

OpenAI Math Digest

CATALOGUE 033

Campana's orbifold Iitaka conjecture and logarithmic subadditivity

Catalogue synthesis & 5 proof overviews

Independent reading guide · English edition

Edition: 2026-10-11

Created by: Masataka Iwai · Osaka University

AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

Contents

The synthesis comes first, followed by all 5 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

00 · Catalogue synthesis	3
01 · Orbifold and logarithmic Iitaka subadditivity	15
02 · Logarithmic Kodaira dimension and whole-fiber variation	34
03 · The reverse logarithmic Kodaira inequality and additivity	59
04 · Projective Hodge lines and ordinary Iitaka subadditivity	77
05 · B-semiampleness for compact log-smooth Kähler fibrations	90

Original-manuscript snapshot: `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`

Catalogue 033 · Synthesis

[Read on the website ↗](#)

CATALOGUE 033

Campana's orbifold Iitaka conjecture and logarithmic subadditivity

Five papers on subadditivity and variation of Kodaira dimension, logarithmic additivity, and moduli-line semiampleness. Start with the official-order inventory, then follow the result-level connections into the relevant proof sections.

Papers in official catalogue order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

Catalogue 033 · 2026-10-11 checked source snapshot

#

01

Manuscript and overview

Orbifold and logarithmic Iitaka subadditivity →

Short name: Orbifold Iitaka

2026-09-26 · 55 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Orbifold subadditivity for class-C SNC pairs, including coefficient one; logarithmic and ordinary characteristic-zero consequences.

[Theorem 1.1; Corollaries 6.2–6.3](#) · pp. 2–3, 40–41 ↗

Role and available overview

Section comparison on the relative Iitaka base, Hodge adjoint positivity, and twist cancellation; distinguishes the results supplied to later papers.

[Read the proof overview](#) →

#

02

[Manuscript and overview](#)

Logarithmic Kodaira dimension and whole-fiber variation →

Short name: Whole-fiber variation

2026-09-26 · 75 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

A lower bound involving variation of the whole geometric generic fiber over a base of nonnegative logarithmic Kodaira dimension. Smooth-family consequences use additional inputs.

[Theorem 1.1; Corollaries 1.2–1.3 · pp. 2, 5](#) ↗[Role and available overview](#)

Builds a parameter field from sections and descends the whole fiber using root-cover constancy, markings, and finite-cover descent.

[Read the proof overview](#) →

#

03

Manuscript and overview

The reverse logarithmic Kodaira inequality and additivity →

Short name: Reverse log additivity

2026-09-26 · 51 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

An upper bound for projective pairs whose boundary strata are smooth over the open base; the logarithmic lower bound then gives additivity.

[Theorem 1.1](#); [Corollary 1.2](#); [Proposition 1.3](#) · pp. 2–3 ↗

Role and available overview

Constructs a base section retaining a prescribed zero and removes a numerical twist for a positive-dimensional period quotient.

[Read the proof overview](#) →

#

04

Manuscript and overview

Projective Hodge lines and ordinary Iitaka subadditivity →

Short name: Projective Hodge lines

2026-09-27 · 40 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

Ordinary characteristic-zero subadditivity and adjoint positivity for variations whose entire highest Hodge piece is a line.

[Theorems 1.1–1.2 · p. 2](#) ↗

Role and available overview

An alternative proof through boundary and finite-ramification rank loss, canonical bundle data, two-base comparison, and interpolation.

[Read the proof overview](#) →

#

05

[Manuscript and overview](#)

B-semiampleness for compact log-smooth Kähler fibrations →

Short name: Kähler B-semiampleness

2026-09-10 · 38 pages

[Original paper \(GitHub\)](#) ↗[Principal claim](#)

Stabilization and semiampness of the moduli b-line for compact Kähler fibrations with relative log-canonical class pulled back from the base.

[Theorem 1.1\(a\)–\(b\) · p. 3](#) ↗[Role and available overview](#)

Identifies thresholds with Hodge extensions and descends global generation of the specified line from semiample product factors.

