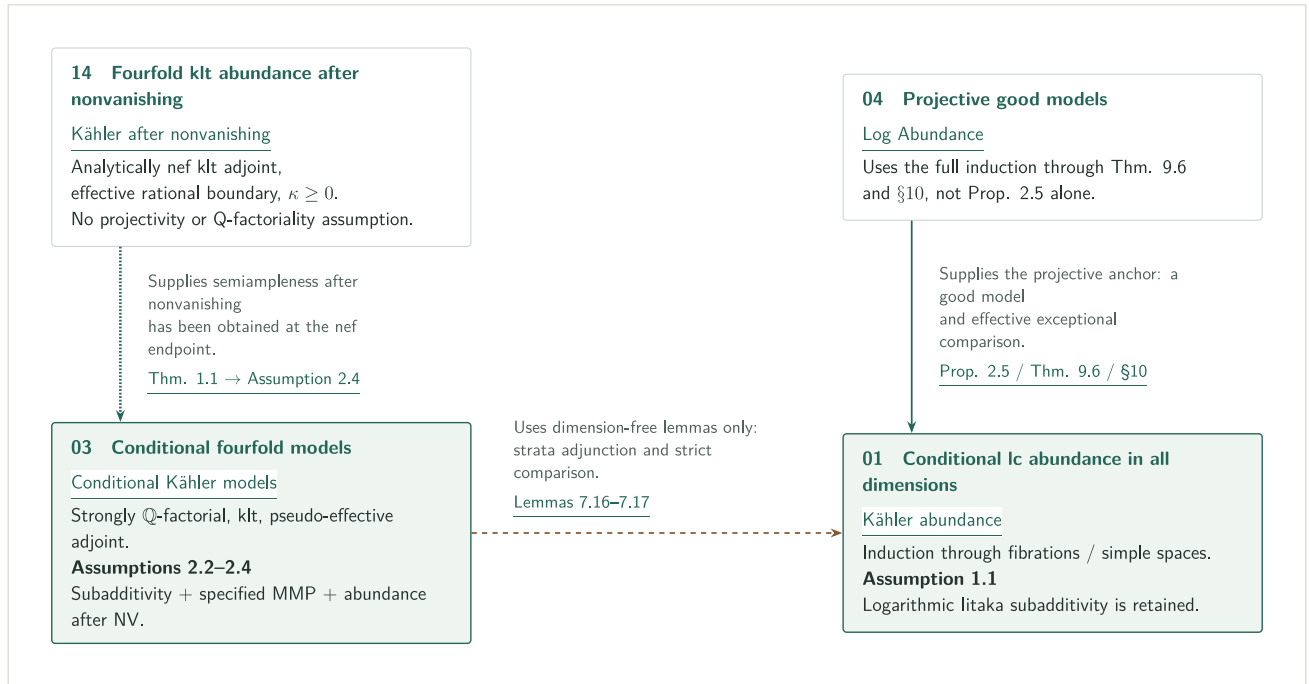
[Log litaka](#) → [slc indices](#) · [Proposition 5.1](#) / [Lemmas 5.2–5.3](#) [Uniform litaka](#) → [slc indices](#) · [Lemma 5.2](#)

[Log Abundance](#) → [slc indices](#) · [Theorem 11.1](#)

[slc indices](#) · [Proof overview](#) →

3. Kähler: retain dimension, singularities and premises

Conditional all-dimensional lc abundance, conditional klt fourfold models, and klt fourfold abundance after nonvanishing have different scopes. Dimension-free auxiliary lemmas are drawn separately from the main theorems of the papers containing them.



Dotted arrows denote explicit premises. The model theorem at lower left retains Assumptions 2.2–2.4; all-dimensional lc abundance at lower right retains Assumption 1.1. The horizontal dashed arrow imports only the strata lemmas.

[Open figure ↗](#) [TeX source ↓](#)

1. In dimension four, separate MMP, nonvanishing and subsequent abundance

Conditional Kähler models proves nonvanishing at the specified nef MMP endpoint, then invokes Kähler after nonvanishing, Theorem 1.1, for semiampleness. This is Assumption 2.4. Orbifold subadditivity (Assumption 2.2) and the specified MMP (Assumption 2.3) remain necessary; the after-nonvanishing theorem alone does not supply all premises.

Connection details: [Kähler after nonvanishing](#) → [Conditional Kähler models](#) · [Theorem 1.1](#)

[Conditional Kähler models](#) · [Proof overview](#) → [Kähler after nonvanishing](#) · [Proof overview](#) →

2. For all-dimensional lc abundance, connect a projective anchor to analytic induction

Kähler abundance, Proposition 2.10, uses the full Log Abundance induction for projective good models and effective exceptional comparison. The argument then splits into a positive-dimensional fibration case and a simple case of algebraic dimension zero. Theorem 6.1 in the latter gives meromorphic nonvanishing, not directly a holomorphic section. Signed-boundary rigidity and lower-dimensional induction return to generation of the original nef adjoint. Assumption 1.1 remains in the main theorem.

Connection details: [Log Abundance](#) → [Kähler abundance](#) · [Proposition 2.5 / Theorem 9.6 / §10](#)

[Kähler abundance](#) · [Proof overview](#) → [Kähler abundance](#) · [Proof overview](#) →

3. The inputs extending beyond dimension four are specified lemmas

Conditional Kähler models, Lemmas 7.16–7.17, give dimension-free strata adjunction and strict comparison, used in Kähler abundance, Theorem 3.13. Torsion-freeness in Kähler after nonvanishing, Lemma 7.1, is another input: Kähler abundance, Lemma 4.3, uses it to pass from residue matching to section extension. Neither connection applies a fourfold main theorem in arbitrary dimension.

Connection details: [Conditional Kähler models](#) → [Kähler abundance](#) · [Lemmas 7.16–7.17](#)

[Kähler after nonvanishing](#) → [Kähler abundance](#) · [Lemma 7.1](#)

[Kähler abundance](#) · [Proof overview](#) → [Kähler after nonvanishing](#) · [Proof overview](#) →

Follow a result into its application

The following records include all 23 selected connections. Each gives the supplied result, its use, and the scope of the editorial check.

23 selected connections

INPUT · 06

Fourfold nonvanishing

[Lemma 8.1](#) · [p. 35](#) ↗

Auxiliary result

Converts a real effective presentation to a rational one using finite rational linear constraints. This linear-algebra lemma uses no nonvanishing theorem and is distinct from the input to Corollary 1.4.

USED IN · 07

Supported lifting

[Theorem 1.3, proof](#) · [p. 27](#) ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 03

Conditional Kähler models

Lemmas 7.16–7.17 · pp. 58–60 ↗

Auxiliary result

Supplies actual adjunction on normalized strata and strict discrepancy comparison over the exceptional locus. Both lemmas are dimension-free; the fourfold main theorem is not applied in all dimensions.

USED IN · 01

Kähler abundance

Theorem 3.13, proof · pp. 29–30 ↗

Source overview → Receiving overview →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 14

Kähler after nonvanishing

Lemma 7.1 · pp. 51–52 ↗

Auxiliary result

Uses torsion-freeness of the first higher direct image when the pair is klt off its floor and an actual positive adjoint multiple is pulled back from the base, to turn generic-fibre residue matching into extension of restricted sections.

USED IN · 01

Kähler abundance

Lemma 4.3, proof · p. 74 ↗

Source overview → Receiving overview →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 04

Log Abundance

Lemmas 7.1–7.5 · pp. 45–52 ↗

Auxiliary result

Independent tools controlling centre degrees, moving multiplicities, slices, and finite covers of a fixed branch complement.

USED IN · 06

Fourfold nonvanishing

§3 / Proposition 6.4 · pp. 8–12, 24 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 10

Uniform Itaka

Lemma 5.2 · pp. 21–22 ↗

Auxiliary result

Supplies the rational-image property of boundary-free nonabelian factors for the Hodge-rank theorem. The source lemma itself uses Log Abundance nonvanishing.

USED IN · 02

slc indices

Lemma 6.6 · pp. 32–34 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 09

Log Iitaka

Proposition 5.1 / Lemmas 5.2–5.3 · pp. 12–13 ↗

Auxiliary result

Uses a quasi-étale product cover and stability under further covers to control the character on the volume form factor by factor.

USED IN · 02

slc indices

Input 6.3 / Lemma 6.4 · pp. 30–32 ↗

Source overview → Receiving overview →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 04

Log Abundance

Corollary 11.2 · pp. 73–74 ↗

Theorem input

Supplies a good minimal model, semiampleness, and effective exceptional comparison for a pseudo-effective canonical divisor.

USED IN · 08

Schnell

Theorem 3.1 / §4.1 · pp. 7–8 ↗

Source overview → Receiving overview →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 05

Minimal metrics

Theorems 1.1–1.2 · pp. 1–2 ↗

Theorem input

Identifies endpoint residual bundles with positive rational multiples of nef klt adjoints, supplying zero-Lelong metrics and interior injectivity. A nonzero section on the whole boundary additionally uses three-dimensional semi-dlt abundance and descent.

USED IN · 06

Fourfold nonvanishing

Theorems 4.1–4.2 / §5 · pp. 13–20 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 04

Log Abundance

Theorem 9.1 · pp. 62–69 ↗

Auxiliary result

Rules out coexistence of the fixed jets, flag, and movable curve constructed in dimension four. This input is the independent Frobenius theorem, not Log Abundance’s abundance theorem.

USED IN · 06

Fourfold nonvanishing

Theorem 7.1 / §7 · pp. 30–34 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 06

Fourfold nonvanishing

Corollary 1.2 · p. 2 ↗

Input to a corollary

Provides the first section on the original variety; combined with the independent Theorem 1.2 to obtain fourfold log abundance.

USED IN · 07

Supported lifting

Corollary 1.4, proof · p. 29 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 07

Supported lifting

Theorem 1.2 · p. 2 ↗

Theorem input

Makes a nef adjoint with known nonvanishing semiample, in the model comparison preserving rounded section spaces in every degree.

USED IN · 13

Fourfold Itaka

Theorem 2.1 / Proposition 2.2 · pp. 6–7 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 11

Relative denominators

Theorem 1.1 / Proposition 6.1 · pp. 2, 15 ↗

Theorem input

Controls moduli denominators and supplies a uniform degree generating the full field of a positive-dimensional big base.

USED IN · 13

Fourfold Itaka

§3 / §10 · pp. 8, 48–49 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 04

Log Abundance

Theorems 1.1, 11.1 / Lemma 2.2 · pp. 2, 73, 8 ↗

Theorem input

Provides the qualitative first section from pseudo-effectivity. Uniform degree comes later from index and relative denominator arguments.

USED IN · 13

Fourfold Itaka

§10 · p. 48 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 04

Log Abundance

Theorem 11.1 · p. 73 ↗

Theorem input

Supplies all-dimensional projective lc good models, including nonvanishing and semiample on the model.

USED IN · 10

Uniform Itaka

Theorem 2.1 · p. 4 ↗

Source overview → Receiving overview →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 12

Stein degree

Theorem 1.1 · p. 1 ↗

Theorem input

Bounds the Stein degree of a coefficient-one component, controlling the index when a principal divisor descends by a field norm.

USED IN · 10

Uniform Itaka

Proposition 3.2, proof · pp. 7–8 ↗

Source overview → Receiving overview →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 13

Fourfold Iitaka

Lemma 5.1 / Corollary 5.2 / Lemma 5.4 · pp. 18, 20–21 ↗

Auxiliary result

Constructs a rational fibration from chains of covering families and bounds section orders and degrees on its leaves. Uses the arbitrary-dimensional chain lemmas, rather than the fourfold main theorem.

USED IN · 10

Uniform Iitaka

Lemma 6.2 · pp. 24–25 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 10

Uniform Iitaka

Theorem 1.2 · p. 2 ↗

Theorem input

Supplies a uniform trivializing degree on the generic fibre, fixing an exact pullback presentation.

USED IN · 09

Log Iitaka

Theorem 2.1 / Proposition 7.1 · pp. 5, 23–24 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 12

Stein degree

Theorem 1.1 · p. 1 ↗

Theorem input

Bounds the field extension from the Stein curve of a boundary component in the residue induction.

USED IN · 09

Log Iitaka

Theorem 2.3 / Lemma 4.6 · pp. 5, 10–11 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 04

Log Abundance

Theorem 11.1 · p. 73 ↗

Theorem input

Passes from nonnegative Iitaka dimension to a good model, where the effective-degree argument uses the semiample fibration.

USED IN · 09

Log Iitaka

Theorem 2.2 / §8 · pp. 5, 29 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 10

Uniform Iitaka

Theorem 1.2 · p. 2 ↗

Theorem input

Trivializes all lc normalization components in one even degree. The receiving paper adds conductor gluing.

USED IN · 02

slc indices

Theorem 2.2 / §7.3 · pp. 5, 37 ↗

Source overview → Receiving overview →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 04

Log Abundance

Theorem 11.1 · p. 73 ↗

Theorem input

Supplies good models of perturbed projective klt pairs in the Hodge-rank argument.

USED IN · 02

slc indices

Theorem 2.3 / Lemma 5.6 · pp. 5, 20–21 ↗

Source overview → Receiving overview →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 04

Log Abundance

Proposition 2.5 / Theorem 9.6 / §10 · pp. 9–12, 69–73 ↗

Theorem input

Uses Log Abundance’s full induction under the stated logarithmic subadditivity assumption to supply projective good models and effective exceptional comparison.

USED IN · 01

Kähler abundance

Proposition 2.10 · p. 12 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

INPUT · 14

Kähler after nonvanishing

Theorem 1.1 · p. 3 ↗

Explicit premise

Gives semiampleness once nonvanishing is obtained for the nef Kähler klt fourfold adjoint. The other two premises remain separate.

USED IN · 03

Conditional Kähler models

Assumption 2.4 · p. 9 ↗

[Source overview](#) → [Receiving overview](#) →

Input statement and the receiving connection compared; the complete input proof is not verified.

[Link to this connection](#)

Read dependencies at the level of results: Fourfold nonvanishing uses Log Abundance’s independent geometric tools and Frobenius theorem. Its abundance main theorem is not that input. In Supported lifting, fourfold nonvanishing enters Corollary 1.4; Theorem 1.2 already assumes a first section. These distinctions matter when tracing apparent back-and-forth citations.

Inputs from outside catalogue 034

033 → Log Abundance: subadditivity

Log Abundance’s Lemma 6.1 uses Corollary 6.2 of the orbifold Iitaka manuscript in catalogue 033, in its zero-boundary projective case. The Log Abundance overview also explains how the existing Hacon–Popa–Schnell result supplies this particular application.

[See the statements and application in Log Abundance →](#)

033 / 056 → conditional Kähler models

The receiving manuscript identifies orbifold subadditivity and the specified Kähler MMP as Assumptions 2.2–2.3. Together with abundance after nonvanishing from 034, they form its three premises. The companion statements and selected application conditions are compared in the conditional Kähler overview. The complete proofs of the premises have not been independently verified.

[Assumptions 2.2–2.4 · pp. 7–9 ↗ Read the conditional overview →](#)

Established inputs such as Hashizume’s reduction, gluing, lower-dimensional abundance, and effective birationality are not exhaustively mapped here. The individual articles identify their selected external inputs separately. The all-dimensional Kähler article also records Hodge-line, relative Iitaka and weak-effectivity inputs from OILS, outside the 14-paper inventory.

What this map checks

- Read the principal statements of all 14 manuscripts and selected proof passages where results are imported. For the recorded arrows, compared the supplied statement with the receiving connection. This does not verify the complete source or receiving proof.
- Manuscripts with a separate proof overview are marked in the inventory above. Each article records its reading scope and unchecked passages; the full proofs and unlisted connections are not certified. No recorded arrow does not mean no dependency.
- Background references are kept separate from proof inputs. For example, Kähler after nonvanishing (p. 6) treats projective lifting and Log Abundance as background while proving its analytic boundary argument internally. The conditional Kähler manuscript retains its three stated premises.

- All 14 PDF hashes match the recorded source snapshot. This page refers to the commit below, checked on 8 October 2026, and does not incorporate subsequent revisions. Expert review and formal verification have not been performed.

[OpenAI Research Catalog · 034 · p. 5 ↗](#)

[Official catalogue at the reference commit ↗](#)

CONTENTS.md: Log abundance for compact Kähler spaces under logarithmic litaka subadditivity

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034 / 01

[Read on the website ↗](#)

CATALOGUE 034 · PAPER 01 / 14 Overview

Log abundance for compact Kähler spaces under logarithmic Iitaka subadditivity

Conditional dimension induction and generation of actual line bundles

Assuming logarithmic Iitaka subadditivity, the manuscript proves semiampleness of nef adjoints for compact Kähler lc pairs in every finite dimension. It constructs good divisorial decompositions by induction, using Hodge lines for fibrations and signed boundaries on simple spaces to reach generation of actual holomorphic lines.

Manuscript author: OpenAI · 2026-10-04 · 148 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

The statements retain the source order: Assumption 1.1, the final target Theorem 1.2, the induction statement Theorem 2.13, and four tools used in the proof. Each retains its own hypotheses and scope.

Assumption 1.1 — Logarithmic Iitaka subadditivity

The paper assumes that for every surjective morphism $f : X \rightarrow Y$ with connected fibres between smooth connected complex projective varieties, and reduced effective SNC divisors D_X, D_Y , including zero divisors,

$$\mathrm{Supp}(f^*D_Y) \subseteq \mathrm{Supp}(D_X) \implies \kappa(X, K_X + D_X) \geq \kappa(F, K_F + D_F) + \kappa(Y, K_Y + D_Y)$$

(Assumption 1.1, p. 3). Here F is a very general smooth fibre and $D_F = D_X|_F$. The source's convention for $\kappa = -\infty$ is retained.

[Assumption 1.1 · p. 3](#)

Theorem 1.2 — Log abundance for compact Kähler spaces

Assume Assumption 1.1. Let X be a normal irreducible compact Kähler complex analytic space, and let $\Delta \geq 0$ be a rational boundary such that (X, Δ) is lc and $J = K_X + \Delta$ is $\mathbb{Q}$ -Cartier. If J is analytically nef, there is a positive integer m such that mJ is Cartier and the evaluation map

$$H^0(X, \mathcal{O}_X(mJ)) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \mathcal{O}_X(mJ)$$

is surjective everywhere. This includes every finite dimension and zero boundary, but remains **conditional on subadditivity**. It asserts generation of an actual holomorphic line bundle, a stronger conclusion than expressing its Chern class as the pullback of a Kähler class.

Theorem 1.2 · p. 3

Theorem 2.13 — Good divisorial decomposition

The inductive assertion G_n concerns a smooth connected compact Kähler n -fold, an SNC boundary B with coefficients in $[0, 1] \cap \mathbb{Q}$, and a pseudo-effective adjoint $J = K_X + B$. It asks for a smooth Kähler modification $\mu : Y \rightarrow X$ and

$$\mu^* J \sim_{\mathbb{Q}} P + R, \quad P \text{ semiample}, \quad R = N(c_1(\mu^* J)) \geq 0$$

(Definition 2.1, p. 6). Here R is a rational divisor, N is the analytic divisorial negative part, and $\sim_{\mathbb{Q}}$ means an isomorphism of actual rational holomorphic lines. Theorem 2.13, p. 13 asserts all G_n under Assumption 1.1.

Definition 2.1; Theorem 2.13 · pp. 6, 13

Theorem 4.1 — Generation on the entire reduced boundary

Assume G_j for $j < n$. Let (V, B) be a normal irreducible compact Kähler dlt n -fold, globally $\mathbb{Q}$ -factorial, with effective rational boundary and analytically nef $J = K_V + B$. Under the lc-strata resolution condition of Definition 3.1, an actual Cartier multiple of $J|_{[B]}$ is generated on the **whole reduced boundary**. The resolution condition asks for a resolution with exceptional crepant coefficients strictly below one, which is an isomorphism at the generic point of every lc centre.

Theorem 4.1 · p. 71

Proposition 5.1 — The rank-one fibration case

Assume Assumption 1.1 and G_j for $j < n$. Let X be a smooth connected compact Kähler n -fold, B a rational SNC boundary, and $J = K_X + B$ pseudo-effective. Suppose there are a modification $\pi : \tilde{X} \rightarrow X$ and a base W , both smooth compact Kähler, such that $\tilde{B} = \pi_*^{-1}B + \text{Exc}(\pi)_{\text{red}}$ has SNC support. Suppose the proper surjective holomorphic map $g : (\tilde{X}, \tilde{B}) \rightarrow W$ has connected fibres, $0 < \dim W < n$, the base is projective or has $a(W) = 0$, and $\kappa(F, K_F + \tilde{B}_F) = 0$ on very general fibres. Proposition 5.1, p. 90 gives G_n in this situation.

Proposition 5.1 · p. 90

Theorem 6.1 — Meromorphic nonvanishing on simple spaces

Theorem 6.1, p. 107 states that a smooth connected simple compact Kähler manifold with $a(X) = 0$ has a nonzero meromorphic section of $K_X^{\otimes m}$ for some $m > 0$. Its proof is independent of Assumption 1.1. It does not assert a holomorphic section or an effective canonical representative.

Theorem 6.1 · p. 107

Theorem 7.1 — Signed rigidity

Theorem 7.1, p. 125 assumes a normal irreducible, globally $\mathbb{Q}$ -factorial, simple compact Kähler space X with $a(X) = 0$, a dlt pair with reduced boundary $D = \sum D_i$, and the resolution condition of Definition 3.1. Suppose $L = K_X + D$ is analytically nef; $L \sim_{\mathbb{Q}} \sum a_i D_i$ as actual rational lines, allowing negative $a_i \in \mathbb{Q}$; the pullback of $\{L - cD\}$ to a projective resolution with smooth compact Kähler source is pseudo-effective for some $c > 0$; and $L|_D$ is semiample on the whole reduced boundary. Then L is torsion.

Theorem 7.1 · p. 125

References on the arrows

Unprefixed numbers refer to this manuscript P. Catalog references LA, CGM4, and ANV4 are used only for the specified results. OILS supplies rank-one Hodge lines and periods; the remaining keys are external tools.

- LA, Proposition 2.5 / Theorem 9.6 / §10
- CGM4, Lemmas 7.16–7.17 / ANV4, §§10–12

- OLS, Proposition 2.7 / Theorem 3.1
- Ou, Theorems 1.1 / 1.4 / FM, Theorem A
- Sai, Theorem 1 / Cam, Lemma 17 / Corollary 18

2. Proofs of Theorems 2.13 and 1.2: the global induction

Proof route. Theorem 1.2 is the target and Theorem 2.13 the induction statement. [Generation on the whole boundary](#) is a shared tool. The [positive-dimensional fibration](#) and the simple case via [meromorphic nonvanishing](#) and [signed rigidity](#) feed back into this diagram.

Assume the lower-dimensional assertions G_j and prove G_n . Split by algebraic dimension and simplicity, then descend generation to the original nef lc adjoint.

01 Fix subadditivity and lower-dimensional induction

Assumption 1.1, G_j ($j < n$), $J = K_X + B$ pseudo-effective.

Use the assumption in the projective input; use an incidence cover in the nonsimple case.

[Prop. 2.10; p. 12](#) [\[LA\] Lemma 6.1; pp. 30–31](#)

02 Separate the routes by algebraic dimension

$a(X) = n$: projective good model.

$0 < a(X) < n$: algebraic reduction.

$a(X) = 0$: simple / covering family and finite descent.

Solve the fibration and simple cases; descend a torsion positive part by norms.

[Props. 2.9, 2.11–2.12; pp. 11–12](#) [Thm. 2.13; p. 13](#)

03 Obtain a decomposition of actual rational lines

$\mu^*J \sim_{\mathbb{Q}} P + R$, P semiample, $R = N(\mu^*J) \geq 0$.

Cancel the exceptional negative part; projection formula preserves generation everywhere.

[Lemma 2.3; p. 8](#) [Prop. 2.7; pp. 9–10](#)

04 Descend generation to the original nef lc adjoint

$K_Y + B_Y = p^*J + E$, $E \geq 0$ exceptional.

$N(p^*J) = 0$, $p_*\mathcal{O}_Y = \mathcal{O}_X \implies mJ$ globally generated.

TeX proof diagram descending a good decomposition to the main theorem

[Open figure ↗](#) [TeX source ↓](#)

1. Locate the use of subadditivity

The assumption enters through the projective good-model statement [Proposition 2.10, p. 12](#). That proposition invokes the induction in [LA](#), [Proposition 2.5](#), [Theorem 9.6](#) and [§10](#), and identifies its subadditivity use in [LA](#), [Lemma 6.1, pp. 30–31](#): a resolved projective Albanese fibration with both boundaries zero. This is the stated entry point of the condition in the present paper.

[Proposition 2.10 · p. 12](#) · [LA, Lemma 6.1 · pp. 30–31](#)

2. Use projective input or algebraic reduction in positive algebraic dimension

If the algebraic dimension is $a(X) = n$, Kähler–Moishezon makes X projective and Proposition 2.10 applies. If $0 < a(X) < n$, algebraic reduction gives a nontrivial fibration.

Propositions 2.10–2.11; Theorem 2.13 · pp. 12–13

3. Use a covering family and norms in the nonsimple case

When $a(X) = 0$ and X is not simple, Cam, Lemma 17 / Corollary 18, p. 10 supplies an incidence space with a nontrivial fibration and a generically finite evaluation onto X . Its algebraic dimension is still zero, so a semiample positive part there is torsion. Finite analytic norms and uniqueness of the negative part descend this purely negative decomposition (Proposition 2.9, p. 11). A space is simple when no positive-dimensional proper compact subvariety passes through a very general point.

Proposition 2.9; Theorem 2.13 · pp. 11, 13 · Cam, Lemma 17 / Corollary 18

4. Close the simple case by signed rigidity

For simple X with $a(X) = 0$, Theorem 6.1 supplies a signed canonical representative. Add a boundary and pass to a nef model; boundary generation and Theorem 7.1 give torsion. Proposition 2.12 subtracts the added boundary from the negative part. Sections 5–6 explain this route.

Proposition 2.12, proof · pp. 144–145

5. Remove the exceptional error and descend generation

The final passage to the main theorem is Proposition 2.7, pp. 9–10. Giving exceptional divisors coefficient one on an lc resolution yields $K_Y + B_Y = p^*J + E$, with $E \geq 0$ exceptional. Lemma 2.3 shows that E is added unchanged to the negative part, so it can be cancelled in the actual line identity. If the original J is nef, its remaining negative part is zero. Normality, $p_*\mathcal{O}_Y = \mathcal{O}_X$, and the projection formula identify sections and descend generation to the original singular space.

Lemma 2.3; Proposition 2.7 · pp. 8–10

3. The interface of §§3–4: models and whole-boundary generation

What is compared on strata. CGM4 Lemmas 7.16–7.17 supply actual adjoint lines and strict comparisons across small dlt steps, entering Theorem 3.13 here. Torsion-freeness from ANV4 Lemma 7.1 enters Lemma 4.3 to pass from residue compatibility to section extension. This is separate from the later use of ANV4 §§10–12 in the lifting argument of §7. Theorem 3.13 · pp. 29–30; Lemma 4.3 · p. 74

Separate contraction of a known negative part from gluing sections on the entire boundary. The first preserves the actual nef line; the second uses induction on proper strata and compatible residues.

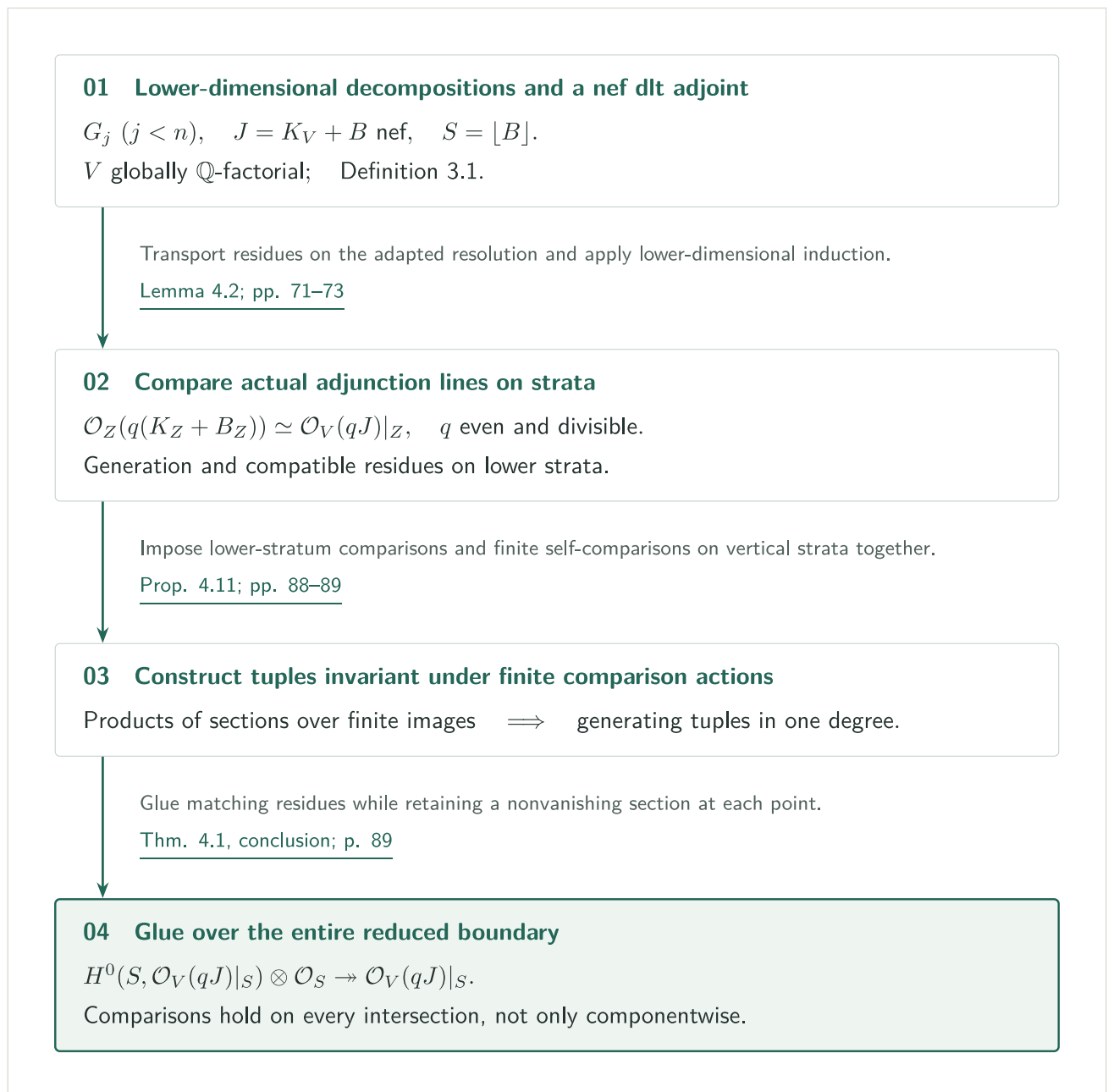


Figure 2. Pass from generation on proper strata through comparisons and finite images to generation on the entire boundary.

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1. Contract the known negative part and preserve the nef line

Proposition 3.8, pp. 23–24 starts with an actual presentation $J \sim_{\mathbb{Q}} P + E$, where P is nef and E remains the entire negative part on a resolution. It contracts E while descending a fixed Cartier multiple of P at every step.

Proposition 3.8 · pp. 23–24

2. Use induction on proper strata for signed presentations

With only a signed presentation, Theorem 3.13 / Corollary 3.14, pp. 29–32 apply. Lower-dimensional G_j supplies one model carrying a nef parameter interval; every divisorial negative multiplicity is affine there. A nontrivial wall changes its slope, so such walls cannot remain in that interval. Curves away from the floor have zero intersection with an adjoint represented on the floor, precluding further negative operations after special termination.

Theorem 3.13; Corollary 3.14 · pp. 29–32

3. Match residues on every lower stratum

Generation on individual adjunction components does not immediately descend to their union. The source uses residue comparisons in divisible even degree, compatibility on every lower stratum, and finite images of self-comparisons on vertical strata. Products over those finite images give invariant generating tuples in a common degree. Gluing them produces sections on the whole boundary (Proposition 4.11 and the conclusion of Theorem 4.1, pp. 88–89). The full auxiliary proofs of the restricted model theory in §3 and boundary theory in §4 are outside this draft's independent audit.

Lemma 4.2; Proposition 4.11; Theorem 4.1, conclusion · pp. 71–73, 88–89

4. Proof of Proposition 5.1: descent along a fibration

Algebraic reduction and relative Iitaka construction reduce general nontrivial fibrations to this rank-one case. If $a(X) = 0$ and the fibre is of log general type, slightly lowering horizontal coefficients below one allows a stable-family comparison that produces a positive-dimensional projective factor, contradicting algebraic dimension zero (pp. 90–91).

Construct the rank-one base presentation, identify the error with the entire negative part, and then generate the positive part.

01 Prepare a fibration with Kodaira-zero fibres

$$J \sim_{\mathbb{Q}} g^*H + A^*, \quad H = K_W + T + p^*P_S.$$

$$\min_{E \rightarrow D} (A^*)_E / \text{ord}_E(g^*D) = 0, \quad (A^*)_E \geq 0.$$

Use the rank-one Hodge line and a zero minimum coefficient to extend psh weights.

[\[OILS\] Prop. 2.7, Thm. 3.1; pp. 10, 17–18](#) [Lemma 5.3; pp. 94–95](#)

02 Subtract the horizontal negative part and descend the metric

$$A^{*,\text{hor}} \leq N(J), \quad H \text{ pseudo-effective.}$$

Use lower-dimensional induction or a conditional generalized MMP preserving the actual nef line.

[Lemmas 5.4–5.7; pp. 95–101](#)

03 Lower the base dimension or reach a stopping case

$$a(W) = 0 : H \sim_{\mathbb{Q}} N(H).$$

W projective : smaller base / nef H_m big or torsion.

The intersection-matrix kernel and mixed Hodge index exclude residual vertical positivity.

[Lemma 5.8; pp. 101–103](#)

04 Identify every vertical error with the negative part

$$J \sim_{\mathbb{Q}} P_0 + A, \quad A = N(J) \geq 0.$$

$$P_0 \sim_{\mathbb{Q}} 0 \quad \text{or} \quad P_0 = b^*H_m \text{ } (H_m \text{ big nef}).$$

The torsion branch ends. In the big nef branch, contract the negative part and extend sections.

[Prop. 3.8; pp. 23–24](#) [Thm. 4.1; p. 71](#) [Prop. 5.9; pp. 103–106](#)

05 Big nef case: extend boundary sections

$$r^*L \sim_{\mathbb{Q}} f^*H_m, \quad L \text{ nef dlt adjoint.}$$

$$\mathbf{B}(L) \cap \lfloor \Delta \rfloor = \emptyset \implies \mathbf{B}(L) = \emptyset.$$

TeX proof diagram constructing a good decomposition along a fibration

[Open figure ↗](#) [TeX source ↓](#)

1. Prepare the Hodge line and error coefficients

The rank-one Hodge line and period positivity from [OILS](#) give a prepared model with

$$J \sim_{\mathbb{Q}} g^*H + A^*, \quad H = K_W + T + M, \quad M = p^*P_S$$

where T is a rational SNC boundary, $p : W \rightarrow S$ contracts onto a smooth projective variety, and P_S is a nef rational line. If S is a point, $M \sim_{\mathbb{Q}} 0$; otherwise $K_S + a_0 P_S$ is big for some $a_0 > 0$. Lemma 5.2, pp. 91–94 retains nonnegative coefficients $(A^*)_E$ at primes dominating a base divisor, with minimum normalized coefficient zero. Signs at primes mapping into codimension at least two are not yet determined.

Lemma 5.2 · pp. 91–94 · OLS, Proposition 2.7 / Theorem 3.1

2. Subtract the horizontal negative part and descend pseudo-effectivity

Fibre induction puts the horizontal part $A^{*,\text{hor}}$ inside $N(J)$. Subtracting it from a positive curvature current leaves a metric whose weight is constant on connected smooth fibres and descends to the base. The zero coefficient above each base prime gives local upper boundedness and extension of the descended weight, proving that H is pseudo-effective (Lemma 5.3, pp. 94–95).

Lemma 5.3 · pp. 94–95

3. Lower the base dimension or reach big or torsion data

If $a(W) = 0$, then S is a point and lower-dimensional induction gives $H \sim_{\mathbb{Q}} N(H)$. For projective W , a chosen conditional generalized MMP either lowers the base dimension or reaches a nef H_m that is big or torsion. In a second program decreasing the nef data, adding a sufficiently large fixed multiple of H_{nef} combines the extremal-ray length bound with Cartier integrality to force every contraction to be H_{nef} -trivial (Lemmas 5.6–5.7, pp. 95–101). Semiampleness of the Hodge line itself is not assumed.

Lemmas 5.4–5.7 · pp. 95–101

4. Identify every vertical error with the negative part

Lemma 5.8, pp. 101–103 proves that the resulting divisor A , including the pulled-back base error, is effective and equals $N(J)$. Mixed Hodge index handles images of codimension at least two. For divisorial images, the intersection matrix has only whole-fibre multiples in its kernel; the zero minimum coefficient excludes the residual vertical part.

Lemma 5.8 · pp. 101–103

5. Extend boundary sections and remove the base locus

A torsion positive part finishes the argument. In the big nef case, Proposition 3.8 contracts the negative part and Proposition 5.9, pp. 103–106 applies. It extends the boundary sections of Theorem 4.1 through analytic injectivity FM, then creates a new floor over any remaining base locus using an lc threshold. Boundary generation for this new pair gives a contradiction, proving semiampleness of the actual adjoint.

Proposition 3.8; Theorem 4.1; Proposition 5.9 · pp. 23–24, 71, 103–106

5. Proof of Theorem 6.1: two diagonals and meromorphic nonvanishing

Assume that no canonical power has a meromorphic section. One diagonal produces high rank, a second forces determinant vanishing, and comparison with the point-pole bound yields a contradiction.

01 Assume there is no meromorphic canonical section

$a(X) = 0$, X simple, $L = c_1(K_X) \geq 0$.

$A \hookrightarrow \Omega_X^{\otimes k} \implies c_1(A) \leq kL$.

Use Ou's foliation criterion for slopes and transfer the bound to a point blowup.

Lemma 6.2; pp. 107–108 [Ou] Thms. 1.1, 1.4; pp. 1, 3

02 Increase volume and obtain high rank from the diagonal

$Z = \mathbb{P}_{X^2}(K_1 \oplus K_2)$, $d = 2n + 1$, $M = P_1 + P_2 + q\xi$.

$\text{vol}(P) = 1$, $\text{vol}(M) \asymp q$, $s \asymp q^{1/d}$.

Restricted-volume differentiation and direct-image estimates give a fixed positive rank fraction.

Lemmas 6.5–6.8; pp. 112–118

03 Force determinant vanishing using a second incidence

$F_j \subset \text{Sym}^{js} \Omega_Z$, $\text{rk } F_j / \text{rk } \text{Sym}^{js} \Omega_Z \geq c_n$.

$b_V \geq s - C_n$, $h_j \geq c_n j R_j^+ s$.

Remove the common zero; cancel determinant classes using slopes and the metric lower bound.

Lemmas 6.9–6.10; pp. 119–123 (6.40)–(6.44); pp. 124–125

04 Contradict the upper bound on point poles

$\frac{c_n s}{2 + (s + 2q)/r} \leq C_n$, $r \geq q^2$, $s \longrightarrow \infty$.

$\implies K_X^{\otimes m}$ has a nonzero meromorphic section.

TeX proof diagram comparing two diagonals and determinant orders

[Open figure ↗](#) [TeX source ↓](#)

1. Bound slopes under the contrary hypothesis

Assuming no canonical power has a meromorphic section, the Albanese map and simplicity give $H^1(\mathcal{O}_X) = 0$. Uniruledness and foliation criteria [Ou](#) make $L = c_1(K_X)$ pseudo-effective and imply $c_1(A) \leq kL$ for every line subsheaf $A \rightarrow \Omega_X^{\otimes k}$. Countably many line bundles can be handled simultaneously, carrying the estimate to a blowup at a very general point ([Lemma 6.2](#), pp. 107–108).

[Lemma 6.2 · pp. 107–108](#) · [Ou, Theorems 1.1 / 1.4](#)

2. Use the first diagonal to produce a high-rank subsheaf

Write K_i, P_i for pullbacks from the two factors of X^2 , and $\xi = c_1(\mathcal{O}_Z(1))$. A big class P normalized to volume one has point-pole threshold $\tau(P, x) \leq C_n$. On the other hand, on $Z = \mathbb{P}_{X^2}(K_1 \oplus K_2)$, with $d = 2n + 1$, the class $M = P_1 + P_2 + q\xi$ has volume comparable to q . Blowing up the diagonal in Z^2 and differentiating restricted volumes produces, for $s \asymp q^{1/d}$, a subsheaf occupying a fixed positive proportion of the rank of $\mathrm{Sym}^{js} \Omega_Z$ (Lemmas 6.5–6.8, pp. 112–118).

Lemmas 6.5–6.8 · pp. 112–118

3. Force determinant vanishing with the second incidence

An incidence arising from the diagonal in X^2 then converts the pole bound $b_V \geq s - C_n$ into vanishing of coefficients. A fixed positive proportion of the rows in each high-rank determinant must vanish to high order.

Lemmas 6.9–6.10 · pp. 119–123

4. Cancel determinant classes and contradict the point-pole bound

After factoring out the common zero, restrict the determinant to a point blowup and apply the cotangent slope estimate again. The normalized determinant classes cancel, giving

$$\frac{c_n s}{2 + (s + 2q)/r} \leq C_n, \quad r \geq q^2, \quad s \asymp q^{1/(2n+1)}$$

(equations (6.40)–(6.44), pp. 124–125). The left side diverges, a contradiction. The order of limits matters: fix the geometric data for each q before taking the divisible integer j large.

(6.40)–(6.44) · pp. 124–125

6. Proofs of Theorem 7.1 and Proposition 2.12: signed boundary and torsion

Exclude intermediate numerical dimension $0 < \nu(L) < n$ through compact deformations leaving the boundary. Treat zero and maximal numerical dimension separately, then recover the simple-case induction from torsion.

01 Start with a nef boundary presentation retaining signs

$L = K_X + D \sim_{\mathbb{Q}} \sum a_i D_i$, $L|_D$ semiample.
 $\mu^*\{L - cD\}$ pseudo-effective, $0 < \nu(L) < n$.

Mixed Hodge index makes the intersection image smaller; take roots and separate supports.

Lemmas 7.2–7.3; pp. 126–131

02 Separate positive boundary fibres from negative and zero components

$\dim \varphi(D_i) = \nu(L) - 1$, $\dim F = n - \nu(L) > 0$.
 $\mathcal{O}_Z(aS) \simeq \omega_Z(S + T)$, $\mathcal{O}_S(S) \simeq g^*R_U$.

Use split residue and localization at residual poles, then proper Kähler direct image on closed strata.

Lemma 7.4, Prop. 7.5; pp. 132–136 [ANV4] Lemma 11.2; p. 80

03 Insert residues with residual poles into a Hodge module

$R^i g_* \mathcal{O}_S(aS) \hookrightarrow R^i h_* \omega_H(A) = F_0 M^i$.
 $\text{Hom}(N, \ker \sigma) = 0$ (N ample on the projective parameter space).

The adjugate identity avoids dividing by the Jacobian and tests the symbol kernel with an ample root.

Prop. 7.6, (7.19)–(7.23); pp. 137–141

04 Kill every finite obstruction in all integral degrees

$I = \mathcal{O}_Z(-S)$, $\delta_k : g_*(I^j/I^{j+1}) \rightarrow R^1 g_*(I^{j+k}/I^{j+k+1})$.
 $\delta_k = 0$ ($j \in \mathbb{Z}$, $k \geq 1$).

Realize deformations by Douady and Artin approximation; use proper cycle families and Baire.

Lemma 7.7; pp. 141–143 Thm. 7.1, proof; p. 143

05 Deformations leaving the boundary contradict simplicity

$F_q \rightsquigarrow \mathcal{F}/\Delta \rightsquigarrow$ covering family.
 $0 < \nu(L) < n$ impossible; $a(X) = 0 \implies L \sim_{\mathbb{Q}} 0$.

TeX proof diagram lifting signed boundary fibres to prove torsion

Open figure ↗ TeX source ↓

1. Separate positive boundary fibres from negative and zero supports

Suppose $0 < \nu(L) < n$. Mixed Hodge index and the intersection matrix of the signed coefficients produce a positive-coefficient component whose boundary-map image has dimension $\nu(L) - 1$. Its general fibres avoid the negative- and zero-coefficient components (Lemma 7.2, pp. 126–127). Taking roots and separating the positive and negative supports gives local neighborhoods with a reduced Cartier divisor S , a residual boundary T , and an actual isomorphism $\mathcal{O}_Z(aS) \simeq \omega_Z(S + T)$.

Lemmas 7.2–7.3 · pp. 126–131

2. Retain residual poles in the Hodge-theoretic residue

Residue on a resolution yields a split injection retaining the residual poles A ,

$$R^i g_* \mathcal{O}_S(aS) \hookrightarrow R^i h_* \omega_H(A)$$

(Lemmas 7.3–7.4, pp. 128–133). The divisors A and H may intersect, so these extra poles must be retained.

Proposition 7.5, pp. 133–136 localizes SNC local cohomology along the residual poles.

Decomposition into proper Kähler closed strata supplies the lowest Hodge filtration and symbol-kernel vanishing.

Lemma 7.4; Proposition 7.5 · pp. 132–136

3. Kill obstructions in every integral degree and finite order

Proposition 7.6, pp. 137–141 makes

$$g_*(I^j/I^{j+k+1}) \longrightarrow g_*(I^j/I^{j+k}) \quad (j \in \mathbb{Z}, k \geq 1)$$

surjective for $I = \mathcal{O}_Z(-S)$, in every integral degree and at every finite order. The obstruction is a degree- k derivation. An adjugate identity on a coordinate-power cover sends it into the symbol kernel without dividing by the Jacobian. Maps from the ample root line to that kernel vanish, killing even obstructions supported at special parameters. This extends the lifting method of ANV4 to residual poles.

Proposition 7.6 · pp. 137–141

4. Contradict simplicity through deformations leaving the boundary

Lift the boundary-fibre equations and normal direction through all finite orders. Douady space and analytic Artin approximation then produce compact deformations leaving the boundary. Scheme-theoretic images under finite maps and cycles of bounded volume allow proper incidence images that include limits. Countability and Baire's theorem give a covering family of

positive-dimensional proper subvarieties, contradicting simplicity (Lemma 7.7 and the conclusion of Theorem 7.1, pp. 141–143). The case $\nu = 0$ gives torsion through the signed identity; $\nu = n > 0$ gives bigness and $a(X) = n$, which is excluded.

Lemma 7.7; Theorem 7.1, conclusion · pp. 141–143

5. Subtract the added boundary and return to the original adjoint

In the proof of Proposition 2.12, pp. 144–145, enlarge to a reduced SNC boundary D containing the signed canonical divisor from Theorem 6.1, then use Corollary 3.14 to reach a nef dlt model. Theorem 4.1 supplies boundary generation; O_U and exceptional negative-part subtraction supply pseudo-effectivity with $c = 1$. After Theorem 7.1 gives torsion, uniqueness of the positive current in Lemma 2.6, p. 9 allows subtraction of the added boundary, recovering a purely negative decomposition of the original J . This closes the simple case, then Theorem 2.13 and finally Theorem 1.2.

Proof of Proposition 2.12; Lemma 2.6 · pp. 144–145, 9

7. Which papers supply which steps

Input

LA Log abundance in characteristic zero, 2026-09-24

Direct use

Proposition 2.10 here; input Proposition 2.5 p. 9, Theorem 9.6 p. 69, §10 p. 72, Lemma 6.1 pp. 30–31

Editorial comparison

Statements, completion of induction, and subadditivity use compared; entire input proof outside scope

Input

[OILS Orbifold and logarithmic Iitaka subadditivity](#), 2026-09-26

Direct use

Lemma 5.2 and §§5.1 / 5.4 here; input Lemma 2.6 p. 9, Proposition 2.7 p. 10, Theorem 3.1 pp. 17–18, Lemma 3.3 p. 18, Lemma 5.2 p. 32

Editorial comparison

Hypotheses, conclusions and uses compared; proofs of Hodge-line construction, positivity and stable-family comparison outside scope

Input

[CGM4 Conditional good minimal models...](#), 2026-10-05

Direct use

Adjunction and strict comparison on strata in §3; input Lemmas 7.16–7.17 pp. 58–60

Editorial comparison

Dimension-free lemma statements compared; its four-dimensional main theorem is not an all-dimensional input

Input

[ANV4 Abundance after nonvanishing...](#), 2026-09-27

Direct use

Root neighborhoods, residue, filtered direct image and obstruction derivations in §7; input Lemma 10.1 p. 66, Proposition 10.4 p. 71, Proposition 11.1 p. 75, Lemma 11.2 p. 80, Lemmas 12.1–12.2 pp. 82–84

Editorial comparison

Statements compared; extension to residual poles is made here in Lemma 7.4 / Proposition 7.5

Input

[HX](#) Hacon–Xie, arXiv:2607.24986v1

Direct use

Theorem 1.3 p. 2: analytic cone and length bound, used here on p. 16

Editorial comparison

Statement compared; the entire restricted model theory here is not independently audited

Input

[TX](#) arXiv:2301.09186v3 / [Xie](#) arXiv:2211.10800v1

Direct use

Proposition 5.5, p. 97. TX Theorems A (=4.2), B (=5.4), D (=5.2), p. 2; Xie Theorem 1.5, pp. 2–3

Editorial comparison

Compared relative smooth-model induction, NQC decompositions, scaling hypotheses and descent of actual lines; input proofs outside scope

Input

[Ou](#) arXiv:2501.18088v1

Direct use

Theorems 1.1 / 1.4 pp. 1, 3; Lemma 6.2 and canonical pseudo-effectivity on p. 144

Editorial comparison

Statements and applications compared; not all slope-related auxiliary inputs compared

Input

[FM](#) Fujino–Matsumura, author version 2021-07-15

Direct use

Theorem A p. 3; section extension in Proposition 5.9, p. 105

Editorial comparison

Curvature lower bound, multiplier ideal and multiplying-section hypotheses compared

Input

Sai arXiv:2204.09026v5

Direct use

Theorem 1 p. 1; applied to smooth closed strata in Proposition 7.5, p. 135

Editorial comparison

Constant-source proper Kähler direct-image statement compared; independent verification of mixed/localized transitions remains open

Input

Cam arXiv:2605.19713v2

Direct use

Lemma 17 / Corollary 18 p. 10; Theorem 2.13, p. 13 here

Editorial comparison

Generically finite incidence statement and use compared

8. Return to the sources

The editorial reading covers the main theorem and the selected routes in §§2, 5 and 7; the two contraction/termination connections in §3; the generating-tuple conclusion in §4; and the slope, two-diagonal and determinant-cancellation passages in §6. **This is not a record of verifying all 148 pages.** The cited passages above provide entry points into the source.

Further editorial checks concern the restricted model induction and finite geography in §3.8; the full residue-comparison and finite-image proofs in §§4.2–4.4; the direct-image estimate in §6.3 and all local calculations of Lemma 6.10; and strictness of the residual-pole filtered direct image in §7.3 and its connection to Sabbah–Schnell vanishing. Uses of Boucksom's negative-part theory, mixed Hodge index and Douady/Artin/Fujiki results were read here, but independent comparison of all their original hypotheses is incomplete. Uninvestigated catalog relations remain uninvestigated, rather than being recorded as absent.

Uniform indices for semi-log-canonical log Calabi–Yau pairs

Descend normalization trivializations by controlling residue cycles

A common index on normalization components does not by itself glue across the conductor. The argument bounds klt characters using a Hodge rank, then kills residue cycles with a uniform exponent independent of the number of components.

Manuscript author: OpenAI · 2026-10-05 · 42 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Cartier trivialization independent of component count

Fix $d \geq 4$ and a finite set $\Phi \subset [0, 1] \cap \mathbb{Q}$. Theorem 1.1, p.2 asserts that an integer $a = a(d, \Phi) > 0$ works for every connected, projective, equidimensional slc pair (X, B) of dimension d over any algebraically closed field of characteristic zero satisfying

$$K_X + B \sim_{\mathbb{Q}} 0, \quad \{\text{nonzero coefficients of } B\} \subset \Phi :$$

the divisor $a(K_X + B)$ is Cartier and

$$\mathcal{O}_X(a(K_X + B)) \simeq \mathcal{O}_X.$$

The theorem's stated range $d \geq 4$ is distinct from the consequence obtained by adding established lower-dimensional results. The exponent is independent of the numbers of normalization components and lc strata.

[Theorem 1.1 · p. 2](#)

Theorem 5.1 — Top-form Hodge rank (input to the character bound)

The character bound uses a particular Hodge rank, not all Betti numbers. For a smooth projective n -fold U , let $T(U) \subset H^n(U, \mathbb{Q})$ be the smallest rational Hodge substructure containing $H^{n,0}(U)$, and let $r(U) = \dim_{\mathbb{Q}} T(U)$. In [Theorem 5.1, p.16](#), $n \geq 2$ and T is a complex projective $\mathbb{Q}$ -factorial terminal n -fold with $K_T \sim 0$. A smooth resolution U satisfies $h^1(U, \mathcal{O}_U) = 0$, and every rational image Y with $0 < \dim Y < n$ has rationally connected smooth projective resolution. The asserted bound is $r(T) := r(U) \leq R_n$.

[Theorem 5.1 · p. 16](#)

Theorem 6.1 — A uniform klt pluricanonical character exponent

For $n \geq 0, m \geq 1$, [Theorem 6.1, p.29](#) asserts an integer $L(n, m) > 0$ for integral complex projective klt pairs (V, Δ) of dimension n , with $\Delta \geq 0$ and $m(K_V + \Delta)$ an integral principal divisor. If a rational m -canonical form satisfies $\operatorname{div}(\theta) = -m\Delta$, then every B -birational self-map satisfies

$$f^*\theta = c\theta \implies c^{L(n,m)} = 1.$$

The map f need not have finite order.

[Theorem 6.1 · p. 29](#)

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to this manuscript. The principal external inputs use the following keys.

- [UI Uniform Pluricanonical Iitaka Fibrations · October 3, 2026 · PDF.](#)
- [LA Log abundance in characteristic zero · September 24, 2026 · PDF.](#)
- [ULI Uniform log Iitaka fibrations and bounded moduli denominators · October 4, 2026 · PDF.](#)
- [HJ Han–Jiang · Theorem 1.2 · arXiv v2.](#)
- [JL Jiang–Liu · Theorem 3.2 · arXiv.](#)
- [K Kollár, Sources of log canonical centers · arXiv v3.](#)

2. Proof of Theorem 1.1

Proof route. The Hodge-rank bound in Theorem 5.1, together with the factorwise arguments, gives the character exponent in Theorem 6.1. That exponent kills the residue cycles here; normal lc indices and conductor descent are additional inputs to Theorem 1.1.

First trivialize the normalization components over the complex numbers. The remaining tasks are to kill all conductor residue ratios simultaneously and show that the descended form is an actual Cartier generator.

01 Trivialize normalization components

$$\operatorname{div}(\theta_i) = -m\Delta_i, \quad m = m(d, \Phi) \text{ even}$$

Compare the normal index with the original principal multiple

UI: Thm. 1.2, p. 2 §7.3, pp. 37–38

02 Make conductor ratios constant

$$r_e = \tau^* \theta_T / \theta_S, \quad r_e^{M/m} = b_i / b_j$$

The nonuniform M proves constancy only

Preserve internal residues and transport branch multipliers

K: Prop. 14, pp. 8–9 Lems. 7.1–7.2, pp. 35–37

03 Move a cycle to one minimal klt stratum

$$f^* \theta_A = (\prod_e r_e) \theta_A, \quad \dim A \leq d - 1$$

Apply the uniform character exponent and a spanning tree

Thm. 6.1, p. 29; (7.10)–(7.11), p. 38

04 Kill all cycles with one exponent

$$L = \operatorname{lcm}_{q < d} L(q, m), \quad a = mL$$

$$c_i = c_j r_e^L$$

Extend generators from codimension one

§§7.2–7.3, pp. 37–39

05 Descend node generators to a Cartier divisor

$$\mathcal{O}_X(a(K_X + B)) \simeq \mathcal{O}_X$$

Even residue matching and S_2 extension

From normalization through conductor cycles to slc trivialization

Open figure ↗ TeX source ↓

1. Trivialize normalization components in a common even degree

For the normalization $\nu : \coprod X_i \rightarrow X$, write $K_{X_i} + \Delta_i = \nu^*(K_X + B)|_{X_i}$. The conductor enters Δ_i with coefficient one. The normal lc index theorem, [UI Theorem 1.2, p.2](#), supplies an even common integer m , depending only on d, Φ , and rational m -canonical forms with $\text{div}(\theta_i) = -m\Delta_i$. This trivializes the normalization components; descent remains to be proved.

[Theorem 2.2; §7.3 · pp. 5, 37–38](#)

2. Use the existing nonuniform trivialization to make ratios constant

Form the finite graph whose vertices are normalization components and whose edges are generic conductor nodes. For an edge $e : i \rightarrow j$, the branch correspondence τ compares the residues by $r_e = \tau^*\theta_T/\theta_S$. The original rational linear triviality supplies a possibly nonuniform principal Cartier degree M , divisible by m . Comparing its trivialization with the θ_i gives

$$r_e^{M/m} = b_i/b_j,$$

which proves that r_e is constant. This step does not bound M . [§7.3, pp.37–38](#)

[§7.3; \(7.8\) · p. 38](#)

3. Transport residue ratios to minimal klt strata

Uniformity comes from transporting these constants to minimal klt strata. Within one dlt component, the $\mathbb{P}^1$ -links of [Kollár Theorem 10 and Proposition 14, pp.6,8–9](#) preserve residues: evenness removes the sign. Across the conductor, [Lemma 7.2](#) constructs a B -birational map of minimal strata carrying the same multiplier r_e . It does not simply restrict an arbitrary birational map to a stratum. On a common SNC model, it compares residue fields and proves that the relevant local monomial matrix has determinant ± 1 . [Lemmas 7.1–7.2, pp.35–37](#)

[Lemmas 7.1–7.2 · pp. 35–37](#)

4. Kill every cycle character with one exponent

A closed walk therefore gives a B -birational self-map of one minimal klt stratum with character $\prod_e r_e$. [Theorem 6.1](#) below gives

$$L = \text{lcm}_{0 \leq q \leq d-1} L(q, m), \quad a = mL,$$

so every closed walk satisfies $\prod_e r_e^L = 1$. Constants chosen along a spanning tree then satisfy $c_i = c_j r_e^L$ on every edge, and the forms $c_i \theta_i^L$ glue. Loops and multiple edges are included. No factor equal to the length of a cycle enters the exponent. [Equations \(7.10\)–\(7.11\), p.38](#)

[\(7.10\)–\(7.11\) · p. 38](#)

The uniform character exponent is constructed in the [proof of Theorem 6.1](#). It allows the following node-gluing step without introducing the cycle length into the exponent.

5. Match generators at nodes and extend by S_2

One must obtain a Cartier generator, not merely a descended section. At the split node $F[[x, y]]/(xy)$, the canonical generator has branches $(dx/x, -dy/y)$; matching even-degree residues matches generators. Nonsplit nodes are checked after a quadratic extension and descended. The S_2 extension of these codimension-one generators identifies the degree- a reflexive extension itself with $\mathcal{O}_X$. Finally, descent to a finitely generated field and faithfully flat base change transfer the conclusion from $\mathbb{C}$ to any algebraically closed characteristic-zero field.

§§7.2–7.3, pp.37–39

§§7.2–7.3 · pp. 37–39

3. Proof of Theorem 5.1

Assume $r(T)$ is unbounded. Organize the small-order covering families from the diagonal estimate by a chain quotient, then contradict volume asymptotics with an upper bound for sections.

01 Large rank forces small local positivity

$$r(T) \rightarrow \infty \Rightarrow \varepsilon(h; x) \rightarrow 0$$

Use tracking, chains and litaka fibres

Lems. 4.1, 5.5, pp. 9,18–20 UI: Lems. 6.2–6.4, pp. 25–26

02 Small-order families and their chain field

$$\kappa(C) = 0, \quad \gamma(P|_C) \leq \eta \operatorname{vol}(P)^{1/n}$$

Use LA termination and a semiample endpoint

LA: Thm. 11.1, p. 73 Lems. 5.6–5.7, pp. 20–25

03 Good models and volume calibration

$$h : X \rightarrow Y, \quad b = \dim Y, \quad s = n - b$$

$$v = \operatorname{vol}(D)^{1/n} \asymp \delta, \quad \ell_X(D) \leq C\delta$$

Make families vertical on the maximal chain quotient

§5.5, Lem. 5.8, pp. 25–28

04 Vertical jets and base slopes contradict volume

$$h^0(mD) \leq C(m\eta v)^s (m\delta)^b$$

$$h^0(mD) \sim \frac{m^n v^n}{n!}, \quad \eta \ll 1$$

Hodge-rank contradiction and the use of LA

[Open figure ↗](#) [TeX source ↓](#)

1. Turn large rank into covering families with small orders

Lemma 4.1 bounds $r(T)$ from a Seshadri lower bound for a normalized ample class on a canonical model with $K \sim 0$. Its diagonal argument uses the quantitative Hodge norm gap of Lemma 3.2. Unbounded rank would force small local positivity. Tracking, generic chains, and the small-volume litaka-fibre estimate of UI Lemmas 6.2–6.4, pp.25–26 then produce covering families of $\kappa = 0$ subvarieties with small vanishing order for their complete section systems. Lemma 5.5, pp.18–20

Lemma 4.1 · pp. 9–16; Lemma 5.5 · pp. 18–20

2. Realize the chain field as a morphism

The chain field is realized as a morphism on a small terminal model. Applying LA Theorem 11.1, p.73 to $(T, \epsilon J)$ for a movable J supplies termination and a semiample endpoint. This is an actual use of LA abundance. Lemma 5.6, pp.20–21

Lemma 5.6 · pp. 20–21 · LA, Theorem 11.1 · p. 73

3. Calibrate volume and degree over a maximal base

The proof maximizes the dimension of a proper chain quotient and bounds its base using the rational-image hypothesis and UI's moduli-denominator and boundedness inputs. It then removes divisors mapping into codimension at least two in the base. Lemma 5.7 calibrates a big divisor D and a linear functional ℓ_X . With $v = \text{vol}(D)^{1/n}$, $b = \dim Y$, and $s = n - b$, the scale is

$$c\delta \leq v \leq C\delta, \quad \ell_X(D) \leq C\delta.$$

pp.21–25

Lemma 5.7; §5.4 · pp. 21–25

4. Compare vertical jets and horizontal sections with volume

Comparison with a base hyperplane makes the small-order families vertical; maximality makes their chain leaves fill the generic fibre. Vertical jets bound the generic rank by $C(m\eta v)^s$. Determinants and cotangent slopes on the bounded base give

$$h^0(X, mD) \leq C(m\eta v)^s (m\delta)^b.$$

For sufficiently small η , this contradicts the volume leading term $m^n v^n / n!$. The section estimate on $\mathbb{P}^b$ in Lemma 5.8 must be linear in the rank for this comparison. §5.5, pp.25–28

Lemma 5.8; §5.5 · pp. 25–28

4. Proof of Theorem 6.1

Estimate the multiplier on a volume form rather than the order of the self-map. Control the exponent lost in lifting by dimension alone, then bound the factor characters using their respective inputs.

01 Fixed-degree form and a birational self-map

$$m(K_V + \Delta) \sim 0, \quad f^*\theta = c\theta$$

Use product decomposition and a characteristic subgroup

ULI: Prop. 5.1, Lems. 5.2–5.3, pp. 12–13 Lem. 6.4, pp. 30–32

02 Fix factor directions on a product cover

$$c^{n!} = \prod_i d_i^{m/p_i}$$

The cover degree does not enter the exponent

Bound RC representations and the volume-form Hodge structure

HJ: Thm. 1.2, pp. 1–2 JL: Thm. 3.2, p. 8

03 Control factors by RC boundedness and Hodge rank

$$d_i^{E_i} = 1, \quad \varphi(e) \leq b(r)$$

Take the least common multiple of possible character orders

Thm. 5.1, p. 16; Lems. 6.5–6.6, pp. 32–34

04 An exponent depending only on dimension and degree

$$E = E(n, m), \quad L(n, m) = n!E, \quad c^{L(n, m)} = 1$$

Decomposing characters on a product cover

[Open figure ↗](#) [TeX source ↓](#)

1. Lift to a product cover and fix its factors

After crepant terminalization, ULI Proposition 5.1 and Lemmas 5.2–5.3, pp.12–13 provide a quasi-étale product cover. It has a rationally connected factor carrying all the boundary, an abelian factor, and boundary-free Calabi–Yau or symplectic factors. Lemma 6.4 uses a characteristic subgroup of the smooth-locus fundamental group to lift the map, and its $n!$ -th power fixes the factor directions. The cover degree need not be bounded. For factor multipliers d_i and form degrees p_i , the comparison is

$$c^{n!} = \prod_i d_i^{m/p_i};$$

the cover degree does not enter this exponent. [pp.30–32](#)

Lemma 6.4 · pp. 30–32 · ULI, Proposition 5.1; Lemmas 5.2–5.3 · pp. 12–13

2. Bound the RC character in a bounded family

For the RC factor, fixed-index boundedness from Han–Jiang Theorem 1.2, pp.1–2 leads to the bounded-family representation theorem Jiang–Liu Theorem 3.2, p.8.

Lemma 6.5 · pp. 32–33 · HJ, Theorem 1.2 · pp. 1–2 · JL, Theorem 3.2 · p. 8

3. Bound the remaining character orders by Hodge rank

For the nonabelian boundary-free factors, UI Lemma 5.2, pp.21–22 supplies the rational-image property needed in Theorem 5.1. Abelian factors have the cohomology rank bound $\binom{2r}{r}$. The action on an integral polarized Hodge structure constrains a character of order e by $\varphi(e) \leq b(r)$; take the least common multiple of the possible orders. Combining this with the RC bound yields $L(n, m) = n!E(n, m)$. Lemmas 6.5–6.6, pp.32–34

Lemma 6.6; Theorem 6.1 proof · pp. 33–34

The uniform rank bound comes from the proof of Theorem 5.1. Return the resulting character exponent to the cycle step of Theorem 1.1.

5. Which papers supply which steps

Input

UI Thm.1.2, p.2

Role here

Thm.2.2 and §7.3: common normal lc index

Check performed

Input statement and application compared

Input

UI Thm.1.1, Props.4.1,4.5

Role here

Prop.2.4: bounded chain-quotient bases

Check performed

Target's input description read; proofs outside scope

Input

UI Lems.6.2–6.4, pp.25–26

Role here

Inputs 5.2–5.4 and Lemma 5.5: chains and small volumes

Check performed

Input statements and uses compared

Input

LA Thm.11.1, p.73

Role here

Thm.2.3, Lemma 5.6 and §5.4: good models

Check performed

Statement and rational-boundary applications compared; proof outside scope

Input

ULI Prop.5.1, Lems.5.2–5.3, pp.12–13

Role here

Input 6.3 and Lemma 6.4: product structure stable under further covers

Check performed

Original statements and uses compared

Input

HJ Thm.1.2 and JL Thm.3.2

Role here

Lemma 6.5: RC character bound

Check performed

Original statements compared; promotion from bounded varieties to log bounded pairs read only here

Input

UI Lem.5.2, pp.21–22

Role here

Lemma 6.6: applying Thm.5.1 to nonabelian factors

Check performed

Input and application compared; also an indirect dependence on LA nonvanishing

Input

K Thm.10, Prop.14

Role here

Lemma 7.1: residue-preserving maps within a component

Check performed

Original statements and even-degree sign calculation compared

U4 is also cited in Inputs 5.2–5.4 as the source of geometric lemmas used through UI. Its original lemmas were not rechecked for this draft. A bibliography entry alone is not a dependency edge.

6. Return to the sources

Theorem 5.1; §5 · pp. 16–28 · Theorem 6.1; §6 · pp. 29–34 · §7 · pp. 35–39

Editorial reading covered the main statements, §§2–3, the chain quotient, calibration statement and vertical–horizontal jet comparison in §5, factor characters in §6, and residues, cycles, nodes and S_2 descent in §7. It did not check all diagonal calculations in Lemma 4.1, every calibration estimate in Lemma 5.7, or independently validate the fundamental-group lift and local valuation argument. Original slope inputs such as Campana–Păun, the underlying singular Beauville–Bogomolov literature, and the proofs of the UI and LA input theorems remain unchecked. This draft traces citations and applications; it does not certify the theorem.

The revision rechecked the statements and descent in §7, and organized proof introductions, stages and citations. The recorded scope for external inputs is retained from the initial draft.

Source: October 5, 2026 version, PDF pagination, snapshot

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Conditional good minimal models for compact Kähler fourfolds

Return nonvanishing to the specified nef endpoint under three premises

The premises are full orbifold subadditivity, the specified fourfold MMP, and abundance after nonvanishing. The central task is producing a first section: through calculations on the base in positive algebraic dimension, and through whole-boundary extension and foliations in algebraic dimension zero.

Manuscript author: OpenAI · 2026-10-05 · 138 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Conditional good minimal models

Assumptions 2.2, 2.3 and 2.4 are all retained. [Theorem 1.1, p.4](#) concerns a normal connected compact Kähler fourfold X and an effective rational Weil divisor B such that (X, B) is klt. The space is globally strongly $\mathbb{Q}$ -factorial, and the actual adjoint $D = K_X + B$ is $\mathbb{Q}$ -Cartier and analytically pseudo-effective.

The conclusion is a normal connected globally strongly $\mathbb{Q}$ -factorial compact Kähler fourfold Y and a bimeromorphic map $\phi : X \dashrightarrow Y$ such that:

1. ϕ extracts no prime divisor and $B_Y = \phi_* B$.
2. (Y, B_Y) is klt, and the actual $\mathbb{Q}$ -Cartier adjoint $D_Y = K_Y + B_Y$ is analytically nef.
3. Every prime divisor over the models satisfies $a(E; Y, B_Y) \geq a(E; X, B)$, strictly for primes on X contracted by ϕ .
4. Some mD_Y is Cartier and $\mathcal{O}_Y(mD_Y)$ is globally generated.

There is no uniform bound on m . Global strong $\mathbb{Q}$ -factoriality in Definition 2.1, p.7, means that **every coherent rank-one reflexive sheaf on the whole space has an invertible positive reflexive power**. It is not replaced by local analytic $\mathbb{Q}$ -factoriality. Likewise, $\sim_{\mathbb{Q}}$ denotes an actual holomorphic line-bundle isomorphism after taking powers; equality of Bott–Chern classes is insufficient.

Theorem 1.1 · p. 4; Definition 2.1 · p. 7

Theorem 1.2 — Nonvanishing on the nef endpoint

Under the same three premises, Theorem 1.2, p.4 asserts $\kappa(Y, J) \geq 0$ for an ordinary klt pair (Y, Δ) on a normal connected globally strongly $\mathbb{Q}$ -factorial compact Kähler fourfold, with Δ effective and rational and actual $\mathbb{Q}$ -Cartier adjoint $J = K_Y + \Delta$ analytically nef. Apply this theorem to the pair produced by the initial MMP, then apply Assumption 2.4.

Theorem 1.2 · p. 4

Assumptions 2.2–2.4 — The three retained premises

Premise

Assumption 2.2, pp.7–8

Required scope

Full orbifold Iitaka subadditivity in arbitrary dimensions, Fujiki class $\mathcal{C}$, rational SNC coefficients in $[0, 1]$, holomorphic surjections with connected fibres

Supplier and check

OI, catalog 033, Thm.1.1 p.3 and invariant-base definitions pp.2–3 compared; proof unchecked

Premise

Assumption 2.3, pp.8–9

Required scope

A program starting on the specified pseudo-effective ordinary klt fourfold, permitting any negative extremal ray and continuation after every finite prefix, with termination of every such program

Supplier and check

FM, catalog 056, Thm.6.20 p.47; Prop.3.4 pp.12–14; Lem.3.2 p.11 and relevant proof passages compared; full termination proof unchecked

Premise

Assumption 2.4, p.9

Required scope

Generation of an actual nef $\mathbb{Q}$ -Cartier klt adjoint once $\kappa \geq 0$; no strong factoriality requirement

Supplier and check

AN, item 14 of catalog 034, Thm.1.1 p.3 compared; a first section is a hypothesis

The inequality in Assumption 2.2 is

$$\kappa(Z, K_Z + \Gamma) \geq \kappa(F, K_F + \Gamma|_F) + \kappa(g | \Gamma).$$

It uses the invariant in equations (2)–(5), rather than the orbifold divisor of an arbitrary base model: $m(g, \Gamma; P) = \min_{E \mapsto P} \{\text{ord}_E(g^*P)/(1 - \gamma_E)\}$ and

$B(g, \Gamma) = \sum_P (1 - 1/m(g, \Gamma; P))P$, followed by the infimum of $\kappa(K_S + B)$ over the stipulated orbifold-model equivalence class. Coefficient one means infinite multiplicity.

Existence of suitable neat models and computation there are included. The fibre powers in §4 require dimensions larger than four. [pp.7–8,18 OI pp.2–3](#)

Assumption 2.3 inserts no initial modification. It requires a nef supporting class with a Kähler margin, projective bimeromorphic contractions with connected fibres, negative-side relative Bott–Chern dimension one, and, for small contractions, the relative analytic Proj of the actual canonical algebra. It preserves klt singularities, the Kähler and global strong categories, and pseudo-effectivity, with nonextraction and discrepancy comparison. It does

not grant all non-pseudo-effective, dlt or generalized-pair programs. Comparing the suppliers' statements does not turn this exposition into an unconditional certification of the three premises.

Assumptions 2.2–2.4 · pp. 7–9

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to this manuscript. The principal external inputs use the following keys.

- [OI](#) Orbifold and logarithmic Iitaka subadditivity · catalogue 033 · September 26, 2026 · [PDF](#).
- [FM](#) Finite ordinary minimal model programs on compact Kähler fourfolds · catalogue 056 · October 5, 2026 · [PDF](#).
- [AN](#) Abundance after nonvanishing for compact Kähler fourfolds · catalogue 034 · September 27, 2026 · [PDF](#).
- [Ou](#) [Ou](#) · Theorem 1.1 · [arXiv v1](#).
- [DO](#) [Das](#)–[Ou](#) · Corollary 1.3 · [arXiv v4](#).
- [DPS](#) [Demailly](#)–[Peternell](#)–[Schneider](#) · hard Lefschetz · [arXiv v2](#).
- [CCE](#) [Campana](#)–[Claudon](#)–[Eyssidieux](#) · Theorem 1 · [arXiv v3](#).

2. Reduction for Theorems 1.1–1.2

Proof route. The specified MMP leads to nonvanishing (Theorem 1.2), then to semiampleness (Assumption 2.4). Nonvanishing splits into [positive algebraic dimension](#) and [algebraic dimension zero](#); the latter uses [extension from the whole reduced boundary](#). All three Assumptions 2.2–2.4 remain in force.

First take the MMP endpoint and prove nonvanishing there using Theorem 1.2. Separate the uses of the three premises and track sections of the same adjoint line to the final step.

01 Start with all three premises retained

(X, B) klt, $D = K_X + B$ pseudo-effective
Global strong $\mathbb{Q}$ -factoriality; actual adjoint line

Assumption 2.3: every negative-ray program terminates

FM: Thm. 6.20, p. 47 Ass. 2.3, p. 9

02 The initial program reaches its nef endpoint

(Y, B_Y) , $J = K_Y + B_Y$ nef
Retain nonextraction and discrepancy inequalities

Prove nonvanishing under the three premises

Prop. 3.5; Prop. 4.1; Thm. 9.4, pp. 13–14, 83–84

03 Return canonical sections to the same adjoint

$\kappa(Y, J) \geq 0$
Projective, uniruled, irregular; then $a(W) > 0$ or $a(W) = 0$

Apply Assumption 2.4 after nonvanishing

AN: Thm. 1.1, p. 3 §9.4, p. 84

04 Generate the actual line on the same endpoint

$H^0(Y, \mathcal{O}_Y(mJ)) \otimes \mathcal{O}_Y \rightarrow \mathcal{O}_Y(mJ)$

From the three premises to generation on the specified endpoint

[Open figure ↗](#) [TeX source ↓](#)

1. Fix the nef endpoint supplied by the MMP

Assumption 2.3 supplies an endpoint satisfying (i)–(iii) of Theorem 1.1. It assumes termination for arbitrary choices of negative rays; retain the actual adjoint line on this endpoint. Property (iv) still requires a first section, which termination alone does not provide.

Assumption 2.3 · p. 9; §9.4 · p. 84

2. Reduce nonvanishing to two cases on a smooth model

In the projective branch, Lemma 3.1 derives logarithmic Iitaka subadditivity from Assumption 2.2 and applies the manuscript's Theorem A.2, p.85. Appendices A–I prove that conditional projective abundance statement; it is not another independent premise. The uniruled branch uses the rational quotient, and the irregular branch the Albanese map, with nonvanishing in dimension at most three. [Ou Theorem 1.1, p.1](#) relates non-uniruledness to canonical pseudo-effectivity for smooth compact Kähler manifolds. [§3, pp.10–14](#)

The remaining case is a smooth non-uniruled, non-Moishezon fourfold W with $q(W) = 0$. Proposition 4.1 handles $a(W) > 0$ and Theorem 9.4 handles $a(W) = 0$. The following proof sections trace these two branches.

[Proposition 3.5 · pp. 13–14](#); [Proposition 4.1 · p. 14](#); [Theorem 9.4 · pp. 83–84](#)

3. Return the section to the original adjoint and use the final premise

Push a canonical section on W down to Y and multiply by the section of the effective boundary. This yields nonvanishing for J on the specified nef pair. Only now is the hypothesis $\kappa(Y, J) \geq 0$ of Assumption 2.4 available. Applying it to that same actual J gives global generation.

[Proposition 3.5 · pp. 13–14](#); [§9.4 · p. 84](#) · [AN, Theorem 1.1 · p. 3](#)

3. Proof of Proposition 4.1: positive algebraic dimension

Let W be a smooth non-uniruled, non-Moishezon fourfold with $q(W) = 0$ and $0 < a(W) = d < 4$. Transfer the canonical line to the base of algebraic reduction, then obtain a first section according to its dimension.

01 Obtain sections after ample twisting first

$$q(W) = 0, \quad 0 < a(W) = d < 4$$

$$\kappa(K_W + Nb^*H) \geq 0 \quad (N \gg 0)$$

Use local rationality and arbitrary-choice termination

Lem. 4.2, Prop. 4.3, pp. 14–16

02 Use one empty-boundary program

$$N > 5k_i \Rightarrow H_i \cdot R_i = 0$$

Make $K_{V_i} + NH_i$ nef, then semiample

Apply Assumption 2.4 to existing sections, then Stein factorization

AN: Thm. 1.1, p. 3 (19)–(20), p. 16

03 Retain an actual canonical pullback

$$K_V \sim_{\mathbb{Q}} g^*A_T, \quad K_M \sim_{\mathbb{Q}} f^*A + R$$

$$R \geq 0, \quad A \text{ pseudo-effective}, \quad q(S) = 0$$

Use fibre powers for surfaces and modular orders for threefolds

Ol: Thm. 1.1, p. 3 Lems. 4.4–4.5, pp. 17–23

04 Obtain nonvanishing in each base dimension

$$d = 1 : S = \mathbb{P}^1; \quad d = 2 : K_S \cdot P_0 \leq 0$$

$$d = 3 : K_T + \Xi \sim_{\mathbb{Q}} A_T, \quad (T, \Xi) \text{ klt}$$

$$\kappa(S, A) \geq 0 \Rightarrow \kappa(W, K_W) \geq 0$$

Positive algebraic dimension and nonvanishing on the base

[Open figure ↗](#) [TeX source ↓](#)

1. Construct an actual canonical pullback from twisted sections

For $0 < a(W) = d < 4$, resolve algebraic reduction and choose a very ample H on its projective base S_0 . Lower-dimensional fibre nonvanishing and ample twisting first give $\kappa(K_W + Nb^*H) \geq 0$ for every sufficiently large integer N . At each stage choose $N > 5k_i$, where k_i is the current canonical Cartier index. Local rationality forces $H_i \cdot R_i = 0$ for a ray negative for $K_{V_i} + NH_i$, so its step also belongs to a single empty-boundary K -program. Rechoosing N does not change that program's boundary. Arbitrary termination and Assumption 2.4 make the twist semiample; Stein factorization yields actual line-bundle identities

$$K_V \sim_{\mathbb{Q}} g^* A_T, \quad K_M \sim_{\mathbb{Q}} f^* A + R, \quad R \geq 0.$$

Here $S \rightarrow T$ is a smooth projective resolution, A is the pullback of A_T , $q(S) = 0$, and A is pseudo-effective. Proposition 4.3, pp.15–16

Proposition 4.3 · pp. 15–16

2. Obtain nonvanishing on bases of dimension one and two

For $d = 1$, $S = \mathbb{P}^1$. For $d = 2$, apply Assumption 2.2 to r -fold fibre powers over a curve. Fix the local form $t = x^e$ and the exceptional coefficient h before varying r ; the base coefficient has pole bound $m r h / e$. Letting $r \rightarrow \infty$ gives

$$\left(A - K_S + \sum_{P \text{ exceptional}} c_P P \right) \cdot H' \geq 0$$

(Lemma 4.4, pp.17–19). For the Zariski decomposition $A = P_0 + N_0$, the remaining case $P_0^2 = 0$ gives $K_S \cdot P_0 \leq 0$; Riemann–Roch and $q(S) = 0$ then produce sections.

Lemma 4.4; §4.3 · pp. 17–19

3. Produce a klt threefold adjoint from genus-one fibres

For $d = 3$, the generic fibre has genus one. Identify its Hodge line by $\mathcal{H}^{\otimes m} \simeq \mathcal{O}_S(m(A - K_S))$, and use the modular sections $E_4^{m/4}$ and $\Delta_{\text{mod}}^{m/12}$ with $12 \mid m$. Their common divisorial order ℓ_P satisfies $\ell_P / m = 1 - t_P$, where

$$t_P = \min_{E \mapsto P} \frac{1 + \text{coeff}_E R}{\text{ord}_E(f^* P)}.$$

The integrability calculation proves this **on every required model, including exceptional base valuations**. A general linear combination supplies an effective Ξ on the original normal base T such that (T, Ξ) is klt and $K_T + \Xi \sim_{\mathbb{Q}} A_T$. No separate $\mathbb{Q}$ -Cartier hypothesis on K_T is imposed. Projective threefold nonvanishing supplies the section to pull back. Lemma 4.5 and §4.4, pp.19–23

Lemma 4.5; §4.4 · pp. 19–23

4. Proof of Proposition 9.3: extension from the whole reduced boundary

In algebraic dimension and irregularity zero, first produce a nonzero section on the whole reduced boundary. Assuming nonvanishing fails, kill the obstructing H^1 and extend that section to the ambient space, obtaining a contradiction.

01 Construct a reduced boundary and a nef klt interval

$$h : (V, D) \rightarrow (T, G), \quad J = K_V + D = h^*(K_T + G)$$

$$D \neq 0, \quad K_T + tG \text{ nef klt} \quad (t_0 \leq t < 1)$$

Obtain the metric by an interval limit and sections by cluster gluing

Lem. 5.1; Props. 7.20, 8.1, pp. 23, 62–81

02 Supply the metric and whole-boundary section separately

$$\nu(h_{\min}, x) = 0, \quad 0 \neq s_D \in H^0(D, m_0 J|_D)$$

Retain fractional differentials, even residues and the ambient class

Compute the ideal on a resolution and use low-degree Leray

(176)–(178), p. 82–83

03 Compare extension obstructions under the contrary assumption

$$\kappa(T, K_T + G) = -\infty, \quad K_M + F \sim_{\mathbb{Q}} L = p^* J$$

$$H^1(V, \mathcal{I}_D(mJ)) \hookrightarrow H^1(M, K_M + L_m)$$

Use the trivial multiplier ideal and fixed-bundle vanishing

DPS: Thm. 2.1.1, p. 8 Lem. 9.2; (179), pp. 81–83

04 Use hard Lefschetz and eventual vanishing

$$L_m \sim_{\mathbb{Q}} (m-1)L + \{F\}, \quad \mathcal{I}(h_{L_m}) = \mathcal{O}_M$$

$$0 = H^0(M, \Omega_M^3 \otimes L_m) \rightarrow H^1(M, K_M + L_m)$$

Preserve nonzero powers by reducedness and push to T

Prop. 9.3, p. 83

05 Extend a high power to obtain nonvanishing

$$H^0(V, mJ) \twoheadrightarrow H^0(D, mJ|_D)$$

$$\kappa(T, K_T + G) \geq 0$$

A whole-boundary section and extension by vanishing of H1

Open figure ↗ [TeX source](#) ↓

1. Obtain a nef interval and metric from the reduced boundary

Proposition 7.20, pp.62–65, takes a reduced-boundary pair (T_0, G_0) to a nonextracting model (T, G) with

$$A = K_T + G \text{ nef}, \quad K_T + tG \text{ nef and klt} \quad (t_0 \leq t < 1),$$

and a projective crepant dlt modification $h : (V, D) \rightarrow (T, G)$ with $J = K_V + D = h^*A$. It uses the special-termination arguments of §7 and reduction to ordinary klt programs; it does not add fourfold dlt termination to the premises.

Apply the zero-Lelong minimal-metric Lemma 5.1, p.23, inside this interval, add $(1 - t)G$, and let $t \rightarrow 1$ to obtain a zero-Lelong metric on the pullback of J .

Lemma 5.1 is proved analytically in §5. Volume-normalized capacity, differentiation of a Monge–Ampère equation and a Bochner estimate retaining its residual term are combined. Equation (89) cancels the residual measure on the same sublevel set; after taking the limit at fixed t , the Lelong upper bound (90) contradicts the positive lower bound (93). pp.25–33 This is not a direct application of the projective MM theorem to a nonprojective space.

Proposition 7.20 · pp. 62–65; Lemma 5.1; (89)–(93) · pp. 23–33

2. Match component sections within each cluster

Proposition 8.1, pp.65–81, gives $H^0(D, \mathcal{O}_D(mJ)) \neq 0$ when $D \neq 0$. Full differentials, including fractional coefficients, are retained on the normalization components. Das–Ou Corollary 1.3, p.4 supplies their semiample systems, but componentwise abundance alone does not glue sections. Choose components with maximal system dimension; match even residues on clusters of dominant conductor branches and arrange vanishing on nondominant branches. Lemma 8.4 multiplies all transported sections under a finite loop action, preserving nonzeroness and the required zeros.

Proposition 8.1; Lemmas 8.3–8.4 · pp. 65–72 · DO, Corollary 1.3 · p. 4

3. Control cycles using one ambient Kähler class

For a nonprojective torsion cluster, transport **one ambient Kähler class on V** to the relevant surfaces. For the cycle automorphism φ and a class v of positive square, the construction gives $\varphi^*v - v \in \text{NS}(Q)_{\mathbb{R}}$. Hodge index and integral lattice actions make the top-form multiplier a root of unity, permitting matching after a common power. This does not assert finite characters for arbitrary automorphisms of Kähler surfaces. Lemma 8.9, pp.73–74 Extend by zero on other clusters, and descend to the entire D by codimension-one matching and S_2 . Lemma 8.2 and §8.6, pp.66–67,81

Lemma 8.9 · pp. 73–74; §8.6 · p. 81

4. Kill the extension obstruction under the contradiction hypothesis

Work on smooth models with $a = q = 0$. Lemma 9.1, p.81, gives finitely many prime divisors with linearly independent real cohomology classes. If $\kappa(M, L) = -\infty$, then every fixed vector bundle $\mathcal{E}$ satisfies $H^0(M, \mathcal{E} \otimes \mathcal{O}_M(mL)) = 0$ for all sufficiently large divisible m (Lemma 9.2, pp.81–82). Take determinants of sections of maximal generic rank and compare their nonnegative integral coefficients on the finite set of primes. No global meromorphic frame is assumed at the start.

Proposition 9.3 assumes $\kappa(T, A) = -\infty$ to kill the extension obstruction. On $p : M \rightarrow V$, write $K_M + F \sim_{\mathbb{Q}} L = p^*J$, where F has SNC support and coefficients at most one, allowing negative coefficients. The key comparisons are

$$p_*\mathcal{O}_M(mL - \lfloor F \rfloor) = \mathcal{I}_D \otimes \mathcal{O}_V(mJ),$$

$$H^1(V, \mathcal{I}_D(mJ)) \hookrightarrow H^1(M, K_M + L_m), \quad L_m \sim_{\mathbb{Q}} (m-1)L + \{F\}.$$

The injection is low-degree Leray and needs no higher-direct-image vanishing.

Zero Lelong numbers and the integrability margin of $\{F\}$ make the multiplier ideal trivial. [DPS hard Lefschetz](#) gives $H^0(M, \Omega_M^3 \otimes L_m) \rightarrow H^1(M, K_M + L_m)$. Lemma 9.2 kills the source. High powers of the nonzero section on the whole reduced D therefore extend to V , contradicting the assumption. [Equations \(176\)–\(179\), pp.82–83](#)

[Lemmas 9.1–9.2; Proposition 9.3; \(176\)–\(179\) · pp. 81–83](#) · [DPS, Theorem 2.1.1](#) · [arXiv PDF p. 8](#)

5. Proof of Theorem 9.4: completing algebraic dimension zero

Separate the presence and absence of a nonzero meromorphic pluricanonical tensor. The diagram shows the obstruction on the no-tensor branch; the final stage treats a signed canonical frame.

01 Assume there is no meromorphic canonical frame

K_Y nef, Y has no prime divisors
 $p : M \rightarrow Y$, $a(M) = 0$, K_M pseudo-effective

Use zero-Lelong metrics, hard Lefschetz and forms

DPS: Thm. 2.1.1, p. 8 Lems. 6.2–6.4, pp. 34–37

02 Construct a conormal line from holomorphic forms

$\chi(\mathcal{O}_M) = 0$, $h^{2,0} \geq 1$, $h^{3,0} \geq 2$
 $E \subset \Omega_M^1$ integrable and saturated

Use Hodge index to control every current

Lems. 6.8–6.10, pp. 41–43

03 Take a fixed point on all positive currents

$\alpha^2 = 0$, $\nu(T, x) = 0$ ($\forall T \in \mathcal{C}_\alpha$)
 $\mathcal{F} : \mathcal{C}_\alpha \rightarrow \mathcal{C}_\alpha$ is continuous

Treat meridians, compactification and the Shafarevich base

CCE: Thm. 1, p. 2 §§6.5–6.6, pp. 43–47

04 Kill spherical holonomy to obtain a contradiction

$\rho_0 : \pi_1(M_0) \rightarrow \mathrm{PU}(2)$, $\rho_0 = 1$
 $M_0 \dashrightarrow \mathbb{P}^1$ nonconstant, $a(M_0) = 0$

The foliation contradiction when no meromorphic canonical frame exists

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1. Reduce the absence of a frame to a model with no prime divisors

Theorem 9.4, pp.83–84, separates the existence of a nonzero meromorphic pluricanonical tensor from its absence. In the latter case, include every prime in G_0 and use Proposition 7.20. A nonzero G would supply a section that, after dividing out divisor factors, gives the prohibited tensor. Thus $G = 0$, leaving a model with no prime divisors and nef canonical line; Proposition 6.1 excludes it.

Theorem 9.4 · pp. 83–84

2. Build a foliation and spherical structure from differential forms

Proposition 6.1 starts with a zero-Lelong metric and DPS to obtain $\chi(\mathcal{O}_M) = 0$, $h^{2,0} \geq 1$ and $h^{3,0} \geq 2$. Two- and three-forms produce an integrable saturated conormal line. Hodge index for positive currents gives $\alpha^2 = 0$ and, under the contradiction hypothesis, zero Lelong numbers for **every** positive current in α . This permits a continuous self-map of the full compact convex set of positive currents. Its fixed point supplies a transverse spherical structure. §§6.1–6.4, pp.34–43

Proposition 6.1; §§6.1–6.4 · pp. 34–43

3. Use holonomy to contradict algebraic dimension zero

Index charts give finite holonomy for boundary meridians. A Selberg cover and compactification produce a torsion-free $\mathrm{PU}(2)$ representation. In the Zariski-dense case, the projective Shafarevich base from Campana–Claudon–Eyssidieux Theorem 1, p.2 contradicts $a = 0$. A proper closure is virtually solvable; $q = 0$ on every finite étale cover makes the image finite and hence trivial. Extending the developing map meromorphically by local charts and trace produces a nonconstant function, again contradicting $a = 0$. §§6.5–6.6, pp.43–47

§§6.5–6.6 · pp. 43–47 · CCE, Theorem 1 · p. 2

4. Remove the negative part of a frame and return to a smooth model

If a meromorphic frame exists, write $K_{T_0} \sim_{\mathbb{Q}} P - N$, with $P, N \geq 0$ and disjoint supports, and use $G_0 = (\mathrm{Supp} P)_{\mathrm{red}}$. If $G = 0$, pseudo-effectivity forces $N_T = 0$. Otherwise the boundary-extension section and uniqueness of its meromorphic divisor when $a(T) = 0$ give the same conclusion. Run the MMP on $(T, 0)$ and use Assumption 2.4 to make the canonical line actually torsion. On a resolution, the signed exceptional divisor is effective by the support of a positive current and independence of prime classes. This returns canonical nonvanishing to the smooth model. Equations (180)–(181), p.84

Theorem 9.4; (180)–(181) · p. 84

6. Which papers supply which steps

Auxiliary input in arbitrary dimension. Lemmas 7.16–7.17 (pp. 58–60) identify actual adjoint lines on normalized lc strata and give an effective comparison across a small dlt step, with strict discrepancy increase over the exceptional locus. These two lemmas have no fourfold restriction and enter Theorem 3.13 of Kähler abundance. They are distinct from this paper’s conditional fourfold main theorem. Lemmas 7.16–7.17 · pp. 58–60

Result

OI Thm.1.1, p.3

Use

Assumption 2.2, §§3–4, Lemmas 6.2 and 8.3: subadditivity with invariant orbifold base

Check performed

Statement, base definition and stated uses compared; neat-model theory and input proof unchecked

Result

FM Thm.6.20, p.47; Prop.3.4, pp.12–14

Use

Assumption 2.3, Props.4.3 and 7.20, §9.4

Check performed

Arbitrary-ray termination, canonical algebra and preservation of the global strong condition compared; full termination dependencies outside scope

Result

AN Thm.1.1, p.3

Use

Assumption 2.4, Prop.4.3, Thm.9.4 and final step of Thm.1.1

Check performed

Applications after obtaining $\kappa \geq 0$ compared; AN proof outside scope

Result

Ou Thm.1.1, p.1

Use

Prop.3.5 and §8: smooth Kähler canonical pseudo-effectivity

Check performed

Specified v1 statement compared

Result

DO Cor.1.3, p.4

Use

Lemma 8.3: semiample systems on normal threefold components

Check performed

Specified v4 statement compared; companion dlt-modification theorem not compared

Result

DPS hard Lefschetz

Use

Lemma 6.3 and Prop.9.3: twisted forms surject onto cohomology

Check performed

Compared in arXiv v2, unnumbered introductory theorem p.2 / Thm.2.1.1 p.8; target cites published Thm.0.1

Result

CCE Thm.1, p.2

Use

Lemma 6.13: projective base for semisimple holonomy

Check performed

arXiv v3 statement and torsion-free condition compared

7. Return to the sources

§4 · pp. 14–23 · Proposition 7.20 · pp. 62–65 · §8 · pp. 65–81 · §9 · pp. 81–84

Reading covered the main statements, three premises, §3 reductions, §4 pullback and base calculations, principal capacity/residual passages in §5, principal forms/Hodge/fixed-point/holonomy passages in §6, Proposition 7.20, clusters and the ambient class in §8, and all of §9. It did not verify all relative finite-generation, extraction and special-termination arguments in §7, every projective cluster case in §8, or Appendices A–I in full. All metric limit operations,

singular-chart extensions, and original inputs such as Fujino's local rationality and relative analytic MMP have not been independently checked. These are unverified steps, not absent dependencies or discharged conditions.