[Read the proof overview →](#)

Connections within 033

Orbifold Itaka supplies two different inputs to the main proof of Whole-fiber variation. Reverse log additivity first proves its upper bound independently; the lower bound enters only its additivity corollary. Each paper name below links to the relevant article section; result numbers link to the pinned original.

Source result

Orbifold Iitaka · Corollary 6.2 · pp. 40–41

Recipient and use

Whole-fiber variation · Theorem 2.1; Lemmas 2.8–2.9 · pp. 6, 10–12

Content supplied

Direct input. Supplies the logarithmic lower bound and forces two nonnegative terms on an Iitaka fiber to vanish, enabling the parameter-field construction.

Source result

Orbifold Iitaka · Theorem 3.1 · pp. 17–18

Recipient and use

Whole-fiber variation · Theorem 3.8 / Proposition 3.9; Proposition 3.12 · pp. 21, 23

Content supplied

Direct input. Gives adjoint positivity for the restricted Hodge line. The section bound forces its adjoint base to be a point and makes M rationally trivial.

Source result

Orbifold Iitaka · Corollary 6.2 · pp. 40–41

Recipient and use

Reverse log additivity · Corollary 1.2; Theorem 7.8 / §7.6 · pp. 2, 50

Content supplied

Input to a consequence. Supplies the lower bound for the same pair and very general fiber; combining it with Reverse log additivity’s upper bound gives additivity.

The first two inputs enter the parameter-field and restricted-line stages; neither by itself gives constancy of the whole fiber. The [overview in Whole-fiber variation](#) retains the subsequent constancy, marking, and descent steps. The [upper-bound argument in Reverse log additivity](#) treats all signs of the Kodaira dimensions before introducing the lower bound.

The additional relative-Iitaka construction in [Whole-fiber variation §9](#) uses three further inputs. It is separate from the main proof, which ends in §7.4.

Source result

Orbifold Iitaka · Proposition 2.7 · pp. 10, 16–17

Recipient and use

Whole-fiber variation · Proposition 9.1(i), (iii) · pp. 69–70

Content supplied

Input within an additional result. Preserves relative and absolute section systems and their ratios on the fixed original X , giving relative bigness in the additional ordinary construction.

Source result

Orbifold Iitaka · Lemma 7.1 / Corollary 7.2 · pp. 43–46

Recipient and use

Whole-fiber variation · Proposition 9.1(iv), (v) · pp. 69–70

Content supplied

Input within an additional result. For a reduced boundary, the least-index root cover makes the entire top piece rank one and realizes the normalized M as a line in a pure integral variation.

Source result

Orbifold Iitaka · Lemma 7.4 · pp. 46–47

Recipient and use

Whole-fiber variation · Proposition 9.1: initial normalized boundary · pp. 69–70

Content supplied

Input within an additional result. Normalizes B by comparing the minimum over actual source components with the lct over all valuations, on the same model as the section and Hodge-line comparison.

These inputs preserve sections, identify the entire highest line **for a reduced boundary**, and normalize the boundary on a common model. Their stated uses were compared; this does not verify the full valuation arguments or all of §§8–9.

Connections with catalogue 034

Different results travel in the two directions. The return from Log Abundance supplies good models to a smooth-family consequence in Whole-fiber variation, not to its main variation theorem.

Source result

Orbifold Iitaka · Corollary 6.2 · pp. 40–41

Recipient and use

Log Abundance · Theorem 1.2 $\rightarrow$ Lemma 6.1 · pp. 6, 30–31

Content supplied

Direct input. Applied with zero boundaries to the Albanese fibration, it forces irregularity zero for a nonvanishing counterexample; this is Log Abundance’s sole use of subadditivity.

Source result

Log Abundance · Corollary 11.2 · pp. 73–74

Recipient and use

Whole-fiber variation · Corollary 1.3 proof · p. 5

Content supplied

Input to a consequence. Provides a semiample good model and effective exceptional comparison for each closed fiber whose canonical class is pseudo-effective by BDPP; canonicity then permits Taji’s theorem.

Source result

Orbifold Iitaka · Lemma 2.6 · p. 9

Recipient and use

Kähler abundance · §5.1; reused in Lemma 5.7 · pp. 91, 98

Content supplied

Direct input. Passes to the relative Iitaka base with Kodaira-zero general fiber so that lower-dimensional induction can continue.

Source result

[Orbifold Iitaka · Lemma 5.2 · p. 32](#)

Recipient and use

[Kähler abundance · §5.1 · p. 91](#)

Content supplied

Direct input. Compares the big fibers, after lowering the horizontal boundary, with a projective stable family, excluding the big case at algebraic dimension zero.

Source result

[Orbifold Iitaka · Proposition 2.7 / Theorem 3.1 · pp. 10–17, 17–18](#)

Recipient and use

[Kähler abundance · Lemma 5.2 · pp. 91–94](#)

Content supplied

Direct input. Compares section spaces and the actual rational Hodge line, giving $M=p^*P$ and adjoint bigness on the projective base; signs above codimension two require separate work.

Source result

[Orbifold Iitaka · Lemma 3.3 · pp. 18–19](#)

Recipient and use

[Kähler abundance · Lemma 5.7, interpolation · pp. 98–99](#)

Content supplied

Direct input. Produces a section after a big base twist; interpolation with another effective divisor forces positive Kodaira dimension.

The Albanese application in Log Abundance uses zero boundaries; its manuscript also allows Hacon–Popa–Schnell for that application. The Kähler rows concern specified auxiliary results, with the receiving manuscript’s own assumptions retained. No transitive path through Log Abundance is recorded as a direct input to Schnell.

Explicit premise. [Orbifold Iitaka, Theorem 1.1 · pp. 2–3](#) matches [Conditional Kähler models, Assumption 2.2 · pp. 7–8](#): arbitrary-dimensional class-C orbifold subadditivity, including coefficient one and the invariant base convention. The other Kähler MMP and abundance premises remain. See the [premise statements](#).

Alternative proofs, inclusion, and background

Projective Hodge lines, Theorem 1.1 · p. 2 and Orbifold Iitaka, Corollary 6.3 · p. 41 prove the same ordinary characteristic-zero subadditivity statement by different routes. The latter also includes the logarithmic statement. The ordinary claim is a zero-boundary case of that logarithmic conclusion; this inclusion of results does not create a proof dependency. Compare the two-base interpolation with the characteristic-zero transfer.

Projective Hodge lines, Lemma 8.4 / Theorem 8.1 · pp. 28–29 is an alternative adjoint comparison in Whole-fiber variation, Remark 3.17 · p. 25. Tensor replacement must be performed again on the restriction; the receiving proof uses the direct comparison. Separately, additivity and the variation lower bound recover the nonnegative-base branch for smooth families with finite nonnegative fiber Kodaira dimension (comparison · p. 5); they do not cover negative bases or every special base.

Weak positivity and interpolation in Orbifold Iitaka and Projective Hodge lines are similar methods with different Hodge-line hypotheses. The root-line calculations in Orbifold Iitaka and Kähler B-semiampleness also use different boundaries: orbifold inf-multiplicity versus the lct discriminant. Kähler B-semiampleness · p. 3 explicitly avoids the orbifold Iitaka theorem; this is not a claim that all possible inter-paper dependencies have been investigated. Its whole proof overview follows its own stabilization and product-factor route.

The connection-form Ax–Schanuel argument in Reverse log additivity is recorded as a method background: the manuscript gives the semisimple case it needs. The external-input discussion distinguishes this from importing an external theorem.