The revision rechecked the main statements and three premises, whole-boundary extension, and the final canonical-frame cases, and organized explanations and citations. External-input checking scope is retained from the initial draft. Source: October 5, 2026 version, PDF pagination, snapshot `adc7f1241b42e322a6451854ab7e4b4c146bf78a`.

Log abundance in characteristic zero

Nonvanishing, boundary deformations, and dimension induction

The manuscript derives smooth nonvanishing from incompatible jet and Frobenius estimates, then combines it with boundary geometry to obtain good minimal models and log abundance.

Manuscript author: OpenAI · 2026-09-24 · 85 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

The following are the manuscript's claims, retaining their original numbering, hypotheses, and scope. Theorem 9.6 is the conditional induction step; Theorem 11.1 records the result after induction closes.

Theorem 1.1 (Log abundance)

Let (X, B) be a normal projective lc pair over an algebraically closed field of characteristic zero. Assume that B is an effective rational divisor and $K_X + B$ is $\mathbb{Q}$ -Cartier. If $K_X + B$ is nef, some positive Cartier multiple is globally generated.

$$K_X + B \text{ nef} \implies K_X + B \text{ semiample.}$$

[p. 2 · GitHub ↗](#)

Theorem 9.6 (The smooth nonvanishing step)

Assume that every projective complex lc pair of dimension less than n , with effective real boundary and pseudo-effective adjoint, has a good log minimal model (Assumption 2.1). Then, for a smooth projective complex n -fold X ,

$$K_X \text{ pseudo-effective} \implies \kappa(X, K_X) \geq 0.$$

[pp. 69–72 · GitHub ↗](#)

Theorem 11.1 (Real-boundary good models over the complex numbers)

Every projective lc pair (X, B) over $\mathbb{C}$, with effective real boundary and pseudo-effective $\mathbb{R}$ -Cartier adjoint $K_X + B$, has a good log minimal model.

Real semiampleness means $\mathbb{R}$ -linear equivalence to the pullback of an ample real divisor under a contraction. This real-boundary conclusion is the hypothesis passed to the next dimension.

p. 73 · [GitHub](#) ↗

How to read the citations

Numbers without a prefix refer to this LA manuscript. The arrows state both the cited result and what it supplies. Page numbers refer to the pinned PDF; GitHub opens the file preview.

- [Hash] Kenta Hashizume · On the non-vanishing conjecture and existence of log minimal models — arXiv:1609.00121v4 (2017); Theorem 1.4. Proposition 2.5: smooth nonvanishing supplies real-boundary lc nonvanishing and ordinary minimal models, with no abundance premise.
- [FG14] Osamu Fujino · Yoshinori Gongyo · Log pluricanonical representations and the abundance conjecture — Compositio Mathematica 150 (2014), author final PDF; Theorem 1.5 · PDF p. 2. Proposition 2.3: descend semiample from the normalization of an slc pair to the whole reduced boundary.
- [FG17] Osamu Fujino · Yoshinori Gongyo · On log canonical rings — author manuscript PDF (log-ring-aspm); Lemma 3.5 · p. 5. Proposition 2.5: rational $\mathbb{Q}$ -factorial dlt pairs with $\kappa \geq 1$; smaller-dimensional good models supply a good model in the current dimension.
- [Ambro] Florin Ambro · The moduli b-divisor of an lc-trivial fibration — arXiv:math/0308143v1 (2003); Theorem 0.2 · p. 2. The Mori-fiber branch of Proposition 2.5: an effective globally klt total pair and actual rational-linear pullback give an effective klt adjoint on the base.
- [OI] OpenAI · Orbifold and logarithmic Iitaka subadditivity — 2026-09-26 · fixed catalogue-034 snapshot; Corollary 6.2 · pp. 40–41. Lemma 6.1 only: zero-boundary subadditivity excludes positive-dimensional Albanese image. Prior statement comparison retained.
- [HPS] Christopher Hacon · Mihnea Popa · Christian Schnell · Algebraic fiber spaces over abelian varieties: around a recent theorem by Cao and Păun — 2017 · version 2; Theorem 1.1 · p. 2. The maximal-Albanese-dimension base in Lemma 6.1 permits this alternative input. Prior statement comparison retained.
- [BDPP] Sébastien Boucksom · Jean-Pierre Demailly · Mihai Păun · Thomas Peternell · The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension — arXiv:math/0405285v1 (2004); Theorem 2.2 · p. 6. §9.2.2: detect a non-pseudo-effective class on a smooth projective blowup by negative pairing with an actual strongly movable curve.
- [CP] Frédéric Campana · Mihai Păun · Orbifold generic semi-positivity: an application to families of canonically polarized manifolds — Annales de l'Institut Fourier 65 (2015); Theorem 2.1 · print p. 844 / PDF p. 11. §9.2.1: apply to smooth W with zero boundary and pseudo-effective K_W for generic semipositivity of the cotangent bundle. The source also uses a restriction theorem to fix the complete-intersection flag.
- [Langer] Adrian Langer · Semistable sheaves in positive characteristic — Annals of Mathematics 159 (2004); Corollary 2.5 · print p. 258 / PDF p. 8. Lemma 9.4: the rank- n bundle on the fixed curve and a nef twist of fixed degree generating its tangent bundle give a uniform Frobenius slope error.

These are the external interfaces expanded below. The manuscript also uses established MMP, rational-polytope and restriction results, Hodge-module strictness/vanishing, and canonical Frobenius filtrations. The relevant source locations are identified below; their full proofs and all auxiliary application details are outside this revision's independent checks.

2. Theorems 1.1 / 11.1: the induction

The induction retains good models for pseudo-effective real-boundary lc pairs in all smaller dimensions. This supplies both boundary adjunction and the reductions to lower-dimensional bases.

Statements above: [Theorem 11.1](#) · [Theorem 1.1](#) · [Manuscript proof](#) ↗

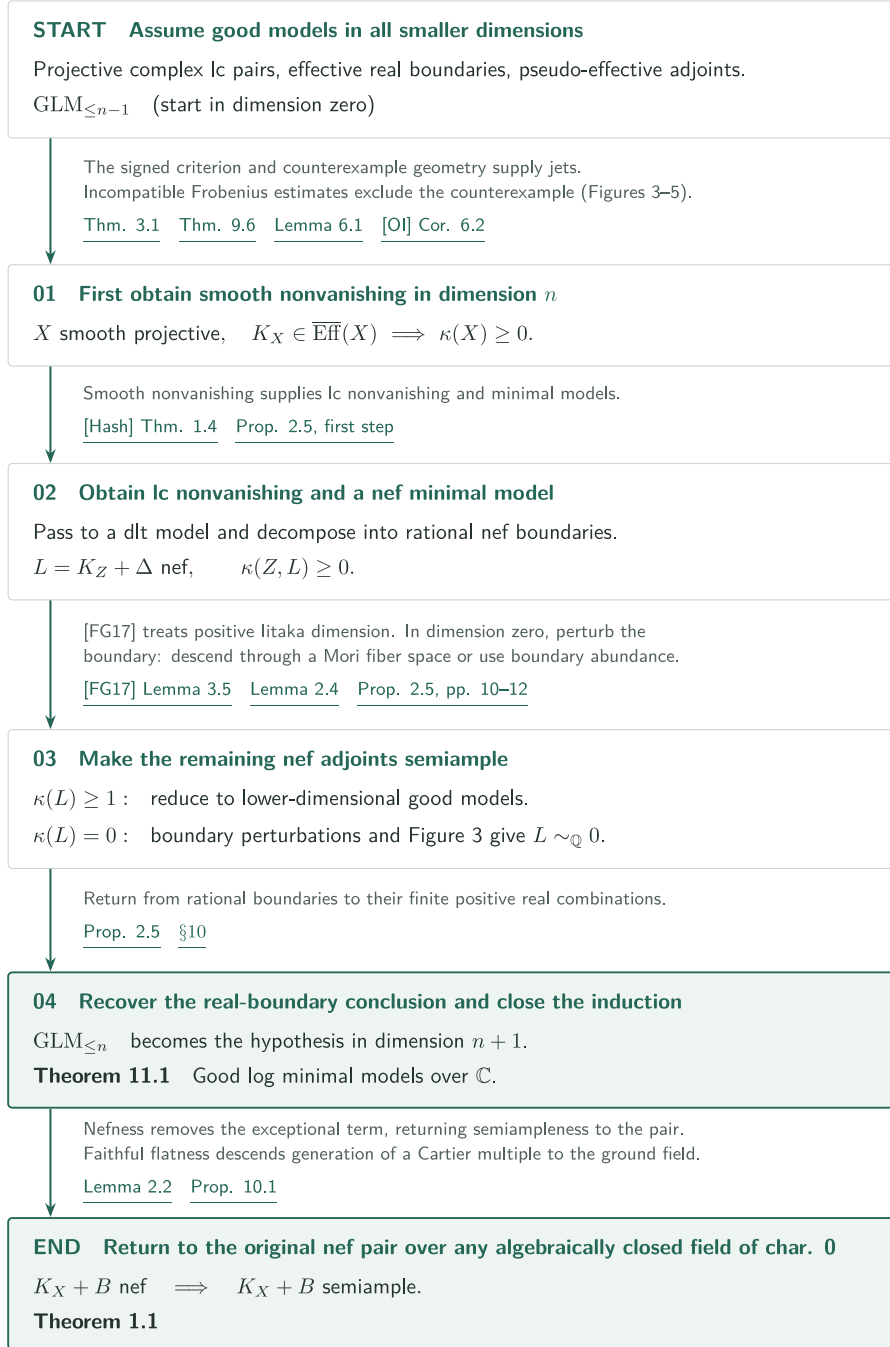


Figure 1. Prove nonvanishing first, then good models in the same dimension; pass the result to the next dimension.

[Open figure ↗](#) [TeX source ↓](#)

1. Establish nonvanishing first

The hypothesis $\text{GLM}_{\leq n-1}$ concerns projective complex lc pairs in dimensions below n , with effective real boundary and pseudo-effective adjoint. Starting from a point, first prove NV_n , Theorem 9.6. General good models in dimension n are not yet available. Boundary semiampness in Proposition 2.3 and special termination in Proposition 6.9 use good models only for smaller-dimensional pairs.

Theorem 3.1 is a geometric criterion assuming semiampness on the whole boundary; it does not assume NV_n . Sections 6–8 use this criterion and the lower-dimensional hypothesis to construct fixed data from a counterexample. The independent Theorem 9.1 excludes their coexistence. Only its application establishes NV_n .

[Assumption 2.1 · p. 7 ↗](#) · [Theorem 9.6 · pp. 69–72 ↗](#)

How Theorem 9.6 is proved →

2. Make the nef model good

After Theorem 9.6, apply Hashizume Theorem 1.4. It supplies real-boundary lc nonvanishing and ordinary log minimal models through dimension n , not yet good models. Proposition 2.5 decomposes the boundary on this nef model into a finite convex combination of rational nef boundaries and proves semiampness for each rational adjoint. Positive Iitaka dimension is handled by [FG17, Lemma 3.5], using lower-dimensional good models.

[\[Hash\] Theorem 1.4 ↗](#) · [\[FG17\] Lemma 3.5 · p. 5 ↗](#) · [Proposition 2.5 · pp. 9–12 ↗](#)

The branches of Proposition 2.5 →

3. Resolve the zero-Iitaka case by boundary perturbation

When $\kappa(L) = 0$, split according to pseudo-effectivity after perturbing the boundary. The non-pseudo-effective branch descends through an L -trivial MMP. The pseudo-effective klt branch constructs an effective representative on the full reduced boundary and applies Theorem 3.1. The remaining non-klt branch reduces to two klt perturbations. The next section separates these arguments.

Once each rational adjoint is semiample, the product of its contractions recovers semiampness of the original real adjoint. This establishes $\text{GLM}_{\leq n}$ and supplies the hypothesis in dimension $n + 1$. Theorem 11.1 records this output in all dimensions; it is not fed back as an input to Theorem 9.6.

[Lemma 2.4 · p. 9 ↗](#) · [Proposition 2.5 · pp. 10–12 ↗](#)

The boundary criterion used here →

4. Return to the original pair and change fields

If the original adjoint is nef, the effective exceptional comparison term vanishes by the negativity lemma (Lemma 2.2). Semiampness on the good model therefore returns to the original pair. Proposition 10.1 spreads the pair, boundary, and a Cartier multiple over a finitely generated field, then descends surjectivity of the actual evaluation map by faithful flatness. This gives Theorem 1.1 over every algebraically closed field of characteristic zero.

[Lemma 2.2 · p. 8 ↗](#) · §10 / [Proposition 10.1 · pp. 72–73 ↗](#)

3. Proposition 2.5: from nonvanishing to good models

Here $\mathbf{NV}_n$ has already been established. In the remaining zero-litaka branch, finish the klt case before returning to the non-klt case.

Proposition 2.5 · pp. 9–12

Assume Assumption 2.1: real-boundary good models for projective complex lc pairs of dimension below n . Also assume that every smooth projective complex n -fold with pseudo-effective canonical divisor has nonnegative Kodaira dimension. Then every projective lc n -fold with effective real boundary and pseudo-effective adjoint has a good log minimal model. In particular, every nef rational lc adjoint in dimension n is semiample.

[Proposition 2.5 · pp. 9–12 · GitHub ↗](#)



Figure 2. Branches of Proposition 2.5. Establish the pseudo-effective klt branch before the final branch with two klt perturbations.

Open figure ↗ TeX source ↓

1. From real to rational boundaries, then to zero Iitaka dimension

Hashizume supplies a $\mathbb{Q}$ -factorial dlt nef model. Fix the support and coefficient-one components; intersect a small rational neighborhood preserving dlt singularities with the rational polytope of nef adjoints. The real boundary lies in a finite simplex of rational dlt boundaries. A positive real combination of their semiample adjoints is the pullback of an ample real divisor from the image of the product of their contractions.

For rational $L = K_Z + \Delta$, fix the finite prime and principal divisors in an effective real representative and solve the rational linear constraints to obtain an effective rational representative. If $\kappa(L) \geq 1$, induction supplies the dimension- $n - 1$ good-model hypothesis in [FG17, Lemma 3.5]. Lemma 2.2 returns semiamplessness from the resulting good model. Thus only nef L with $\kappa(L) = 0$ remains.

[Proposition 2.5 · pp. 9–10 ↗](#) · [\[Hash\] Theorem 1.4 ↗](#) · [\[FG17\] Lemma 3.5 · p. 5 ↗](#)

2. Non-pseudo-effective perturbation: descend with one Cartier index

Put $P = \Delta$ in the klt case and $P = \lfloor \Delta \rfloor$ otherwise, and fix m with mL Cartier. If all sufficiently small perturbations $L - \epsilon P$ are non-pseudo-effective, choose rational $0 < \epsilon < (4nm + 2)^{-1}$. Lemma 2.4 gives $0 \leq L \cdot C < 1/m$ on a curve generating a negative extremal ray. Integrality forces $L \cdot C = 0$, and the same line bundle mL descends through contractions and flips. This uniform index prevents the required smallness of ϵ from changing during the MMP.

The established non-pseudo-effective klt MMP ends in a Mori fiber space $f : Z' \rightarrow T$, with $L' \sim_{\mathbb{Q}} f^*A$, A nef, and $\dim T < n$. For a klt total pair, Ambro Theorem 0.2 supplies an effective klt pair with $A \sim_{\mathbb{Q}} K_T + \Delta_T$, allowing induction. Otherwise P' is relatively ample and a floor component dominates T . Adjunction on its transform S in a crepant dlt blowup makes the lower-dimensional f_S^*A semiample. Equality of sections over the connected-fiber part and norms over the finite part descend this to A .

[Lemma 2.4 · p. 9 ↗](#) · [Proposition 2.5 · p. 10 ↗](#) · [\[Ambro\] Theorem 0.2 · p. 2 ↗](#)

3. The pseudo-effective klt branch: build a full-support representative

If $L - \epsilon \Delta$ is pseudo-effective for a sufficiently small rational $\epsilon > 0$, the available nonvanishing gives $L \sim_{\mathbb{Q}} G_0 = H_0 + \epsilon \Delta$, with $H_0 \geq 0$. On a resolution $p : W \rightarrow Z$, put every exceptional divisor and the strict support of $H_0 + \Delta$ into a reduced SNC divisor D . Then

$K_W + D = p^*L + R$, $R \geq 0$. Positive klt log discrepancies make $G_W = p^*G_0 + R$ supported on all of D . Pushforward comparison of sections preserves $\kappa(K_W + D) = 0$.

On an ordinary minimal model (V, D_V) , the effective exceptional comparison of Lemma 2.2 preserves a representative $K_V + D_V \sim_{\mathbb{Q}} G_V = \sum b_i D_i$ with all $b_i > 0$. Hence $0 < c < \min b_i$ makes $K_V + D_V - cD_V$ pseudo-effective. Proposition 2.3 supplies semiampleness on the entire reduced boundary. Every hypothesis of Theorem 3.1 is now met, giving

$$\nu(K_V + D_V) = \kappa(K_V + D_V) = 0.$$

[Proposition 2.5 · pp. 10–11 ↗](#) · [Theorem 3.1 · p. 12 ↗](#)

4. Return the vanishing representative to the original nef divisor

Intersection gives $G_V \cdot H_V^{n-1} = 0$ for ample H_V . Effectivity forces $G_V = 0$, and full support gives $D_V = 0$. To return to G_0 , use a common resolution $W \xleftarrow{q} U \xrightarrow{v} V$: one has $0 \leq q^* p^* G_0 \leq q^* G_W$ and $v_* q^* G_W = 0$. Thus $q^* p^* G_0$ is effective and v -exceptional, and rationally linearly equivalent to the pullback of the original nef L . The negativity lemma forces it to vanish, proving $L \sim_{\mathbb{Q}} 0$.

Good models for klt pairs in dimension n have now been established. The next branch uses this restricted conclusion, not the general lc conclusion still being proved.

[Proposition 2.5 · p. 11 ↗](#)

5. The remaining non-klt case: two perturbations and numerical dimension

Set $P = \lfloor \Delta \rfloor$ and suppose $L - \delta P$ is pseudo-effective for some $\delta > 0$. Choose rational $\epsilon > 0$ with $2\epsilon \leq \delta$ small enough for the perturbed boundaries to be effective and klt. Convexity makes $L - \epsilon P$ and $L - 2\epsilon P$ pseudo-effective. Nonvanishing and $\kappa(L) = 0$ give Iitaka dimension zero for both. The just-established klt case supplies good models, so both have Nakayama numerical dimension $\kappa_{\sigma} = 0$.

An effective rational representative of $L - 2\epsilon P$, together with the identity below, gives $\kappa_{\sigma}(L) = 0$ by monotonicity and homogeneity. The perturbations have not been assumed nef. The original L is nef, so $\nu(L) = 0$. Available nonvanishing writes $L \sim_{\mathbb{Q}} E \geq 0$; ample intersection forces $E = 0$. Return to the rational simplex for the real-boundary conclusion, and use Lemma 2.2 to descend semiampleness to the original rational nef pair.

$$2(L - \epsilon P) = L + (L - 2\epsilon P), \quad 0 \leq \kappa_{\sigma}(L) \leq \kappa_{\sigma}(2(L - \epsilon P)) = 0.$$

[Proposition 2.5 · p. 12 ↗](#)

4. Theorem 3.1: abundance from the boundary

Theorem 3.1 allows negative coefficients in a representative supported on the boundary. Its conclusion is abundance, $\kappa(L) = \nu(L)$, which is then used in Propositions 2.5 and 6.10.

Theorem 3.1 · p. 12

Let (X, D) be a projective $\mathbb{Q}$ -factorial dlt pair over $\mathbb{C}$ with $D = \sum D_i$ reduced. Assume $L = K_X + D$ is nef, $L - cD$ is pseudo-effective for some $c > 0$, and $L \sim_{\mathbb{Q}} \sum a_i D_i$ with $a_i \in \mathbb{Q}$. If $L|_D$ is semiample on the whole reduced scheme D , then $\kappa(X, L) = \nu(X, L)$.

Theorem 3.1 · p. 12 · [GitHub](#) ↗

START Prepare the signed-boundary criterion

$$L = K_X + D \text{ nef}, \quad L - cD \in \overline{\text{Eff}}(X), \quad c > 0,$$

$$L \sim_{\mathbb{Q}} \sum a_i D_i, \quad D = \sum D_i \text{ reduced}, \quad L|_D \text{ semiample.}$$

Below, $n = \dim X$ and $0 < v = \nu(L) < n$.

Mixed intersections locate the positive image away from the other signs.
Induction and conductor gluing supply semiampness on the entire boundary.

[Lemma 3.2](#) [Prop. 3.3](#) [Prop. 2.3](#) [\[FG14\] Thm. 1.5](#)

01 Separate a positive component from the negative and zero parts

Its semiample image has dimension $v - 1$; take roots over a general point.

Place obstructions in the lowest Hodge piece. Negative-twist vanishing
kills the obstruction in the symbol-map kernel.

[Prop. 4.1](#) [Lemma 4.2](#) [Lemma 4.3](#)

02 Lift image coordinates and a transverse coordinate to every thickening

A formal deformation that holds the base directions fixed and moves off the boundary.

Algebraize the formal boundary-fiber deformation by Grothendieck existence.

[Prop. 5.1, pp. 26–28](#)

03 Obtain compact L -trivial subvarieties off the boundary

$$\dim F = n - v, \quad F \cap D = \emptyset, \quad L|_F \equiv 0.$$

These subvarieties form a dominating family on X .

The nef reduction contracts F , so $b \leq v$. Pseudo-effectivity of $L - cD$
makes the boundary vertical; apply vertical nef-divisor descent.

[Lemma 5.2](#) [§5.1, pp. 29–30](#)

04 Descend the signed divisor to the nef-reduction base

$$\pi^* L \sim_{\mathbb{Q}} f^* G_B, \quad G_B \text{ nef}, \quad \dim B = b.$$

The base dimension equals the numerical dimension, so G_B is big.
Combine $\kappa(L) \geq v$ with the nef inequality $\kappa(L) \leq \nu(L)$.

[Thm. 3.1, end of proof](#)

END The base divisor is big; pull back its sections to obtain abundance

$$v = \nu(G_B) \leq b \leq v \implies \kappa(L) = \nu(L).$$

Theorem 3.1

Figure 3. Formal boundary deformations lead to the nef-reduction base. Whole-boundary semiampness is an input; the conclusion is $\kappa = \nu$.

[Open figure ↗](#) [TeX source ↓](#)

1. Glue sections on the whole boundary

Proposition 2.3 applies lower-dimensional induction after adjunction to each D_i . Fujino–Gongyo’s slc gluing theorem descends semiampleness from the normalization across the conductor, giving semiampleness on all of D . Componentwise semiampleness alone is not the hypothesis of Theorem 3.1.

Separate the supply of this hypothesis from the proof of Theorem 3.1 itself. If $D = 0$, the signed relation gives $L \sim_{\mathbb{Q}} 0$. If $v = \nu(L) = 0$, the inequality $(L - cD)H^{n-1} \geq 0$ forces $D = 0$. If $v = n$, the nef divisor is big. Only the remaining range $0 < v < n$ requires the following deformations.

[Proposition 2.3 · p. 8 ↗](#) · [\[FG14\] Theorem 1.5 · p. 2 ↗](#)

2. What Hodge theory lifts

Put $v = \nu(L)$. Lemma 3.2 finds a positive boundary component with image dimension $v - 1$, while its intersections with the negative and zero parts have smaller image. After the root construction and adjunction data of Proposition 3.3, §4 lifts both local coordinates on that image and a transverse coordinate through every infinitesimal neighborhood. The obstruction lies in a negative twist of the lowest Hodge piece; the graph calculation places it in the symbol-map kernel, where Lemma 4.3 kills it.

[Lemma 3.2 / Proposition 3.3 · pp. 13–18 ↗](#) · [Proposition 4.1 / Lemma 4.3 · pp. 19–25 ↗](#)

3. From formal deformations to a big divisor downstairs

Fix the lifted image coordinates and vary the transverse coordinate to move a general boundary fiber off D . Proposition 5.1 algebraizes this deformation and produces a dominating family of compact L -trivial subvarieties of dimension $n - v$. The nef-reduction base therefore has dimension $b \leq v$. Pseudo-effectivity of $L - cD$ makes D vertical for that reduction, and Lemma 5.2 descends the signed divisor to the base.

$$\pi^*L \sim_{\mathbb{Q}} f^*G_B, \quad v = \nu(G_B) \leq \dim B = b \leq v.$$

The nef divisor G_B has full numerical dimension and is therefore big. Pulling back sections gives $\kappa(L) \geq v$, while the usual inequality for a nef divisor gives the reverse bound. Lemma 3.2 handles $v = 0$ and $v = n$ separately.

Proposition 2.5 applies this to a zero-Iitaka model with a positive full-support representative. In the nonvanishing preparations, Proposition 6.10 uses it in a contradiction proving bigness of $K_W + D$ for the reduced support of a signed canonical representative. There Proposition 6.9 supplies a nef endpoint using lower-dimensional special termination, and $c = 1$ makes $L - D = K_Z$ pseudo-effective. No general good-model assertion in the current dimension is used in advance.

[Proposition 5.1 · pp. 26–28 ↗](#) · [Lemma 5.2 / §5.1 · pp. 28–30 ↗](#) · [Propositions 6.9–6.10 · pp. 41–43 ↗](#)

The conclusion is abundance, not yet the final semiampleness statement. Use it in the zero-litaka branch of the induction and in the signed-representative branch of nonvanishing.

5. Theorem 9.6: construct fixed data from a counterexample

The source handles dimensions zero and one directly. For $n \geq 2$, assume a smooth counterexample with pseudo-effective K_X and $\kappa(X) = -\infty$. Use the exclusions in §§6–8 to realize the three kinds of input to Theorem 9.1 in characteristic zero.

Statements above: [Theorem 9.6](#) · [Manuscript proof](#) ↗



Figure 4. Inputs for Theorem 9.6: choose ϵ , then q , then t and r , then geometric data, and only finally the residual characteristic p .

Open figure ↗ TeX source ↓

1. Counterexample geometry supplies two estimates

Use $q(X) = 0$ from Lemma 6.1, general type from Proposition 6.3, and the exclusion of covering curves of bounded genus and normalized degree in Proposition 6.8. Section 7 tracks moving base components caused by excessive vanishing and bounds their degree and canonical divisor. In Proposition 8.1, general-type components would yield the forbidden curves, establishing a uniform scalar bound.

On the two-slot projective bundle, restrict a component arising from failed jet generation to a slice over one factor. Positive-dimensional branches use curve exclusion and the signed-representative bigness of Proposition 6.10. The branch generically finite over both factors is treated by excluding moving branch divisors, fixing a branch complement, and then fixing finitely many covers. Proposition 6.11 excludes the remaining dominant family of birational correspondences. This gives the large Seshadri constant of Proposition 8.3. The auxiliary semiample models use established big-klt results, not the general abundance theorem being proved.

§§6–7 · pp. 30–51 ↗ · Propositions 8.1, 8.3 · pp. 53–61 ↗

2. Choose the parameters and fix a polarization and flag

On the late terminal model Y , set $K = K_Y$, fix mK Cartier, and put $\mu_t = \text{vol}(K_X + tA_X)^{1/n} \rightarrow 0$. First choose rational $\epsilon > 0$ satisfying both small-constant conditions, next a sufficiently large integer q from Proposition 8.3, and only then small rational $t > 0$ and integers r with $r\mu_t \rightarrow \epsilon$ and $r > qm + 1$.

On a common resolution W , set $L = r(K + tA)$ and $K_W = K + E$, $E \geq 0$. Pull back the semiample positive part of L on its scaling model to obtain H_0 , with $H_0^n \rightarrow \epsilon^n$, $LH_0^{n-1} = H_0^n$, and $K_W H_0^{n-1} \leq H_0^n / r$. Fix t, r, W with strict margins, then make $H = H_0 + \eta H_{\text{amp}}$ ample for small rational $\eta > 0$, preserving the intersection inequalities of Theorem 9.1. Generic semipositivity for pseudo-effective K_W and restriction give one fixed complete-intersection flag with $\mu_{\min}(\Omega_W^1|_C) \geq 0$.

§8.1 / (30)–(32) · p. 52 ↗ · §9.2.1 / (61)–(64) · pp. 69–70 ↗ · [CP] Theorem 2.1 · print p. 844 / PDF p. 11 ↗

3. One movable curve controls every determinant weight

Put $D^+ = 2r(K + 2tA) + 2E$. Proposition 8.1 bounds the order of every nonzero section in every sufficiently divisible degree k at a very general x by $32\epsilon k$. Hence $\pi_x^* D^+ - 40\epsilon J$ is not pseudo-effective on the blowup: otherwise a convex combination with big $\pi_x^* D^+$ would produce excessive vanishing for some $32\epsilon < c < 40\epsilon$.

[BDPP] duality detects this by a strongly movable class. Strictness permits one actual complete intersection $\gamma = f_*(A_1 \cdots A_{n-1})$, on a fixed smooth birational model $f : V \rightarrow \widehat{W}$ with ample integral Cartier divisors A_i . One has $J \cdot \gamma > 0$. Put $S = mK$ and $Q_\lambda = L + K_W/2 + \lambda S$. The difference below is pseudo-effective for every $0 \leq \lambda \leq q$. Thus the same γ bounds all weights, and only the two endpoint inequalities at $0, q$ need to be retained for reduction.

$$D^+ - Q_\lambda = (r - \frac{1}{2} - \lambda m)K + 3rtA + \frac{3}{2}E, \quad \pi_x^* Q_\lambda \cdot \gamma < 40\epsilon J \cdot \gamma.$$

§9.2.2 / (65)–(66) · pp. 70–71 ↗ · [BDPP] Theorem 2.2 · p. 6 ↗

4. Fix finite jets and invoke incompatibility

Proposition 8.3 gives a Seshadri constant greater than $2n + 2$. Choose z_* in the torus open where the semiample contraction is an isomorphism. Fix one denominator-clearing degree k_0 and finitely many sections generating jets through order $(2n + 2)k_0$. Its two base coordinates need not equal the testing point x of the previous step.

All polarization, flag, movable-curve, and jet inputs of Theorem 9.1 now exist in characteristic zero. Inside that theorem, also fix T_0 and finitely many remainder sections before passing to large characteristic. The next section contradicts two bounds for the same determinant. This excludes the counterexample X and proves Theorem 9.6. Only afterwards may Proposition 2.5 be applied.

§9.2.3–9.2.4 · pp. 71–72 ↗

6. Theorem 9.1: incompatibility of fixed data

This theorem assumes neither nonvanishing, abundance, nor minimal models. The preceding counterexample construction supplies its explicit intersection conditions, flag, jets, and movable curve.

Theorem 9.1 · p. 62

Let W be smooth, connected and projective over $\mathbb{C}$, of dimension $n \geq 2$. Let L, H be rational Cartier divisors, H ample, and S integral Cartier. Fix an integer $q > 0$ and real numbers $r > 1, \epsilon > 0$. The following numerical conditions and three kinds of fixed data cannot coexist.

$$\begin{aligned} H^n &\leq (2\epsilon)^n, & K_W H^{n-1} &\leq 2H^n/r, \\ (2L + qS)H^{n-1} &\leq 4H^n, \\ (14\epsilon)^n &< \frac{1}{4}, & 80\epsilon &< \frac{3}{4}. \end{aligned}$$

1. A smooth integral complete-intersection flag from W to a curve C , with divisor classes $h_i = l_i H$ for fixed positive integers l_i , and $\mu_{\min}(\Omega_W^1|_C) \geq 0$.

2. On $Z = \mathbb{P}(\mathcal{O}_W(S)_1 \oplus \mathcal{O}_W(S)_2) \rightarrow W \times W$, put $\xi = c_1(\mathcal{O}_Z(1))$ and $M = L_1 + L_2 + q\xi$, with the symmetric-power direct-image convention. At a smooth point z_* off both axes, finitely many sections of $k_0 M$ generate jets through order $(2n + 2)k_0$, for one integer $k_0 > 0$ with $k_0 L$ Cartier.
3. The blowup $\pi_x : \widehat{W} \rightarrow W$ of a point x , exceptional divisor J , and $\gamma = f_*(A_1 \cdots A_{n-1})$ from a fixed birational morphism $f : V \rightarrow \widehat{W}$, with V smooth integral projective and each A_i ample integral Cartier. Require $J \cdot \gamma > 0$ and $\pi_x^*(L + K_W/2 + \lambda S) \cdot \gamma \leq 40\epsilon J \cdot \gamma$ for every $0 \leq \lambda \leq q$.

Fix all data before varying the residual characteristic p . No relation between x and the base coordinates of z_* is required. This independent theorem assumes neither nefness of L, S, K_W nor pseudo-effectivity of K_W .

Theorem 9.1 · p. 62 · [GitHub](#) ↗

START The fixed inputs of Theorem 9.1

Small polarization and flag, finite jets, and a movable curve on a blowup.

$$(14\epsilon)^n < 1/4, \quad 80\epsilon < 3/4.$$

Thm. 9.1 / (43)–(44), p. 62

Products amplify the fixed jets to make evaluation generically full rank.
Choose a nonzero maximal minor; all constants are fixed before varying p .

Lemmas 9.2–9.3 (46) §9.1.5

01 Reduce to large characteristic p and obtain a nonzero determinant

$$E = F_* \mathcal{O}_W(B_p), \quad \text{rk}(E \boxtimes E) = R = p^{2n}.$$

$$0 \neq \sigma \in H^0(W' \times W', Q_1 \boxtimes Q_2).$$

Count sections on the Frobenius diagonal
by its canonical filtration and slopes:

$$h^0(\Delta_F, \mathcal{L}_p) < R/4.$$

(48)–(49) Lemmas 9.4–9.5

Apply the fixed curve γ to the Chern
class of each Q_i . Independent leading
terms let the factor bounds add.

(44) (57)–(59)

A Lower bound from diagonal rank loss

$$\text{rank}_{(x', x')} < R/4$$

$$\implies \text{ord}_{(x', x')} \sigma \geq 3R/4.$$

B Upper bound from the two factors

$$\text{ord}_{x'} \tau_i \leq R(40\epsilon + O(p^{-1}))$$

$$\implies \text{ord}_{(x', x')} \sigma \leq R(80\epsilon + O(p^{-1})).$$

Fix $80\epsilon < 3/4$, then take p sufficiently large.

Thm. 9.1 / (60) §9.2.4

END Contradictory bounds on one determinant

$$\frac{3}{4} \leq \frac{\text{ord}_{(x', x')} \sigma}{R} \leq 80\epsilon + O(p^{-1}) < \frac{3}{4}.$$

The fixed data cannot coexist. **Theorem 9.1**

Figure 5. Theorem 9.1: fixed jets give one nonzero determinant, whose diagonal order is bounded in two incompatible ways.

Open figure ↗ TeX source ↓

1. Reduce finitely many sections and one flag

Choose one sufficiently ample T_0 and finitely many sections of $T_0 + aS$, $0 \leq a \leq (k_0 - 1)q$, nonzero at the two base coordinates of z_* . Spread all varieties, maps, divisors, flags, points, the complete-intersection witness, and jet and filler sections over one finitely generated $\mathbb{Z}$ -algebra. Shrinking preserves smoothness, the required ampleness, finite jet surjectivity, intersection numbers, and the nonnegative minimum slope on the flag curve.

Only finite data are preserved. Neither a characteristic-zero pseudo-effective cone nor an MMP is transferred to characteristic p . Nonnegative pairing of γ with effective divisors on the reduction follows there by pullback to the fixed birational model and intersection with its ample complete intersection.

§9.1.1 · p. 63 ↗

2. Amplify the fixed jets to obtain a nonzero determinant

Put $N_p = \lfloor p/k_0 \rfloor$ and $B_p = k_0 N_p L + T_0$. Products of N_p copies of the fixed system generate jets through order $(2n + 2)k_0 N_p$. Previously fixed filler sections handle the remainder $(p - k_0 N_p)q$, giving jets of $pq\xi + B_{p,1} + B_{p,2}$. Since $\dim Z = 2n + 1$, for large p these surject onto the local quotient by the p th powers of all regular parameters. Triangularization by leading monomials uses no division by factorials.

For relative Frobenius $F : W \rightarrow W'$, set $E = F_* \mathcal{O}_W(B_p)$, $s = p^n$, and $R = s^2$. Taking the constant coefficient in the torus-coordinate basis $1, z, \dots, z^{p-1}$ retains weights pa . With $U_a = H^0(W, B_p + paS)$, the evaluation below therefore has generic rank R and a nonzero maximal minor σ . The full-rank point need not be the point used to test orders on the diagonal.

$$\bigoplus_{a+b=q} U_a \otimes U_b \otimes \mathcal{O}(-aS'_1 - bS'_2) \longrightarrow E \boxtimes E.$$

Lemmas 9.2–9.3 / (46) · pp. 63–64 ↗

3. Diagonal rank loss forces high vanishing

On the diagonal, every weight $a + b = q$ becomes the same line bundle

$\mathcal{L}_p = \mathcal{O}_{\Delta_F}(B_{p,1} + B_{p,2} + pqS_1)$ on $\Delta_F = W \times_{W'} W$. The diagonal evaluation rank is therefore at most $h^0(\Delta_F, \mathcal{L}_p)$. Filtering by the ideal of the reduced diagonal gives

$G_j = \mathcal{O}_W(2B_p + pqS) \otimes A_j(\Omega_W^1)$ for $0 \leq j \leq n(p - 1)$. Here A_j is the degree- j part of the symmetric algebra modulo p th powers, and its ranks satisfy $\sum_j e_j = p^n$.

The fixed-curve hypothesis $\mu_{\min} \geq 0$ and [Langer, Corollary 2.5] give the slope bound of Lemma 9.4, uniform in j, p . Successive restrictions along the fixed flag bound the sections of every G_j by its rank times the same scalar polynomial. Summing ranks, without an extra factor for the number of pieces, gives $h^0(\Delta_F, \mathcal{L}_p) \leq R(7^n H^n + O(p^{-1})) < R/4$. The last inequality uses $H^n \leq (2\epsilon)^n$ and $(14\epsilon)^n < 1/4$.

$$\text{ord}_{(x', x')} \sigma \geq \frac{3R}{4}.$$

The matrix rank at a diagonal point is less than $R/4$. After constant row operations, more than $3R/4$ rows vanish there, forcing every determinant term into the corresponding power of the maximal ideal.

[Lemmas 9.2–9.5 / equations \(46\)–\(56\) · pp. 63–67 ↗](#) · [Equation \(60\) · p. 69 ↗](#) · [\[Langer\] Corollary 2.5 · print p. 258 / PDF p. 8 ↗](#)

4. Bound the same determinant from above

The determinant is a section of the actual external product $Q_1 \boxtimes Q_2$. The normalized first Chern class of each factor is $L + K_W/2 + \lambda_i S + O(p^{-1})$, with $0 \leq \lambda_i \leq q$. The fixed movable witness bounds the order of every nonzero section of each factor by $R(40\epsilon + O(p^{-1}))$.

Bases adapted to vanishing orders have independent leading terms, so the two factor bounds add for the tensor product.

More precisely, if m_i is the sum of the selected column weights in factor i , then

$Q_i = (\det E)^{\otimes p^n} \otimes \mathcal{O}(m_i S')$ and $\lambda_i = m_i/R \in [0, q]$. Thus

$c_1(Q_i)/R = B_p/p + (p-1)K_W/(2p) + \lambda_i S$. The numerator of the error, $B_p - pL - K_W/2$, ranges over a fixed finite list of divisor classes, making $O(p^{-1})$ uniform in the selected columns.

The movable curve tests the effective divisor of every nonzero section on the reduction itself; that section need not lift to characteristic zero.

$$\frac{3R}{4} \leq \text{ord}_{(x', x')} \sigma \leq R(80\epsilon + O(p^{-1})).$$

Since $80\epsilon < 3/4$ was chosen first and all other data are fixed, large p gives a contradiction. This proves Theorem 9.1. Applying it to the data constructed from a counterexample in the previous section gives Theorem 9.6, returning to the nonvanishing step of the overview. Proposition 2.5 then establishes real-boundary good models in the same dimension.

[§9.1.5 / equations \(57\)–\(60\) · pp. 67–69 ↗](#) · [§9.2.4 · p. 72 ↗](#)

Return Theorem 9.6 to step 1 of the main diagram. Hashizume and Proposition 2.5 then produce good models in the same dimension; induction gives Theorem 11.1.

7. External inputs and consequences

Inputs to the main proof

Hashizume is used only after smooth nonvanishing has been proved. Fujino–Gongyo gluing supplies the whole-boundary condition before that step, using smaller-dimensional models. The positive-Iitaka lemma and Ambro’s descent enter the subsequent good-model argument. BDPP, Campana–Păun and the restriction theorem construct the fixed witnesses of Theorem 9.6; Langer’s slope bound is used inside the independent Theorem 9.1. Their hypotheses and uses are specified in the reference guide and the corresponding proof stages.

[Proposition 2.5](#) · [Theorem 3.1](#) · [Theorem 9.6](#) · [Theorem 9.1](#)

Catalogue 033 → LA: subadditivity in one lemma

[OI], Corollary 6.2 is stated as LA’s Theorem 1.2. Its only use is Lemma 6.1, with both boundaries zero. If the Albanese image had positive dimension, its resolved Stein base would have nonnegative Kodaira dimension, and the smaller-dimensional fiber would have nonvanishing by induction. Subadditivity would contradict the counterexample.

For this specific base of maximal Albanese dimension, the established Hacon–Popa–Schnell Theorem 1.1 supplies the same required inequality, as the LA introduction also records. This alternative covers the application in Lemma 6.1, not the full logarithmic theorem of [OI].

[\[OI\] Corollary 6.2 · pp. 40–41 ↗](#) · [Hacon–Popa–Schnell, Theorem 1.1 · p. 2 ↗](#) · [LA §1 / Lemma 6.1 · pp. 6, 30–31 ↗](#)

Within 034: the independent nonvanishing tools

Lemmas 7.1–7.5 control centres and moving components; Theorem 9.1 compares determinant orders for fixed witnesses. Fourfold nonvanishing imports these tools in its §§3, 6–7. This use does not invoke the abundance main theorem. The receiving argument →

Within 034: the good-model output

Theorem 11.1 supplies the qualitative model before effective degrees are studied. Its uses differ: model existence in Uniform Iitaka, the semiample fibration in Log Iitaka, perturbed klt models in the Hodge-rank argument for slc indices, and nonvanishing in Fourfold Iitaka.

- [Uniform Iitaka · Theorem 2.1](#)
- [Log Iitaka · Theorem 2.2 / §8](#)
- [slc indices · Theorem 2.3 / Lemma 5.6](#)
- [Fourfold Iitaka · §10](#)

Kähler abundance instead invokes the full projective induction in its Proposition 2.10, while retaining its own subadditivity assumption. [The projective input →](#)

Within 034: LA → Schnell

For a smooth projective complex variety X with pseudo-effective K_X , Theorem 11.1 applied to $(X, 0)$, together with the comparison lemma, gives Corollary 11.2: a Q-factorial klt good model V and an effective q-exceptional comparison on a common resolution. This is the input used by Theorem 3.1 of the Schnell manuscript.

$$p^*K_X = q^*K_V + E, \quad E \geq 0, \quad q_*E = 0, \quad K_V \text{ semiample.}$$

[Continue to the mixed-intersection argument in Schnell →](#)

[Theorem 11.1 / Corollary 11.2 · pp. 73–74 ↗](#) · [Catalogue 034 dependency table →](#)

Consequences after the main proof

Sections 11–14 derive rational-linear triviality in Iitaka dimension zero, slc abundance and boundary extension, finite generation of rational lc rings, geometric consequences for rational curves and cotangent tensors, and relative analytic abundance near compact base subsets. These consequences follow after the induction. Section 12 additionally cites the ordinary-pair case of the generalized-minimal-model manuscript in catalogue 036; that companion is not an input to the good-model induction above.

[§§11–14 · pp. 73–81 ↗](#) · [Catalogue 036: generalized minimal models · Corollary 1.2 ↗](#)

8. Source entry points and reading scope

- [§2 / Proposition 2.5 · pp. 7–12](#) — The two induction outputs and the zero-Iitaka cases.
- [Theorem 3.1 / §§3–5 · pp. 12–30](#) — Signed representatives, formal lifting, compact families, and descent.
- [Proposition 2.5 / §10 / Theorem 11.1 · pp. 9–12, 72–73](#) — Close each dimension only after smooth nonvanishing; return from rational to real boundaries.
- [Propositions 8.1, 8.3 · pp. 53–61](#) — The scalar bound and the two-factor jet construction.
- [Theorem 9.1 / Theorem 9.6 · pp. 62–72](#) — The determinant contradiction and construction of its witnesses.

Revision scope · 11 October 2026

Rechecked the pinned 24 September manuscript (85 pages, inventory hash matched): the statements and branch structure of Proposition 2.5, the inputs and completion of Theorem 3.1, the interfaces of Propositions 6.9–6.10 and 8.1/8.3, the fixed-data statement and determinant proof of Theorem 9.1, and the construction and use of its witnesses in Theorem 9.6. The ordering of the induction and the final descent in §§10–11 were compared with the article. External statements rechecked in this revision include Hashizume Theorem 1.4, Fujino–Gongyo Theorem 1.5 and Lemma 3.5, Ambro Theorem 0.2, BDPP Theorem 2.2, Campana–Păun Theorem 2.1, and Langer Corollary 2.5.

The geometry and external inputs of §§6–8 are described at the interface level: the article does not independently establish every current, valuation, moving-component, special-termination, or branch-family estimate. The supplied finite data and the way Theorem 9.1 uses them have been traced; this is not a certification that all of their geometric construction is correct. The statements of the auxiliary BCHM/Birkar, restriction, Saito and Sun inputs have not all been independently rechecked against their own sources in this revision. The OI/Hacon–Popa–Schnell comparison retains the previous reading record.

This overview reconstructs the manuscript’s argument. The full Hodge-module lifting calculation, the valuation/current and moving-component estimates in §§6–8, all cited proofs, and the full deductions in §§11–14 have not been independently verified. The catalogue-036 connection records LA’s citation; the companion proof was not checked. No expert review or formal verification is claimed.

Minimal metrics and interior injectivity for nef adjoints

Extremal sections with a fixed twist and injectivity in ordinary cohomology

The paper supplies two analytic inputs. It proves vanishing of the Lelong numbers of minimal metrics on nef klt adjoints by localizing extremal sections with one fixed ample twist.

Independently, it proves ordinary H^1 injectivity for interior boundaries with zero-Lelong endpoint metrics, returning weighted harmonic estimates to a fixed Hilbert space and ordinary cohomology. The second proof does not use the first theorem.