Sources and checking scope

Official catalogue / OpenAI Research Catalog · 033 · p. 5. The field is Algebraic and complex geometry; the inventory retains the official order. The 033 manuscripts are fixed at commit fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb. Connections to 034 retain its published snapshot adc7f1241b42e322a6451854ab7e4b4c146bf78a (Log Abundance: 2026-09-24; Kähler abundance: 2026-10-04; conditional Kähler models: 2026-10-05).

The revision of 11 October 2026 compared missing proof-target statements and their conclusions against the five pinned PDFs, rechecked their hashes, and matched the recorded connections to the articles. The twelve inputs and one premise match inherit the statement/use checks of 8 October; no new complete verification of external proofs is claimed. Full connection records, including conditions and limits, remain in the [source record](#).

- [Orbifold Itaka](#): rank loss, the full stable-family construction, valuation/BFMT comparisons, and core consequences.
- [Whole-fiber variation](#): all regularization and L^2 estimates, the original Hanamura and SGA1 statements, and §§8–9 in full.
- [Reverse log additivity](#): homogeneous replacement, compact flags, full boundary/representation calculations.
- [Projective Hodge lines](#): multivariable extensions, all auxiliary proofs, and the alternatives in §§8–9.
- [Kähler B-semiampleness](#): the complete parameter-space construction, polarization changes, and external compactification theory.

Unlisted relations remain uninvestigated. Statement/use comparison, a proof overview, and independent verification of a complete proof are distinct. No expert review or formal verification is asserted.

Orbifold and logarithmic Iitaka subadditivity

Exact section comparison on the relative Iitaka base and two positivity inputs

The manuscript asserts orbifold subadditivity for rational SNC pairs in Fujiki class C, including boundary coefficient one. Its argument retains the original logarithmic pluricanonical sections on a relative Iitaka base and combines adjoint positivity of a Hodge line with addition for log-general-type fibers. A fixed section removes the final ample twist by multiplication.

Manuscript author: OpenAI · 2026-09-26 · 55 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Orbifold subadditivity

Let X be a smooth compact connected complex manifold in Fujiki class C, and let Δ be an SNC boundary with coefficients in $\mathbb{Q} \cap [0, 1]$. Let $f : X \rightarrow Y$ be a surjective holomorphic map with connected fibers onto a normal compact irreducible complex space. For a very general smooth fiber F and $\Delta_F = \Delta|_F$, the manuscript proves

$$\kappa(X, K_X + \Delta) \geq \kappa(F, K_F + \Delta_F) + \kappa(f| \Delta).$$

A point has Iitaka dimension zero, and $(-\infty) + a = -\infty$, including $a = -\infty$.

On a smooth-base model, put

$$m_\Delta(E) = \frac{1}{1 - \Delta_E}, \quad m(f, \Delta; D) = \min_{E \rightarrow D} \text{ord}_E(f^*D) m_\Delta(E),$$

$$B(f, \Delta) = \sum_D \left(1 - \frac{1}{m(f, \Delta; D)} \right) D.$$

Here $1/0 = \infty$ and $1/\infty = 0$; no divisibility condition is imposed. The base term in the theorem is the invariant

$$\kappa(f | \Delta) = \inf_{f' \sim f} \kappa(Y', K_{Y'} + B(f', \Delta'))$$

over the orbifold birational equivalence allowed by (1.2), rather than the dimension of the orbifold base on an arbitrary model. For a normal singular base, resolve the base and the main component, using the strict transform of Δ plus the reduced divisor exceptional over the original source.

Theorem 1.1 · (1.1)–(1.4) · pp. 2–3

Corollary 6.2 — Logarithmic subadditivity

Let $f : X \rightarrow Y$ be a surjective morphism with connected fibers between smooth connected projective complex varieties. Let D_X, D_Y be reduced SNC divisors, allowing zero, and assume

$$\text{Supp}(f^*D_Y) \subseteq \text{Supp}(D_X).$$

For a very general fiber F and $D_F = D_X|_F$,

$$\kappa(X, K_X + D_X) \geq \kappa(F, K_F + D_F) + \kappa(Y, K_Y + D_Y).$$

The same statement holds for holomorphic fibrations between smooth compact manifolds in Fujiki class C.