Manuscript author: OpenAI · 2026-09-27 · 21 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Minimal metrics on nef klt adjoints (pp. 1–2)

Let (H, Θ) be a normal connected projective complex klt pair with effective rational divisor Θ , and suppose $D_H = K_H + \Theta$ is nef and $\mathbb{Q}$ -Cartier. On every projective log resolution $\pi : V \rightarrow H$, the actual rational line bundle $N = \pi^* D_H$ admits a semipositive singular Hermitian metric with minimal singularities, and every such metric has zero Lelong number at every point. Neither K_H nor Θ is separately required to be $\mathbb{Q}$ -Cartier. Theorem 1.1, pp. 1–2

Theorem 1.2 — Interior injectivity (p. 2)

Let V be smooth projective over $\mathbb{C}$, L an integral divisor, and $0 \leq C_0 \leq C_2$ effective rational divisors with combined SNC support. Assume that **both actual rational line bundles $L - C_0, L - C_2$** admit semipositive singular Hermitian metrics with zero Lelong numbers everywhere. For rational $0 < \lambda < 1$, put

$$C_1 = (1 - \lambda)C_0 + \lambda C_2, \quad L_i = L - \lfloor C_i \rfloor \quad (i = 0, 1).$$

The natural inclusion induces an injection

$$H^1(V, \mathcal{O}_V(K_V + L_1)) \longrightarrow H^1(V, \mathcal{O}_V(K_V + L_0)).$$

Theorem 1.2, p. 2

Corollary 2.4 — Multiplier ideals (pp. 10–11)

Under Theorem 1.1, every minimal weight φ satisfies $\mathcal{I}(t\varphi) = \mathcal{O}_V$ for every real $t > 0$. This is independent of the minimal metric and of the Cartier multiple used to normalize its weights. Corollary 2.4, pp. 10–11

Corollary 4.1 — Fourfolds with nonzero Euler characteristic (p. 18)

Let (X, Δ) be a projective complex klt fourfold with effective rational boundary and nef $D = K_X + \Delta$. If $\chi(X, \mathcal{O}_X) \neq 0$, then D is semiample. No numerical-dimension restriction is imposed; the Euler hypothesis produces the first section. Corollary 4.1, pp. 18–19

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to this manuscript. The principal external inputs use the following keys.

- [Fuj Fujino](#), Fundamental theorems for the log minimal model program · Theorem 3.2 · PDF.
- [Dem Demailly](#), Analytic Methods in Algebraic Geometry · July 2011 · Theorem 5.1 · PDF.
- [Reg Demailly](#), Regularization of closed positive currents and intersection theory · Main Theorem 1.1 · PDF.
- [LP Lazić–Peternell](#) · Corollary D · arXiv v4.
- [GM Gongyo–Matsumura](#) · Corollary 5.3 · 2017 published version.

2. Proof of Theorem 1.1

Proof route. The zero-Lelong metric theorem and interior injectivity, Theorem 1.2, are independent results. Corollary 4.1 uses the metric theorem with existing nonvanishing and semiamplessness criteria.

Put $N = \pi^*(K_H + \Theta)$ and $n = \dim V$. Assuming positive Lelong number, replace an extremal section with a fixed twist by a section having the same value at the chosen point but smaller mass.

01 Choose a fixed twist from the adjoint structure

$$N = \pi^*(K_H + \Theta) = K_V + B, \quad B_j < 1, \quad B^- \text{ } \pi\text{-exceptional.}$$

$mN + A$ is globally generated for every allowed m .

Klt vanishing gives generation; minimality and compactness give small mass.

[\[Fuj\] Thm. 3.2, p. 734](#) [Claim 2.2, \(2.5\), pp. 6–7](#)

02 Assume positive Lelong number and maximize evaluation

$$Q_m(s_m) = 1, \quad s_m(p) \neq 0, \quad \tau_m = m^{-1} \log |s_m|^2.$$

$$g_m = \varphi + a/m - \tau_m, \quad \varepsilon_m = \int_{\{g_m < -R\}} e^{\tau_m - a/m - b} \rightarrow 0.$$

Fix the cutoff and convex weight; inverse curvature gives a uniform bound.

[\[Dem\] Thm. 5.1, p. 33](#) [Lemmas 2.1, 2.3, pp. 5, 8–9](#)

03 Localize and solve the error with fixed weights

$$\bar{\partial}u_m = \bar{\partial}(\chi(g_m)s_m), \quad w_m = \chi(g_m)s_m - u_m.$$

$$\int |u_m|^2 e^{-S_m - f(g_m)} \leq C\varepsilon_m, \quad C \text{ independent of } m.$$

Hartogs extension in the Cartier bundle downstairs; nonintegrability fixes evaluation.

[§2.4, \(2.13\)–\(2.15\), pp. 9–10](#)

04 Cross negative exceptional components by descent and extension

$$V \setminus \text{Supp } B^- \rightarrow H^\circ \rightarrow H \rightarrow V.$$

$$w_m \in H^0(V, mN + A), \quad w_m(p) = s_m(p).$$

A fixed upper slope d keeps the final Hölder bound uniform.

[\(2.15\)–\(2.17\), Cor. 2.4, pp. 10–11](#)

05 Renormalization contradicts extremality

$$Q_m(w_m)^m \leq C'\varepsilon_m(1 + I)^d \rightarrow 0.$$

$$\nu(\varphi, p) = 0 \quad (\forall p), \quad \mathcal{I}(t\varphi) = \mathcal{O}_V \quad (t > 0).$$

Extremal sections with a fixed twist force zero Lelong numbers

[Open figure ↗](#) [TeX source ↓](#)

1. Choose extremal sections with one fixed twist

Write $N = K_V + B$. Every coefficient of B is less than one, while B^- is π -exceptional; thus e^{-b} is integrable for $b = [B]$. Fix very ample H_0 and put $A = \pi^*((n+1)H_0)$. Klt vanishing and Castelnuovo–Mumford regularity make $mN + A$ globally generated whenever mD_H is Cartier. If a minimal weight φ has positive Lelong number at p , choose s_m maximizing $|s(p)|$ subject to

$$Q_m(s) = \int_V |s|^{2/m} e^{-a/m-b} = 1.$$

Global generation gives $s_m(p) \neq 0$. A smaller-mass section with the same value will contradict this choice.

§2.2 · pp. 5–6 · Fuj, Theorem 3.2 · PDF p. 8

2. Localize on a fixed sublevel and solve the error

A subsequential limit of $\tau_m = m^{-1} \log |s_m|^2$ and minimality $\tau_\infty \leq \varphi + C$ allow one fixed R . For $g_m = \varphi + a/m - \tau_m$,

$$\varepsilon_m = \int_{\{g_m < -R\}} e^{\tau_m - a/m - b} \longrightarrow 0.$$

Positive Lelong number also supplies fixed c for which $e^{-c\varphi-b}$ is nonintegrable at p , retaining the negative coefficients of B . Fix a cutoff χ and a convex function f with $c \leq f' \leq d$. On a smooth affine open avoiding B and the zeros of s_m , Lemma 2.3, with $S_m = (m-1)\tau_m + a/m + b$, gives

$$\bar{\partial}u_m = \bar{\partial}(\chi(g_m)s_m), \quad \int |u_m|^2 e^{-S_m - f(g_m)} \leq C\varepsilon_m.$$

Here C depends only on χ, f , not on m or the open set. The external input is Dem's smooth inverse-curvature estimate. The manuscript passes to singular weights by decreasing approximation and weak limits in spaces with fixed earlier weights.

Claim 2.2; (2.5)–(2.12); Lemmas 2.1, 2.3 · pp. 5–9 · Dem, Theorem 5.1 · p. 33

3. Extend past negative exceptional components and preserve evaluation

The section $w_m = \chi(g_m)s_m - u_m$ is holomorphic on the open set. An upper bound for $G_m = S_m + d \max(0, g_m)$ and its L^2 estimate first extend it to $V \setminus \text{Supp } B^-$. Instead of assuming that bound on B^- , descend to the complement of a codimension-two set in H . The actual Cartier bundle $mD_H + A_H$ and normality of H permit Hartogs extension; pulling back gives $w_m \in H^0(V, mN + A)$.

Near p , the section s_m is nonvanishing, $\chi(g_m) = 1$, and $S_m + f(g_m) = b + c\varphi + O_m(1)$. A nonzero value of $s_m - w_m$ at p would contradict the finite correction norm and the nonintegrability fixed above. Hence $w_m(p) = s_m(p)$. This $O_m(1)$ is a local comparison for each m , distinct from the estimates requiring uniform constants.

§2.4; (2.13)–(2.15) · pp. 9–10

4. Contradict extremality with a uniform Hölder bound

Take $m > d + 1$ and set $I = \int_V e^{\varphi-b} < \infty$. The exponent $d/(m-1)$ in Hölder's inequality cancels the final power $m-1$, giving

$$Q_m(w_m)^m \leq C' \varepsilon_m (1 + I)^d \longrightarrow 0.$$

All of R, c, d, χ, f, C', I are independent of m . For large m , $0 < Q_m(w_m) < 1$, so $Q_m(w_m)^{-m/2} w_m$ has mass one and larger evaluation at p than s_m . This contradiction proves $\nu(\varphi, p) = 0$. Corollary 2.4 then follows from Skoda integrability.

(2.16)–(2.17); Corollary 2.4 · pp. 10–11

3. Proof of Theorem 1.2

This proof takes the two endpoint metrics as inputs and does not use Theorem 1.1. Start with an ordinary class in the kernel of multiplication, estimate weighted representatives, and then kill the original class in one fixed space.

01 Regularize and compare the two endpoint metrics

$$L - C_0, L - C_2 : \nu = 0, \quad 0 < \lambda < 1.$$

$$E = \lfloor C_1 \rfloor - \lfloor C_0 \rfloor, \quad s : L_1 \rightarrow L_0 \text{ is a contraction.}$$

Use bounded local potentials, fixed-cover cocycles and one prescribed seminorm.

Lemmas 3.1–3.2, Prop. 3.3, pp. 11–16

02 Prepare the ordinary class map and uniform primitives

$$\kappa_k : Z_{h_k}^1 \longrightarrow H^1(V, K_V \otimes F).$$

$$f \in \ker \kappa_k \Rightarrow \bar{\partial}v = f, \quad \|v\|_k \leq C\|f\|_k \quad (C \text{ independent of } k).$$

Log-sum weights give contraction and curvature comparison for Bochner.

(3.1)–(3.6), (3.11)–(3.13), pp. 12–17

03 Estimate harmonic representatives of a kernel class

$$A_{1,k} \geq (1 - \lambda)A_{0,k} \geq 0, \quad C_\lambda = (1 - \lambda)^{-1}.$$

$$\|\bar{\partial}_{0,k}^*(su_k)\|_{0,k} = O(\sqrt{\varepsilon_k}).$$

The fixed-cover class map and open mapping supply uniformly bounded primitives.

Prop. 3.3, (3.14), pp. 14–17

04 Pair with uniform primitives to make the image small

$$\kappa_{0,k}(su_k) = 0 \Rightarrow \bar{\partial}v_k = su_k, \quad \sup_k \|v_k\| < \infty.$$

$$\|su_k\|_{0,k}^2 = \langle v_k, \bar{\partial}_{0,k}^*(su_k) \rangle \longrightarrow 0.$$

Pass from varying weights to one fixed Hilbert space and use weak continuity.

(3.15), §3.5, pp. 17–18 (1.1), p. 2

05 Retain the ordinary class in a fixed Hilbert space

$$u_k \rightharpoonup 0, \quad \kappa_*(u_k) = [\gamma] \Rightarrow [\gamma] = 0.$$

$$H^1(K_V + L_1) \hookrightarrow H^1(K_V + L_0) \Rightarrow H^0(K_V + L_0) \twoheadrightarrow H^0(E, (K_V + L_0)|_E).$$

Endpoint metrics give ordinary H1 injectivity

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1. Regularize the endpoints and align contraction with curvature

In [Reg](#), zero Lelong numbers make the singular set of the decreasing smooth approximants empty. Dini's theorem makes the curvature loss uniform, while monotonicity and Skoda integrability bound the integrals at each fixed exponent. The log-sum modification in (3.1)–(3.3) makes multiplication $s : L_1 \rightarrow L_0$ by the canonical section of $E = [C_1] - [C_0] \geq 0$ a contraction. On the complement of the SNC support it gives

$$\theta_{1,k} + \varepsilon_k \omega_c \geq (1 - \lambda)(\theta_{0,k} + \varepsilon_k \omega_c) \geq 0.$$

The strict inequality $\lambda < 1$ keeps the Bochner comparison constant $C_\lambda = (1 - \lambda)^{-1}$ finite. The statement does not extend the argument to the endpoint $\lambda = 1$.

[Lemma 3.1; \(3.1\)–\(3.6\)](#) · pp. 11–13 · [Reg](#), [Main Theorem 1.1](#) · p. 2

2. Construct the ordinary class map and uniform primitives

[Lemma 3.2](#) supplies a fixed complete Kähler metric ω_c on the SNC complement U . Its bounded local potentials can be added to the weights on a fixed finite cover to make curvature positive with norm-comparison constants independent of k . Local primitives from [Dem](#) have holomorphic differences. Uniform upper bounds on the weights allow ordinary L^2 extension of these differences, producing a Čech cocycle in ordinary $H^1(V, K_V \otimes F)$.

The continuous map κ_k of [Proposition 3.3](#) kills the closure of exact forms. To split a cocycle in its kernel, apply one fixed Fréchet open mapping theorem with only a sup norm on the smaller cover prescribed. The cocycle family is uniformly bounded in the finitely many compact seminorms needed by that theorem. This gives splittings bounded in the prescribed sup norm; uniform integrability of the weights then gives $\bar{\partial}v = f$ and $\|v\|_k \leq C\|f\|_k$. No lift bounded in every seminorm, or linear choice of global primitive, is claimed.

[Lemmas 2.1, 3.2; Proposition 3.3; \(3.7\)–\(3.8\)](#) · pp. 5, 13–16

This step supplies primitives bounded uniformly for the varying weights. The [Bochner estimate](#) kills the image norm; only the [comparison with a fixed Hilbert space](#) kills the original ordinary cohomology class.

3. Combine Bochner estimates with the uniform primitive

Project a smooth representative γ of an ordinary kernel class onto u_k , orthogonal to the closure of exact forms. Complete-metric cutoffs and the curvature comparison give

$\|\bar{\partial}_{0,k}^*(su_k)\|_{0,k} = O(\sqrt{\varepsilon_k})$. The manuscript also uses cutoffs to place the form in the Hilbert-adjoint domain. Since the class map kills the closure, $\kappa_{0,k}(su_k) = 0$; the preceding step gives $\bar{\partial}v_k = su_k$ with $\sup_k \|v_k\|_{0,k} < \infty$. Hence

$$\|su_k\|_{0,k}^2 = \langle v_k, \bar{\partial}_{0,k}^*(su_k) \rangle \longrightarrow 0.$$

This is an estimate in varying weights. Eliminating the ordinary class still requires the next comparison.

§§3.4–3.5; (3.9)–(3.14) · pp. 16–17

4. Return to a fixed Hilbert space and kill the original class

For a fixed smooth weight h_* , the bound $h_{1,k} \leq h_* + C_*$ gives bounded inclusions into fixed Hilbert spaces in both degrees zero and one. On compact subsets of U , where s is nonvanishing, the preceding estimate gives strong convergence $u_k \rightarrow 0$. Together with global boundedness in the fixed space, this implies $u_k \rightarrow 0$.

Exact forms approximating $\gamma - u_k$ also pass to the fixed space. Applying Proposition 3.3 to the fixed weight gives a continuous class map with $\kappa_*(u_k) = [\gamma]$ for every k . Weak continuity into finite-dimensional ordinary cohomology forces $[\gamma] = 0$, proving Theorem 1.2. The exact sequence (1.1) then gives surjectivity of $H^0(V, K_V + L_0) \rightarrow H^0(E, (K_V + L_0)|_E)$ when $E \neq 0$. This concerns the entire divisor scheme, including nonreduced structure.

(3.15), §3.5 · pp. 17–18; (1.1) · p. 2

4. Proof of Corollary 4.1

Fix the metric supplied by Theorem 1.1, then apply the external criteria for nonvanishing and semiamplicity in that order.

01 Use one metric on one resolution

(X, Δ) projective klt, $\dim X = 4$, $D = K_X + \Delta$ nef.

$\pi^* \mathcal{O}_X(rD) : \Theta(h) \geq 0, \quad \nu(h, \cdot) = 0.$

Use zero divisorial part; set the auxiliary nef divisor and parameter to zero.

Thm. 1.1, pp. 1–2 [LP] Cor. D, p. 4

02 Use Euler characteristic to obtain the first section

$\chi(X, \mathcal{O}_X) \neq 0 \implies \kappa(X, D) \geq 0.$

Apply the fourfold criterion to the positive Cartier multiple and nonvanishing.

[GM] Cor. 5.3, p. 499 Cor. 4.1 proof, p. 19

03 Use the same zero-Lelong metric for semiample

$D = K_X + \Delta$ semiample.

Nonzero Euler characteristic yields fourfold abundance

[Open figure ↗](#) [TeX source ↓](#)

1. Obtain nonvanishing from the metric and Euler characteristic

Take a tensor power of Theorem 1.1's metric on the actual bundle $\pi^* \mathcal{O}_X(rD)$. It satisfies LP's generalized algebraic singularities condition with zero divisorial part. Set the auxiliary nef divisor and parameter to zero; $\chi(X, \mathcal{O}_X) \neq 0$ then gives $\kappa(D) \geq 0$.

[Theorem 4.2; Corollary 4.1 proof · p. 19](#) · [LP, Corollary D · p. 4](#)

2. Apply the semiample criterion to the same metric

For a nef adjoint on a klt fourfold, GM gives semiample from a zero-Lelong metric on a positive Cartier multiple and nonvanishing. Supply the latter from the preceding step and use the same metric. Theorem 1.2 is not used in this corollary.

[Theorem 4.3; Corollary 4.1 proof · p. 19](#) · [GM, Corollary 5.3 · printed p. 499 / PDF p. 23](#)

5. Which papers supply which steps

An application using both theorems. Proposition 5.1 in Fourfold nonvanishing combines endpoint metrics from Theorem 1.1 with interior injectivity from Theorem 1.2 to obtain surjectivity of restriction to the whole floor. A nonzero boundary section is supplied separately by the following three-dimensional semi-dlt abundance step.

Input

Fuj, Theorem 3.2, p. 734 (PDF p. 8)

Use in the paper

§2.2, p. 6: the klt multiplier ideal is trivial; the nef adjoint and ample difference give vanishing and, by regularity, the fixed globally generated twist.

Checking scope

Statement and substitution checked. The original regularity theorem was not checked.

Input

Dem, Theorem 5.1, p. 33

Use in the paper

Lemma 2.1 $\rightarrow$ Lemma 2.3 and Proposition 3.3: inverse-curvature solvability on complete Kähler manifolds.

Checking scope

Compared the author's July 2011 statement and its $(n, 1)$ specialization, including completeness, positivity, closedness and integrability. Singular limits are additional arguments here.

Input

Reg, Main Theorem 1.1, p. 2 and Skoda integrability

Use in the paper

Lemma 3.1, pp. 11–12: smooth approximants, small curvature loss and fixed-exponent integral bounds.

Checking scope

Compared decreasing approximation, curvature bounds and Lelong level sets in the author's PDF, and Dini's theorem in the application. Skoda's original statement remains unchecked.

Input

LP, Corollary D, p. 4

Use in the paper

Theorem 4.2 and Corollary 4.1, p. 19: the first section using nonzero Euler characteristic.

Checking scope

May 7, 2019 v4 statement and the $N = 0, t = 0$ specialization checked; proof excluded.

Input

GM, Corollary 5.3, p. 499

Use in the paper

Theorem 4.3 and Corollary 4.1, p. 19: nonvanishing and a zero-Lelong metric imply semiampleness.

Checking scope

Published 2017 statement and application checked; proof excluded.

Within 034, [Fourfold nonvanishing](#), Theorems 4.1–4.2, restates these two theorems and uses them in the proof of Proposition 5.1, (5.9)–(5.10), pp. 18–19. It identifies the endpoint residual bundles on a resolution with **positive rational multiples of nef klt adjoints**, applies Theorem 1.1, and pushes down Theorem 1.2's ordinary H^1 injection using multiplier-ideal local vanishing. This yields restriction surjectivity to the entire reduced non-klt locus. Producing a nonzero section there additionally requires threefold semi-dlt abundance, gluing and descent in [FN](#); these are not supplied by the present analytic theorem alone.

6. Return to the sources

Start with [Theorems 1.1–1.2 · pp. 1–2](#), then [§§2.2–2.4 · pp. 5–10](#) for localization, extension and extremality, [§§3.1–3.5 · pp. 11–18](#) for class maps and the fixed space, and [§4 · pp. 18–19](#) for the external criteria.

This revision reread those proof passages and newly compared the statements and application hypotheses of [Dem](#), Theorem 5.1 (July 2011, p. 33), and [Reg](#), Main Theorem 1.1 (p. 2). Statement-and-application checks for [Fuj](#), [LP](#) and [GM](#) are retained from the initial draft.

The full limiting arguments, harmonic-operator domains and all uniform constants, unchecked original sources for Skoda integrability, envelopes and regularity, and the full external proofs have not been independently verified. Explaining the fixed-space bridge does not certify the entire analytic proof. Other catalogue connections remain unexamined. No expert review or formal verification has been completed.

The manuscript is the September 27, 2026 version; page numbers are PDF pages and the snapshot is fixed at `adc7f1241b42e322a6451854ab7e4b4c146bf78a`.

Fourfold nonvanishing by minimal metrics and moving jets

Incompatible scalar vanishing orders and two-factor jets

A hypothetical minimal fourfold counterexample has positive canonical degree on every curve through a very general point. This gives small scalar vanishing orders, while a projective bundle over two factors has large jet generation. The proof fixes finite witnesses for both and forces incompatible orders on a Frobenius evaluation determinant. Zero-Lelong metrics and interior injectivity enter the intermediate signed-boundary argument.

Manuscript author: OpenAI · 2026-09-27 · 38 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Smooth fourfold nonvanishing (p. 2)

Let X be a smooth connected projective complex fourfold. If K_X is pseudo-effective, then for some positive integer m ,

$$H^0(X, mK_X) \neq 0.$$

There is no assumption on numerical dimension, irregularity or Euler characteristic. The result does not give a uniform bound on m . [Theorem 1.1, p. 2](#)

Corollary 1.2 — The original lc pair and a prescribed Cartier index (p. 2)

Let (X, Δ) be a connected normal projective complex lc pair, $\dim X \leq 4$, with effective rational boundary. Suppose $D = K_X + \Delta$ is nef and $\mathbb{Q}$ -Cartier. For every positive integer r with rD Cartier, some positive integer m satisfies

$$H^0(X, \mathcal{O}_X(mrD)) \neq 0.$$

No $\mathbb{Q}$ -factoriality is required, and the section is on the original X . [Corollary 1.2](#), p. 2

Proposition 5.1 — Signed support (an intermediate proposition)

The hypotheses are that Y is a projective $\mathbb{Q}$ -factorial terminal fourfold, K_Y is nef with $\kappa(K_Y) = -\infty$, and every proper positive-dimensional subvariety through a very general locus is of general type on resolution. Let $J \sim_{\mathbb{Q}} K_Y$ be a rational Weil divisor with either sign, $p : W \rightarrow Y$ a log resolution, and D_W a reduced SNC divisor containing the strict support of J and every exceptional divisor. Then $K_W + D_W$ is big. [Proposition 5.1](#), p. 14

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to this manuscript. The principal external inputs use the following keys.

- [MM](#) Minimal metrics and interior injectivity for nef adjoints · September 27, 2026 · [PDF](#).
- [LA](#) Log abundance in characteristic zero · September 24, 2026 · [PDF](#).
- [CT](#) · Theorem 1.1 · [arXiv v2](#).
- [Fuj](#) Fujino, Abundance theorem for semi log canonical threefolds · [PDF](#).
- [GM](#) Gongyo–Matsumura · Corollary 5.3 · [arXiv v2](#).
- [Hash](#) Hashizume · Theorem 1.4 · [arXiv v4](#).

2. Proof of Theorem 1.1

Proof route. This section establishes smooth fourfold nonvanishing; [Corollary 1.2](#) passes it to lc pairs. The signed-support branch is isolated in [Proposition 5.1](#), whose conclusion returns to the construction of two-factor jets.

Assume nonvanishing fails and construct both small scalar vanishing orders and large two-factor jets. Feed them into the independent Frobenius tool from [LA](#). Here A is a fixed ample divisor on the counterexample model, and $\varepsilon > 0$ will be chosen sufficiently small.

01 Fix one minimal counterexample

$$K = K_Y \text{ nef, } \kappa(K) = -\infty, \quad K^4 = 0, \quad n(K) = 4.$$

$K \cdot C \geq 1/\iota$; proper subvarieties through very general points are general type.



Bounded volume and diverging curve degrees exclude moving scalar centers.

Props. 2.1–2.2, 3.5, pp. 5–12 [LA] Lemmas 7.1–7.3, pp. 45–49

02 Bound scalar orders on one factor

$$L_t = r_t(K + tA), \quad L_t^4 \rightarrow \varepsilon^4, \quad r_t \rightarrow \infty.$$

$$P_s = 2r_t(K + 2tA), \quad \text{ord}_x(s) \leq 32k\varepsilon.$$



Exclude slice centers; use the signed-support diagram for non-axis multisections.

Prop. 5.1, §6.4, Cor. 6.5, pp. 14–30

03 Generate a finite system of jets on the two-slot bundle

$$Z = \mathbb{P}(\mathcal{O}(S_0)_1 \oplus \mathcal{O}(S_0)_2), \quad S_0 = \iota K.$$

$$\epsilon(P; z) > 10 \quad \Rightarrow \quad \text{surjection onto jets through order } 10k_0.$$



The scalar bound supplies the curve; retain the finite two-slot jet system.

Lemma 7.2, (7.2)–(7.7), pp. 31–33

04 Fix the resolution, flag and movable curve first

$$\mu_{\min}(\Omega_W^1|_C) \geq 0, \quad J_x \gamma > 0.$$

$$\pi_x^*(L + K_W/2 + \lambda S_0)\gamma \leq 40\varepsilon J_x \gamma \quad (0 \leq \lambda \leq q).$$



Only now vary the characteristic; the input is the independent finite-data theorem.

[LA] Thm. 9.1, p. 62 §7.3, pp. 33–34

05 Compare the incompatible determinant orders

$$R = p^8, \quad \text{rk}_{\text{gen}} \Phi_p = R, \quad \text{rk}_{\Delta} \Phi_p < R/4.$$

$$3R/4 \leq \text{ord } \sigma \leq R(80\varepsilon + O(p^{-1})), \quad 80\varepsilon < 3/4.$$

A minimal counterexample yields incompatible scalar and two-slot jets

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1. Fix the minimal counterexample and its curve degrees

Proposition 2.1 replaces a counterexample by one fixed projective $\mathbb{Q}$ -factorial terminal fourfold Y , with $K = K_Y$ nef, $\kappa(K) = -\infty$, $K^4 = 0$, $q(Y) = 0$, and nef dimension $n(K) = 4$. A fixed Cartier index ι gives $K \cdot C \geq 1/\iota$ for every curve through a very general point. Nef dimension here is distinct from numerical dimension $\nu(K)$. Using abundance in dimensions at most three, Proposition 2.2 makes every proper positive-dimensional subvariety through that locus of general type on resolution.

Propositions 2.1–2.2 · pp. 5–7

2. Bound vanishing using bounded volume and diverging degree

For fixed small $\varepsilon > 0$, take $L_t = r_t(K + tA)$ with $L_t^4 \rightarrow \varepsilon^4$ and $r_t \rightarrow \infty$. Curve degrees satisfy $L_t \cdot C \geq r_t/\iota \rightarrow \infty$ although volume stays bounded. Moving high-order sections create base components. The moving-multiplicity input from LA controls ordinary powers of their ideals for the **entire high-jet kernel**. A general-type center with bounded degree and canonical intersection produces a curve of bounded degree, contradicting the diverging curve degrees. Proposition 3.5 therefore gives, for every allowed Cartier degree,

$$P_s = 2r_t(K + 2tA), \quad \text{ord}_x(s) \leq 4k(P_s^4)^{1/4} \leq 32k\varepsilon$$

(pp. 10–12).

§§3.4–3.5; Proposition 3.5 · pp. 10–12

3. Exclude slices and correspondences to generate two-factor jets

For the two-factor argument, set $S_0 = \iota K$ and

$$Z = \mathbb{P}(\mathcal{O}(S_0)_1 \oplus \mathcal{O}(S_0)_2) \longrightarrow Y \times Y, \quad P = L_{t,1} + L_{t,2} + q\xi.$$

The analytic SNC hypothesis is not imposed directly on this bundle. Restrict moving base components to slices and exclude, in turn, a proper generically finite image, the entire bundle over a proper image, and a non-axis multisection over all of Y (§6.4, pp. 26–29). In the last case, intersection with the axes gives

$$D_0 - D_\infty \sim e\iota K,$$

and Proposition 5.1 below controls its signed support. Remaining correspondences with both projections dominant and generically finite are excluded by Proposition 6.4. First rule out sweeping branch divisors; then LA, Lemma 7.5, p. 51 gives finitely many finite étale covers of a fixed open. A degree bound alone is not used to assert finiteness. Birational automorphism groups would then produce a dominant map from an abelian variety to Y , again contradicting positive canonical curve degree.

Propositions 6.3–6.4; Corollary 6.5 · pp. 23–30

The proof of Proposition 5.1 separates metric-based surjectivity of restriction from the nonzero boundary section supplied by three-dimensional semi-dlt abundance.

4. Fix the finite data before varying the characteristic

This gives $\epsilon(P; z) > 10$ in Proposition 6.3 and one finite $10k_0$ -jet surjection in Corollary 6.5. Section 7 now fixes t, r_t , a resolution W , an ample perturbation H , a complete-intersection flag with $\mu_{\min}(\Omega_W^1|_C) \geq 0$, and a concrete strongly movable curve γ on the blowup of a point. The numerical conditions are

$$H^4 \leq (2\epsilon)^4, \quad K_W H^3 \leq 2H^4/r_t, \quad (2L + qS_0)H^3 \leq 4H^4, \quad (14\epsilon)^4 < 1/4, \quad 80\epsilon < 3/4.$$

They are supplemented by $J_x \gamma > 0$ and

$$\pi_x^*(L + K_W/2 + \lambda S_0)\gamma \leq 40\epsilon J_x \gamma \quad (0 \leq \lambda \leq q).$$

The curve is the pushforward of an ample complete intersection on one fixed birational model. This permits testing effective divisors after reduction without assuming specialization of an effective cone.

Theorem 7.1; §§7.1–7.2 · pp. 31–33

5. Compare two orders of the same determinant

Theorem 7.1 restates LA, Theorem 9.1, p. 62, which rules out coexistence of these fixed data. In §7.3 the characteristic- p evaluation has generic rank $R = p^8$ but diagonal rank less than $R/4$. Testing the two actual determinant factors with γ gives the conflicting inequalities

$$\text{ord}_{(x', x')} \sigma \leq R(80\epsilon + O(p^{-1})), \quad \text{ord}_{(x', x')} \sigma \geq 3R/4$$

for a nonzero determinant σ . The jet point z_* need not lie over the independently chosen point x . All finite data must be fixed over $\mathbb{C}$ before varying p .

§7.3 · pp. 33–34

3. Proof of Proposition 5.1

Assume $K_W + D_W$ is not big, where the reduced SNC boundary contains the signed support. Construct and lift a section from a nonempty floor before applying the klt semiampleness criterion.

01 Assume the signed support is not log big

$$J \sim_{\mathbb{Q}} K_Y, \quad D_W \supset \text{Supp } J_{\text{strict}} \cup \text{Exc}(p).$$

$$\kappa(K_W + D_W) \leq 0.$$

Use a dlt MMP and a Cartier-index-preserving perturbation; use the maximal coefficient.

(5.2)–(5.7), pp. 15–17

02 Construct a nef klt interval and a nonempty floor

$$M' - uD' \text{ nef klt adjoint} \quad (0 < u \leq \delta).$$

$$N = K' + B \sim_{\mathbb{Q}} \alpha M' - sD', \quad S = \lfloor B \rfloor \neq 0, \quad \mathcal{J}(B) = \mathcal{I}_S.$$

Apply [MM] to both endpoints; threefold abundance separately gives a floor section.

[MM] Thms. 1.1–1.2, pp. 1–2 (5.8)–(5.12), pp. 17–19

03 Lift whole-floor sections using endpoint metrics

$$L_R - C_i \sim_{\mathbb{Q}} t_i(m) h^*(K' + (1 - u_i(m))D') \quad (i = 0, 2).$$

$$H^1(\mathcal{O}(mN) \otimes \mathcal{I}_S) \hookrightarrow H^1(\mathcal{O}(mN)), \quad H^0(S, mN|_S) \neq 0.$$

After descent and lifting of a section, apply [GM] to the klt adjoint.

[GM] Cor. 5.3, p. 19 Prop. 5.1, pp. 19–20

04 Apply semiampleness after nonvanishing

$$\kappa(N) = 0, \quad N/\alpha \sim_{\mathbb{Q}} K' + (1 - s/\alpha)D' \text{ semiample.}$$

$$N/\alpha \equiv 0, \quad (K' + (1 - s/\alpha)D')A^3 > 0.$$

Signed support, whole-floor lifting, and the final contradiction

[Open figure ↗](#) [TeX source ↓](#)

1. Construct a nef boundary with nonempty floor

Assume it is not big. Positive Iitaka dimension, combined with general-type general fibers, would force bigness, so $\kappa(K_W + D_W) \leq 0$. A dlt MMP and a small crepant perturbation preserving a Cartier index produce $M' \sim_{\mathbb{Q}} G'$, $\kappa(M') \leq 0$, and a nef klt interval $M' - uD'$ for $0 < u \leq \delta$. The signed divisor G' is supported on D' . Its largest positive coefficient a and small $s > 0$ give

$$B = (1 - s)D' + (s/a)G', \quad \alpha = 1 + s/a, \quad N = K' + B \sim_{\mathbb{Q}} \alpha M' - sD', \quad S = \lfloor B \rfloor \neq 0.$$

Here $0 \leq B \leq D'$, (Z', B) is lc, N is nef, and $\mathcal{J}(Z', B) = \mathcal{I}_S$.

§5; (5.2)–(5.7) · pp. 15–17

2. Use endpoint metrics to obtain restriction onto the whole floor

Fix $0 < b_0 < 1 < b_2$ and construct effective SNC boundaries C_i on a resolution. The endpoint residuals in (5.9) satisfy

$$L_R - C_i \sim_{\mathbb{Q}} t_i(m) h^*(K' + (1 - u_i(m))D'), \quad t_i(m) > 0, \quad u_i(m) \rightarrow s/\alpha \in (0, \delta).$$

MM, Theorem 1.1 supplies zero-Lelong metrics on the actual endpoint bundles. MM, Theorem 1.2 and local vanishing then give

$$H^1(Z', \mathcal{O}(mN) \otimes \mathcal{I}_S) \hookrightarrow H^1(Z', \mathcal{O}(mN)).$$

Thus $H^0(Z', mN) \rightarrow H^0(S, mN|_S)$ is surjective for sufficiently large divisible m .

(5.9)–(5.10) · p. 18 · MM, Theorems 1.1–1.2 · pp. 1–2

3. Descend and lift a nonzero section of the whole floor

A nonzero boundary section is constructed separately. Apply adjunction and threefold semi-dlt abundance to the entire floor T of a dlt blowup. Local vanishing in (5.12), giving $g_*\mathcal{O}_T = \mathcal{O}_S$, descends a **whole-floor** section, including conductor compatibility, to S and then lifts it. Only now does $\kappa(N) = 0$ follow.

(5.11)–(5.12) · p. 19

4. Apply klt semiampness after obtaining nonvanishing

Finally apply GM, Corollary 5.3, p. 19 to $N/\alpha \sim_{\mathbb{Q}} K' + (1 - s/\alpha)D'$. Semiampness and Iitaka dimension zero give numerical triviality, contradicting its positive intersection with an ample cube because K' is pseudo-effective, $D' \neq 0$, and $1 - s/\alpha > 0$. The nonvanishing assumption of GM is obtained before its use.

§5, final klt adjoint · pp. 19–20 · GM, Corollary 5.3 · p. 19

This contradiction handles the signed-support branch. Return to the exclusion of slices and correspondences, then to two-factor jets and the Frobenius comparison that closes smooth nonvanishing.

4. Proof of Corollary 1.2

Transfer smooth canonical nonvanishing to the lc pair, then rationalize the effective real representative. The aim is a section on the original variety with the prescribed index r retained.

01 Establish smooth fourfold nonvanishing

X smooth projective, $\dim X = 4$, K_X pseudo-effective $\Rightarrow \kappa(X) \geq 0$.

Smooth nonvanishing implies lc real-boundary nonvanishing through dimension four.

[\[Hash\] Thm. 1.4, pp. 1–2](#) [§8, p. 35](#)

02 Obtain an effective real representative on the original lc pair

$\dim X \leq 4$, $D = K_X + \Delta$ nef, $D \sim_{\mathbb{R}} G \geq 0$.

Use a finite rational linear system, keeping zero coefficients and preserving positivity.

[Lemma 8.1, Cor. 1.2 proof, p. 35](#)

03 Rationalize with the same support and clear the prescribed index

$D \sim_{\mathbb{Q}} G_{\mathbb{Q}} \geq 0$, rD Cartier.

$\exists m > 0 : H^0(X, \mathcal{O}_X(mrD)) \neq 0$.

Smooth nonvanishing, rational feasibility, and a prescribed-index section

[Open figure ↗](#) [TeX source ↓](#)

1. Pass from smooth to lc nonvanishing

Hash, Theorem 1.4, p. 2 takes smooth fourfold canonical nonvanishing to nonvanishing for projective lc pairs of dimension at most four. Since nef D is pseudo-effective, it gives $D \sim_{\mathbb{R}} G \geq 0$ on the original normal variety.

[§8 · pp. 34–35](#) · [Hash, Theorem 1.4 · pp. 1–2](#)

2. Rationalize with fixed support and clear the prescribed index

Lemma 8.1 writes this as a rational linear system involving finitely many prime and principal divisors. Fixing the zero coefficients and approximating the positive ones by a rational solution gives $D \sim_{\mathbb{Q}} G_{\mathbb{Q}} \geq 0$ with the same support. Clearing denominators together with the prescribed r produces a section of mrD on X itself, rather than only on a $\mathbb{Q}$ -factorial model.

[Lemma 8.1; Corollary 1.2 proof · pp. 34–35](#)

5. Which papers supply which steps

Two inputs to Supported lifting. Corollary 1.2 → Corollary 1.4 supplies the first section in dimension four. Lemma 8.1 → Theorem 1.3 supplies only rationalization of an effective real presentation; it does not replace the premise of smooth nonvanishing in all dimensions.

Input

MM, Theorems 1.1–1.2, pp. 1–2

Use and supplied content

Theorems 4.1–4.2, pp. 13–14, and Proposition 5.1, pp. 18–20: endpoint metrics, ordinary injectivity and the final klt metric.

Checking scope

Statements, endpoint formulas and application compared; MM proof passages were also read for the preceding draft, without certification of the full analytic proofs.

Input

LA, Lemmas 7.1–7.5, pp. 45–52

Use and supplied content

§3 center/degree estimates and §6 slices and fixed branch complement.

Checking scope

Input statements and principal uses compared; complete proofs excluded.

Input

LA, Theorem 9.1, p. 62

Use and supplied content

Theorem 7.1 and §§7.1–7.3, pp. 31–34: finite-data Frobenius incompatibility.

Checking scope

Hypotheses compared; the present flag, curve, jets and determinant comparison read. Full LA proof unverified.

Input

CT, Theorem 1.1, p. 2

Use and supplied content

Proposition 2.1 and §5: termination of the MMPs.

Checking scope

v2 statement and ordinary-pair specialization with zero nef part checked.

Input

Fuj, Corollary 4.10

Use and supplied content

§5, p. 19: semi-dlt threefold abundance on the entire reduced floor.

Checking scope

Target adjunction and descent read. The specified external corollary could not be retrieved and remains pending.

Input

GM, Corollary 5.3, p. 19

Use and supplied content

End of Proposition 5.1, pp. 19–20: klt semiampness after nonvanishing.

Checking scope

v2 statement and application compared.

Input

Hash, Theorem 1.4, pp. 1–2

Use and supplied content

§8, p. 35: smooth nonvanishing implies lc real-boundary nonvanishing.

Checking scope

v4 statement, Conjectures 1.1–1.3 and application compared.

The abundance main theorem of LA and logarithmic litaka subadditivity are not inputs here. The introduction, p. 3, combines Corollary 1.2 with the results after nonvanishing in Supported lifting as a **consequence**; this must not become a reverse dependency in the proof of smooth nonvanishing.

6. Return to the sources

§§2–3 · pp. 5–12 · §§4–5 · pp. 13–20 · §§6–7 · pp. 21–34 · §8 · pp. 34–35

Passages read: §2, pp. 5–7, fixed counterexample; §§3.4–3.5, pp. 10–12, degree obstruction and scalar orders; §§4–5, pp. 13–20, analytic inputs and signed support; §§6.3–6.5, pp. 23–30, correspondences, case analysis and finite jets; §7, pp. 30–34, finite data and determinant; §8, pp. 34–35, rationalization. All numerical conditions of Theorem 7.1 were also compared with the image of PDF p. 31.

Outstanding items include the full local ideal calculations for moving centers, external proofs about covers and birational groups, original statements of HMX effective birationality, generic semipositivity and the restriction theorem for the flag, Fujino's specified threefold corollary, and complete Frobenius/slope arguments. Their validity is not certified by this draft. The revision retains these limits while aligning results, proof introductions, numbered explanations and citations.

The manuscript is the September 27, 2026 version. Page numbers are PDF pages; the snapshot is `adc7f1241b42e322a6451854ab7e4b4c146bf78a`.

034 / 07

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CATALOGUE 034 · PAPER 07 / 14 Overview

Lifting sections from the reduced support of an adjoint

From generation on the whole reduced support to finite-order lifting and section growth

The paper seeks ambient sections from generation on the reduced support of an effective adjoint multiple. It retains gluing across the entire support and uses positive frame weight to eliminate obstructions on finite neighborhoods. Supported lifting holds in every dimension; fourfold nonvanishing is neither a hypothesis nor an input to its proof.

Manuscript author: OpenAI · 2026-09-27 · 40 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Supported boundary lifting

Let (V, C) be a normal projective $\mathbb{Q}$ -factorial dlt pair over an algebraically closed field of characteristic zero, with effective rational boundary C , and put $A = K_V + C$. For a sufficiently divisible positive integer q , suppose a nonzero effective Cartier divisor G satisfies

$$0 \neq G \geq 0, \quad G \sim qA, \quad \text{Supp } G \subseteq \text{Supp } [C].$$

If $\mathcal{O}_V(G)|_{G_{\text{red}}}$ is semiample on the **whole reduced scheme**, then $\kappa(V, A) > 0$.

[Theorem 1.1 · p. 2](#)

No additional nefness is assumed. The support inclusion can be strict: writing $C = S + H + C_0$, with $S = G_{\text{red}}$, the extra coefficient-one divisor H may meet S and its strata. Semiampleness separately on normalized components cannot replace the whole-support hypothesis.

Theorem 1.2 — Abundance after nonvanishing

For a projective lc pair (X, Δ) over $\mathbb{C}$ of dimension at most four with effective rational boundary, suppose $D = K_X + \Delta$ is $\mathbb{Q}$ -Cartier, nef, and $\kappa(X, D) \geq 0$. Then D is semiample; if $\kappa(X, D) = 0$, then $D \sim_{\mathbb{Q}} 0$. No uniform index is asserted. The extension to arbitrary algebraically closed characteristic-zero fields is separately stated as Corollary 7.2 (p. 29).

Theorem 1.2 · p. 2; Corollary 7.2 · p. 29

Theorem 1.3 — Reduction to smooth canonical nonvanishing

Assume, **in every dimension**, that every smooth connected projective complex variety W with pseudo-effective K_W has $H^0(W, mK_W) \neq 0$ for some $m > 0$. Then every normal projective lc pair (X, Δ) over any algebraically closed characteristic-zero field, with effective rational boundary and nef $\mathbb{Q}$ -Cartier adjoint $D = K_X + \Delta$, has semiample D . The premise is not restricted to nef canonical divisors.

Theorem 1.3 · p. 2

Corollary 1.4 — Fourfold log abundance

For a normal projective lc pair (X, Δ) over an algebraically closed characteristic-zero field, of dimension at most four with effective rational boundary and nef $\mathbb{Q}$ -Cartier adjoint $D = K_X + \Delta$, the divisor D is semiample. If $\kappa(X, D) = 0$, then $D \sim_{\mathbb{Q}} 0$. This is where NV Corollary 1.2 adds fourfold lc nonvanishing.

Corollary 1.4 · p. 2; proof · p. 29

References on the arrows

Unprefixed references denote this manuscript. External inputs use the following keys.

- **KS**: Popa, Kodaira–Saito vanishing and applications (author PDF).
- **Saito**: Mixed Hodge Modules (1990).
- **FG**: Fujino–Gongyo, Log pluricanonical representations and abundance conjecture (final author PDF).
- **Bir**: Birkar, On existence of log minimal models (arXiv:0706.1792).
- **Hash**: Hashizume, On the non-vanishing conjecture and existence of log minimal models (arXiv:1609.00121v4).

- [NV](#): Fourfold nonvanishing by minimal metrics and moving jets (September 27, 2026).

2. Proof of Theorem 1.1

Proof route. Lifting in [Theorem 1.1](#) enters abundance after nonvanishing in [Theorem 1.2](#). The [last diagram](#) separates the conditional all-dimensional Theorem 1.3 from fourfold Corollary 1.4 and their distinct inputs.

Start from generation on the entire reduced support $S = G_{\text{red}}$ and show that $h^0(V, \mathcal{O}_V(NG))$ is unbounded. To eliminate finite-neighborhood obstructions, treat the horizontal symbol and the vertical Euler action separately.

01 Move whole-support generation to the frame bundle

$$0 \neq G \sim q(K_V + C), \quad S = G_{\text{red}}, \quad L|_S = f^*\mathcal{O}(1).$$

Retain every cyclic-cover component and poles over the extra boundary.

Lemmas 2.3, 3.1, pp. 9–13

02 Insert the full obstruction group by residues

$$R^1 g_* \mathcal{O}_E \hookrightarrow F_0 \mathcal{N} = R^1 h_* \omega_R(D_h), \quad \text{wt}(\Omega) = a > 0.$$

Positive weight gives an ample source, forcing the horizontal part to vanish.

Lemma 3.2; Prop. 4.1, pp. 13–17 [KS] Thm. 8.2, pp. 14–15

03 Separate horizontal symbol and actual Euler action

$$\text{symbol}(v_k) = 0, \quad \text{wt}(v_k) = a + k > 0, \quad s\varepsilon = -(a + k)s.$$

Use Euler action in the graph identity, then return to arbitrary F .

(5.5), (5.11)–(5.12), pp. 18–20

04 Kill the obstruction for every boundary function

$$\delta_k(F) = 0, \quad g_*(\mathcal{O}_Z/I^{k+1}) \twoheadrightarrow g_*(\mathcal{O}_Z/I^k).$$

Use finite invariants, frame degree zero, and periodic positive twists.