Corollary 6.2 · pp. 40–41

Corollary 6.3 — Ordinary subadditivity in characteristic zero

Let k be an algebraically closed field of characteristic zero, and let $f : X \rightarrow Y$ be a surjective morphism with connected fibers between smooth connected projective k -varieties. For its geometric generic fiber F ,

$$\kappa(X) \geq \kappa(F) + \kappa(Y).$$

More generally, the logarithmic inequality holds over k under the same boundary hypotheses as Corollary 6.2, again replacing a very general complex fiber by the geometric generic fiber.

Corollary 6.3 · p. 41

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Orbifold and logarithmic Iitaka subadditivity](#). External inputs use the following keys.

- [FF](#) Osamu Fujino; Taro Fujisawa — Variation of mixed Hodge structure and its applications: Hodge extensions and relative logarithmic top forms. author version 0.55, 2025-03-11, 36 pages.
- [Vil](#) Mads Bach Villadsen — Hodge modules and Kähler morphisms: Compactification of period quotients or finite covers; see the checked scope below. [arXiv:2401.09544v1](#), 2024-01-17, 26 pages.
- [BC](#) Yohan Brunebarbe; Benoît Cadorel — Hyperbolicity of varieties supporting a variation of Hodge structure: Log-general type from an immersive period map. [arXiv:1707.01327v1](#), posted 2017-07-05, manuscript dated 2017-07-06, 7 pages.
- [BT](#) Benjamin Bakker; Jacob Tsimerman — The Ax–Schanuel conjecture for variations of Hodge structures: Ax–Schanuel input to boundary rank loss. [arXiv:1712.05088v1](#), 2017-12-14, 12 pages.
- [KP](#) Sándor J. Kovács; Zsolt Patakfalvi — Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension: Positivity of projective stable families and variation bounds for marked models. 62-page author manuscript; PDF metadata creation date 2016-09-26 UTC (2016-09-27 JST).
- [Fuj](#) Osamu Fujino — Notes on the weak positivity theorems: Weak positivity for producing sections with a negative ample twist. author version 0.54, 2015-06-30, 44 pages.
- [Cam](#) Frédéric Campana — Orbifolies géométriques spéciales et classification biméromorphe des variétés kählériennes compactes: Computing the invariant orbifold base term on neat models. [arXiv:0705.0737v8](#), 155-page PDF; [arXiv stamp](#) 2009-07-15, title page date 2018-10-25.
- [BFMT](#) Benjamin Bakker; Stefano Filipazzi; Mirko Mauri; Jacob Tsimerman — Baily–Borel compactifications of period images and the b-semiampleness conjecture: Hodge/moduli identification and projective semiampleness; the results used and checking limits are distinguished below. [arXiv:2508.19215v2](#), 2025-12-18 (as specified in [OI](#) reference [7]).

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — Transfer to the relative Iitaka base

Proof target : [Theorem 1.1](#) · pp. 2–3 See the hypotheses and conclusion.

Proof route. The main proof combines exact section comparison on the relative Iitaka base with two positivity inputs. The next section expands the final [cancellation of the ample twist](#); the resulting lower bound then yields [logarithmic subadditivity](#) and the [characteristic-zero statement](#). The [additional inputs to WV](#) form a separate part of the article.

Exclude the automatic case where either right-hand term is $-\infty$, and put

$r = \kappa(F, K_F + \Delta_F) \geq 0$ and $b = \kappa(f \mid \Delta) \geq 0$. Below, X, W, Y denote allowed common models; (X_0, Δ_0) is the fixed smooth reference pair to which sections descend. The map $h : W \rightarrow Y$ retains the original base, while $p : W \rightarrow S$ is a separate map arising from periods.

Relative litaka reduction with a fixed reference

$$X \xrightarrow{g} W \xrightarrow{h} Y, \quad \dim W_y = r, \quad \kappa(G, K_G + \Delta_G) = 0$$

$$C = B(f, \Delta), \quad \kappa(Y, K_Y + C) \geq b$$

Construct the Hodge line and boundary from the Kodaira-zero generator
Lem. 2.6, p. 9; Prop. 2.7, pp. 10–17

Compare boundary orders and transfer sections

$$B(g, \Delta) \leq T \leq 1, \quad D_W = K_W + T$$

$$H^0(W, q(D_W + M)) \simeq H^0(X_0, q(K_{X_0} + \Delta_0))$$

Products and ratios are preserved, also over Y .

Relative comparison and multiplicities
Prop. 2.7(iv); Lem. 2.12
pp. 10, 17; (6.4)–(6.5), pp. 39–40

Adjoint positivity of the highest line
Thm. 3.1, pp. 17–18
Prop. 4.5; Lem. 4.7, pp. 25–31

Use the map to the original base

$$h : W \rightarrow Y$$

$$(D_W + M)|_{W_y} \text{ big}$$

$$B(h, T) \geq C$$

$$T_E = 1 \quad (\text{codim}_Y h(E) \geq 2)$$

Pass to the separate period base

$$p : W \rightarrow S, \quad S \text{ projective}$$

$$M = p^*P, \quad P \text{ nef}$$

$$K_S + a_0P \text{ big}, \quad a_0 > 0$$

No map between Y and S is assumed.

General-type addition and cancellation by one fixed section
Prop. 3.4, pp. 19–20; (6.6), p. 40

Combine the two inputs

$$\kappa(W, D_W + M) \geq r + \kappa(Y, K_Y + C)$$

Descent to the reference preserves ratios and image dimensions
Prop. 2.7; Lem. 2.1, pp. 5–6, 16–17; §6, p. 40

Return to the original pair

$$\kappa(X, K_X + \Delta) \geq r + b$$

Main theorem: the relative litaka reduction combines boundary control with Hodge-line positivity

Open figure ↗ TeX source ↓

1. Preserve exceptional coefficients and fix the original base

Resolve the base and flatten before resolving the source. Source primes mapping into base codimension at least two become exceptional over a fixed smooth reference. Taking the strict transform plus reduced exceptional boundary makes the difference of log canonical divisors effective and exceptional; pluricanonical sections, products, and ratios are preserved. Fix the resulting smooth base Y and set $C = B(f, \Delta)$. Further changes of W retain C and the coefficient-one condition, together with $\kappa(Y, K_Y + C) \geq b$.

Lemma 2.1 · Corollaries 2.3, 2.5 · Lemmas 2.4, 2.6 · pp. 5–9; §6 · p. 39

2. Replace the Kodaira-zero fiber by an actual Hodge line

The relative Iitaka factorization $X \xrightarrow{g} W \xrightarrow{h} Y$ has $\dim W_y = r$ and logarithmic Iitaka dimension zero on a very general g -fiber G . Take a root of a relative generating form s in a sufficiently divisible degree and use the highest Hodge part of its distinguished character. This does not assert that the entire highest Hodge space is a line. Coefficient-one boundary requires the mixed Hodge structure of an open fiber and passage to the unique nonzero pure weight grade. The extension theorem FF retains this identification at the boundary.

Lemmas 2.6, 2.8–2.10 · pp. 9–15 · FF, Theorem 1.1 (ii), (iv) · p. 2

The boundary T compares the original form's allowed poles with its Hodge-extension order. In the notation of §2.6.3,

$$\alpha_D = \min_{E \mapsto D} \frac{l_E + \Delta_E}{a_E}, \quad \beta_D = \min_{E \mapsto D} \frac{l_E + 1 - a_E}{a_E}, \quad T_D = \alpha_D - \beta_D.$$

Here $a_E = \text{ord}_E(g^*D)$ and $l_E = m^{-1} \text{ord}_E(s \wedge g^*\omega^m)$, where ω is a local ordinary base volume form. The minima include every component of the prepared SNC source. This gives

$B(g, \Delta) \leq T \leq 1$ and, in sufficiently divisible degrees q ,

$$H^0(W, q(K_W + T + M)) \simeq H^0(X_0, q(K_{X_0} + \Delta_0)).$$

The untested primes mapping into base codimension at least two are exceptional over the reference. Descend first to X_0 , then extend across codimension two. Specifying this reference is the bridge from a generic formula to actual global section spaces.