Lemma 5.2; Prop. 1.5, pp. 21–23

05 Count finite layers to obtain ambient section growth

$$h^0(V, \mathcal{O}_V(NG)) \rightarrow \infty \implies \kappa(V, K_V + C) > 0.$$

Positive frame weight and residues kill finite-neighborhood obstructions

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1. Retain the obstruction group through cyclic covers and residues

Replace G by a multiple so that $L = \mathcal{O}_V(G)$ satisfies $L|_S = f^*\mathcal{O}_{\mathbb{P}^t}(1)$. Pass to the nonzero-frame bundle $P = L^\times$ and construct two cyclic covers, retaining every component when the algebras split. On a resolution there is a logarithmic form σ of positive frame weight a , with residue Ω .

The polar divisor D_h retains poles above intersections with H . Lemma 2.3 inserts the obstruction group into

$$\iota : R^1 g_* \mathcal{O}_E \hookrightarrow R^1 h_* \omega_R(D_h) = F_0 \mathcal{N} \hookrightarrow \mathcal{N}.$$

Here $g : E \rightarrow B$, $h : R \rightarrow B$, and $B = \mathcal{O}_{\mathbb{P}^1}(1)^\times$. The lowest piece $F_0 \mathcal{N}$ of a mixed Hodge module on the boundary graph retains cohomology concentrated on smaller supports.

§2, Lemmas 2.3, 3.1, pp. 5–13

2. Kill the horizontal part using the identity for every boundary function

With $I = (y)$, divide the obstruction from order k to $k+1$ by y^k . This gives a derivation $\delta_k : g_* \mathcal{O}_E \rightarrow R^1 g_* \mathcal{O}_E$: the product of two lifting errors vanishes modulo y^{k+1} . The graph-residue computation gives, for **every** boundary function F ,

$$\iota(\delta_k(F)) + \sum_i \iota(F \delta_k(t_i)) \partial_{t_i} = 0,$$

using the right $\mathcal{D}$ -module action, as in (5.5). Set $F = 1$ and take the first symbol. The resulting zero-symbol tensor has weight $a + k > 0$. A transverse finite cover and the adjugate of its differential turn a nonzero horizontal part into a nonzero map from an ample line into the lowest symbol kernel. Lemma 3.2, derived from Kodaira–Saito vanishing, excludes it.

Proposition 4.1, Lemma 4.2, Proposition 5.1, pp. 13–20

3. Kill the vertical part using the actual Euler action

For the vertical part, vanishing of the symbol alone is insufficient. The **actual** Euler action $s\varepsilon = -(a + k)s$ and the residue identity kill that part; returning to arbitrary F then gives $\delta_k(F) = 0$. This separation is the core of the lifting argument.

§4.3, (5.11)–(5.12), pp. 17, 20

4. Turn finite-layer lifting into ambient section growth

Finite-group invariants and frame degree zero descend the construction to divisorial layers on V . The periodicity $Q_{j-r} = Q_j \otimes \mathcal{O}(1)$ accumulates positive twists. Higher cohomology remains bounded while sections grow. Passing from finite neighborhoods to the ambient variety loses at most the fixed number $h^1(V, \mathcal{O}_V)$:

$$h^0(V, \mathcal{O}_V(NG)) \geq h^0(V, \mathcal{O}_V(NG)/\mathcal{O}_V) - h^1(V, \mathcal{O}_V) \longrightarrow \infty.$$

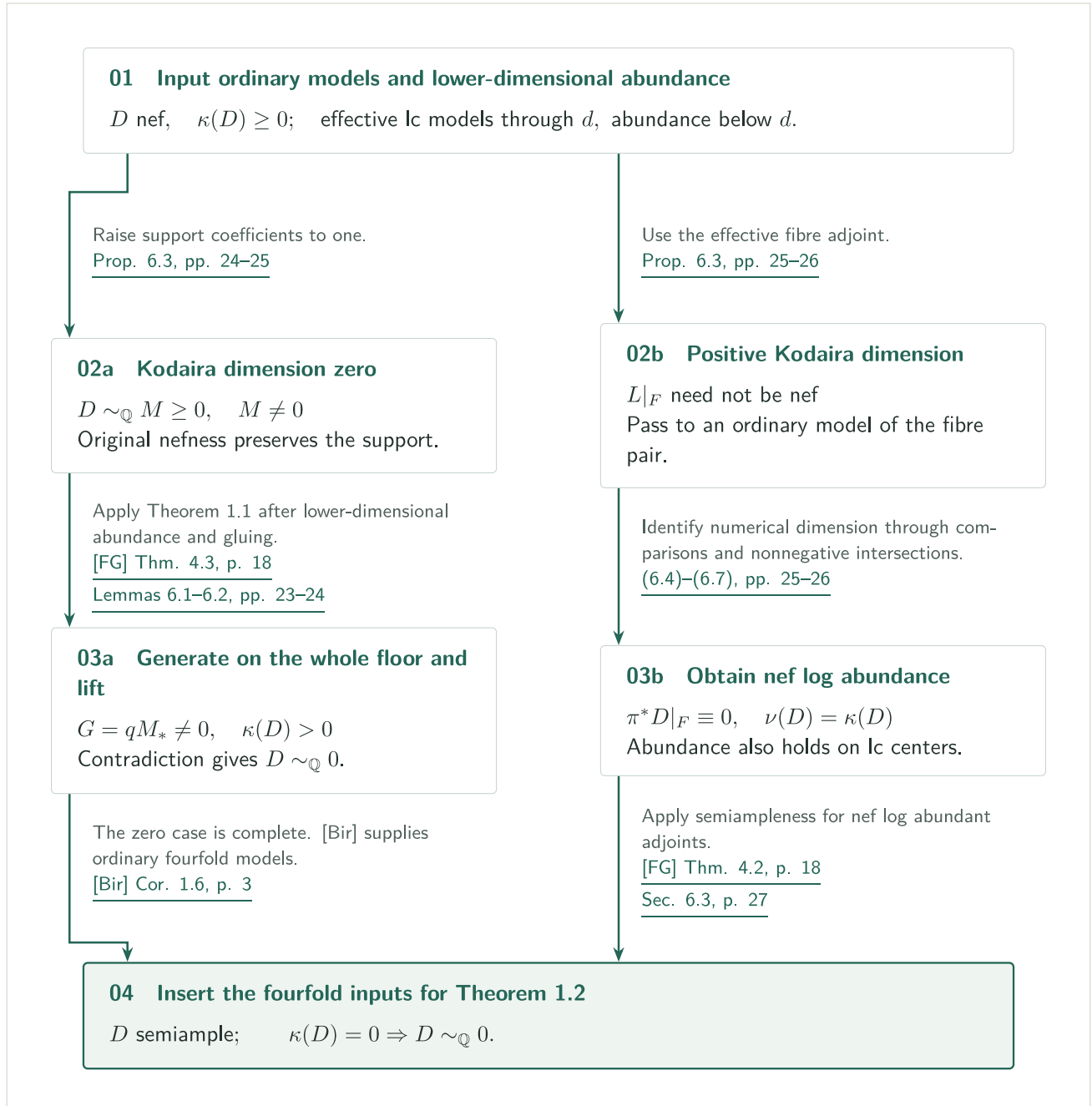
When $\ell = 0$, the number of layers still grows. No convergent formal neighborhood or extension of the boundary morphism to a whole neighborhood is assumed.

Lemma 5.2 and proof of Proposition 1.5, pp. 21–23

3. Proof of Theorem 1.2

Proposition 6.3 assumes **ordinary** minimal models for effective lc pairs through dimension d , and full lc abundance below d . It deduces abundance after nonvanishing in dimension d ; good minimal models are not assumed at the outset.

Proposition 6.3 · pp. 23–24



Ordinary minimal models and whole-floor gluing give abundance

Open figure ↗ TeX source ↓

1. Exclude a nonzero effective representative in Kodaira dimension zero

For $\kappa(D) = 0$, suppose $0 \neq M \geq 0$ and $M \sim_{\mathbb{Q}} D$. Raise support coefficients to one on a resolution, then pass to an ordinary minimal model (V_*, C_*) . Kodaira dimension zero and an effective representative M_* are retained. Nefness of the original D and negativity ensure M_* does not disappear. Lower-dimensional abundance and FG normalization gluing give semiampleness on the **whole floor**. Diagram 1 applied to $G = qM_*$ contradicts Kodaira dimension zero, so $D \sim_{\mathbb{Q}} 0$.

Lemmas 6.1–6.2 and Proposition 6.3, pp. 23–25

2. Use fibre minimal models in positive Kodaira dimension

For $\kappa(D) = k > 0$, on a resolution write $L = K_W + \Gamma = \pi^*D + N$, with effective exceptional N . The fibre adjoint $L|_F$ need not be nef. Take an ordinary minimal model of that effective fibre pair and apply lower-dimensional abundance there. Comparison (6.6) and nonnegative intersections yield $\pi^*D|_F \equiv 0$, and then $\nu(D) = \kappa(D)$. Whole-floor semiampleness also gives abundance on normalized lc centers.

FG Theorem 4.2 now applies.

Proposition 6.3, pp. 25–27

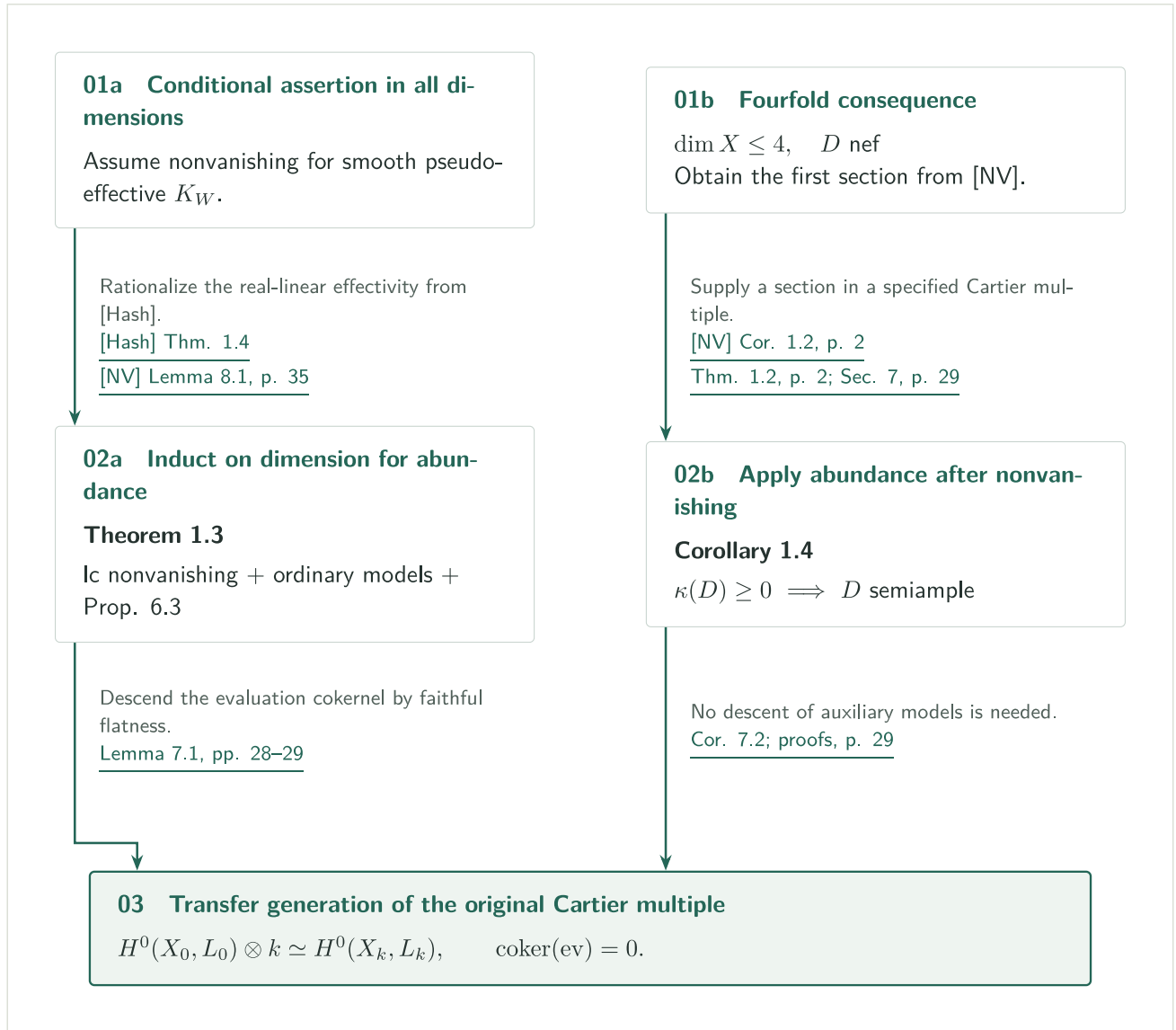
3. Insert the fourfold inputs and transfer the ground field

For $d = 4$, insert known lower-dimensional abundance and Birkar's ordinary minimal-model theorem for effective lc fourfolds to obtain Theorem 1.2. Transfer to other characteristic-zero fields descends and extends a specified Cartier multiple of the original divisor and its evaluation map; it does not assert preservation of $\mathbb{Q}$ -factoriality under field descent.

§§6.3, 7, pp. 27–29

4. Proofs of Theorem 1.3 and Corollary 1.4

The conditional all-dimensional assertion of Theorem 1.3 and the fourfold consequence of Corollary 1.4 receive nonvanishing from different sources. Both routes finally transfer generation of a Cartier multiple of the original adjoint between ground fields.



Distinct inputs for the conditional all-dimensional theorem and the fourfold corollary

[Open figure ↗](#) [TeX source ↓](#)

1. Supply ordinary models from the all-dimensional premise

For Theorem 1.3, assume smooth canonical nonvanishing **in all dimensions**.

Hash Theorem 1.4 supplies lc nonvanishing and ordinary minimal models. The passage from real-linear to rational-linear effectivity uses only the finite rational linear algebra in NV Lemma 8.1, separately from that paper's fourfold nonvanishing theorem. Proposition 6.3 then closes dimension induction.

§6.3, p. 27

2. Obtain the first fourfold section from a separate input

For Corollary 1.4, NV Corollary 1.2 supplies a nonzero section in a multiple of a prescribed Cartier multiple on the original normal lc variety; Theorem 1.2 and field transfer finish the argument. That nonvanishing input does not enter the proofs of Theorems 1.1–1.2. Appendix A gives a separate conormal–period method, not an input to diagram 1.

§7, p. 29; Appendix A, pp. 30–38

5. Which papers supply which steps

Use in effective Iitaka. Theorem 1.2 makes a nef adjoint semiample after nonvanishing in the qualitative model step of Fourfold Iitaka. Uniformity of the degree is a separate subsequent argument.

Input

KS Kodaira–Saito vanishing

Use

Lemma 3.2, p. 13: negative hypercohomology with an inverse ample twist

Scope checked

Compared Popa author's PDF Theorem 8.2, PDF pp. 14–15. The manuscript cites published Theorem 28, a different numbering scheme

Input

Saito Projective direct-image strictness for mixed Hodge modules

Use

Lemma 3.1, pp. 12–13: the entire coherent lowest piece

Scope checked

Read the application and local calculation; rechecking Saito 1990 Theorem 2.14 / Proposition 2.15 is pending

Input

FG Theorems 4.3 / 4.2, final author PDF p. 18

Use

Whole-floor gluing in Lemma 6.2 and nef log abundance in Proposition 6.3

Scope checked

Compared statements, hypotheses, and uses; proofs excluded

Input

Bir Corollary 1.6, accessed arXiv PDF p. 3

Use

§6.3, p. 27: ordinary minimal models of effective lc fourfolds

Scope checked

Statement checked; distinguished from good models

Input

Hash Theorem 1.4, arXiv v4 §1

Use

Theorem 1.3 only: smooth nonvanishing gives lc nonvanishing and ordinary models

Scope checked

Compared Conjectures 1.1–1.3 and the implication

Input

NV Corollary 1.2, p. 2; Lemma 8.1, p. 35

Use

The former feeds Corollary 1.4; the latter rationalizes Theorem 1.3

Scope checked

Statements and uses on pp. 27, 29 checked; full nonvanishing proof excluded

Theorem 1.2 also occurs in the final assembly of fourfold log Iitaka in [Ufour](#), §10, pp. 48–49; its detailed use belongs to that article. Classical threefold abundance and its correction, and all external splitting/vanishing theory used for residue insertion, are not collectively certified here.

6. Return to the sources

pp. 8–13 treat extra coefficient-one poles, residue injection, and the entire coherent lowest piece.

pp. 14–23 treat transverse covers, Euler action, the graph identity, finite-layer lifting, and growth.

pp. 23–29 treat ordinary model comparison, the whole floor, Iitaka fibres, and field transfer.

These core passages were read and the main statements and listed direct inputs compared.

Pending work includes all details of the cyclic covers and split residue insertion in §2, independent reconstruction of each Hodge-filtration identification from external theory, proofs of classical lower-dimensional abundance, Appendix A, and full external proofs. Diagram arrows present the manuscript's logical connections; they do not certify those connections independently. Japanese and English text, diagrams, references, and limitations are aligned.

Schnell fiber spaces and good canonical models

Good minimal models, mixed intersections, and Schnell's reduction

The results in manuscript order, followed by proof diagrams with cited inputs and an explanation of each argument.

Manuscript author: OpenAI · 2026-09-24 · 11 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

The statements follow the manuscript's numbering, order, hypotheses, and conclusions.

Theorem 1.1 (Schnell's zero-Kodaira fiber-space conclusion)

Suppose X and Y are smooth connected projective complex varieties and $f: X \rightarrow Y$ is surjective with connected fibers. Assume its smooth geometric generic fiber F has $\kappa(F) = 0$. If H is an ample Cartier divisor on Y and m_0 a positive integer such that $m_0 K_X - f^* H$ is pseudo-effective, then

$$\kappa(X) = \dim Y.$$

Corollary 1.2 (Campana–Peternell)

For a smooth connected projective complex variety X , an effective Cartier divisor D on X , and a positive integer m_0 , pseudo-effectivity of $m_0 K_X - D$ implies

$$\kappa(X) \geq \kappa(D).$$

Corollary 1.3 (Schnell’s general fiber-space conclusion)

Suppose $f: X \rightarrow Y$ is surjective with connected fibers, with X and Y smooth connected projective complex varieties. Let F be a very general smooth fiber. For an ample Cartier divisor H on Y and a positive integer m_0 , assume $m_0 K_X - f^* H$ is pseudo-effective. Then

$$\kappa(X) = \kappa(F) + \dim Y.$$

There also exist positive integers r, ℓ_0 such that, for every integer $\ell \geq \ell_0$,

$$H^0(X, \mathcal{O}_X(\ell r K_X - f^* H)) \neq 0.$$

The account below traces the arguments for these claims. No uniform values of r, ℓ_0 are asserted in Corollary 1.3.

References on the arrows

Unprefixed theorem, lemma, section, and equation numbers refer to the OpenAI manuscript above. External inputs use the following keys.

- [LA] Log abundance in characteristic zero · Corollary 11.2 · pp. 73–74 · GitHub · [Read the LA overview](#) →
- [Sch22] Christian Schnell, Singular metrics and a conjecture by Campana and Peternell · arXiv v1
- [PS14] Popa–Schnell, On direct images of pluricanonical bundles · Lemma 4.6

2. Proof of Theorem 1.1

If $\dim Y = 0$, then $F = X$ and the conclusion is the hypothesis. Write $X \xleftarrow{p} W \xrightarrow{q} V$ for the common resolution, and put $d = \dim X$, $y = \dim Y > 0$, and $h = fp$. In the diagram, A is a very ample Cartier divisor on V .

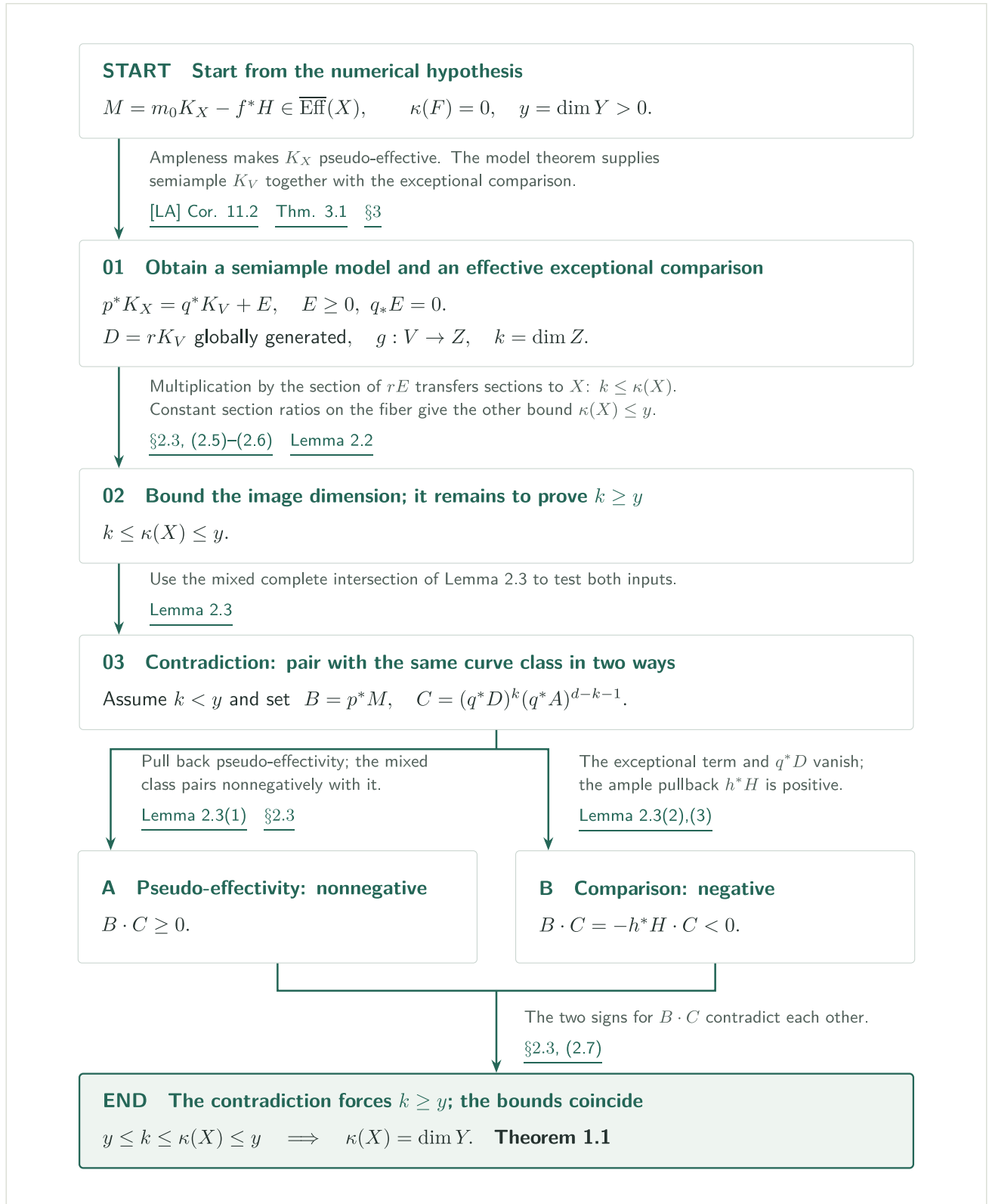


Figure 1. Construct the model, bound the dimensions, and compare two pairings with the same curve class.

Open figure ↗ [TeX source](#) ↓

1. Two outputs of model existence

Ampleness makes f^*H pseudo-effective, so the numerical hypothesis implies pseudo-effectivity of K_X . Theorem 3.1 gives a K_X -negative birational contraction $X \dashrightarrow V$, with V normal, projective, $\mathbb{Q}$ -factorial and klt, and K_V semiample. It also supplies the comparison below on a common resolution. This theorem is [LA, Corollary 11.2]; constructing the model is an external input.

$$X \xleftarrow{p} W \xrightarrow{q} V, \quad p^*K_X = q^*K_V + E, \quad E \geq 0, \quad q_*E = 0.$$

Theorem 3.1 / §3 · pp. 7–8 ↗ · [LA] Corollary 11.2 · pp. 73–74 ↗

2. Effectivity transfers the sections

Choose $D = rK_V$ Cartier and globally generated, and write $g: V \rightarrow Z$ for its morphism and $k = \dim Z$. Since rE is effective Cartier, multiplication by its canonical section gives $H^0(V, D) \hookrightarrow H^0(X, rK_X)$. The section ratios are unchanged, hence $k \leq \kappa(X)$. Independently, Lemma 2.2 uses $\kappa(F) = 0$ to give $\kappa(X) \leq y$. It remains to prove $k \geq y$.

§2.3, (2.5)–(2.6) · pp. 6–7 ↗ · Lemma 2.2 · pp. 4–5 ↗

3. Incompatible pairings with the same curve class

Assume $k < y$ and use the curve class $C = (q^*D)^k(q^*A)^{d-k-1}$ of Lemma 2.3. The pullback $B = m_0p^*K_X - h^*H$ is pseudo-effective, so part (1) gives $B \cdot C \geq 0$. Part (2) uses $q_*E = 0$ to give $E \cdot C = 0$, while part (3) gives $q^*D \cdot C = 0$ and $h^*H \cdot C > 0$. The canonical comparison therefore yields

$$0 \leq B \cdot C = m_0 \left(\frac{1}{r} q^*D + E \right) \cdot C - h^*H \cdot C = -h^*H \cdot C < 0.$$

For strict positivity, $y > k$ supplies a cotangent direction from h outside those from gq ; directions from A complete a transverse intersection. Pulling all factors of C back from V kills the exceptional term. Thus $k \geq y$ and $y \leq k \leq \kappa(X) \leq y$. This is the good-model argument of Theorem 2.1; Theorem 3.1 supplies its model, completing Theorem 1.1.

Lemma 2.3(1)–(3) · pp. 5–6 ↗ · Theorem 2.1, §2.3 / (2.7) · pp. 6–7 ↗

3. Proof of Corollary 1.2

Theorem 3.1 and section transfer also give $K_T \in \overline{\text{Eff}}(T) \Rightarrow \kappa(T) \geq 0$ for smooth projective complex T (§4.1). This nonvanishing settles $\kappa(D) = 0$. Figure 2 treats $\kappa(D) > 0$.

START Seek the lower bound coming from the effective divisor D

$$D \geq 0, \quad m_0 K_X - D \in \overline{\text{Eff}}(X), \quad \kappa(D) > 0.$$

Resolve the linear system, take Stein factorization, and resolve the base.
Preserve pseudo-effectivity and choose an ample Cartier divisor on the base.

[Sch22] §§4–6 §4.2, (4.2)

01 Realize the litaka dimension as a base dimension

$$f_0 : X_0 \rightarrow Y_0, \quad \dim Y_0 = \kappa(D), \quad X_0 \sim_{\text{bir}} X.$$

$$a_0 K_{X_0} - f_0^* H_0 \in \overline{\text{Eff}}(X_0), \quad H_0 \text{ ample Cartier.}$$

Restrict the numerical hypothesis to a very general fiber.
Apply nonvanishing, supplied by model existence and section transfer.

Thm. 3.1 §4.1, (4.1)

02 The fiber has nonnegative Kodaira dimension at every stage

$$K_{F_i} \in \overline{\text{Eff}}(F_i) \implies \kappa(F_i) \geq 0.$$

If $\kappa(F_i) = 0$, skip the iteration and go directly to 05.

Schnell's construction uses a sufficiently positive base twist to obtain
an effective divisor of litaka dimension $\kappa(F_i) + \dim Y_i$.

[Sch22] Lemma 7.1 §4.2

03 Iteration A: add a positive base twist to obtain an effective divisor

Run this step only while $\kappa(F_i) > 0$.

$$L_i \geq 0, \quad L_i \sim r_i K_{X_i} + f_i^*(b_i H_i).$$

$$\kappa(L_i) = \kappa(F_i) + \dim Y_i.$$

The identity preserves pseudo-effectivity; reapply step 01 to L_i .

$$(a_i b_i + r_i) K_{X_i} - L_i \sim b_i (a_i K_{X_i} - f_i^* H_i).$$

§4.2 [Sch22] §§4–6

04 Iteration B: preserve the hypothesis and increase the base dimension

$$f_{i+1} : X_{i+1} \rightarrow Y_{i+1}, \quad a_{i+1} K_{X_{i+1}} - f_{i+1}^* H_{i+1} \in \overline{\text{Eff}}(X_{i+1}).$$

$$\dim Y_{i+1} = \dim Y_i + \kappa(F_i) > \dim Y_i.$$

Replace i by $i + 1$ and repeat 02–04.

The strictly increasing integers $\dim Y_i$ are bounded by $\dim X$.
Thus iteration with positive fiber Kodaira dimension cannot continue forever.

§4.2

05 Reach a fiber of Kodaira dimension zero after finitely many steps

$$\kappa(F_j) = 0, \quad \dim Y_j \geq \dim Y_0 = \kappa(D).$$

Very general and geometric generic fibers have the same Kodaira dimension.
Apply Theorem 1.1 and birational invariance of Kodaira dimension.

§4.1 Theorem 1.1 §4.2

END Apply the main theorem and return to the original X

$$\kappa(X) = \kappa(X_j) = \dim Y_j \geq \kappa(D). \quad \text{Corollary 1.2}$$

Figure 2. The numerical hypothesis is preserved while the base dimension increases; iterate only while the fiber has positive Kodaira dimension, then apply Theorem 1.1.

Open figure ↗ TeX source ↓

1. Realize the Iitaka dimension as a base dimension

Resolve $|nD|$, take Stein factorization, and resolve the base as in [Sch22, §§4–6]. This gives $f_0: X_0 \rightarrow Y_0$ with $\dim Y_0 = \kappa(D)$ and pseudo-effective $a_0 K_{X_0} - f_0^* H_0$ for an ample Cartier divisor H_0 . On resolving the base, a sufficiently divisible big and nef pullback minus an effective divisor supplies the required ample divisor. Restriction to very general fibers, followed by nonvanishing, gives $\kappa(F_i) \geq 0$ at every stage.

[Sch22] §§4–6 · p. 2 ↗ · §4.1–4.2, (4.2) · pp. 8–9 ↗

2. Absorb positive fiber Kodaira dimension into the base

While $\kappa(F_i) > 0$, the construction in [Sch22, Lemma 7.1] gives effective $L_i \sim r_i K_{X_i} + f_i^*(b_i H_i)$ with $\kappa(L_i) = \kappa(F_i) + \dim Y_i$. The numerical hypothesis survives because

$$(a_i b_i + r_i) K_{X_i} - L_i \sim b_i (a_i K_{X_i} - f_i^* H_i).$$

Apply the divisor-to-fiber-space reduction to L_i again: $\dim Y_{i+1} = \dim Y_i + \kappa(F_i)$. These dimensions strictly increase and are bounded by $\dim X$, so some finite stage has $\kappa(F_j) = 0$. Section 4.1 identifies the Kodaira dimensions of very general and geometric generic fibers, allowing Theorem 1.1 to apply. Birational invariance gives $\kappa(X) = \dim Y_j \geq \dim Y_0 = \kappa(D)$.

[Sch22] Lemma 7.1 / §9 · p. 3 ↗ · §4.2 · p. 9 ↗

4. Proof of Corollary 1.3

First use Corollary 1.2 to obtain equality of Kodaira dimensions. Feed that equality into the Fujita–Mori criterion, then construct a section after subtracting the original ample divisor H .

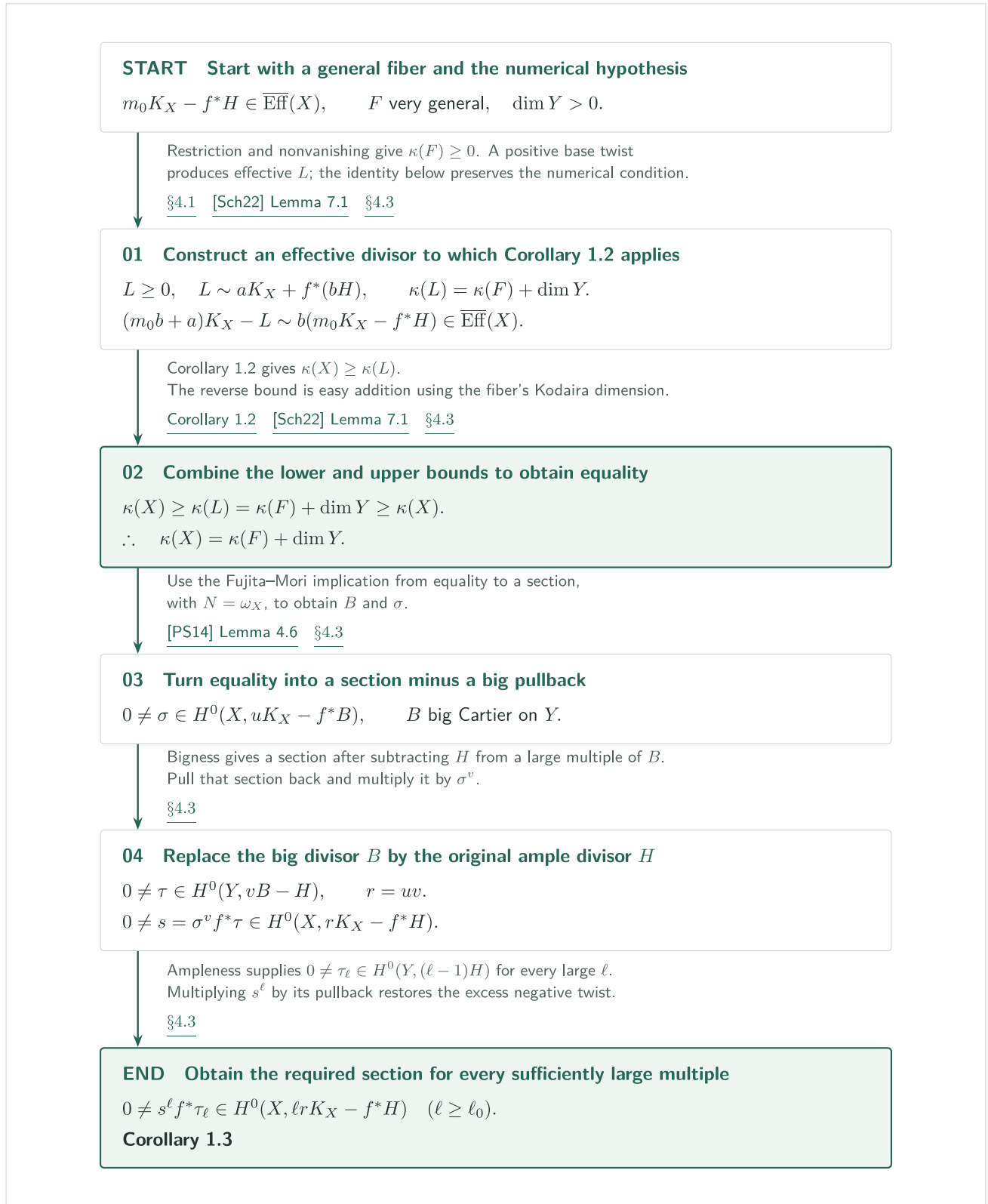


Figure 3. Campana–Peternell, equality of Kodaira dimensions, a big negative twist, and the original ample negative twist.

Open figure ↗ TeX source ↓

1. Apply Corollary 1.2 to the effective twist

After §4.1 gives $\kappa(F) \geq 0$, use the construction in [Sch22, Lemma 7.1] to choose effective $L \sim aK_X + f^*(bH)$ with $\kappa(L) = \kappa(F) + \dim Y$. Since $(m_0b + a)K_X - L \sim b(m_0K_X - f^*H)$ is pseudo-effective, Corollary 1.2 gives $\kappa(X) \geq \kappa(L)$. Easy addition gives the reverse bound, hence $\kappa(X) = \kappa(F) + \dim Y$.

[Corollary 1.2 / §4.3 · pp. 2, 10 ↗](#) · [\[Sch22\] Lemma 7.1 · p. 3 ↗](#)

2. Convert equality into a section and adjust the twist

Apply the Fujita–Mori criterion in [PS14, Lemma 4.6] with $N = \omega_X$. It supplies a big Cartier divisor B on Y and $0 \neq \sigma \in H^0(X, uK_X - f^*B)$. Choose $0 \neq \tau \in H^0(Y, vB - H)$ using bigness; then $r = uv$ and $s = \sigma^v f^* \tau$ give a nonzero section of $rK_X - f^*H$. Ampleness of H supplies $0 \neq \tau_\ell \in H^0(Y, (\ell - 1)H)$ for every sufficiently large ℓ , and $s^\ell f^* \tau_\ell$ is the required section.

When Y is a point, the equality is tautological and the section assertion follows from nonvanishing in §4.1 and powers of a nonzero section.

[\[PS14\] Lemma 4.6 · p. 14 ↗](#) · [§4.3 · p. 10 ↗](#)

5. Which papers supply which steps

This article records the LA input to Schnell. The catalogue summary records the connections among all 14 papers. The external rows list the principal reduction tools; routine background references are omitted.

Within catalogue 034 · Direct proof input

Source result

Log abundance in characteristic zero

Corollary 11.2 · pp. 73–74

Used in

Schnell: Theorem 3.1 $\rightarrow$ Theorem 2.1; §4.1 · pp. 7–8

Input and role

For smooth projective complex varieties with pseudo-effective K , supplies a semiample good minimal model and the effective exceptional comparison. Used for both the main theorem and nonvanishing.

Editorial check

Statement, comparison, and application hypotheses matched; the full model-existence proof is not verified.

Outside catalogue 034 · Reduction and effectivity

Source result

Christian Schnell, Singular metrics and a conjecture by Campana and Peternell

arXiv:2202.01295v1 · §§4–10, Lemma 7.1 · pp. 2–4

Used in

Schnell: §4.2–4.3 · pp. 9–10

Input and role

With nonvanishing as input, passes from an effective divisor to fiber spaces and uses a positive base twist to reduce to $\kappa(F) = 0$.

Editorial check

Reduction, construction in Lemma 7.1, and use matched; deeper cited proofs are not verified.

Source result

Mihnea Popa–Christian Schnell, On direct images of pluricanonical bundles

arXiv:1405.6125v2 · Lemma 4.6 · p. 14

Used in

Schnell: §4.3 · p. 10

Input and role

The Fujita–Mori criterion turns equality of Kodaira dimensions into a pluricanonical section minus a big pullback.

Editorial check

Lemma statement and application with $N = \omega_X$ matched; the original Fujita–Mori proofs are not verified.

An earlier conclusion, a different proof route

Zou’s Theorems 1.3 and 5.1 state the same zero-Kodaira conclusion. That proof uses the canonical bundle formula. The feature emphasized here is the mixed-intersection route through a good minimal model of the total space. Zou is a comparison, not a proof input in this argument; only the statements and the beginning of the proof were compared. Zou, arXiv:2409.19981v4 · pp. 4, 24–25 ↗

6. Return to the sources

- Lemma 2.3 · pp. 5–6 — The construction and strict positivity of the curve class.
- Theorem 3.1 · pp. 7–8 — Where the external model-existence theorem enters.
- §4.2–4.3 · pp. 9–10 — Iteration to zero-Kodaira fibers and the final section construction.

Scope of this revised draft · 8 October 2026

Read the main manuscript’s statements and the arguments in §§2–4. Compared the statement and proof paragraph of Log abundance, Corollary 11.2 (pp. 73–74), Schnell’s reduction (§§4–10 in v1), and Popa–Schnell, Lemma 4.6. The dependency table records the specific interfaces checked.

The full 85-page Log abundance proof and the proofs behind its inputs have not been independently verified in preparing this article. This is an AI-generated account of the argument, with no expert review or formal verification completed.

Pinned manuscript snapshot: `adc7f1241b42e322a6451854ab7e4b4c146bf78a`

Page numbers refer to the linked PDFs; external arXiv versions are fixed in the dependency table.

← [Catalogue 034: papers and dependencies](#)

Uniform log Iitaka fibrations and bounded moduli denominators

Residue induction, inertia characters, and recovery of the entire Iitaka field

The manuscript seeks one degree, depending only on dimension and a finite rational coefficient set, that determines the Iitaka fibration of an lc pair. It bounds curve-weight denominators through horizontal residues and klt block characters, then uses an exact canonical bundle formula to recover the full field of section ratios.

Manuscript author: OpenAI · 2026-10-04 · 32 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Generating the entire Iitaka field in one degree

Fix $d \geq 5$ and a finite set $\Phi \subset [0, 1] \cap \mathbb{Q}$. Let X be a normal integral projective d -fold over an algebraically closed field k of characteristic zero, and let B be an effective rational boundary. Assume that (X, B) is lc, the nonzero coefficients of B belong to Φ , $D = K_X + B$ is $\mathbb{Q}$ -Cartier, and $\kappa(X, D) \geq 0$. The authors assert that there is a positive integer m , depending only on d, Φ , such that

$$V_m(D) := H^0(X, \mathcal{O}_X(\lfloor mD \rfloor)) \neq 0, \quad F_m(D) = K(D) \subset k(X)$$

(Theorem 1.1, p. 2). Here $F_m(D)$ is generated by all ratios of degree- m sections, and $K(D)$ by section ratios in all positive degrees. The conclusion identifies subfields of $k(X)$, rather than merely their transcendence degrees. It is an existence statement for m , without a numerical bound or a uniform Cartier index for $K_X + B$ itself. The coefficient set in the main theorem is finite.

[Theorem 1.1 · p. 2](#)

Theorem 4.2 — Valutive weights over a curve

Fix integers $s \geq 0$ and $m \geq 1$. Let C be a smooth complex projective curve, $K = \mathbb{C}(C)$, and z a rational uniformizer at $c \in C$. Suppose that V/K is normal, geometrically integral, projective of dimension s , that (V, B) is geometrically lc with an effective rational boundary, and that a rational m -canonical form ϕ satisfies $\operatorname{div}_V \phi + mB = 0$. Taking all divisorial valuations E over c , set

$$w_z(\phi) = \inf_E \frac{\operatorname{ord}_E(\phi \wedge (dz/z)^{\otimes m}) + m}{m \operatorname{ord}_E z}.$$

Theorem 4.2, pp. 7–8 asserts the existence of a positive integer $q(s, m) > 0$ with $q(s, m)w_z(\phi) \in \mathbb{Z}$. This integer is independent of the curve, pair and form; no separate coefficient set is fixed. The assumptions already make mB integral.

Theorem 4.2 · pp. 7–8

Theorem 7.2 — Moduli denominators with an exact presentation fixed

Fix $d \geq 1$ and a finite rational coefficient set Φ . Let $f : X \rightarrow Z$ be a contraction of normal complex projective varieties, with $\dim X \leq d$ and $\dim Z > 0$. Assume that (X, B) is lc, $B \geq 0$ has nonzero coefficients in Φ , and the $\mathbb{Q}$ -Cartier divisor $K_X + B$ is $\mathbb{Q}$ -linearly equivalent to f^*L for a $\mathbb{Q}$ -Cartier divisor L . Proposition 7.1, pp. 23–24 supplies a uniform $p_0(d, \Phi)$ and a rational function ψ such that

$$K_X + B + \frac{1}{p_0} \operatorname{div} \psi = f^* D_Z, \quad D_Z \sim_{\mathbb{Q}} L, \quad D_Z = K_Z + B_Z + M_Z$$

For the moduli b-divisor defined by an exact presentation of this degree, Theorem 7.2, p. 25 asserts that a uniform $p(d, \Phi)$, divisible by p_0 , makes pM_W nef Cartier on every smooth projective determination of the moduli part. This does not bound denominators of representatives modified by arbitrary rational principal divisors, nor assert b-semiampleness.

Proposition 7.1; Theorem 7.2 · pp. 23–25

References on the arrows

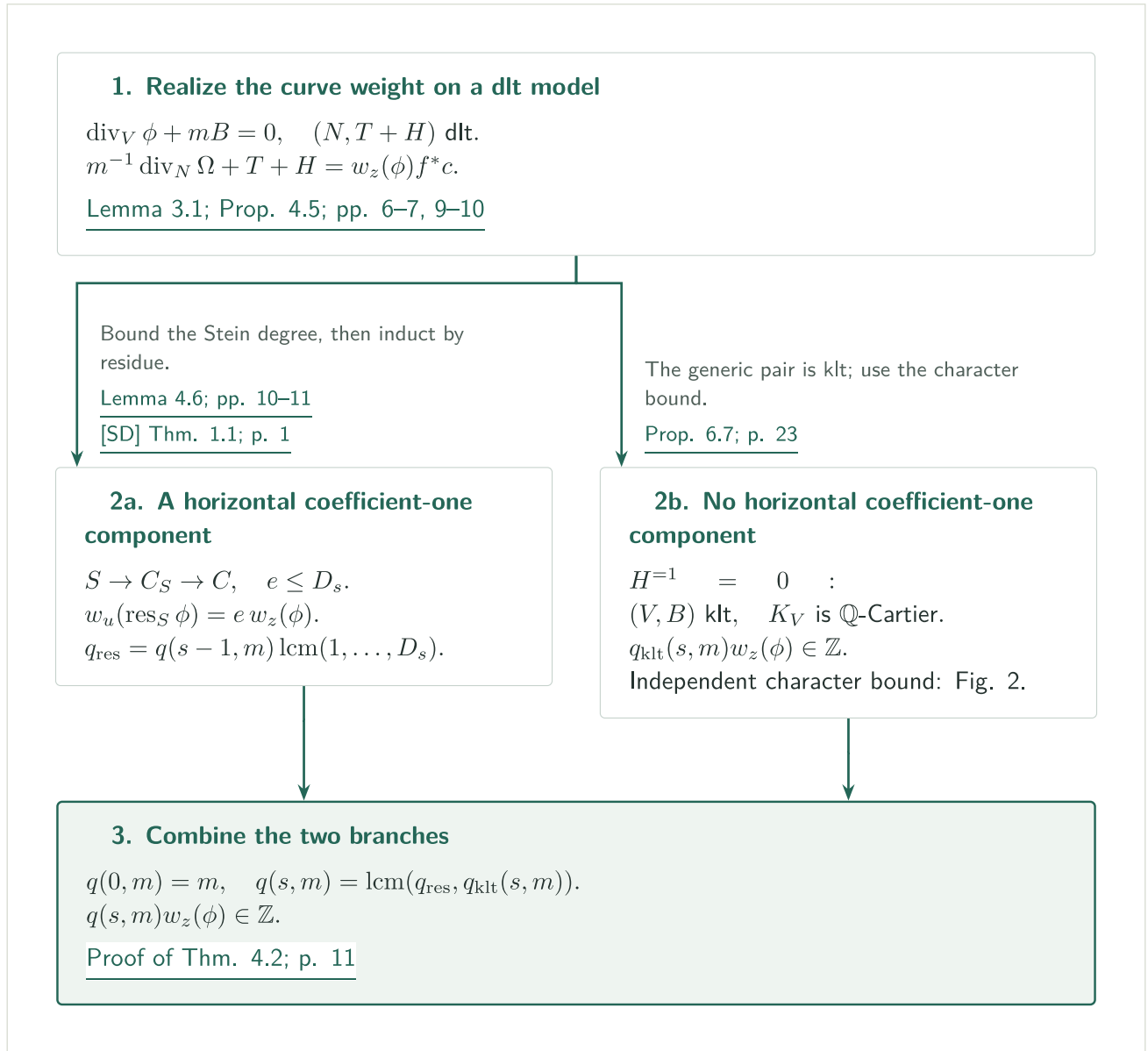
Unprefixed numbers refer to this manuscript [P](#). Direct catalog inputs are [U](#) for normal lc indices, [LA](#) for projective good models, and [SD](#) for boundary-component Stein degrees. These are distinct from the external decomposition and effectivity inputs.

- [U](#), Theorem 1.2 · p. 2
- [LA](#), Theorem 11.1 · p. 73
- [SD](#), Theorem 1.1 · p. 1
- [MW](#), Theorem 1.3 / [TX](#), Theorem A
- [FG](#), Theorem 3.6 / [HMX](#), Theorem 1.1 / [BZ](#), Theorem 1.3

2. Proof of Theorem 4.2: induction along horizontal boundary

Proof route. The [Theorem 1.1 diagram](#) closes the argument. Normal lc indices on the generic fibre fix an exact presentation. Curve weights use Stein degrees and [klt characters](#); [moduli denominators](#) and [effective systems on the base](#) then recover the full Iitaka field.

Induct on relative dimension s , starting with $q(0, m) = m$. For $s > 0$, realize the weight as a divisor identity on a dlt model. A horizontal coefficient-one component permits residue induction; its absence leads to the independent klt bound.



TeX proof diagram reducing lc curve weights to the klt case

[Open figure ↗](#) [TeX source ↓](#)

1. Realize the weight on a relative dlt model

Proposition 4.5, pp. 9–10 first produces a relative dlt model $(N, T + H)$, where T is the reduced special fibre and H the horizontal boundary, on which the weight appears in an actual divisor identity. Removing negative crepant coefficients introduces an effective exceptional error. After a relative MMP this error is vertical; comparison with a relative principal divisor makes it a scalar multiple of the fibre, and the threshold defining the weight forces the scalar to vanish. The relative MMP is justified through Lemma 3.1, pp. 6–7: add a sufficiently positive divisor from the base, use projective log abundance, and use the length bound for lc extremal rays to force every contraction to be vertical.

Lemma 3.1; Proposition 4.5 · pp. 6–7, 9–10

2. Bound the Stein degree before taking a horizontal residue

If H has a coefficient-one component S , dlt adjunction and residue reduce the dimension.

Applying SD on the generic fibre bounds the finite degree in the Stein factorization $S \rightarrow C_S \rightarrow C$ by D_s . With ramification index $e \leq D_s$, the residue weight is $e w_z(\phi)$, so $q(s-1, m) \operatorname{lcm}(1, \dots, D_s)$ clears the original denominator (Lemma 4.6, pp. 10–11).

[Lemma 4.6 · pp. 10–11](#) · [SD, Theorem 1.1 · p. 1](#)

3. Use the klt bound in the remaining case and combine the integers

When no coefficient-one horizontal component remains, the prepared generic fibre is $\mathbb{Q}$ -Gorenstein klt. That case requires a separate character bound.

Choose $q(s, m)$ as a common multiple of the two branch bounds to close the induction.

[Proof of Theorem 4.2 · p. 11](#)

3. Proof of Proposition 6.7: uniform klt character bounds

Work in the setting of Theorem 4.2 with $s \geq 1$, geometrically klt generic pair, and $\mathbb{Q}$ -Cartier K_V . Give the block characters a common exponent, then cancel the covering degree introduced by decomposition and semistable reduction.

01 Split into blocks after a finite cover

$$(F, H) \times A \times \prod Y_i \times \prod Z_j, \quad b = s!.$$

$$\phi = a \bigwedge_i \eta_i^{\otimes m/p_i}, \quad p_i = m \text{ or } 1.$$

Recover blocks through central idempotents and stop permutations before semistable reduction.

Prop. 5.4, Cor. 5.5; pp. 14–17 (6.2)–(6.3); p. 19

02 Normalize weights to zero on a comparison cover

$$\ell w_z(\phi) = \text{ord}_{c'} a / m.$$

$$\zeta^{b \text{ord}_{c'} a} \prod_i \lambda_i^{m/p_i} = 1.$$

For nonabelian blocks, pass from Lefschetz fixed points to the normal lc index of a cyclic quotient.

Lemmas 6.3–6.6; pp. 19–22 [U] Thm. 1.2; p. 2

03 Give all block characters a common exponent

$$\lambda_i^{Q(s,m)} = 1.$$

Integral cohomology / fixed points and residues on normal components.

Raise the inertia relation to the power Q and return to the weight identity.

Prop. 6.7; p. 23

04 Cancel the uncontrolled ramification degree

$$\ell \mid bQ \text{ord}_{c'} a.$$

$$m s! Q(s, m) w_z(\phi) = bQ \text{ord}_{c'} a / \ell \in \mathbb{Z}.$$

TeX proof diagram controlling inertia characters after product decomposition

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1. Transfer block decompositions to actions on a comparison cover

The decomposition in MW gives, after a finite quasi-étale cover, a rationally connected block carrying the boundary, an abelian variety, Calabi–Yau blocks and irreducible symplectic blocks. The other blocks carry no boundary. Vanishing of logarithmic tangent vector fields on the rationally connected block (Lemma 5.2, pp. 12–13) supports the comparison of product decompositions. The argument does not assume that a finite Galois closure is itself a product. Proposition 5.4, pp. 14–15 recovers block function fields from central idempotents in the endomorphism algebra of the tangent sheaf, kills permutations by an $s!$ -th power, and constructs a common finite comparison cover with actions on its blocks.

Lemmas 5.2–5.3; Proposition 5.4 · pp. 12–15 · MW, Theorem 1.3

2. Normalize to weight zero and obtain the inertia relation

After semistable base change and normalization to weight zero, write $\phi = a \bigwedge_i \eta_i^{\otimes m/p_i}$, with $p_i = m$ on the rationally connected block and $p_i = 1$ otherwise. If ℓ is the ramification index and $b = s!$, equations (6.2)–(6.3), p. 19 give

$$\ell w_z(\phi) = \frac{\text{ord}_{c'} a}{m}, \quad \zeta^{b \text{ord}_{c'} a} \prod_i \lambda_i^{m/p_i} = 1$$

(6.2)–(6.3) · p. 19

3. Kill characters through fixed points and residues on normal components

For abelian blocks, finite-rank integral cohomology bounds the character order. On the other blocks, Du Bois base change compares $H^j(\mathcal{O})$ on the generic and special fibres, and coherent Lefschetz traces produce a fixed point for a bounded power. A further bounded power fixes a coefficient-one component through that point. Adjunction and residue on this **normal component**, followed by the normal lc index theorem U on its cyclic quotient, kill the residue character uniformly (Lemmas 6.5–6.6, pp. 21–22). An slc index theorem for the whole special fibre is not the input here.

Lemmas 6.3–6.6 · pp. 19–22 · U, Theorem 1.2

4. Eliminate ramification using the divisibility relation

A common $Q(s, m)$ with $\lambda_i^Q = 1$ for every block yields $\ell \mid bQ \text{ord}_{c'} a$. Consequently

$$q_{\text{klt}}(s, m) = m s! Q(s, m), \quad q_{\text{klt}}(s, m) w_z(\phi) \in \mathbb{Z}.$$

This is Proposition 6.7, p. 23. The decisive connection is the cancellation of the uncontrolled ramification degree introduced by semistable reduction.

Proposition 6.7 · p. 23

4. Proofs of Theorem 7.2 and Proposition 8.2: denominators and complete section spaces

After fixing the generic-fibre index p_0 , use curve weights to bound moduli denominators. When the base adjoint is big, combine effective birationality with exact section comparison to recover the entire base field.

01 Descend generic-fibre trivialization in a uniform degree

$$K_X + B + p_0^{-1} \operatorname{div} \psi = f^* D_Z.$$

$$D_Z = K_Z + B_Z + M_Z, \quad \text{coeff } B_Z \text{ DCC.}$$

Use the normal lc index, Hilbert 90 and moduli descent; slice while preserving the threshold.

[Prop. 7.1; pp. 23–24](#) [Lemma 7.3; pp. 25–27](#)

02 Read a moduli coefficient as a transverse-curve weight

$$w_z(\phi) = \alpha - 1 + t_P, \quad \text{coeff}_P M_W = w_z(\phi) - \text{coeff}_P K_W.$$

$$p = \operatorname{lcm}(p_0, q(0, p_0), \dots, q(d-1, p_0)).$$

Replace negative exceptional coefficients and use complete rounded section spaces.

[\[BZ\] Thm. 1.3; p. 3](#) [Lemma 8.1, Prop. 8.2; pp. 27–29](#)

03 Apply effective birationality on a big base

$$pM_W \text{ nef Cartier}, \quad K_W + A + M_W \text{ big}, \quad A \geq 0.$$

$$H^0(X, \lfloor n(K_X + B) \rfloor) = \psi^{n/p_0} f^* H^0(Z, \lfloor nD_Z \rfloor).$$

Cancel the common factor in ratios; move ratios in arbitrary degrees to multiples.

[Lemma 8.1, Prop. 8.2; pp. 27–29](#)

04 Recover the base field from all section ratios

$$F_m(K_X + B) = K(K_X + B) = \mathbb{C}(Z).$$

Big base adjoint case. Main theorem: Figure 4.

TeX proof diagram from curve denominators to the full litaka field

[Open figure ↗](#) [TeX source ↓](#)

1. Descend in the same index and choose an exact presentation

The normal lc index theorem [U](#) on the geometric generic fibre, followed by Hilbert 90 in the same degree, supplies the uniform p_0 . Qualitative moduli descent uses [FG](#), while the DCC property of discriminant coefficients uses [HMX](#).

[Proposition 7.1 · pp. 23–24](#) · [U, Theorem 1.2](#) · [FG, Theorem 3.6](#)

2. Measure the moduli coefficient as a curve weight

A transverse curve through the generic point of a prime divisor $P \subset W$ preserves its coefficient α and threshold t_P , including negative coefficients in the crepant subboundary. The relative form has weight $w_z(\phi) = \alpha - 1 + t_P$, differing from the moduli coefficient only by the integral coefficient of K_W (Lemma 7.3, pp. 25–27). A common multiple of $q(s, p_0)$ over all relative dimensions makes pM_W integral and hence Cartier on smooth W .

Theorem 7.2; Lemma 7.3 · pp. 25–27

3. Combine base effectivity with complete section spaces

When L is big, effective birationality BZ applies on the base. Negative exceptional crepant coefficients are first replaced by the reduced exceptional boundary to produce an effective generalized pair. For $p_0 \mid n$, the complete section spaces satisfy

$$H^0(X, \lfloor n(K_X + B) \rfloor) = \psi^{n/p_0} f^* H^0(Z, \lfloor nD_Z \rfloor)$$

Lemma 8.1, pp. 27–28 proves this identification and its birational invariance for rounded divisors, without introducing the Cartier index of the total space. A ratio in any degree can be moved to a degree divisible by p_0 by multiplying numerator and denominator by a power of one section. Thus the entire Iitaka field equals the base field (Proposition 8.2, pp. 28–29).

Lemma 8.1; Proposition 8.2 · pp. 27–29 · BZ, Theorem 1.3

5. Proof of Theorem 1.1: from a good model to the full Iitaka field

Return to the main theorem using the index, denominator, and effectivity bounds. Over the complex numbers, choose a good model and split by the dimension of its semiample base. Then use invariance of the ratio-field equality under field extension.

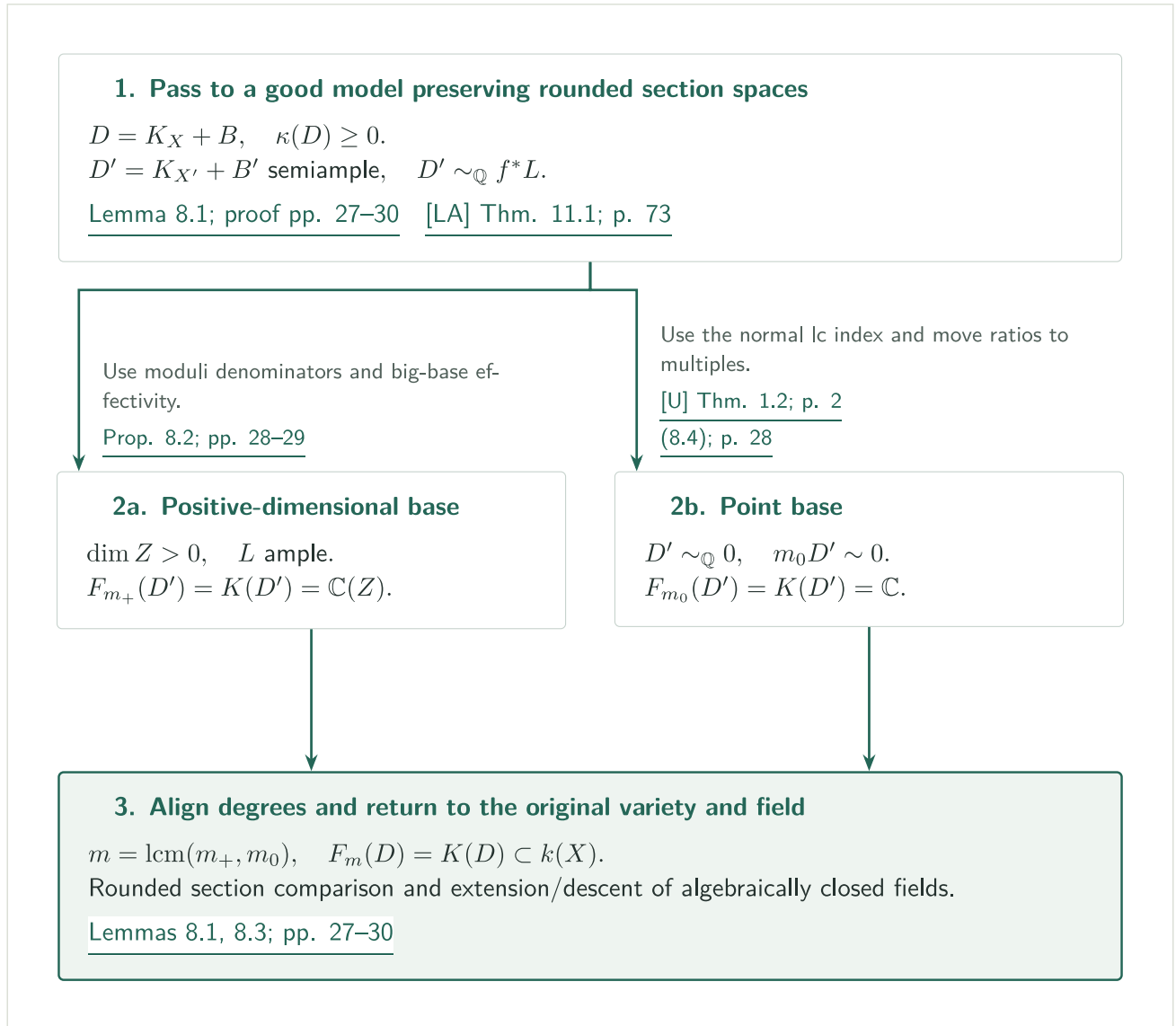


Figure 4. Pass to a good model, handle positive and zero litaka dimension separately, and return to the original field.

[Open figure ↗](#) [TeX source ↓](#)

1. Preserve rounded section spaces on the way to a good model

The assumption $\kappa(X, D) \geq 0$ makes D pseudo-effective. After a crepant dlt modification, LA and TX give a good model with semiample $D' = K_{X'} + B'$. Its coefficients remain in $\Phi \cup \{1\}$. On a common resolution, the difference between the pullback of the original adjoint and the pullback of D' is effective and exceptional. Lemma 8.1 identifies all rounded section spaces inside the common function field.

Lemma 8.1(ii); proof of Theorem 1.1 · pp. 27–28, 29 · LA, Theorem 11.1 · TX, Theorem A

2. Use big-base effectivity in positive Iitaka dimension

If the semiample contraction $f : X' \rightarrow Z$ has positive-dimensional base, then $D' \sim_{\mathbb{Q}} f^*L$ with L ample. Apply Proposition 8.2 with the fixed coefficient set $\Phi \cup \{1\}$. The complete ratios in one uniform degree generate $k(Z) = K(D')$. This identifies the embedded function field, including the finite information of the map.

Proposition 8.2; proof of Theorem 1.1 · pp. 28–30

3. Use the index theorem in dimension zero and align degrees

If Z is a point, then $D' \sim_{\mathbb{Q}} 0$. The normal lc index theorem U gives a uniform principal multiple. Its section ratios are constants. A ratio in any other degree can be moved to a common multiple by multiplying numerator and denominator by powers of a section, so $K(D') = \mathbb{C}$. Take a common multiple of the two branch degrees and return to X by rounded section comparison. No uniform Cartier index of the total space is needed.

(8.4); proof of Theorem 1.1 · pp. 28, 30 · U, Theorem 1.2

4. Transfer the ratio-field equality to arbitrary characteristic-zero fields

The algebraic closure k_0 of a finitely generated field of definition embeds into $\mathbb{C}$. Lemma 8.3 uses base change of rounded divisorial sections and an intersection identity for intermediate fields $F \subset k(X)$. It shows that nonemptiness and $F_m(D) = K(D)$ hold if and only if they hold after an extension of algebraically closed fields. Descend the complex conclusion to k_0 and extend to the original k , retaining the same integer.

Lemma 8.3, (8.5)–(8.6); proof of Theorem 1.1 · pp. 29–30

6. Which papers supply which steps

Input

U Uniform Pluricanonical Iitaka Fibrations

Original statement and role here

2026-10-03, Theorem 1.2, p. 2: uniform integral principal multiples for normal lc pairs with rational DCC coefficients and rationally trivial adjoint divisor. Used in Theorem 2.1, Lemma 6.5, Proposition 7.1 and the $\kappa = 0$ case

Editorial check

Statement and application passages compared; the input's proof is outside scope

Input

LA Log abundance in characteristic zero

Original statement and role here

2026-09-24, Theorem 11.1, p. 73: good models for complex projective lc pairs with pseudo-effective adjoint divisor. Used in Lemmas 3.1, 6.4 and §8

Editorial check

Statement and uses compared, retaining the rational-boundary semiampleness conclusion needed here

Input

SD Arithmetic Stein-degree bounds...

Original statement and role here

2026-09-25, Theorem 1.1, p. 1; bounds the Stein-curve degree of a coefficient-one horizontal component in Lemma 4.6

Editorial check

Statement and use compared; a separate catalog article outlines its proof

Input

MW Matsumura–Wang

Original statement and role here

arXiv:2105.14308v3, Theorem 1.3, p. 4; quasi-étale product decomposition in Proposition 5.1

Editorial check

Statement compared; proof outside scope

Input

TX Tsakanikas–Xie

Original statement and role here

arXiv:2301.09186v3, Theorem A, p. 2; existence of a terminating scaling MMP when a minimal model exists

Editorial check

Statement compared; ordinary pairs here have zero nef part

Input

FG, HMX, BZ

Original statement and role here

FG Theorem 3.6 (PDF p. 7, journal p. 1726), moduli descent; HMX Theorem 1.1 (p. 2), ACC for thresholds; BZ Theorem 1.3 (p. 3), effective birationality; used in §§7–8

Editorial check

Statements and application passages compared; input proofs outside scope

7. Return to the sources

The editorial reading covers the pinned main statement and the proof route in §§3–8, pp. 6–30, including equations (6.2)–(6.3), cancellation of the ramification degree in Proposition 6.7, the coefficient comparison in Lemma 7.3, and the argument with rounded section ratios. The TeX diagrams compress this route. This revision rechecked the two induction branches and the cases after passing to a good model, aligning diagrams with the numbered explanations.

Remaining checks include independent scrutiny of the transition from logarithmic-volume rigidity in Lemma 5.2 to the comparison-cover decomposition; reducedness of the marked special fibre after the quotient MMP in Lemma 6.4; and comparison with the original Du Bois base-change and coherent Lefschetz statements invoked in Lemma 6.6. Their uses in this paper were read, but input proofs and every auxiliary citation were not recursively verified. The background reference HP was located without a statement-level comparison. The low-dimensional Relative denominators... paper is a methodological predecessor for curve preparation; its low-dimensional conclusion is not recorded as an input proving the higher-dimensional denominator theorem.

Uniform Pluricanonical Iitaka Fibrations

Simultaneous induction on Iitaka degrees and normal lc indices

The manuscript claims that one pluricanonical degree depending only on dimension defines every Iitaka fibration of a smooth projective variety of nonnegative Kodaira dimension. The induction also proves a common exponent killing the **actual linear divisor class** of a normal lc log Calabi–Yau pair. Its central mechanisms are bounds on volume-form characters rather than on semistable ramification itself, and a contradiction between positivity on high-index canonical covers and jet estimates on diagonal products.

Manuscript author: OpenAI · 2026-10-03 · 44 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Uniform pluricanonical degree (p. 2)

For every integer $d \geq 1$ there is an integer $m(d) > 0$ such that, over every algebraically closed field of characteristic zero, the complete system $|m(d)K_X|$ defines the Iitaka fibration of every smooth integral projective d -fold X with $\kappa(X) \geq 0$.

The system is nonempty and its section-ratio field **equals** the full Iitaka field $K(K_X) \subset k(X)$ generated over all pluricanonical degrees. Having an image of dimension $\kappa(X)$ alone is weaker. The statement includes nonvanishing in a common degree when $\kappa = 0$, and birationality when $\kappa = d$. Every positive multiple works; no practical numerical value of $m(d)$ is supplied.

[Theorem 1.1](#) and §9, pp. 2, 40–42

Theorem 1.2 — Uniform log canonical index (p. 2)

Fix $d \geq 0$ and a rational DCC set $\Phi \subset [0, 1] \cap \mathbb{Q}$. There is an integer $a(d, \Phi) > 0$ such that every normal integral projective lc pair (X, B) over an algebraically closed field of characteristic zero, with $\dim X = d$, effective boundary having coefficients in Φ , and $\mathbb{Q}$ -Cartier adjoint $K_X + B \sim_{\mathbb{Q}} 0$, satisfies

$$a(d, \Phi)(K_X + B) \sim 0.$$

This multiple is an integral principal divisor. The statement controls more than numerical triviality or the Cartier index, and does not cover nonnormal slc pairs.

Theorem 1.2 and §§2, 9.1, pp. 2, 4–5, 41–42

References on the arrows

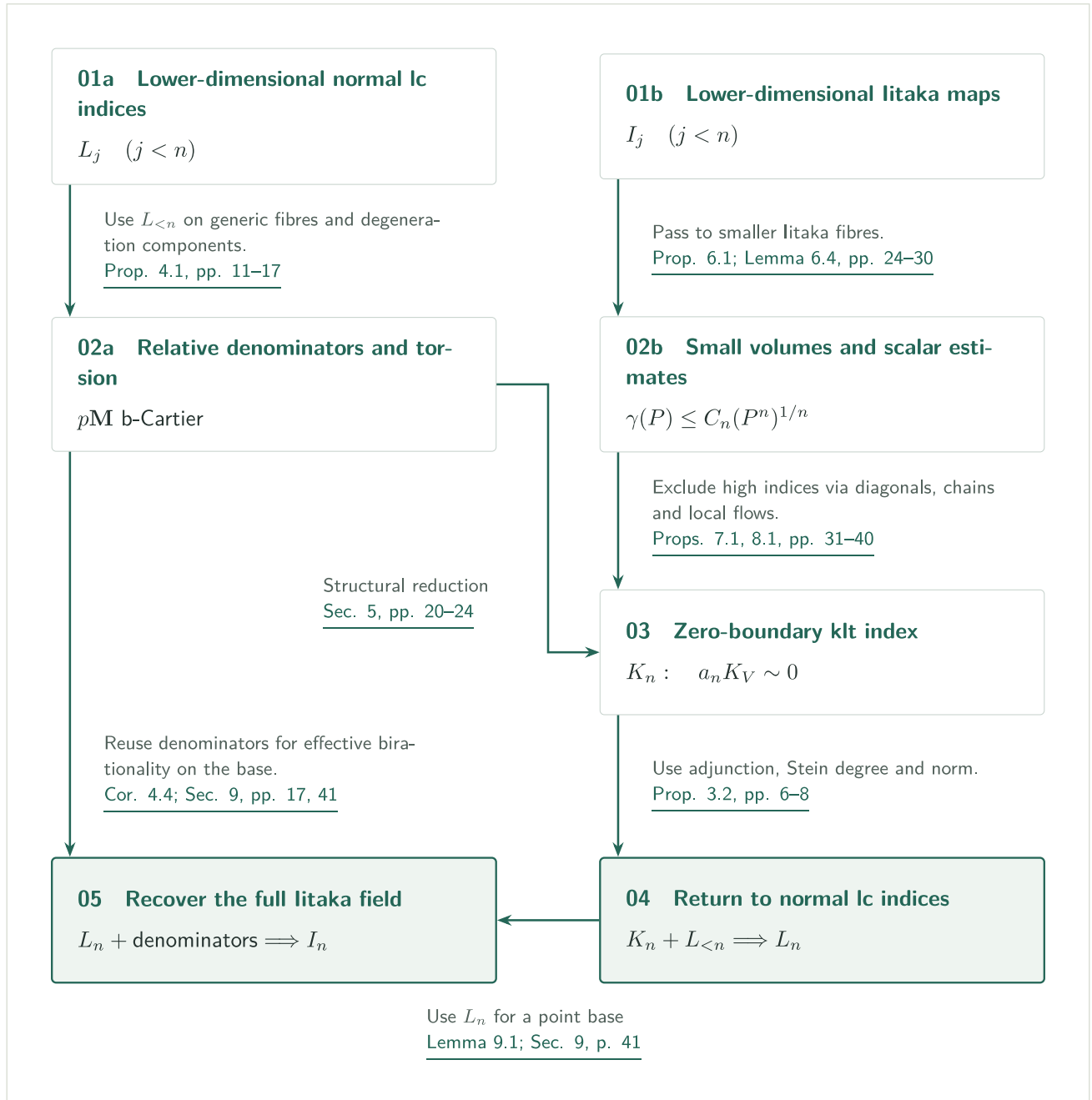
Unprefixed references denote results of this manuscript. External references use the following keys.

- **LA**: Log abundance in characteristic zero, September 24, 2026.
- **SD**: Arithmetic Stein-degree bounds for log Calabi–Yau pairs, September 25, 2026.
- **Ufour**: Uniform effective log Iitaka fibrations for fourfolds, September 26, 2026. Its chain and tracking lemmas used here are stated in arbitrary dimension.
- **BZ**: Birkar–Zhang, Effectivity of Iitaka fibrations and pluricanonical systems of polarized pairs, arXiv:1410.0938v2.

2. Organizing the simultaneous induction

Proof route. Lower-dimensional indices control **moduli denominators**, and lower-dimensional Iitaka maps exclude **high-index sequences**. Returning to **normal lc indices** proves Theorem 1.2; **recovery of the full Iitaka field** then proves Theorem 1.1.

Let I_j denote Theorem 1.1 in dimension j , and L_j Theorem 1.2 in dimension j for every rational DCC coefficient set. Write K_n for the zero-boundary klt index assertion: one integer makes $a_n K_V \sim 0$ for every projective klt n -fold with $K_V \sim_{\mathbb{Q}} 0$. Starting in dimension zero, the step assumes I_j, L_j for $j < n$ and proves K_n, L_n, I_n in that order.



Two lower-dimensional inputs supply the index and litaka assertions in the next dimension

[Open figure ↗](#) [TeX source ↓](#)

1. Control relative denominators with lower-dimensional indices

Apply $L_{<n}$ to generic fibres and degeneration components over positive-dimensional bases. The resulting moduli denominator and torsion control over rationally connected bases support the structural reduction for K_n . The same denominator bound is reused for I_n , as the separate route in the diagram records.

Notation 2.2 and Figure 1 · pp. 4–5; Proposition 4.1 and §5 · pp. 11–24

2. Exclude high indices using lower-dimensional Iitaka maps

The hypotheses $I_{<n}$ enter the passage from small-volume covering members to smaller Iitaka fibres in Lemma 6.4 and the scalar estimates. Apply these estimates to canonical covers and diagonal products; the degree and local-flow argument in §8 then gives K_n . The assertion I_n is not assumed at this stage.

Proposition 6.1, Lemma 6.4 and §§7–8 · pp. 24–40

3. Prove the normal lc index before returning to Iitaka maps

Adjunction, Stein degree and norm give L_n from K_n and $L_{<n}$. On a good model the Iitaka base is either a point, where L_n applies, or positive-dimensional, where denominators and effective birationality apply. Global ACC reduces DCC coefficients to finite sets for L_n ; descent to arbitrary algebraically closed characteristic-zero fields is performed at the end.

Proposition 3.2 and §9 · pp. 6–8, 40–42

3. Bounding moduli denominators

For a boundary-free klt contraction over a positive-dimensional base, seek a common Cartier multiple of the base moduli part. The input is $L_{<n}$: this bound is obtained before the index assertion in the current dimension.

01 Fix lower-dimensional indices

L_j ($j < n$), $f : X \rightarrow Z$, X klt, $B_X = 0$, $\dim Z > 0$.
 $K_X \sim_{\mathbb{Q}} f^* D_0$.

Use the generic-fibre index and qualitative bundle formula.

Prop. 4.1, pp. 11–12

02 Construct the exact canonical bundle formula

$$K_X + p_0^{-1} \operatorname{div}(\psi) = f^*(K_Z + B_Z + M_Z).$$

Take residues on dlt components after semistable reduction; apply L_j .

Lemmas 3.3, 4.3, pp. 8, 14–17 [LA] Thm. 11.1, p. 73

03 Bound degeneration characters using residues

$$w_u(\eta_i) = 0, \quad \lambda_i^N = 1, \quad j = \dim S < n.$$

Kill volume-form characters to clear coefficients; ramification need not be bounded.

(4.6)–(4.7), p. 14; Sec. 4.4, p. 17

04 Fix the Cartier denominator of the moduli part

$$p = p_0 n! N, \quad p\mathbf{M} \text{ b-Cartier.} \quad \textbf{Proposition 4.1}$$

Lower-dimensional normal lc indices bound degeneration characters and moduli denominators

[Open figure ↗](#) [TeX source ↓](#)

1. Turn the generic-fibre index into an exact pullback equality

Write L_j for Theorem 1.2 in dimension j . This step uses only $j < n$. Consider a contraction $f : X \rightarrow Z$ from a boundary-free klt n -fold, with $\dim Z > 0$ and K_X rationally linearly pulled back from the base. A common p_0 kills the canonical class of the geometric generic fibre. Descending that trivialization yields an equality of **actual rational divisors**

$$K_X + \frac{1}{p_0} \operatorname{div}(\psi) = f^* D_Z, \quad D_Z = K_Z + B_Z + M_Z.$$

Qualitative canonical bundle formula theory supplies DCC coefficients for B_Z and b-nefness of $\mathbf{M}$. The remaining issue is a uniform Cartier denominator.

[Proposition 4.1, pp. 11–12](#)

2. Reduce denominators to volume-form characters

Slice transversely to a prime divisor on the base and take semistable degenerations of the Beauville–Bogomolov factors of the generic fibre. Normalize the weights of their volume forms to zero. Equation (4.7) relates the ramification degree ℓ to central characters λ_i . A common exponent N killing these characters makes $p = p_0 n! N$ clear the moduli coefficients: no bound on ℓ itself is needed.

Lemma 4.2 and (4.5)–(4.7), pp. 13–14

3. Kill degeneration characters with lower-dimensional residues

Lemma 4.3 uses LA to obtain a relatively semiample dlt degeneration, and applies L_j to residues on normal components of its special fibre. Structure-sheaf cohomology traces and a Lefschetz argument produce a bounded power preserving a component. Neither L_n nor an slc index theorem enters this step. Equivariant semistable reduction and coherent Lefschetz details have not been independently verified here.

Lemma 4.3 and §4.4, pp. 14–17

4. Excluding a high-index sequence

This is the high-index exclusion needed for K_n . Read the argument in three groups: scalar and Frobenius estimates (1–2), chain degrees and degree-one forgetting (3–5), and the rational-flow contradiction (6).

After the structural reduction of Proposition 5.1, consider canonical covers $\pi : Y \rightarrow V$ with indices $r \rightarrow \infty$. Set $G = \mu_r$, $Z_t = Y^t / G_{\text{diag}}$, $\theta_t : Z_t \rightarrow V^t$ and $P_t = \theta_t^*(L^{\boxplus t})$. This argument takes place over the complex numbers, as required for its later use of local flows.

01 Obtain local positivity after structural reduction

$\text{Bir}(V)$ countable, $\varepsilon(L) \geq c$, $\varepsilon(\pi^*L) \geq cr^{1/n}$.

Compare scalar upper orders with Frobenius-diagonal rank loss.

Props. 6.1, 7.1, pp. 24, 31–37

02 Use the determinant contradiction on diagonal products

$\text{ord}(\det) \geq 3R/4$ and $< 3R/4 \implies \varepsilon(P_t) \leq C_2 t^2$.

Close low-cost families under group actions, permutations and projections; form chain quotients.

Lemma 6.2; (8.1), pp. 25, 37–38

03 Fix finitely many sizes and construct compatible leaves

$2 \leq t \leq T$, $B = C_2 T^2 + 1$, $P_t^{h_t} \cdot H_t \leq (h_t B)^{h_t}$, $W_t = \theta_t(H_t)$.

Fix T , increase r , and compare fibre positivity with the leaf degree bound.

(8.2)–(8.3), p. 38

04 Exclude positive-dimensional fibres over a coordinate

$a = h_t - j > 0 \implies P_t^{h_t} \cdot H_t \geq c^j (cr^{1/n})^a \rightarrow \infty$.

Thus $1 \leq h_t \leq n$.

On a long constant-dimensional stretch, rule out exponential degree growth.

(8.4)–(8.6), p. 39

05 Force degree-one forgetting by a polynomial degree bound

$d_t \leq K T^{2n}$, $h_t = h_{t-1} \Rightarrow d_t/d_{t-1} = \deg(W_t \dashrightarrow W_{t-1})$.

Some $W_k \dashrightarrow W_{k-1}$ is birational.

Use separability and intersections of product function fields to descend tangent transport.

(8.7), p. 39

06 Transport tangent directions rationally

$R(y, z)R(x, y) = R(x, z)$, $v_1, \dots, v_h$ rational, $h > 0$.

Geometric integrality gives rational recovery; apply it also to inverse flows.

(8.8); Sec. 9, p. 40

07 Contradict countability through local flows

$E_z(y) = \text{ev}(E_z(u), \xi(u, y))$, $E_z \in \text{Bir}(V)$.

Uncountably many $E_z \implies K_n$.

Cover positivity, diagonal estimates, chain leaves and degree-one forgetting contradict countability

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1. Supply the structural hypotheses for scalar estimates

The terminal variety V supplied by the reduction has countable $\text{Bir}(V)$. Every G -equivariant rational fibration of Y with positive-dimensional base and fibre has a general-type smooth geometric generic fibre. Proposition 6.1 uses this condition and $I_{<n}$. Normalize $1 \leq L^n \leq 2$, and apply it to V and to Y with its G -invariant polarization π^*L :

$$\gamma(L; V) \leq C_n, \quad \varepsilon(L) \geq c_n, \quad \varepsilon(\pi^*L) \geq c_n r^{1/n}.$$

Here γ measures normalized orders of **all sections** on resolutions in sufficiently divisible degrees, not just restrictions of ambient sections. The chain and tracking lemmas from [Ufour](#) are statements in arbitrary dimension; its fourfold main theorem is not being applied in all dimensions.

Propositions 5.1, 6.1 and Lemmas 6.2–6.4 · pp. 20, 24–30 ·

[Ufour](#), Lemmas 5.1, 5.4, 6.3–6.4 · pp. 18, 21, 25, 27

2. Turn rank loss along the Frobenius diagonal into a determinant contradiction

Section 7 deduces $\varepsilon(P_t) \leq C_2 t^2$ from the scalar upper bound. Under the opposite inequality, specialize to positive characteristic and construct a nonzero determinant from evaluation along the Frobenius diagonal. If its rank is R , at least $R - \lfloor R/4 \rfloor$ rows can be made to vanish on the diagonal, giving order at least $3R/4$. Scalar estimates on the coordinate sections bound the same order by $Rb_0(C_1 + 1) < 3R/4$. This contradiction gives the quadratic diagonal-product bound; it does not yet bound the canonical index.

Proposition 7.1, (7.18)–(7.19) and the proof's conclusion · pp. 31, 36–37

3. Form projection-compatible leaves from low-cost curves

First fix a finite T and take a tail where the estimates hold simultaneously for $2 \leq t \leq T$. With $B = C_2 T^2 + 1$, the strict inequality $\varepsilon(P_t) < B$ supplies marked covering curves whose degree divided by marked multiplicity is at most B . Close the finite family lists under deck transformations, coordinate permutations and nonconstant coordinate projections. Projection does not increase this ratio. Retaining generic parameters along chains permits Lemma 6.2 to apply. The leaves H_t and their images $W_t = \theta_t(H_t)$ are projection-compatible, and for $h_t = \dim H_t > 0$ satisfy

$$P_t^{h_t} \cdot H_t \leq (h_t B)^{h_t}.$$

Geometric integrality of generic leaves will also be needed for the later field-degree comparison.

Lemma 6.2 and Proposition 8.1, (8.1) · pp. 25, 37–38 · [Ufour](#), Corollary 5.2 and Lemma 5.4 · pp. 20–21

4. Force every single-coordinate projection to be generically finite

Suppose one coordinate image is I , with $j = \dim I$ and $a = h_t - j > 0$. The Seshadri bound at a very general mark gives $L^j \cdot I \geq c^j$. Fixing that coordinate identifies the geometric generic fibre of Z_t with Y^{t-1} , where the remaining polarization has lower bound $cr^{1/n}$. The projection formula and nefness therefore give

$$P_t^{h_t} \cdot H_t \geq c^j (cr^{1/n})^a.$$

For fixed T , this contradicts the preceding upper bound as $r \rightarrow \infty$. Every single-coordinate projection is thus generically finite onto its image, and $1 \leq h_t \leq n$.

Proposition 8.1, (8.2)–(8.3) · p. 38

5. Extract a degree-one forgetting map from a polynomial bound

Let d_t be the degree of W_t over its first-coordinate image. The previous bounds give $d_t \leq KT^{2n}$, with K independent of T . Projection compatibility and single-coordinate finiteness imply $h_t \leq h_{t-1}$. When equality holds, $W_t \dashrightarrow W_{t-1}$ is dominant of degree d_t/d_{t-1} . Since $1 \leq h_t \leq n$, a large T provides a long constant-dimensional stretch. If all its adjacent degrees were at least two, exponential growth would contradict the polynomial bound. Thus for some $k \geq 3$, every single-coordinate omission has degree one.

Proposition 8.1, (8.4)–(8.6) · p. 39

6. Convert rational coordinate recovery into birational local flows

Degree one and separability make the leaf tangent distribution inject into each coordinate tangent space and map isomorphically to every omitted-coordinate distribution. Intersecting function fields of independent product variables shows that tangent transport is a rational map $R(x, y)$ in two variables, satisfying $R(y, z)R(x, y) = R(x, z)$. Transport a basis from a reference point to obtain rational vector fields; commutativity is not required.

Geometric integrality of the generic leaf preserves the forgetting-map degree after algebraically closing the generic base field. The last coordinate can therefore be recovered rationally from the remaining coordinates and the leaf's name. Write ev for this recovery map, ξ for the quotient, and E_z for an ordered composition of local flows. Then

$$E_z(y) = \text{ev}(E_z(u), \xi(u, y)).$$

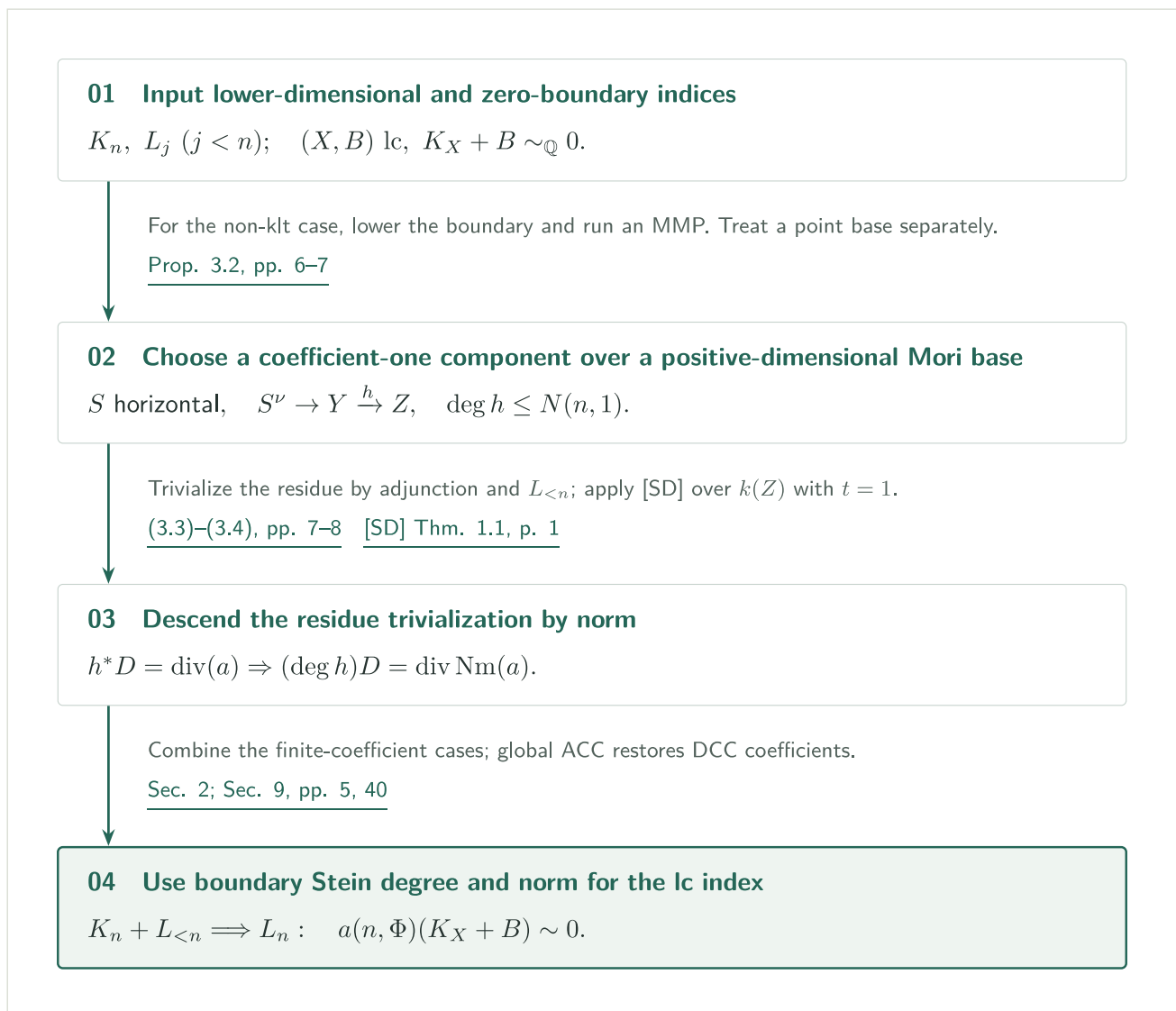
For fixed small z the right side is rational in y , and the reversed negative-time flows give a rational inverse. Their inverse identities on an analytic open imply $E_z \in \text{Bir}(V)$. The positive-rank distribution makes $E_z(y_0)$ assume uncountably many values, contradicting the countability supplied by the structural reduction.

Proposition 8.1, (8.7)–(8.8) and the start of §9 · pp. 39–40

5. Returning to normal lc indices

Now combine the zero-boundary assertion K_n with $L_{<n}$ to prove L_n . The diagram focuses on the non-klt case with positive-dimensional Mori base. The klt reduction and the log Fano case over a point are separate cases in Proposition 3.2.

Two branches close before the residue argument. For klt pairs, the zero-boundary index K_n enters Xu's klt induction theorem. For a point Mori base, bounded complements give $B^+ \geq B$; the effective numerically trivial difference $B^+ - B$ vanishes. The diagram treats the remaining non-klt branch with positive-dimensional base. Proposition 3.2 · pp. 6–8



Adjunction to a coefficient-one component and bounded Stein degree permit norm descent

Open figure ↗ TeX source ↓

1. Trivialize residues on a horizontal component

After proving K_n , Proposition 3.2 recovers L_n . In the non-klt case an MMP with a slightly reduced boundary produces a Mori fibre space and a horizontal coefficient-one component S . Adjunction to S^ν and lower-dimensional indices trivialize its residue.

Proposition 3.2 · pp. 6–7

2. Use the bounded Stein degree as the norm multiplier

For the Stein factorization $S^\nu \rightarrow Y \xrightarrow{h} Z$, the norm gives

$$h^*D = \operatorname{div}(a) \implies (\deg h)D = \operatorname{div} \operatorname{Nm}(a).$$

Apply SD Theorem 1.1 over $k(Z)$ with coefficient threshold $t = 1$ to bound $\deg h$. This is not a gluing-index argument on the entire reduced boundary. Global ACC reduces the DCC coefficient statement to finite coefficient sets.

Proposition 3.2, pp. 6–8; §9, p. 40

6. Recovering the full Iitaka field in one degree

Use the Iitaka contraction $f : V \rightarrow Z$ on a good model. The complete system must generate the entire Iitaka field, rather than merely have the correct image dimension.

01 Represent the litaka fibration on a good model

$$X \dashrightarrow V \xrightarrow{f} Z, \quad K_V \text{ semiample}, \quad H^0(X, mK_X) = H^0(V, mK_V).$$

Use the good model from [LA] and split the point and big-base cases.

[\[LA\] Thm. 11.1, p. 73](#) [Lemma 9.1; proof of Thm. 1.1, p. 41](#)

02 Choose a degree according to the base dimension

$$\dim Z = 0 : L_n; \quad \dim Z > 0 : D_Z \text{ big}, \quad p\mathbf{M} \text{ b-Cartier.}$$

Apply effective birationality with fixed coefficients and nef denominator.

[\[BZ\] Thm. 1.3, p. 3](#) [Cor. 4.4, \(4.10\), p. 17](#)

03 Recover the full base field from complete-system ratios

$$H^0(V, mK_V) = \psi^{m/p_0} f^* H^0(Z, \lfloor mD_Z \rfloor), \quad p_0 \mid m.$$

Positive multiples preserve section ratios; faithful flatness gives descent.

[Sec. 9, pp. 41–42](#)

04 Take a common multiple and descend the field

$$k(\text{ratios of } |m(n)K_X|) = K(K_X). \quad \textbf{Theorem 1.1}$$

Good models and effective birationality prove Theorem 1.1

[Open figure ↗](#) [TeX source ↓](#)

1. Transfer every integer degree to the good model

LA supplies a terminal good minimal model V of the smooth variety X . Lemma 9.1 identifies pluricanonical section spaces in every integer degree $m \geq 0$, interpreting the spaces on V divisorially even when mK_V is not Cartier. If the litaka base is a point, the already established L_n supplies a common nonvanishing degree.

[Lemma 9.1, p. 41](#)

2. Apply effective birationality on the base with fixed denominator

For a positive-dimensional base, apply BZ Theorem 1.3 to the big base adjoint with the fixed DCC coefficients and fixed nef denominator from the denominator diagram. On a smooth determination it is an ordinary lc pair with a nef Cartier multiple, which verifies the effective theorem's hypotheses. For $p_0 \mid m$, equation (4.10) gives

$$H^0(X, mK_X) = \psi^{m/p_0} f^* H^0(Z, \lfloor mD_Z \rfloor).$$

The birational system downstairs therefore recovers its full function field. Passing to a common multiple preserves ratios because $s/s_0 = (ss_0^{q-1})/s_0^q$. Descent to a field of definition and faithful flatness extend the conclusions to all algebraically closed fields of characteristic zero.