Proposition 2.7 · (2.12), (2.14), (2.16) · pp. 10, 13–17

3. Recover the hypotheses for addition from relative section comparison

Proposition 2.7(iv) also identifies direct images over Y and preserves section ratios. The relative Iitaka image has dimension $r = \dim W_y$, so $K_W + T + M$ is big on very general h -fibers. Independently, the multiplication of weighted orders in Lemma 2.12 gives $B(h, T) \geq C$ and coefficient one on every prime of W mapping into codimension at least two in Y . Both are hypotheses required for addition.

Proposition 2.7(iv) · Lemma 2.12 · pp. 10, 17; (6.3)–(6.5) · pp. 39–40

4. Supply adjoint positivity on the period base

Theorem 3.1 starts with the highest Hodge line of a complex direct summand of a pure real-polarizable variation. After modification it supplies a smooth projective S with $M = p^*P$, P nef, and $K_S + a_0P$ big. The integral lattice belongs to the ambient variation; the chosen complex summand need not be rational.

The proof constructs a quotient from the full periods of rational adjoint factors. Compactification Vil and the Moishezon criterion BC provide a projective model. If D_S marks nonidentity local monodromy, including finite nonidentity monodromy, then $K_S + D_S$ is big. Lemma 4.6 uses BT and other inputs to assert $\nu(P|_{D_i}) < \nu(P)$ for each component. In Lemma 4.7, the available section count has degree $\nu(P)$ in the coefficient t , whereas the boundary loss has smaller degree. This makes $K_S + a_0P$ big. The rank of the chosen-line map is distinguished from the rank of the full period map.

Theorem 3.1 · pp. 17–18; Proposition 4.5 · Lemmas 4.6–4.7 · pp. 25–31 · Vil, Theorem 2.19 · p. 16 · BC, Theorem 1.1 / Corollary 1.3 · pp. 1–2 · BT, Theorem 1.1 · p. 2

5. Apply addition and return the sections to the original source

All hypotheses of Proposition 3.4 now hold. The section operation explained below gives

$$\kappa(W, K_W + T + M) \geq r + \kappa(Y, K_Y + C) \geq r + b.$$

Proposition 2.7 and exceptional descent return these systems to the original (X, Δ) . Ratios and image dimensions are preserved throughout, transferring the lower bound to the theorem's left side. If S is a point, M is rationally trivial and Proposition 3.2 applies directly.

Propositions 3.2, 3.4 · pp. 18–20; §6 · (6.6) · p. 40

Section 4 transfers this lower bound to a logarithmic base. The intermediate Theorem 3.1 also supplies WV's restricted Hodge comparison.

Ol, §6 · pp. 39–41 · WV, Theorem 3.8 / Proposition 3.9 · p. 21

3. The core of Proposition 3.4 — Cancel the ample twist with a fixed section

Proof target : Proposition 3.4 · pp. 19–20

Write $D_W = K_W + T$ and $M = p^*P$. Start from bigness of $D_W + M$ on the fibers of h . The argument fixes rational coefficients and multiplies two systems; it makes no limiting-continuity assertion for κ .

Proposition 3.4

Let $h : W \rightarrow Y$ be a surjective map with connected fibers between smooth compact manifolds in Fujiki class C. Let T, C be rational SNC boundaries (coefficients in $[0, 1]$) on W, Y , with $C \leq B(h, T)$. Assume $T_E = 1$ for **every** prime divisor E of W mapping to codimension at least two in Y . Let $p : W \rightarrow S$ be surjective with connected fibers, with S smooth projective, and let P be a nef rational line on S , $M = p^*P$. If $K_S + a_0P$ is big for some rational $a_0 > 0$, and $K_W + T + M$ is big on very general h -fibers, then

$$\kappa(W, K_W + T + M) \geq \dim(W/Y) + \kappa(Y, K_Y + C).$$

The inequality is automatic when the last term is $-\infty$. If S is a point, $M \sim_{\mathbb{Q}} 0$.