Corollary 4.4, p. 17; §9, pp. 41–42

7. Which papers supply which steps

Input or relation

LA Theorem 11.1, p. 73

Content and use

Good models for complex projective lc pairs with pseudo-effective adjoint; used in Theorem 2.1, Lemma 4.3, Lemma 9.1

Check made here

Input statement and application compared; LA's full proof is outside scope

Input or relation

SD Theorem 1.1, p. 1

Content and use

Constant-field degree of a coefficient-one component of a normal integral lc log CY pair over any characteristic-zero field; Proposition 3.2, pp. 7–8

Check made here

Checked $H^0(X_\eta, \mathcal{O}) = k(Z)$, $t = 1$, and the norm step

Input or relation

Ufour Lemmas 5.1, 5.4, Corollary 5.2 (pp. 18, 20–21), Lemmas 6.3–6.4 (pp. 25, 27)

Content and use

Chain quotients, degree bounds, tracking and covering; Lemmas 6.2–6.3 and §8

Check made here

These statements have arbitrary dimension; the fourfold main theorem is not used in arbitrary dimension

Input or relation

[BZ Theorem 1.3](#), [arXiv v2 p. 3](#)

Content and use

Effective birationality for big polarized lc adjoints with fixed dimension, DCC coefficients and nef Cartier denominator; Corollary 4.4, p. 17

Check made here

Original statement and use on the smooth determination compared; proof not reviewed

Input or relation

[R §§3–7](#)

Content and use

Earlier fourfold method for denominators and RC torsion, developed in §§4–5 here

Check made here

Methodological predecessor; no edge asserting that a fourfold conclusion holds in every dimension

Input or relation

This paper → [Log Theorem 2.1](#) (p. 5), [SLC Theorem 2.2](#) (p. 5)

Content and use

They restate and use the normal lc index Theorem 1.2

Check made here

Recipient input statements checked; recipient full proofs not investigated

Other inputs include Xu's klt index induction, Birkar's complements and RC boundedness, Ambro's canonical bundle formula, global ACC, and weak positivity. Their original statements have not all been cross-checked in this draft. They remain unresolved entries in the verification record, rather than being treated as absent dependencies.

8. Return to the sources

Start with §3 (pp. 6–8), §4 (pp. 11–19), §6 (pp. 24–30), and §§7–9 (pp. 31–42). We read the main statements, the normal-lc norm reduction, the weight/residue/component-fixing parts of the denominator argument, scalar and diagonal statements and their main connections, the leaf/degree-one/flow argument in §8, and the section comparison and closing induction in §9. We did not independently verify the entire structural reduction in §5, all weak-positivity and flattening calculations in §6, all positive-characteristic estimates in §7, or the proofs of external inputs.

Relative denominators and effective systems for log Calabi-Yau fibrations

Exact pullback presentations, whole-fibre residues, and complete section spaces

For lc-trivial fibrations with total dimension at most four, the manuscript uniformly controls a generic-fibre trivializing degree and a Cartier denominator of the moduli b-divisor. The key is a form on the **whole reduced special fibre** after reduction to a curve. Its conductor compatibility kills the cyclic character recording the denominator. Separate applications recover complete systems over a big base and control torsion over a rationally connected base.

Manuscript author: OpenAI · 2026-09-27 · 23 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 (RD, p. 2)

Exact presentation and moduli denominators

Fix a finite set $\Phi \subset [0, 1] \cap \mathbb{Q}$ and an integer $1 \leq d \leq 4$. There are positive integers $p_0 \mid p$, with p_0 clearing Φ , and a rational DCC set $\mathcal{B} \subset [0, 1]$, depending only on d, Φ . Suppose (X, B) is a projective complex lc pair of dimension d with $B \geq 0$ and coefficients in Φ , and $f : X \rightarrow Z$ is a contraction onto a positive-dimensional normal projective base, with

$$K_X + B \sim_{\mathbb{Q}} f^* D$$

for a $\mathbb{Q}$ -Cartier divisor D . There exist $\psi \in \mathbb{C}(X)^*$ and $D_Z \sim_{\mathbb{Q}} D$ giving an equality of actual divisors

$$K_X + B + \frac{1}{p_0} \operatorname{Div}(\psi) = f^* D_Z, \quad D_Z = K_Z + B_Z + M_Z.$$

The effective boundary B_Z has coefficients in $\mathcal{B}$; $(Z, B_Z + M_Z)$ is generalized lc; $\mathbf{M}$ is b-nef and $p\mathbf{M}$ is b-Cartier. If (X, B) is klt, the base pair is generalized klt.

The bound concerns the **chosen exact presentation**, not every rationally linearly equivalent representative of D_Z . The integer p_0 trivializes the generic adjoint and controls the principal correction; p makes the moduli part b-Cartier. No uniform basepoint-freeness follows as part of this statement.

Theorem 1.1 · p. 2

Proposition 6.1 (RD, pp. 15–17)

The full function field of a big base

If, in addition, D is big, there is $m = m(d, \Phi) > 0$ such that for every positive multiple $l \in m\mathbb{Z}_{>0}$ the complete system

$$|l(K_X + B)|$$

is nonempty and its section ratios generate exactly $\mathbb{C}(Z) \subset \mathbb{C}(X)$. Its rational map is birationally equivalent to f and to the Iitaka fibration of $K_X + B$. This recovers the embedded field, including its finite part. Sections belong to the reflexive divisorial sheaf; $l(K_X + B)$ need not be Cartier.

Proposition 6.1 · pp. 15–17

Proposition 7.1 (RD, p. 17; proof pp. 17–21)

Principal multiples over a rationally connected base

Fix $1 \leq e \leq 4$, a rational DCC set $I \subset [0, 1]$, and an integer $p > 0$. Let Z be a normal projective complex variety of dimension e with a rationally connected smooth projective resolution. Assume $(Z, B + M_Z)$ is generalized klt, $B \geq 0$ has coefficients in I , and $\mathbf{M}$ is rational b-nef with $p\mathbf{M}$ b-Cartier. There is $\ell = \ell(e, I, p) > 0$ such that

$$D = K_Z + B + M_Z \sim_{\mathbb{Q}} 0 \implies \ell D \text{ is an integral principal divisor.}$$

Thus every $a\ell D$, $a \geq 1$, is principal and $h^0(Z, \mathcal{O}_Z(a\ell D)) = 1$. One can take a common integer for any fixed subset of the stated dimensions.

Proposition 7.1 · pp. 17–21

Corollary 7.2 (RD, p. 21)

For a finite rational coefficient set Φ , there is $r = r(\Phi) > 0$ with the following property. Let (X, B) be a projective klt fourfold pair, $B \geq 0$, with coefficients in Φ and $K_X + B \sim_{\mathbb{Q}} 0$. If a contraction $f : X \rightarrow Z$ has positive-dimensional base admitting a rationally connected smooth projective resolution, then $r(K_X + B)$ is an integral principal divisor. This includes $B = 0$, and every positive multiple is principal.

Corollary 7.2 · p. 21

References on the arrows

Unprefixed numbers refer to this manuscript RD. The principal external inputs are:

- JL, Corollary 1.6 · slc indices
- FG, Theorem 3.6 · canonical bundle formula
- CT, Theorems 1.1–1.2 · termination
- HMX, Theorems 1.1 / 1.5 · ACC
- BZ, Theorems 1.3 / 1.6 · effective birationality / generalized ACC
- Bir, Theorem 1.7 · boundedness

2. Theorem 1.1: exact presentation and special-fibre residues

Proof route. Theorem 1.1 fixes an exact presentation and moduli denominators. The applications split into Proposition 6.1, recovering the full function field of a big base, and Proposition 7.1 / Corollary 7.2, giving principal multiples over a rationally connected base.

Fix an exact presentation in degree p_0 on the generic fibre, then prove integrality of pM_W at each prime of a smooth base model. The local denominator is detected by a cyclic character on residues along the whole special fibre.

01 Trivialize the generic fibre in the same degree

$$K_X + B + p_0^{-1} \operatorname{Div}(\psi) = f^* D_Z, \quad p_0 \mid p.$$

Hilbert 90 retains the degree; after qualitative descent, slice to a curve.

[§3; pp. 6–7](#) [§4, \(4.6\); pp. 7–10](#)

02 Test each coefficient on a transverse curve

$$\beta = \alpha + t_P, \quad \operatorname{coeff}_P M_W \equiv \beta \pmod{\mathbb{Z}}.$$

$$p_0(K_N + T + H) + \operatorname{Div}(\psi_C) = p_0 \beta f_N^*[c].$$

Adjunction, depth and global ACC permit the slc fibre index theorem.

[Lemma 5.1; pp. 11–13](#) [\[JL\] Cor. 1.6; p. 2](#)

03 Construct one form on the whole special fibre

$$(T, B_T) \text{ slc}, \quad \dim T \leq 3, \quad p(K_T + B_T) \sim 0.$$

$$u \text{ global}, \quad p \text{ even}, \quad p_0 \mid p.$$

An auxiliary divisible degree makes each ratio constant; even-degree next residues identify them.

[\(5.9\)–\(5.12\); pp. 14–15](#)

04 Glue the residue ratios on the cyclic cover

$$p_0 \beta = a/m, \quad w^m = z, \quad s \mapsto \zeta^{-a} s.$$

$$\operatorname{res}_{T_{Y,j}}(s^{p/p_0}) = r_j \rho_j^* u_i, \quad r_j = r \in \mathbb{C}^*.$$

Connectedness and conductor matching give invariance, including component permutations.

[Lemma 5.1, conclusion; p. 15](#)

05 Kill the character and obtain the Cartier denominator

$$\zeta^{-ap/p_0} = 1 \implies m \mid p/p_0 \implies p\beta \in \mathbb{Z}.$$

$$pM_W \text{ integral on smooth } W \implies p\mathbf{M} \text{ b-Cartier.}$$

Cyclic residue comparison clearing the denominator

[Open figure ↗](#) [TeX source ↓](#)

1. Descend in the same degree and fix the exact presentation

The geometric generic fibre has dimension at most three. The low-dimensional slc index theorem principalizes its adjoint in a uniform degree. Ratios of conjugate principalizing functions are constants, so Hilbert 90 descends the principalization to the fibre over $\mathbb{C}(Z)$ in the **same degree** p_0 . Comparison with the original pullback relation produces the exact equality (1.1), since a function with vertical divisor is constant on the generic fibre. The effective lc boundary gives the rank-one condition for an lc-trivial fibration. The qualitative canonical bundle formula then descends $\mathbf{M}$ to a nef divisor on a smooth model W (§3, pp. 6–7).

§3 · pp. 6–7 · JL, Corollary 1.6 · FG, Theorem 3.6

2. Slice to detect the denominator and remove the error

For $P \subset W$, put $\alpha = \text{coeff}_P D_W$ and let t_P be the threshold for the crepant sub-pair. The coefficient of M_W differs from $\alpha + t_P$ by an integer. It suffices to show $p(\alpha + t_P) \in \mathbb{Z}$. A transverse curve slice retains α by specifying canonical representatives through residues. After a relative dlt MMP, the error E_N is vertical by generic-fibre negativity, and nefness plus fibre connectedness makes it proportional to the fibre. The original threshold-attaining component has zero error coefficient, forcing that proportionality constant to vanish. Locally one obtains

$$p_0(K_N + T + H) + \text{Div}(\psi_C) = p_0\beta f_N^*[c], \quad \beta = \alpha + t_P.$$

Pseudo-effectivity is used in the four-dimensional termination input (§4, pp. 7–10).

§4, (4.1)–(4.6) · pp. 7–10 · CT, Theorems 1.1–1.2

3. Obtain a form on the entire special fibre

Lemma 5.1 (pp. 10–15) clears the denominator in this identity. A depth theorem and conductor adjunction make the reduced fibre T slc. Global ACC puts the different coefficients in a finite set. The JL index theorem supplies a nowhere-vanishing form u of uniform even degree p on the log pluricanonical line of **all of** T .

Lemma 5.1, (5.1)–(5.4) · pp. 10–13 · JL, Corollary 1.6

4. Make each residue ratio constant on the cyclic cover

Write $p_0\beta = a/m$ in lowest terms and normalize the base change $w^m = z$, obtaining $\pi : Y \rightarrow N$. Coefficient comparison makes every fibre multiplicity divisible by m , so π is étale in codimension one. The form $s = (\pi^*\theta)^{\otimes p_0}\pi^*\psi_C w^{-a}$ carries character ζ^{-a} . On each normalized component, compare $\text{res}(s^{p/p_0})$ with π^*u . Adjunction in a sufficiently divisible auxiliary degree shows their ratio is constant. This auxiliary degree may depend on the pair.

Lemma 5.1, (5.9)–(5.11) · p. 14

5. Match constants at double crossings and kill the character

Back in the uniform degree p , the next residues at a double crossing agree because p is even, forcing the constants to agree across components. Connectedness handles the group action even when it permutes components. Thus $\zeta^{-ap/p_0} = 1$, giving $m \mid p/p_0$ and $p\beta \in \mathbb{Z}$.

Lemma 5.1, (5.12), conclusion · p. 15

3. Proposition 6.1: complete section spaces recover the full base field

The exact presentation identifies complete section spaces; effective birationality on the base identifies their ratio field. Put $L = K_X + B$ and assume $p_0 \mid l$ in the diagram.

01 A big base and a fixed presentation

$$L = K_X + B, \quad D_Z \sim_{\mathbb{Q}} D \text{ big}, \quad p_0 \mid l.$$

Regular functions on the generic fibre come from the base; valuations descend effectivity.

[\(6.1\)–\(6.2\); p. 16](#)

02 Identify the complete section spaces

$$H^0(X, \lfloor lL \rfloor) = \psi^{l/p_0} f^* H^0(Z, \lfloor lD_Z \rfloor).$$

Replace negative exceptional coefficients; apply effective birationality.

[\(6.3\); p. 16](#) [\[BZ\] Thm. 1.3; p. 3](#)

03 Produce birational sections on a smooth base model

$$K_W + A + M_W = q^* D_Z + E_W, \quad E_W \geq 0 \text{ exceptional.}$$

$$\text{coeff } A \subset \mathcal{B} \cup \{1\}, \quad pM_W \text{ nef Cartier.}$$

Pushforward removes the exceptional error; section ratios cancel the common factor.

[Prop. 6.1, conclusion; p. 17](#)

04 Recover the full field and identify the litaka map

$$m = \text{lcm}(p_0, b_1, \dots, b_d), \quad l \in m\mathbb{Z}_{>0}.$$

$$\mathbb{C}(\text{all section ratios}) = \mathbb{C}(Z) \subset \mathbb{C}(X).$$

Effective systems and the entire litaka field

[Open figure ↗](#) [TeX source ↓](#)

1. Transfer the complete section space to the base

Set $L = K_X + B$ and assume $p_0 \mid l$. The exact presentation yields an equality inside $\mathbb{C}(X)$:

$$H^0(X, \mathcal{O}_X(\lfloor lL \rfloor)) = \psi^{l/p_0} f^* H^0(Z, \mathcal{O}_Z(\lfloor lD_Z \rfloor)).$$

For the reverse inclusion, divide a source section by ψ^{l/p_0} . It is regular on the generic fibre and hence belongs to $\mathbb{C}(Z)$. Effectivity descends by testing a pullback component of positive multiplicity dominating each prime of the base. This valuation argument applies to reflexive sheaves ((6.1)–(6.2), p. 16).

[\(6.1\)–\(6.2\) · p. 16](#)

2. Construct birational systems on a base model with effective boundary

On a resolution $q : W \rightarrow Z$, the potentially negative exceptional coefficients of the crepant boundary are replaced as follows. Let A be the strict transform of B_Z plus the reduced exceptional divisor. Then

$$K_W + A + M_W = q^* D_Z + E_W, \quad E_W \geq 0 \text{ is } q\text{-exceptional.}$$

The left side is big, the boundary coefficients lie in a fixed DCC set, and pM_W is nef Cartier. Apply [BZ, Theorem 1.3].

(6.3) · p. 16 · BZ, Theorem 1.3

3. Recover the embedded base field from all section ratios

Push its birational sections to Z , and transfer them to X . The common factor ψ^{l/p_0} cancels in ratios, which generate the entire base field. Taking $m = \text{lcm}(p_0, b_1, \dots, b_d)$ works for every positive multiple (pp. 16–17).

Proposition 6.1, conclusion · pp. 16–17

4. Proposition 7.1 and Corollary 7.2: coefficients and class-group torsion

Construct the integer clearing coefficients separately from the integer killing class-group torsion. ACC and $p\mathbf{M}$ supply the former; boundedness and a rationally connected resolution supply the latter. Multiply them at the end.

01 Generalized klt data and a rationally connected resolution

$$D = K_Z + B + M_Z \sim_{\mathbb{Q}} 0, \quad pM \text{ b-Cartier}, \quad \dim Z \leq 4.$$

Extract specified valuations via ordinary klt boundaries; apply boundedness after ACC.

[Prop. 7.1; pp. 17–19](#) [\[Bir\] Thm. 1.7; p. 6](#)

02 Use extraction and ACC to bound singularities

$$\text{coeff } B \subset I_0 \text{ finite}, \quad a_E \geq \epsilon > 0.$$

$$Z \text{ bounded up to isomorphism in codimension one.}$$

Bound smooth-locus topology; the resolution gives finite generation of the class group.

[\(7.1\) and its preparation; pp. 20–21](#)

03 Bound the torsion exponent through topology

$$\text{Cl}(Z)[n] \simeq \text{Hom}(H_1(Z_{\text{reg}}, \mathbb{Z}), \mu_n).$$

$$H_1 \text{ in a finite list of finite groups,} \quad T \text{Cl}(Z)_{\text{tors}} = 0.$$

Use finite coefficients and integrality of pM_Z , then pull back from the base.

[Prop. 7.1; p. 21](#) [Cor. 7.2, \(7.2\); p. 21](#)

04 Clear actual coefficients before principalizing

$$qD \text{ integral,} \quad \ell = Tq, \quad \ell D \text{ principal.}$$

$$r(K_X + B) = \text{Div}((v \circ f)^{r/\ell} \psi^{-r/p_0}) \quad (\text{Cor. 7.2}).$$

Principal multiples over rationally connected bases

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1. Use extraction and ACC to reach boundedness

Construct ordinary klt boundaries to obtain small $\mathbb{Q}$ -factorializations and extractions of specified valuations for the generalized pair. Applying global ACC also after extraction gives a finite boundary coefficient set and a uniform positive lower bound on generalized log discrepancies. A numerically trivial nef part is treated separately by ordinary global ACC. [Bir, Theorem 1.7] then bounds the bases up to isomorphism in codimension one (pp. 17–19).

[Proposition 7.1 · pp. 17–19](#) [· Bir, Theorem 1.7](#)

2. Bound the torsion exponent through bounded topology

For $U = Z_{\text{reg}}$, the group $H_1(U(\mathbb{C}), \mathbb{Z})$ is unchanged upon removing codimension-two subsets. Semialgebraic triviality in a bounded family gives only finitely many such groups. A rationally connected resolution Y has $\text{Pic}^0(Y) = 0$, so $\text{Cl}(Z)$ is finitely generated. Kummer theory gives

$$\text{Cl}(Z)[n] \simeq \text{Hom}(H_1(U(\mathbb{C}), \mathbb{Z}), \mu_n).$$

For fixed Z , the orders on the left stay bounded as n varies. Hence H_1 has no free summand. The finite list of finite groups supplies a uniform torsion exponent T .

Proposition 7.1, (7.1) · pp. 20–21

3. Clear actual coefficients before principalizing

Separately, the finite boundary coefficient set and integrality of pM_Z give an integral actual divisor qD . Its class is torsion, so $\ell = Tq$ makes it principal (pp. 20–21).

Proposition 7.1, conclusion · p. 21

4. Pull the principal divisor back to the total space

Corollary 7.2 applies Theorem 1.1 with $D = 0$ and pulls back the resulting principalization, choosing r divisible by ℓ, p :

$$r(K_X + B) = \text{Div}((v \circ f)^{r/\ell} \psi^{-r/p_0}).$$

Corollary 7.2, (7.2) · p. 21

5. Which papers supply which steps

Separate the uses in Fourfold Iitaka. The exact presentation of Theorem 1.1 and the complete systems of Proposition 6.1 enter the final effectivity step. Principal multiples over rationally connected bases from Proposition 7.1 instead enter the index bound in Lemma 4.6 of that paper (pp. 15–16).

Input

JL, Corollary 1.6, v1 p. 2

Use

§§2–3 and Lemma 5.1: uniform indices for projective slc log Calabi–Yau pairs of dimension at most three, including a global form with gluing.

Verification

Statement and application compared.

Input

FG, Theorem 3.6, p. 1726 / PDF p. 7

Use

§3: qualitative b-nefness and descent after checking rank one.

Verification

Statement and use compared.

Input

CT, Theorems 1.1–1.2, v2 p. 2

Use

§4, p. 9: flip termination for pseudo-effective four-dimensional NQC lc pairs, and for three-dimensional pairs.

Verification

Dimension and pseudo-effectivity conditions compared; full details of Remark 2.11 not compared.

Input

HMX, Theorems 1.1 / 1.5, PDF pp. 2–3

Use

Threshold ACC in §3, finite different coefficients in §5, and ordinary global ACC in §7.

Verification

Statements and uses compared.

Input

BZ, Theorem 1.3, p. 3; Theorem 1.6, pp. 4–5

Use

Effective birationality for big polarized pairs in §6 and generalized global ACC in §7.

Verification

Statements, rounding convention, and applications compared.

Input

Bir, Theorem 1.7, v2 p. 6

Use

§7: boundedness up to codimension-one isomorphism for rationally connected generalized ϵ -lc Calabi–Yau varieties.

Verification

Statement and application compared.

The manuscript's uses of Kollár's depth, adjunction and gluing theorems, Delfs–Knebusch semialgebraic triviality, Kummer theory, and Riemann existence were read; the individual external theorem texts were not separately compared.

Section 3 (pp. 8–9) of the [fourfold Iitaka manuscript](#) takes this paper's Theorem 1.1, Proposition 6.1, and Proposition 7.1 as Theorem 3.1, Proposition 3.2, and Proposition 3.3. All three restatements were compared; all downstream applications were not investigated. The low-dimensional slc input here is [JL](#); it must not be replaced by the high-dimensional slc index manuscript in Catalog 034.

6. Return to the sources

Read the proof text of §§2–7 (pp. 4–21), tracing the exact presentation, disappearance of the dlt-model error, the character argument in Lemma 5.1, and the complete-section-space equality (6.1). In §7, read the extraction–ACC–topology–torsion sequence. Visually checked the character exponent, $p\beta$, and Proposition 6.1 on PDF p. 15.

The complete proofs of external inputs, external foundations for depth and pluriresidue gluing, and an independent audit of all topological hypotheses in §7 remain outside this check. Japanese and English versions retain the same statements, formulas, section structure, and verification status. Source links use the fixed-commit GitHub viewer; pages are stated in the labels.

Arithmetic Stein-degree bounds for log Calabi–Yau pairs

MMP induction, Galois orbits of valuations, and adjunction in the same index

The manuscript bounds the degree of the constant field of a boundary component using only the ambient dimension and a positive lower bound for its coefficient. Its proof combines induction through Mori fibre spaces with two additional arguments: converting geometric Fano boundedness into arithmetic bounds on valuation orbits, and making a vertical component accessible to divisorial adjunction.

Manuscript author: OpenAI · 2026-09-25 · 15 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 (SD, p. 1)

Fix an integer $d \geq 1$ and a real number $t > 0$. Let (X, B) be a projective lc $\mathbb{Q}$ -pair over any characteristic-zero field k , with X normal and integral, $B \geq 0$, and

$$\dim X = d, \quad H^0(X, \mathcal{O}_X) = k, \quad K_X + B \sim_{\mathbb{Q}} 0.$$

If S is a prime component with $\text{coeff}_S B \geq t$, and k_S is the relative algebraic closure of k in $k(S)$, there is an integer $N(d, t)$ depending only on d, t such that

$$c(S/k) := [k_S : k] \leq N(d, t).$$

For the normalization S^ν , this implies

$$\text{sdeg}(S/\text{Spec } k) = \dim_k H^0(S, \mathcal{O}_S) \leq \dim_k H^0(S^\nu, \mathcal{O}_{S^\nu}) = c(S/k) \leq N(d, t).$$

This is the contraction-to-a-point statement for ordinary pairs. It is not a statement for generalized pairs or arbitrary relative Stein degrees. The case $t > 1$ is vacuous; no fixed coefficient set is assumed.

Theorem 1.1 · p. 1

Proposition 3.5 (arithmetic input to the proof; SD, pp. 6–7)

Fix $q \geq 1$, $\epsilon > 0$, and $n \geq 1$. Let Q/k be a normal geometrically integral projective ϵ -lc Fano variety of dimension q , and assume (Q, Λ) is lc, $\Lambda \geq 0$, and $n(K_Q + \Lambda) \sim 0$. Every divisorial valuation with $a(v, Q, \Lambda) < 1$ has constant-field degree bounded by $A(q, \epsilon, n)$. The valuation need not survive as a divisor on Q .

Proposition 3.5 · pp. 6–7

References on the arrows

Unprefixed result numbers refer to this manuscript SD. External inputs use Bir19 for complements, Bir21 for BAB, LM26 for MMP and extraction, and Kol10 for differentials and residues. The final section distinguishes checking statements and applications from checking entire proofs.

- Bir19, Theorem 1.7 · arXiv v4 / Bir21, Theorem 1.1 · arXiv v2
- LM26, arXiv v4
- Kol10, Canonical models · June 1, 2010 draft

2. Proof of Theorem 1.1: the first MMP

Proof route. The first MMP closes by induction over a positive-dimensional base; only a point base leads to the second MMP. Its exits are the arithmetic valuation-orbit bound, induction for horizontal components, and adjunction for vertical components. The four bounds are combined at the end.

Induct on dimension, treating all positive coefficient thresholds simultaneously. For $d = 1$, $\deg B = 2$ gives $tc(S/k) \leq 2$. Henceforth let $d \geq 2$, write $M_{<d}(u) = \max_{1 \leq r < d} N(r, u)$, and fix a rational $0 < b = b(t) < t$. The first MMP preserves S and makes it relatively ample.

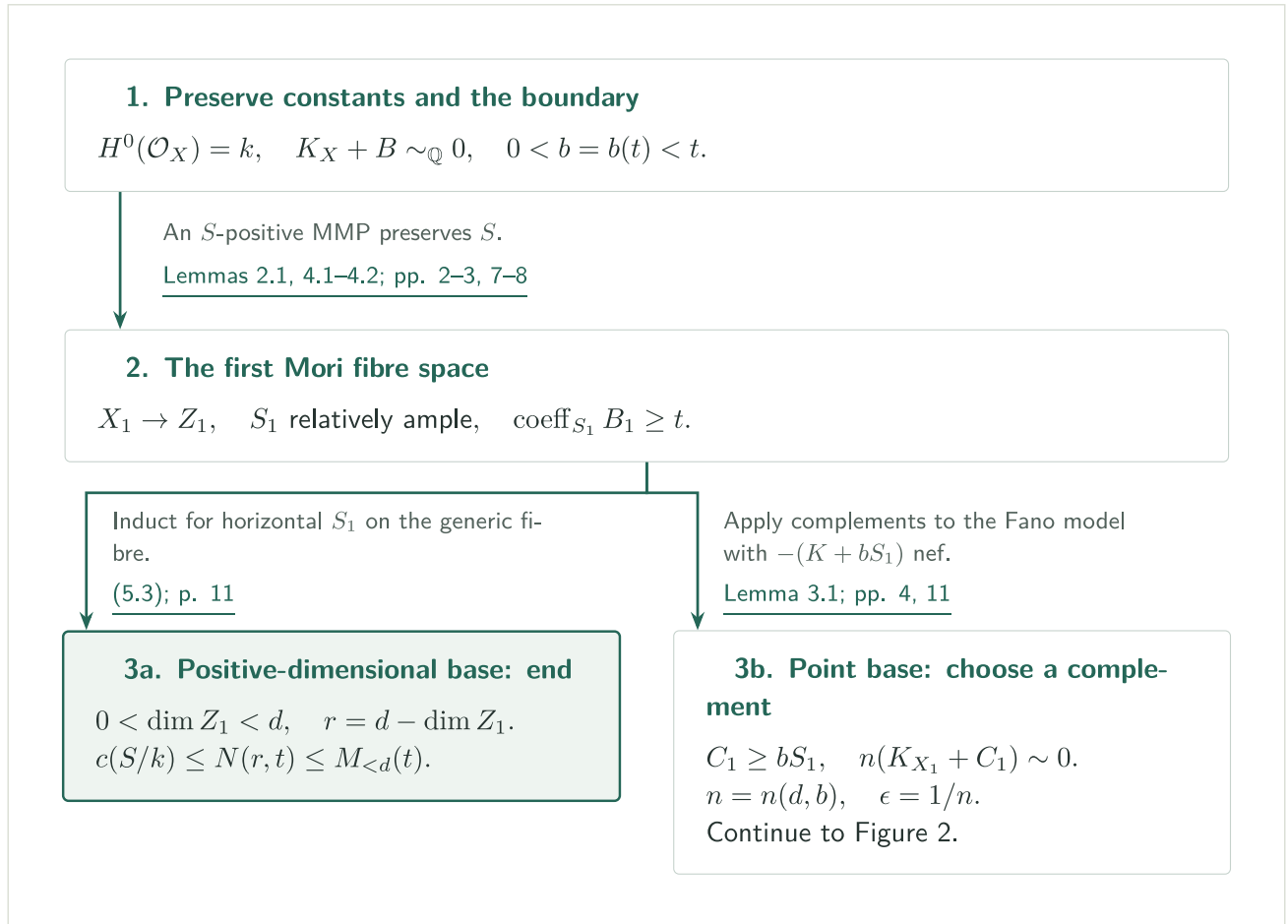


Figure 1. The first MMP: a positive-dimensional base ends by induction; only a point base proceeds to complements and the second MMP.

[Open figure ↗](#) [TeX source ↓](#)

1. Preserve the constants and the boundary component

Lemma 2.1 identifies $c(S/k)$ with the number of geometric components, so tracking the same divisorial valuation through birational operations preserves this degree. Lemma 4.1 supplies a $\mathbb{Q}$ -factorial klt model with effective crepant boundary. Run the MMP for $K + B - bS \sim_{\mathbb{Q}} -bS$. Every contracted extremal ray is positive for S , and negativity prevents S from being contracted. On the resulting $X_1 \rightarrow Z_1$, the transform S_1 is relatively ample and retains its coefficient lower bound t .

Lemmas 2.1, 4.1–4.2 · pp. 2–3, 7–8

2. End on the generic fibre when the base has positive dimension

If $\dim Z_1 > 0$, relative ampleness makes S_1 horizontal. The generic fibre has global function field $k(Z_1)$ and is an lc log Calabi–Yau pair of dimension $r = d - \dim Z_1 < d$ with threshold t .

Induction and the comparison of constants for horizontal components give $c(S/k) \leq M_{<d}(t)$.

This branch needs neither a complement nor the second MMP.

Lemma 2.1; §5.2, (5.3) · pp. 2–3, 11

3. Choose a bounded-index complement when the base is a point

If $Z_1 = \text{Spec } k$, then X_1 is Fano and $-(K_{X_1} + bS_1)$ is nef. Lemma 3.1 applies bounded complements with the fixed coefficient b , then chooses a general member of a linear system over k . It gives an lc boundary $C_1 \geq bS_1$ with $n(K_{X_1} + C_1) \sim 0$. First choose $b = b(t)$, then $n = n(d, b)$. The next step uses this n to control singularities without depending on the original boundary denominators.

Lemma 3.1; §5.2, (5.4) · pp. 4, 11 · Bir19, Theorem 1.7 · p. 4

3. Proof of Theorem 1.1: three exits from the second MMP

Set $\epsilon = 1/n$. Extract the low-discrepancy valuations before running a K -MMP, producing an ϵ -lc Mori fibre space. Tracking the original valuation v_S leads to three cases, even if S no longer survives as a divisor.

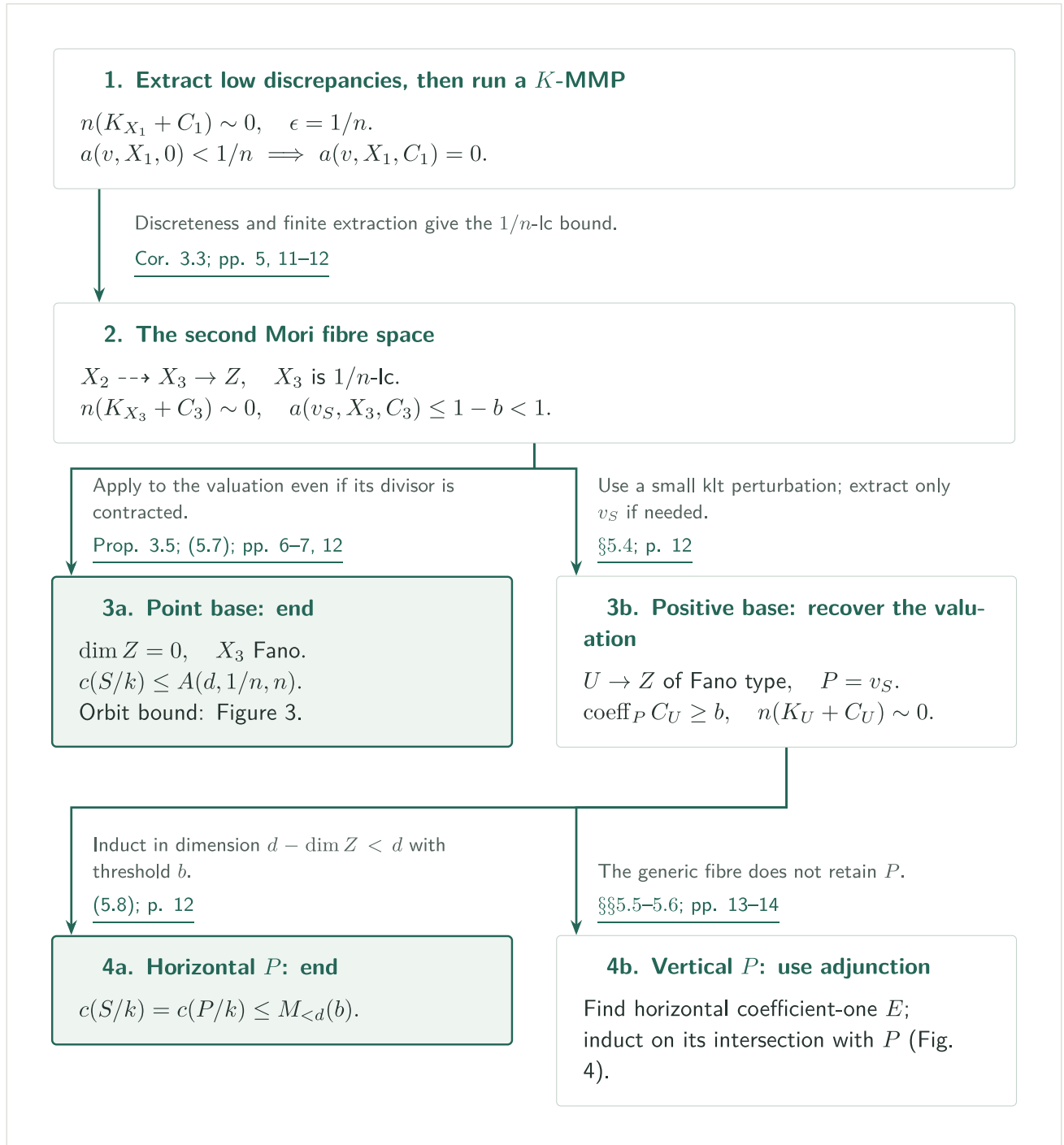


Figure 2. The second MMP: Proposition 3.5 handles a point base; over a positive-dimensional base the recovered P is horizontal or vertical.

[Open figure ↗](#) [TeX source ↓](#)

1. Turn index discreteness into a uniform discrepancy bound

Corollary 3.3 makes the valuations with $a(v, X_1, 0) < 1/n$ a finite set. Their log discrepancies for the complement are nonnegative multiples of $1/n$, no larger than their original discrepancies, so they are all lc places. Extract exactly these valuations to obtain a $1/n$ -lc variety X_2 ; every extracted divisor has coefficient one in C_2 . After a K -MMP, X_3 remains $1/n$ -lc, retains an lc complement of the same index, and satisfies $a(v_S, X_3, C_3) \leq 1 - b < 1$.

Corollary 3.3; §5.3, (5.5)–(5.6) · pp. 5, 11–12

2. End with the valuation-orbit bound over a point

If $\dim Z = 0$, then X_3 is a $1/n$ -lc Fano variety. Proposition 3.5 applies with $q = d$, $\epsilon = 1/n$, and boundary C_3 , giving $c(S/k) \leq A(d, 1/n, n)$. Its input is a valuation satisfying $a(v_S, X_3, C_3) < 1$, so the strict transform of S may already have been contracted. Section 4 explains this arithmetic bound.

Proposition 3.5; §5.3, (5.7) · pp. 6–7, 12

This exit uses the arithmetic proof of Proposition 3.5.

3. Recover the original valuation over a positive-dimensional base

If $\dim Z > 0$, choose $\Gamma = (1 - \lambda)C_3$ for sufficiently small $\lambda > 0$. This is klt with relatively ample anti-log-canonical divisor. Extract only v_S if it is exceptional; otherwise use the divisor already present. On the resulting model $U \rightarrow Z$ of relative Fano type, the prime P satisfies $c(P/k) = c(S/k)$ and has coefficient at least b . The relation $n(K_U + C_U) \sim 0$ is preserved.

§5.4 · p. 12 · LM26, Theorem 22.1 · p. 87

4. Use induction for a horizontal prime and adjunction for a vertical prime

If P is horizontal, apply induction with threshold b on the generic fibre of dimension $d - \dim Z < d$, obtaining $c(S/k) \leq M_{<d}(b)$. A vertical P disappears on that fibre, so the same induction cannot apply. Section 5 constructs a different horizontal coefficient-one component and transfers the intersection with P to its normalization.

§5.4, (5.8); §§5.5–5.6 · pp. 12–14

For the vertical case, continue to adjunction retaining the same index.

4. Proof of Proposition 3.5: from geometric boundedness to arithmetic orbits

Geometric boundedness from BAB must be converted into a bound on the constant field of a valuation over k . Descend a polarization after a bounded extension, then construct a finite Galois set using a bounded SNC resolution.

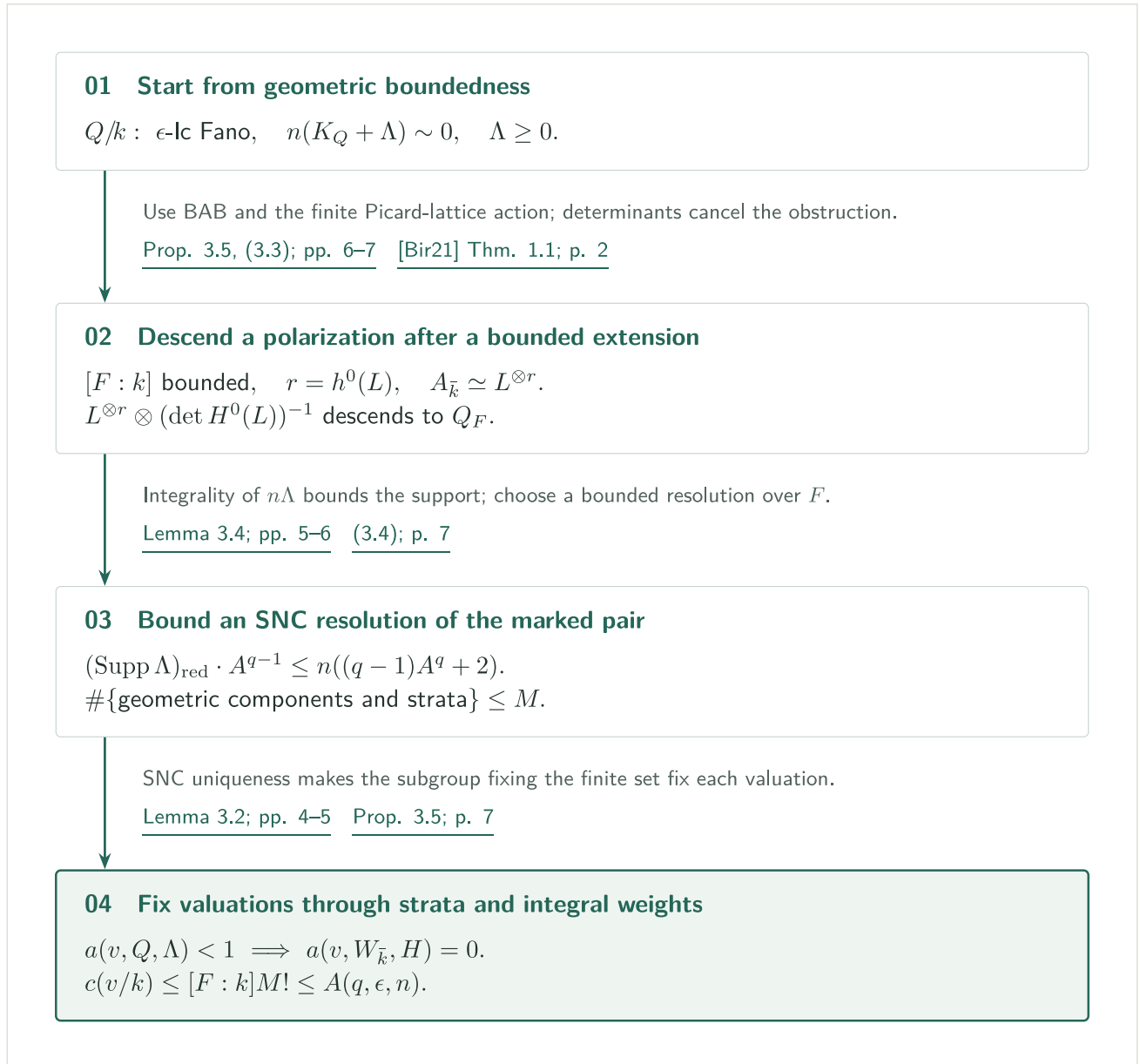


Figure 3. Descending a polarization, bounding a resolution, and fixing valuations through strata and integral weights.

[Open figure ↗](#) [TeX source ↓](#)

1. Remove the polarization's descent obstruction after a bounded extension

BAB supplies a bounded very ample bundle L over $\bar{k}$. The manuscript embeds the Picard group into a lattice of bounded rank and bounds its finite Galois image, obtaining a bounded extension F/k fixing the class of L . Fixing a class does not finish descent of the bundle. With $r = h^0(L)$, the following determinant factor cancels the scalar two-cocycle and supplies a bounded very ample bundle A on Q_F .

$$L^{\otimes r} \otimes (\det H^0(L))^{-1}.$$

Proposition 3.5, (3.3) · pp. 6–7 · Bir21, Theorem 1.1 · p. 2

2. Bound a resolution containing the boundary support

Integrality of $n\Lambda$ and $n(K_Q + \Lambda) \sim 0$ bound the degree of the boundary support with respect to A . Apply Lemma 3.4 to this embedding and support to obtain a resolution over F and a reduced SNC divisor H containing the boundary. A uniform bound M on its geometric components and strata supplies the finite set needed in the group-action argument.

Lemma 3.4; Proposition 3.5, (3.4) · pp. 5–7

3. Fix each valuation by fixing components and strata

If $a(v, Q, \Lambda) < 1$, then $a(v, W_{\bar{k}}, H)$ is a nonnegative integer less than one, hence zero. Lemma 3.2 uniquely determines this valuation from its centre stratum and positive integral weights. Thus the Galois subgroup fixing every element of the finite set fixes each valuation, and its orbit has size at most $[F : k]M! \leq A(q, \epsilon, n)$. There is no need to bound the total number of valuations.

Lemma 3.2; Proposition 3.5 · pp. 4–5, 7

5. Proof of Theorem 1.1: bringing the vertical component into adjunction

Consider the vertical branch of the second MMP. Construct a nonzero intersection J of P with a horizontal coefficient-one component E , combine the two lower-dimensional bounds for E and J , and return to the number of conjugates of P .

01 A horizontal coefficient-one component from bigness

$\pi^*S_1 \text{ big} \implies (\pi^*S_1)_{X_3} \text{ big}.$
 $E_3 \subset [C_3] \text{ horizontal, } c(E_3/k) \leq M_{<d}(1).$

The relative $-P$ MMP preserves horizontal E , which surjects onto the base.

Lemma 4.3; §§5.5–5.6; pp. 8–9, 13

02 Force intersection through an actual pullback

$mP' = f^*T \neq 0, \quad E \rightarrow Z' \text{ surjective.}$
 $0 \neq J \subset P'|_{E^\nu}.$

Compare powers of residues to retain n ; adding P contributes a positive term.

Lemma 4.4; pp. 9–10 [Kol10] (122.7); p. 62

03 Apply adjunction in the same index n

$n(K_{E^\nu} + C_{E^\nu}) \sim 0, \quad nC_{E^\nu} \text{ integral.}$
 $\text{coeff}_J C_{E^\nu} > 0 \implies \text{coeff}_J C_{E^\nu} \geq 1/n.$

Induct in dimension $d-1$, with threshold $1/n$ and global constants k_E .

Lemma 2.1; (5.11); pp. 2–3, 14

04 Induct over the constants and multiply degrees

$c(J/k) = [k_E : k]c(J/k_E)$
 $\leq M_{<d}(1)N(d-1, 1/n).$

At most $2/b$ components of coefficient at least b pass through each image.

Lemma 4.5; (5.12); pp. 10, 14

05 Count the original conjugates at codimension two

$c(S/k) = c(P'/k) \leq \frac{2}{b} c(J/k)$
 $\leq \frac{2}{b} M_{<d}(1)N(d-1, 1/n).$

Figure 4. Bigness supplies a horizontal component; intersection, same-index adjunction, a tower of constants, and codimension-two counting finish the proof.

Open figure ↗ TeX source ↓

1. Recover a horizontal coefficient-one component from earlier bigness

Since $\mathcal{S}_1$ is ample on the Fano variety X_1 , the divisor $\pi^*\mathcal{S}_1$ on X_2 is effective and big. Its pushforward to X_3 remains big: the map extracts no divisors and preserves the required growth of sections. A divisor supported only in vertical components cannot restrict to a big class on the positive-dimensional generic fibre. The original $\mathcal{S}$ has vertical centre, so a horizontal component E_3 comes from an extracted exceptional divisor and has coefficient one in $\mathcal{C}_3$. Generic-fibre induction for this component gives $c(E_3/k) \leq M_{<d}(1)$.

§5.5, (5.9)–(5.10) · p. 13

2. Use a relative MMP to force a nonzero intersection

On the relative Fano type model U , verticality of P makes $-P$ relatively pseudo-effective. The big-boundary MMP of Lemma 4.3 preserves P and the horizontal E , and basepoint freeness makes $-P'$ semiample. For its contraction $f: U' \rightarrow Z'$, the map $Z' \rightarrow Z$ is birational. Descending the canonical section gives the actual divisor identity $mP' = f^*T$, where $T \neq 0$ is effective Cartier. Since E is proper and horizontal, it surjects onto Z' ; hence $P'|_{E^\nu} \neq 0$. Choose a prime component J of this restriction.

Lemma 4.3; §5.6 · pp. 8–9, 13

3. Preserve the index by residue and discretize positive coefficients

The effective lc pair (U', C') has coefficient one along E , coefficient $\beta \geq b$ along P' , and $n(K_{U'} + C') \sim 0$. Lemma 4.4 gives an effective lc different with $n(K_{E^\nu} + C_{E^\nu}) \sim 0$ for the same n . The manuscript's additional argument takes the residue of a rational n -canonical form over k and recovers the degree- n divisor identity from an identity for a sufficiently divisible power.

Equation (122.7) of Kol10 supplies the residue isomorphism used in this comparison. Equation (122.10) also assumes that $K + E$ itself is $\mathbb{Q}$ -Cartier, so that formula alone does not settle the general case. The manuscript compares residues before and after adding $\beta P'$ at a common Cartier multiple, obtaining $C_{E^\nu} \geq \beta P'|_{E^\nu}$. Thus J has positive coefficient; integrality of nC_{E^ν} makes it at least $1/n$.

Lemma 4.4, (4.2) · pp. 9–10 · Kol10, Definition 122, Proposition 123, Lemma 125 · pp. 61–64

4. Induct over the component's own constants and multiply degrees

Let k_E be the constant field of E^ν . Lemma 2.1 makes E^ν/k_E normal and geometrically integral with $H^0(\mathcal{O}_{E^\nu}) = k_E$. This finite separable extension preserves the canonical divisor, so induction in dimension $d - 1$ with threshold $1/n$ applies to the adjunction pair. Since $k_E \subset k(J)$, the bound for the horizontal component gives

$$c(J/k) = [k_E : k]c(J/k_E) \leq M_{<d}(1)N(d - 1, 1/n).$$

Lemma 2.1; §5.6, (5.11) · pp. 2–3, 13–14

5. Count at codimension two and return to the original component

Over $\bar{k}$, the image of a component of J has codimension two in U' . The geometric components of P' form one Galois orbit, and each contains the image of a conjugate of J . Lemma 4.5 bounds by $2/b$ the number of components of coefficient at least b through each image. It cuts down to a klt surface quotient and blows up a point on a smooth cover, bounding the boundary multiplicity by two. Consequently,

$$c(S/k) = c(P'/k) \leq \frac{2}{b} M_{<d}(1)N(d-1, 1/n).$$

Lemma 4.5; §5.6, (5.12) · pp. 10, 14

6. Combine the four bounds and close the induction

The positive-dimensional base of the first MMP, the point base of the second, horizontal P , and vertical P give four bounds. Choose an integer satisfying

$$N(d, t) \geq \max \left\{ M_{<d}(t), M_{<d}(b), A(d, 1/n, n), \frac{2}{b} M_{<d}(1)N(d-1, 1/n) \right\}.$$

First fix $b = b(t)$, then $n = n(d, b)$; every occurrence of N on the right is already known in lower dimension. The original boundary denominators, the field, and the number of extracted divisors do not enter the dependence of the bound.

§5.7, (5.13) · p. 14

6. Which papers supply which steps

Use in normal lc indices. Proposition 3.2 of Uniform Itaka (pp. 7–8) applies Theorem 1.1 over the generic-fibre field $k(Z)$ with threshold $t = 1$. The bound on the finite Stein degree enters the norm identity, giving a uniform principal multiple of the original adjoint.

Input

Bir19, Theorem 1.7, arXiv p. 4

Use and hypotheses

Lemma 3.1 (p. 4): bounded complements for fixed rational $\mathbf{b}$, Fano type, lc singularities, and nef $-(K + \mathbf{b}S)$. The full original coefficient set need not be fixed.

Checks performed

Statement and application compared; proof not independently checked.

Input

Bir21, Theorem 1.1, arXiv p. 2

Use and hypotheses

Proposition 3.5 (pp. 6–7): geometric boundedness of ϵ -lc Fanos. Arithmetic descent is supplied inside this manuscript.

Checks performed

Statement and use compared.

Input

LM26, Corollaries 21.9–21.10, pp. 86–87; Theorem 22.1, p. 87

Use and hypotheses

Lemmas 4.1–4.3 and §§5.3–5.4: termination of klt MMPs and extraction of specified valuations.

Checks performed

Relevant statements compared. LM26 discrepancy ≤ 0 corresponds to log discrepancy ≤ 1 here. The basepoint-free input from Theorem 11.1 was located only through this manuscript's citation.

Input

Kol10, Definition 122, (122.7)–(122.10), Proposition 123, Lemma 125, pp. 61–64

Use and hypotheses

Lemma 4.4 (pp. 9–10): differentials and pluriseculars.

Checks performed

Compared the June 1, 2010 draft's statements and application. Retaining the same index and adding one component are additional arguments in this manuscript; the external proofs are not independently verified.

Input

BMT11 and KM98

Use and hypotheses

Lemma 3.4: resolution over the field of definition; Lemma 4.5: quotient description of klt surface singularities.

Checks performed

Use located in the manuscript; external theorem texts not checked.

Within Catalog 034, [Uniform log Itaka](#), Theorem 2.3 (p. 5), restates this theorem. Lemma 4.6 (pp. 10–11) applies it over $k = \mathbb{C}(C)$ with coefficient lower bound one. It bounds the degree of the Stein curve $C_S \rightarrow C$ of a horizontal coefficient-one component. After residue, weights change by $w \mapsto ew$; the resulting denominators are cleared using $\text{lcm}(1, \dots, N(s, 1))$. This direct application was compared with the theorem. The norm-descent use in Uniform Itaka described above was also checked at the level of the input statement and application. The Birkar–Qu strategy is background, not an input theorem in a dependency arrow here.

7. Return to the sources

Read Theorem 1.1, Lemmas 2.1–2.2, and the proof text of §§3–5 (pp. 2–14), with attention to Proposition 3.5, Lemmas 4.3–4.5, and §§5.5–5.7. Visually checked the constant-field product, the factor $2/b$, and the recursion on PDF p. 14. This revision rechecked the MMP branches and the vertical argument, and compared the residue construction, effectivity, and discrepancy statements in Kol10, pp. 61–64. The Japanese and English versions retain the same hypotheses, formulas, diagrams, and verification scope.

These checks concern reductions and cited applications. They do not independently establish the construction of resolution families, the external adjunction and quotient-singularity theorems, or every MMP input. Some internal cross-references in the source call lemmas or propositions “Theorem”; this article uses the actual headings, such as Lemma 3.1 and Proposition 3.5. Catalog-source links use the fixed-commit GitHub viewer, with pages in their labels.

Uniform effective log litaka fibrations for fourfolds

From canonical indices to effective log litaka systems in dimension four

The manuscript bounds canonical indices of boundary-free klt fourfolds and incorporates that bound into effective log litaka systems for lc pairs with finite rational coefficients. The index argument contrasts local positivity on covers with jet estimates on diagonal products. The final uniform degree combines qualitative nonvanishing, semiample models and relative denominators.

Manuscript author: OpenAI · 2026-09-26 · 52 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Uniform canonical index

There is a positive integer N_4 such that every normal projective klt complex fourfold X with $K_X \sim_{\mathbb{Q}} 0$ satisfies

$$N_4 K_X \sim 0.$$

This is a trivialization as an integral principal Weil divisor. The theorem has no boundary and does not assume that X is smooth.

[Theorem 1.1 · p. 1](#)

Theorem 1.2 — Uniform effective log litaka fibrations

Fix a finite set $\Phi \subset [0, 1] \cap \mathbb{Q}$. There is a positive integer m depending only on Φ such that, for every normal integral projective complex fourfold X and effective rational Weil divisor Δ with nonzero coefficients in Φ , if (X, Δ) is lc and $D = K_X + \Delta$ is rational Cartier and

pseudo-effective, the complete system $|\lfloor mD \rfloor|$ is nonempty and its section ratios generate the entire embedded Iitaka field $K(D) \subset \mathbb{C}(X)$. The same m works for every canonical representative.

Here $\mathcal{O}_X(\lfloor mD \rfloor)$ is a rank-one reflexive divisorial sheaf; there is no additional requirement that mD be Cartier. The conclusion includes birationality when $\kappa(D) = 4$ and nonvanishing in a uniform degree when $\kappa(D) = 0$. The coefficient hypothesis is a **finite rational set**, not an arbitrary DCC set or real coefficients.

Theorem 1.2 · p. 2

References on the arrows

Unprefixed references denote this manuscript. External inputs use the following keys.

- **LA**: Log abundance in characteristic zero (September 24, 2026).
- **SL**: Lifting sections from the reduced support of an adjoint (September 27, 2026).
- **RD**: Relative denominators and effective systems for log Calabi–Yau fibrations (September 27, 2026).
- **CT**: fourfold flip termination (arXiv:2011.02236v2, Theorem 1.1).
- **JL**: non-klt fourfold index (arXiv:2002.11928v1, Corollary 1.7).
- **Xu**: klt fourfold index with nonzero boundary (arXiv:1905.00297v2, Theorem 1.14).

2. Proof of Theorem 1.1: structural reduction

Proof route. First organize the structural reduction, then exclude the remaining obstruction using [chains](#), [jets](#) and [Frobenius estimates](#). The [closing proof](#) combines that structure with relative effective systems and aligns coefficient sets, fields and degrees.

Reduce the index problem in Theorem 1.1 to terminal fourfolds with general-type equivariant fibres and countable birational automorphism groups. In the structural cases using relative denominators, the crucial point is to preserve the boundary-free canonical index.

01 Boundary-free canonical index

$$K_X \sim_{\mathbb{Q}} 0, \quad \dim X = 4$$

Handle product, abelian and noncanonical cases; take a crepant terminalization in the remainder.

Prop. 4.1–Lem. 4.5, pp. 10–15

02 Residual structural case

$$V \text{ terminal}, \quad q(\tilde{V}) = 0, \quad K_V \sim_{\mathbb{Q}} 0$$

Preserve the canonical index along the small-boundary MMP; apply denominators with zero boundary and bound the RC-base index.

Lem. 4.6, pp. 15–16 RD Thm. 1.1 / Prop. 7.1, pp. 2, 17

03 An intermediate divisor bounds the index

$$1 \leq \kappa(A) \leq 3 \implies N_{\text{mov}} K_V \sim 0$$

Restrict to the case without an intermediate divisor; use big exceptional divisors and terminal discrepancies on fibres.

Lem. 4.7, pp. 16–17

04 Residual equivariant fibres are of general type

$$\pi : Y \rightarrow V, \quad r = \text{ord}(K_V)$$

Structural reduction of the canonical index

[Open figure ↗](#) [TeX source ↓](#)

1. Separate product and rationally connected cases using the cover decomposition

The first reduction uses a decomposition on a finite quasi-étale cover and characters on volume forms. After separating product and abelian cases, the remaining single nonflat four-dimensional factor has the property that its positive-dimensional proper rational images are rationally connected (Ufour, Proposition 4.1–Lemma 4.2, pp. 10–14). In the noncanonical case the canonical divisor on a resolution is not pseudo-effective. Uniruledness, the MRC quotient, and the restriction on rational images then imply rational connectedness; the passage from non-pseudo-effectivity to uniruledness follows Corollary 4.4 (p. 15).

Proposition 4.1–Corollary 4.4 · pp. 10–15

2. Use relative denominators when an intermediate divisor exists

In the canonical case a crepant terminalization preserves the index. If an effective Weil divisor A has $1 \leq \kappa(A) \leq 3$, use a semiample model for $(V, \delta A)$ with $\delta > 0$ small. The proof checks **crepancy for the boundary-free canonical divisor separately** along this MMP. Pushing forward $rK_V = \text{Div}(h)$ preserves the index, and the relative denominator theorem is applied to $(V', 0)$, not to the varying coefficient δ . On the rationally connected base, [RD, Proposition 7.1, p. 17](#) gives an integral principal multiple of the generalized adjoint. The denominator of δ therefore does not enter the constant ([Ufour, Lemma 4.6, pp. 15–16](#)).

[Lemmas 4.5–4.6 · pp. 15–16](#)

3. Make residual equivariant fibres general type

For the remaining terminal fourfold and its canonical cyclic cover, the relevant intermediate equivariant rational fibrations have general-type smooth geometric generic fibres. Lemma 4.7 (pp. 16–17) uses bigness of a divisor pulled back from a descended base and positive terminal discrepancies to make the canonical divisor of the fibre big. This is the hypothesis needed for the next uniform estimates.

[Lemma 4.7 · pp. 16–17](#)

3. Proof of Theorem 1.1: excluding the residual high indices

The chain tools Lemma 5.1, Corollary 5.2 and Lemma 5.4 used here are stated in arbitrary dimension and also enter [high-index exclusion in Uniform Iitaka](#). They supply covering-family, chain and leaf estimates; this is not an application of the fourfold main theorem in arbitrary dimension.

Assume that the index r is unbounded in the residual case. Compare the Seshadri lower bound on the canonical cover $\pi : Y \rightarrow V$ with the upper bound on the diagonal product $Z_t = Y^t / \mu_{r, \text{diag}}$ through projection-compatible chain leaves.

The chain quotient, projection compatibility and degree estimates (Lemmas 5.1, 5.4 and Corollary 5.2), and the tracking lemmas (Lemmas 6.3–6.4), are stated in arbitrary dimension. They must be distinguished from **the four-dimensional Theorem 6.1 and main theorems**, which use volume estimates on lower-dimensional fibres.

[Lemmas 5.1, 5.4 · Corollary 5.2 · Lemmas 6.3–6.4 · pp. 18–27](#)

01 Assume an unbounded residual index sequence

$$r \rightarrow \infty, \quad 1 \leq L^4 \leq 2$$

The general-type fibre condition gives scalar estimates; rank deficit on the Frobenius diagonal gives the quadratic bound.

Thm. 6.1, p. 23; Prop. 7.1, pp. 33–42

02 Lower bounds on covers; upper bounds on diagonal products

$$\varepsilon(L) \geq c, \quad \varepsilon(\pi^*L) \geq cr^{1/4}, \quad \varepsilon(P_t) \leq Ct^2$$

Fix the number of products first; polynomial degree bounds exclude exponential growth of forgetting degrees.

Lem. 5.4, p. 21; Prop. 8.1, pp. 42–45

03 Leaf degrees force degree-one forgetting

$$1 \leq h_t \leq 4, \quad D_t \leq KT^8, \quad W_k \dashrightarrow W_{k-1} \text{ birational}$$

Degree one gives rational evaluation; uncountably many local flows contradict countability of the birational group.

Prop. 8.1 / Lem. 8.2, pp. 45–47

04 Rational evaluation and local flows give a contradiction

$$\{E_z\} \subset \text{Bir}(V) \implies r \text{ bounded}$$

Growing indices contradict rational transport

[Open figure ↗](#) [TeX source ↓](#)

1. Apply scalar bounds using the general-type condition

Let $\gamma(P; V)$ be the supremum of section order divided by degree at a general point, using sufficiently divisible degrees. The scalar bounds of Theorem 6.1 (p. 23) assume the preceding general-type condition on equivariant fibrations. Applied to a normalized polarization $1 \leq L^4 \leq 2$ and the canonical cyclic cover $\pi : Y \rightarrow V$ of index r , they give $\varepsilon(L) \geq c$ and $\varepsilon(\pi^*L) \geq cr^{1/4}$.

[Definition 5.3](#) · [Theorem 6.1](#) · pp. 21, 23

2. Turn diagonal rank loss into contradictory vanishing orders

In contrast, Proposition 7.1 (pp. 33–42) bounds $\varepsilon(P_t) \leq Ct^2$ for the external-sum polarization P_t on $Z_t = Y^t / \mu_{r, \text{diag}}$. In characteristic p , a surplus of ordinary jets produces a matrix of full rank at a general point, but of rank at most $R/4$ on the Frobenius diagonal. A nonzero maximal minor

must vanish there to order at least $3R/4$; slotwise scalar bounds and Lemma 7.4 give a strictly smaller order. LA §9 is a methodological predecessor here; its abundance conclusion is not an input to this jet calculation.

Proposition 7.1 · Lemma 7.4 · pp. 33–42

3. Obtain degree-one forgetting from chain leaves

Proposition 8.1 (pp. 42–46) applies the two Seshadri estimates to chain leaves. Fix finitely many products $2 \leq t \leq T$ before increasing the covering order r . Each leaf then maps generically finitely to any single slot, so its dimension lies between 1 and 4. Leaf degrees are bounded by KT^8 . If every forgetting map along a constant-dimensional stretch had degree at least two, the degrees would grow exponentially. Some forgetting map must consequently have degree one.

Proposition 8.1 · pp. 42–45

4. Extend local flows birationally and bound the index

Degree one is essential: it converts evaluation of the missing coordinate from a finite correspondence into a rational map. Tangent transport and local flows then produce birational self-maps. A positive-dimensional flow gives uncountably many such maps, contradicting Lemma 8.2 (pp. 46–47), which proves countability of $\mathbf{Bir}(V)$ using irregularity zero on a smooth model. Section 9 (pp. 47–48) excludes unbounded index sequences and takes a common multiple of the bounded orders in all cases.

Proposition 8.1 · Lemma 8.2 · §9 · pp. 45–48

4. Proof of Theorem 1.2

For lc fourfold pairs with coefficients in the fixed finite set Φ , first take a semiample model preserving sections in every integer degree. A positive-dimensional Iitaka base calls for relative denominators and effective systems; a point base calls for index theorems.

01 An lc fourfold pair with fixed finite coefficients

$$D = K_X + \Delta \text{ pseudo-effective, } \text{coeff}^+(\Delta) \subset \Phi$$

LA supplies qualitative nonvanishing; SL makes the nef model semiample, and pole inequalities compare every integer degree.

[LA Thm. 11.1, p. 73](#) [SL Thm. 1.2, p. 2](#) [Prop. 2.2, pp. 6–7](#)

02 A semiample model preserving sections in every degree

$$D' \sim_{\mathbb{Q}} f^*A, \quad A \text{ ample, } \Phi' = \Phi \cup \{1\}$$

Use RD for a positive-dimensional base; separate non-klt, klt with boundary, and boundary-free klt over a point.

[RD Prop. 6.1, p. 15](#) [JL Cor. 1.7, p. 2](#) [Xu Thm. 1.14, p. 3](#) [Thm. 1.1, p. 1](#)

03 Choose uniform degrees by base dimension

$$\dim Z > 0 : m_+; \quad \dim Z = 0 : m_{\text{nklt}}, m_{\Delta \neq 0}, N_4$$

Take a common multiple and use the every-multiple conclusion and all-degree comparison; canonical representatives preserve ratios.

[eq. \(2.1\), p. 6; Sect. 10, pp. 48–49](#)

04 Recover the litaka field with the original complete system

$$\mathbb{C}(s_i/s_j : s_i, s_j \in H^0(X, \mathcal{O}_X(\lfloor mD \rfloor))) = K(D)$$

All-degree comparison and the uniform litaka degree

[Open figure ↗](#) [TeX source ↓](#)

1. Construct a semiample model after qualitative nonvanishing

Initial nonvanishing comes from good models and semiampleness in LA, Theorems 11.1, 1.1.

The degree of the resulting nonzero section depends on the input and is not yet uniform (Ufour, §10, p. 48). Proposition 2.2 (pp. 6–7) next uses a crepant dlt modification, termination of fourfold flips, and abundance after nonvanishing from SL, Theorem 1.2, p. 2 to obtain a model (X', Δ') with coefficient set $\Phi' = \Phi \cup \{1\}$.

[Proposition 2.2 · §10 · pp. 6–7, 48](#)

2. Identify rounded section spaces in every integer degree

The comparison in every integer degree is

$$H^0(X, \mathcal{O}_X(\lfloor kD \rfloor)) = H^0(X', \mathcal{O}_{X'}(\lfloor kD' \rfloor)) \quad (k \geq 0).$$

This is equation (2.1), checked through the pole inequality $\text{Div}(v) + kD \geq 0$ in (2.2) and an effective exceptional difference on a common resolution. A comparison only in selected Cartier degrees would not suffice. For the semiample contraction $f : X' \rightarrow Z$, one has $D' \sim_{\mathbb{Q}} f^*A$ with A ample. Relative algebraic closedness for the contraction places every section ratio in $\mathbb{C}(Z)$.

(2.1)–(2.2) · Lemma 2.3 · pp. 6–8

3. Use effective systems with fixed denominators on a positive-dimensional base

When $\dim Z > 0$, RD, Theorem 1.1, p. 2 gives an exact canonical bundle formula with fixed denominators, and RD, Proposition 6.1, pp. 15–17 recovers the entire base field in **every positive multiple** of a uniform degree. These appear as Ufour Theorem 3.1 and Proposition 3.2 (p. 8). The birational contraction case $\dim Z = 4$ is included.

Theorem 3.1 · Proposition 3.2 · §10 · pp. 8, 48–49

4. Separate three cases over a point base

When Z is a point, $D' \sim_{\mathbb{Q}} 0$. The index is bounded by JL, Corollary 1.7, p. 2 in the non-klt case, by Xu, Theorem 1.14, p. 3 when the pair is klt with $\Delta' \neq 0$, and by the present Theorem 1.1 when it is klt with $\Delta' = 0$. Xu's hyperstandard convention includes the given finite set by using $\{1 - b : b \in \Phi'\}$ and denominator one. Take a common multiple of these degrees and the positive-base degree; the latter permits this because its conclusion holds for every positive multiple. Equation (2.1) returns to the original complete systems.

§10 · p. 49

5. Preserve section ratios under a change of canonical representative

Replacing the canonical divisor by $K_X + \text{Div}(h)$ changes the floor divisor by $m \text{Div}(h)$. Multiplication of sections by h^{-m} identifies the systems without changing their ratios, so the uniform degree is independent of the representative (Ufour, p. 49).

§10 · p. 49

5. Which papers supply which steps

- **LA** Theorem 11.1 (p. 73), Theorem 1.1 (p. 2): qualitative nonvanishing in Ufour §10 (p. 48). The reference to its Frobenius method in §7 is a separate methodological relation.
- **SL** Theorem 1.2 (p. 2): semiampleness of the nef model in Ufour Theorem 2.1 and Proposition 2.2 (pp. 6–7). The main Fourfold nonvanishing theorem is not added as an input to that proposition.

- **RD** Theorem 1.1, Propositions 6.1, 7.1 (pp. 2, 15, 17): Ufour §3 (pp. 8–9). The first two supply effective systems; the last controls the generalized index over the rationally connected base in Lemma 4.6.
- **CT** Theorem 1.1 (arXiv v2, p. 2): flip termination for pseudo-effective NQC lc generalized fourfold pairs. Proposition 2.2 applies it with zero nef data. Its statement and applicability were compared.
- **JL / Xu** The index statements above: the two point-base cases in Ufour §10 (p. 49). Their original statements were compared; their proofs are outside this check.

6. Return to the sources

The source is the [fixed Ufour PDF](#), commit `adc7f1241b42e322a6451854ab7e4b4c146bf78a`. Page references use printed PDF pages. GitHub's PDF viewer does not retain page positions, so use the displayed page and result numbers after following a link.

The editors read the main statements, all-degree comparison in §2, inputs in §3, the intermediate-fibration reductions in §4, the statement of Theorem 6.1, the setup and final determinant comparison in §7, the leaf-degree, degree-one forgetting and rational-flow connections in §8, and the assembly in §§9–10. The directly used RD, SL and LA statements and the three existing index/termination statements above were compared with their sources.

Unresolved: original-source checking of all decomposition and holonomy inputs in §4; all local calculations in the scalar estimates of §6; all spread, semicontinuity and Frobenius-bundle constructions in §7; proofs of external results including Matsumura's theorem. These connections are not claimed to be independently verified.

Abundance after nonvanishing for compact Kähler fourfolds

From generation on the entire boundary to lifting in finite orders

The manuscript treats nef klt adjoints already possessing a nonzero section. Positive Iitaka dimension reduces to an existing Kähler theorem. In Iitaka dimension zero, generation on the entire supported dlt boundary and subsequent growth of ambient sections give a contradiction.

Manuscript author: OpenAI · 2026-09-27 · 92 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Abundance after nonvanishing

Let X be a normal connected compact Kähler complex space of dimension $\dim X = 4$. Let Δ be an effective rational Weil divisor such that (X, Δ) is klt and the actual adjoint $D = K_X + \Delta$ is $\mathbb{Q}$ -Cartier. If D is analytically nef and $\kappa(X, D) \geq 0$, there is an integer $m > 0$ such that mD is Cartier and

$$H^0(X, \mathcal{O}_X(mD)) \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow \mathcal{O}_X(mD)$$

is surjective everywhere. If $\kappa(X, D) = 0$, one can moreover arrange $\mathcal{O}_X(mD) \simeq \mathcal{O}_X$.

There is no projectivity, $\mathbb{Q}$ -factoriality or numerical-dimension assumption on the original X . Nonvanishing is a hypothesis. The “actual adjoint” uses the holomorphic line $(\omega_X^{[r]} \otimes \mathcal{O}_X(r\Delta))^{**}$, for a coefficient-clearing index r , and its tensor powers (equation (2.1), p. 6). Neither K_X nor Δ need be separately $\mathbb{Q}$ -Cartier. Comparisons concern this line itself: equality of numerical classes would leave a possible flat-line difference.

Theorem 1.1 · (2.1) · pp. 3, 6

Theorem 1.2 — Supported lifting

Let (V, B) be a normal irreducible compact Kähler dlt pair of positive dimension, with effective rational boundary and actual $\mathbb{Q}$ -Cartier adjoint $A = K_V + B$. Suppose an effective nonzero rational $\mathbb{Q}$ -Cartier divisor P satisfies

$$P \sim_{\mathbb{Q}} A, \quad \text{Supp } P = \text{Supp} \lfloor B \rfloor.$$

If the actual restriction $A|_S$ to the **entire** reduced subspace $S = \lfloor B \rfloor$ is semiample, then $\kappa(V, A) \geq 1$. This theorem is dimension-free and assumes neither nefness of A nor $\mathbb{Q}$ -factoriality of V . The support equality is essential and is not replaced by inclusion.

Theorem 1.2 · p. 5

References on the arrows

Unprefixed references denote this manuscript. External keys and checked versions are as follows.

- **HLL**: Höring–Lazić–Lehn (arXiv:2508.14634v2).
- **DHP**: Das–Hacon–Păun (arXiv:2205.12205v3).
- **DO**: Das–Ou (arXiv:2306.00671v4).
- **FT**: Fujino, Vanishing theorems for projective morphisms between complex analytic spaces (arXiv:2205.14801v7).
- **Sai**: Saito, Some remarks on decomposition theorem for proper Kähler morphisms (arXiv:2204.09026v5).
- **SS**: Sabbah–Schnell, mixed Hodge module project, Version 2, Chapter 16.
- **FG**: Fujino–Gongyo, log pluricanonical representations (final author PDF).

2. Proof of Theorem 1.1

Proof route. The zero-Kodaira branch uses [generation on the whole boundary](#), [Theorem 6.1](#), followed by [supported lifting](#), [Theorem 1.2](#). Both return to [step 5](#) here to give semiampness of the original nef adjoint.

Use the assumed nonvanishing to choose an effective representative, and separate positive Iitaka dimension from zero. If its zero divisor is nonzero in the latter case, construct a supported nef model with surviving support; boundary generation and [Theorem 1.2](#) then give a contradiction.

01 An actual nef adjoint after nonvanishing

$$D = K_X + \Delta, \quad \kappa(X, D) \geq 0, \quad \dim X = 4$$

Apply HLL on a crepant $\mathbb{Q}$ -factorial model and descend generation to the original space.

[DHP Thm. 6.1, p. 43](#) [HLL Thm. 4.1, pp. 17–18](#) [Prop. 5.2, p. 34](#)

02 Settle positive litaka dimension; take a zero divisor in the remainder

$$\kappa = 0, \quad M = \operatorname{div}(s_0)/m_0; \quad M \neq 0 \text{ assumed}$$

Run the supported program with exact floor support; original nefness and exceptional negativity preserve nonzero support.

[DHP Thm. 7.2, p. 46](#) [Prop. 5.1, p. 33; pp. 45–46](#)

03 A supported nef model with surviving support

$$A = K_V + B \sim_{\mathbb{Q}} P \neq 0, \quad \operatorname{Supp} P = \operatorname{Supp} S, \quad \kappa(A) = 0$$

The special resolution gives generation on the whole floor, then lifting; the contradiction forces the original zero divisor to vanish.

[Thm. 6.1, p. 46; Thm. 1.2, p. 5; Sect. 13, pp. 88–89](#)

04 Boundary generation and lifting give a contradiction

$$A|_S \text{ semiample} \implies \kappa(A) \geq 1; \quad \mathcal{O}_X(m_0 D) \simeq \mathcal{O}_X$$

Reduction of the main theorem to the boundary

[Open figure ↗](#) [TeX source ↓](#)

1. Descend generation to the original space in positive litaka dimension

For $\kappa(X, D) \geq 1$, take the crepant ordinary $\mathbb{Q}$ -factorial Kähler model of DHP, Theorem 6.1, p. 43. Since the original pair is klt, so is the output; HLL, Theorem 4.1, pp. 17–18 supplies semiamplessness. Normality and the projection formula show that every section is pulled back from X , and evaluation surjectivity descends to X (K4N, Proposition 5.2, p. 34). Generation on a birational model alone is not the endpoint.

[Proposition 5.2 · p. 34](#)

2. Match the zero divisor support with a dlt floor

In the remaining case, let $M = \text{div}(s_0)/m_0$ be the normalized zero divisor of a nonzero section. Suppose $M \neq 0$. On a log resolution, raise the coefficients of its strict transforms and the exceptional primes to one. With G_Y the crepant boundary, the effective representative in Lemma 5.3 (pp. 34–35) is $P_Y = p^*M + (B_Y - G_Y)$. Its support is exactly the floor, and comparison with exceptional divisors preserves litaka dimension zero.

Lemma 5.3 · pp. 34–35

3. Ensure that contraction targets remain Kähler

Kählerness of the targets of projective MMP steps cannot be obtained from projectivity of the original space. Following the supported construction of DHP, Theorem 7.2, p. 46, the manuscript prepares threefold floor contractions (Proposition 3.1, p. 9) and their extension to the ambient space (Proposition 4.3, p. 24). The latter uses conormal vanishing and **finite analytic thickenings**.

Claim 5.4 (pp. 37–42) verifies nefness of the descended Bott–Chern class, a current dominating a positive Hermitian form, and positive top intersections on every positive-dimensional subspace, thereby obtaining a Kähler target. The full analytic contraction constructions are outside the checks completed for this draft.

Propositions 3.1, 4.3 · Claim 5.4 · pp. 9, 24, 37–42

4. Preserve nonzero support using the original nefness

Proposition 5.1 (p. 33) produces an ordinary Q-factorial Kähler dlt fourfold (V, B) , a nef $A = K_V + B$, nonzero $P \sim_{\mathbb{Q}} A$, $\text{Supp } P = \text{Supp } S$, and $\kappa(V, A) = 0$. Survival of P uses **nefness of the original D** : if it disappeared, the pullback of M on a common resolution would be nonzero, effective, nef and exceptional, contradicting negativity (pp. 45–46).

Proposition 5.1 · §5.3 · pp. 33, 45–46

5. Combine whole-floor generation with supported lifting

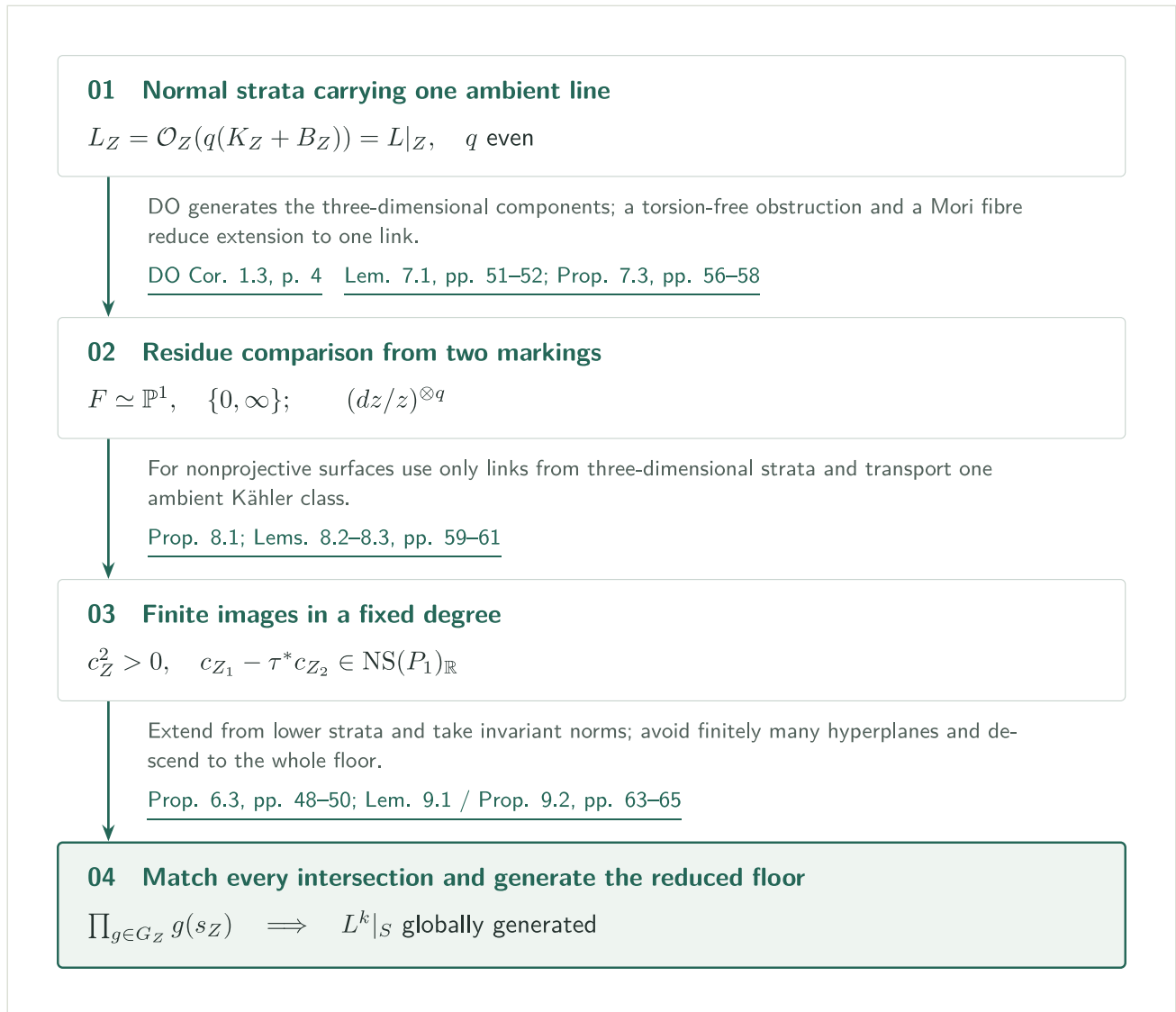
Lemma 5.6 also gives a special projective resolution: its strict boundary and exceptional support have globally smooth distinct SNC components, exceptional crepant coefficients are below one, and it is generically an isomorphism on the image of each intersection component of distinct strict floor primes. **Under this resolution condition**, Theorem 6.1 (p. 46) generates $A|_S$. Theorem 1.2 then yields $\kappa(V, A) \geq 1$, a contradiction. Thus $M = 0$; normality makes s_0 nowhere vanishing, and it trivializes the original holomorphic line (§13, pp. 88–89).

Lemma 5.6 · Theorem 6.1 · §13 · pp. 43–46, 88–89

3. Generating the entire boundary

The result here is **Theorem 6.1**. This section isolates the passage from generation on strata to generation on the whole reduced floor.

Work with the dlt fourfold model possessing the special resolution in Theorem 6.1. Threefold abundance supplies sections componentwise. The goal is to make them agree on every intersection through actual residue identifications.



Residue links and generation on the entire floor

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1. Identify stratum adjoint lines with ambient restrictions

Proposition 6.3 (pp. 48–50) identifies adjoints on normal Kähler dlt strata indexed by intersections with restrictions of one ambient holomorphic line. Sufficiently divisible even degrees remove iterated-residue signs. Sections agreeing on every subordinate stratum descend

uniquely to the entire reduced floor. Apply [DO, Corollary 1.3, p. 4](#) to three-dimensional strata through small models; lower strata inherit generation by restriction ([K4N §7.1, pp. 50–51](#)). Componentwise generation does not supply agreement on intersections.

[Proposition 6.3 · §7.1 · pp. 48–51](#)

2. Reduce extension to the residue comparison of two markings

[Proposition 7.3](#) (pp. 56–58) reduces extension from the floor of a stratum to at most one residue comparison. Torsion-freeness of $R^1 f_* \mathcal{O}_T(-[G])$ in [Lemma 7.1](#) (pp. 51–52) extends generic-fibre matching over every parameter. When a comparison is necessary, a general $\mathbb{P}^1$ fibre of a perturbed Mori contraction has two coefficient-one markings. When both come from one prime, use the degree-two exchange on the **normal finite Stein space of its normalization**. On a common graph, the invertible adjoint subsheaves inside meromorphic pluriforms agree; abstract $\mathbb{Q}$ -linear equivalence would not suffice.

[Lemma 7.1 · Proposition 7.3 · pp. 51–52, 56–58](#)

3. Use the common Kähler class for fixed-degree finite images

On nonprojective surfaces, the action of all birational self-maps on pluriforms need not have finite image. [Section 8](#) restricts to the groupoid generated by links of three-dimensional strata and their inverses. Restrictions of a single ambient Kähler form give classes $c_Z^2 > 0$ on smooth minimal surfaces. [Lemma 8.2](#) (p. 60) shows that links preserve these classes modulo the real Néron–Severi space. In algebraic dimension zero, [Lemma 8.3](#) (p. 61) bounds volume-character orders by $\varphi(n) \leq b_2$; algebraic dimension one is treated separately through the elliptic fibration. [Proposition 8.1](#) (p. 59) asserts finite image on section spaces in **each fixed degree**, not finiteness of the groupoid.

[Proposition 8.1 · Lemmas 8.2–8.3 · pp. 59–61](#)

4. Build compatible sections by norm products and descend to the floor

[Proposition 9.2](#) (pp. 64–65) proceeds from points to three-dimensional strata. Extend an already compatible lower collection, then in the curve and surface stages take norm products of finitely many transports to enforce invariance. [Lemma 9.1](#) preserves lower residues even when a surface link contracts a floor curve to a point. Avoiding finitely many hyperplanes chooses a section satisfying all required nonvanishing conditions, preserving generation. [Proposition 6.3](#) finally descends the tuples to the floor. No new ambient section on V has yet been constructed.

[Lemma 9.1 · Proposition 9.2 · pp. 63–65](#)

Generation on the whole floor enters [step 5 of Theorem 1.1](#) and supplies the hypothesis of the following lifting argument.

4. Proof of Theorem 1.2

For Theorem 1.2 return to a standard dlt pair in arbitrary dimension. No special resolution from the preceding application is added as a hypothesis. Starting from generation on the entire reduced support $\mathcal{S}$, prove growth of ambient sections by connecting root neighborhoods, split residues and vanishing over a projective parameter space.

01 From the generated whole floor to root neighborhoods

$$f : S \rightarrow \mathbb{P}^b, \quad E \text{ reduced Cartier}, \quad I = \mathcal{O}_Z(-E)$$

A second canonical root and derived retraction retain first-cohomology classes supported at special parameters.

[Prop. 10.4, pp. 71–72; Prop. 11.1, p. 75](#)

02 Split residue insertion retains every obstruction

$$R^1 g_* A_{-a} \hookrightarrow R^1 h_* \omega_H = F_0 \mathcal{M}^1$$

Use strict direct images of constant strata and vanishing on pure quotients; apply the adjunction on a compact graph unbranched at the chosen point.

[Sai Thm. 1, p. 1](#) [SS Thm. 16.3.10, p. 734](#) [Cor. 11.3, pp. 81–82](#)

03 Kill the ample-line map into the symbol kernel

$$\mathcal{O}_{\mathbb{P}^b}(a+k) \longrightarrow \ker \sigma, \quad \text{Hom}(N, \ker \sigma) = 0$$

Descend at an arbitrary prescribed point; the local identity and Laurent derivation rule kill the remaining obstruction.

[Lem. 12.2, pp. 84–85; Sect. 12.3, pp. 86–87](#)

04 Lift in all integer degrees and finite orders

$$\delta_k = 0, \quad g_*(I^j / I^{j+k+1}) \twoheadrightarrow g_*(I^j / I^{j+k})$$

Apply Serre vanishing to coherent invariant layer quotients; the final loss is bounded by fixed first cohomology.

[Sect. 12.4, pp. 87–88](#)

05 Recurring invariant layers grow ambient sections

$$F_{j-r} = F_j(1), \quad h^0(V, \mathcal{O}_V(NG)) \geq N - C - h^1(V, \mathcal{O}_V)$$

Finite-order lifting and section growth

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1. Formulate the finite-neighborhood lifting target precisely

The generated restriction $\mathcal{O}_V(G)|_S$, with $G = qP$, gives $f : S \rightarrow T = \mathbb{P}^b$ and an actual identity $\mathcal{O}_V(G)|_S \simeq f^* \mathcal{O}_T(1)$. Near each compact fibre, a root pair and a full normalized cyclic cover produce a reduced Cartier divisor E , with $I = \mathcal{O}_Z(-E)$ and $g : E \rightarrow U$. Proposition 10.2 (p. 68) seeks

$$g_*(I^j/I^{j+k+1}) \longrightarrow g_*(I^j/I^{j+k}) \quad \text{surjective} \quad (j \in \mathbb{Z}, k \geq 1).$$

These are sheaves of complex vector spaces on the underlying support of E . No holomorphic map from the thickening to U , or $\mathcal{O}_U$ -coherence of its pushforward, is assumed. Neighborhoods for lifting a section germ may depend on the order.

Proposition 10.2 · p. 68

2. Retain special-support obstructions through split residues

A second canonical root and residue give a **split injection** from the obstruction sheaf $R^1 g_* A_{-a}$ into $R^1 h_* \omega_H$ for an SNC support (Proposition 10.4, pp. 71–72). The derived retraction retains classes supported at special parameters. Proposition 11.1 (p. 75) identifies the latter with the lowest step $F_0 \mathcal{M}^1$ of a right filtered $\mathcal{D}$ -module. Apply Sai, Theorem 1, p. 1 to constant real Hodge modules on smooth SNC strata and assemble a finite strict filtration with pure quotients. The proof does not assume a blanket Kähler direct-image theorem for arbitrary mixed Hodge modules.

Propositions 10.4, 11.1 · pp. 71–72, 75–80

3. Exclude maps from ample lines on the projective base

Lemma 11.2 (pp. 80–81) supplies the filtered comparison needed to transfer vanishing from SS, Theorem 16.3.10, printed p. 734 / chapter PDF p. 16.

Corollary 11.3 (pp. 81–82) then gives, on the projective parameter space,

$$\mathrm{Hom}(N, \ker \sigma) = 0, \quad \sigma : F_0 \mathcal{M}^1 \otimes T_T \longrightarrow \mathrm{gr}_1^F \mathcal{M}^1$$

for every ample N , including torsion in the kernel.

Lemma 11.2 · Corollary 11.3 · pp. 80–82

4. Kill the obstruction at an arbitrary prescribed point

Assuming shorter lifting orders, the current connecting map becomes a degree- k derivation δ_k on the Laurent graded algebra (Lemma 12.1, pp. 82–83). The adjugate identity in Lemma 12.2 (pp. 84–85) turns its values on base coordinates into a map to the symbol kernel **without inverting the Jacobian** of the parameter change.

Proposition 10.5 (pp. 73–74) supplies a projective coordinate-power cover, unbranched over an arbitrarily prescribed t_* , and one compact SNC graph. Section 12.3 (pp. 86–87) kills the resulting ample-line map by vanishing and descends the result at t_* . Since that point was arbitrary, special-support obstructions disappear as well.

Proposition 10.5 · Lemmas 12.1–12.2 · §12.3 · pp. 73–74, 82–87

5. Turn lifting in all integer degrees into section growth

The order-zero part of the same local identity and the split injection then kill negative-degree obstructions. The derivation rule for an invertible Laurent frame extends this to every integer degree, closing finite lifting induction. Finally cyclic invariants yield coherent layer quotients of $J_j = \mathcal{O}_V(-\lceil jG/r \rceil)$. Their direct images $F_j = f_*(J_j/J_{j+1})$ satisfy $F_{j-r} = F_j \otimes \mathcal{O}_T(1)$. Serre vanishing is applied to these coherent layer quotients. Euler characteristics of finite filtrations give at least $N - C$ quotient sections, and the final loss in passing to ambient sections is bounded by the fixed $h^1(V, \mathcal{O}_V)$. Some degree therefore has two independent sections, proving $\kappa(V, A) \geq 1$ (pp. 87–88). Convergence of an infinite thickening is unnecessary.

§§12.3–12.4 · pp. 87–88

Return the resulting growth of sections to [step 5 of the main proof](#), contradicting the zero-Kodaira counterexample.

5. Which papers supply which steps

Two outgoing results. Theorem 1.1 supplies [Assumption 2.4 in Conditional Kähler](#) after nonvanishing. Separately, Lemma 7.1 (pp. 51–52) gives torsion-freeness of $R^1 f_* \mathcal{O}_T(-\lceil G \rceil)$ for an lc pair klt off its floor when an actual adjoint multiple comes from the base of a map with connected fibres. It enters [Lemma 4.3 of Kähler abundance](#); this does not apply the fourfold main theorem in arbitrary dimension. [Theorem 1.1 · p. 3](#); [Lemma 7.1 · pp. 51–52](#)

- [HLL Theorem 4.1 \(v2, pp. 17–18\)](#): semiampness for Q-factorial Kähler klt pairs of positive Iitaka dimension, unconditional through dimension four. Used in [K4N Proposition 5.2 \(p. 34\)](#).
- [DHP Theorem 6.1 \(v3, p. 43\)](#), [Theorem 7.2 \(p. 46\)](#): crepant dlt models and the supported fourfold program, used in [K4N §§5.1, 5.3](#). Its floor contraction and Kähler targets are discussed together with [K4N §§3–4](#) and [Claim 5.4](#).
- [DO Corollary 1.3 \(v4, p. 4; actual-adjoint conventions pp. 5–6\)](#): semiampness of nef lc Kähler threefold adjoints, applied to strata in [K4N §7.1 \(pp. 50–51\)](#).

- **FT** Theorem 2.9, Proposition 2.11 (v7, pp. 6–7): canonical direct-image torsion-freeness from smooth Kähler sources and local Kählerness, used in K4N Lemma 7.1 and Proposition 10.4.
- **Sai / SS** Constant-source strict direct image and projective vanishing as above: inputs to K4N §11. The filtered comparison for pure real Hodge modules is K4N's own Lemma 11.2.
- **FG** Theorem 1.1 (final author manuscript, p. 1): finiteness of pluricanonical representations of projective lc pairs with semiample adjoint. Applied only to projective surfaces in K4N §8.1 (p. 59).
- **SL and LA** are cited as projective context on K4N p. 6, not as direct inputs to its two analytic boundary theorems. Swapnajt Das's slc Kähler threefold abundance is likewise prior methodological work; K4N p. 5 explicitly excludes its main theorem as an input.

6. Return to the sources

The source is the [fixed K4N PDF](#), commit `adc7f1241b42e322a6451854ab7e4b4c146bf78a`. Unless specified otherwise, page references use printed PDF pages. GitHub links do not retain page positions; use the stated page and result numbers.

The editors read the main statements, actual-line conventions, supported-model preparation and survival of the support, adjunction and descent on strata, two-marking comparisons, the common Kähler class and fixed-degree finite images, norm-product induction, split residue insertion, filtered direct-image and vanishing statements and connections, the adjugate identity and its application to obstructions, layer growth, and final assembly. The stated HLL, DHP, DO, FT, Sai, SS and FG inputs were compared with their source statements and use sites.

Unresolved: complete analytic contraction constructions in §§3–4 and Claim 5.4; all external relative MMP, rationality and current-descent inputs; all root-neighborhood gluing and functorial principalization; independent reconstruction of the filtered comparison between real Hodge-module theories in §11; the full Čech differentiation calculation in Lemma 12.2; and proofs of external theorems. These connections are not claimed to be independently verified.

Version note: Sai v5 Theorem 1 displays weight $j - n$, whereas K4N p. 78 treats this as a sign typo, using constant-module conventions and the identity map. This discrepancy is retained in the verification record. SS was checked in the Version 2 chapter PDF available on the access date, not a fixed-commit source.