Proposition 3.4 · pp. 19–20

Retain fiberwise bigness with a smaller coefficient

$$D_W = K_W + T, \quad M = p^*P, \quad 0 < c < 1$$

$$(D_W + cM)|_{W_y} \text{ big}, \quad A \text{ ample on } S$$

Add a general boundary Q_η and apply general-type-fiber addition
Prop. 3.2, p. 18; (3.5), p. 19; §5, pp. 31–38

Obtain enough ratios with a positive twist

$$E_+(\eta) = D_W + cM + \eta p^*A$$

$$\kappa(W, E_+(\eta)) \geq r + \kappa(Y, K_Y + C) \quad (\eta \in \mathbb{Q}_{>0})$$

Restrict to a general p -fiber, then use the big base twist
Lem. 3.3, pp. 18–19; (3.6)–(3.7), p. 20
[Fuj] Thm. 1.1, pp. 1–2

Produce one section with a negative twist

$$a > \max\{1, a_0\}, \quad H = K_S + aP - \lambda A \text{ big}$$

$$E_- = D_W + aM - \lambda p^*A, \quad 0 \neq e \in H^0(W, qE_-)$$

Set the coefficient of M to one and that of p^*A to zero
(3.8), p. 20

Fix rational coefficients and cancel exactly

$$\theta = \frac{1-c}{a-c}, \quad \eta = \frac{(1-c)\lambda}{a-1}$$

$$D_W + M = (1-\theta)E_+ + \theta E_-, \quad 0 < \theta < 1$$

Take sufficiently divisible n ; the common factor cancels in ratios
Prop. 3.4, following (3.8), p. 20

Preserve image dimension by fixed-section multiplication

$$H^0(W, n(1-\theta)E_+) \xrightarrow{\cdot e^{n\theta/q}} H^0(W, n(D_W + M))$$

$$\kappa(W, D_W + M) \geq r + \kappa(Y, K_Y + C)$$

Adjoint addition: combine a positively twisted system with one fixed negatively twisted section

Open figure ↗ TeX source ↓

1. Construct the positively twisted systems by general-type addition

Openness of the big cone gives $0 < c < 1$ with $D_W + cM$ still big on very general h -fibers. For an ample divisor A on S and any rational $\eta > 0$, the divisor $cP + \eta A$ is ample. A general rational member $Q_\eta \sim_{\mathbb{Q}} cM + \eta p^* A$ leaves $T + Q_\eta$ an SNC boundary, preserves old coefficient-one components, and retains the orbifold base lower bound. Proposition 3.2 gives

$$\kappa(W, D_W + cM + \eta p^* A) \geq r + \kappa(Y, K_Y + C).$$

To prove Proposition 3.2 over a nonprojective base, §5 slightly decreases only the horizontal boundary, obtaining klt general fibers. Lemma 5.2 compares these with a projective stable family of maximal variation. Theorem 8.1 and Corollary 8.3 of KP supply bigness of its relative log canonical divisor. Forms obtained on a finite cover descend through finite-group invariants; Lemma 5.4 retains their fiberwise image dimension. For the products with base sections, $C \leq B(h, T)$ controls poles above base divisors and $T_E = 1$ handles higher-codimension images. The original projective positivity theorem is thus applied to an auxiliary stable family.

Proposition 3.2 · p. 18; (3.5) · p. 19; Lemmas 5.2–5.5 · pp. 32–38 · KP, Theorem 8.1 / Corollary 8.3 · p. 40

2. Obtain one negatively twisted section by weak positivity

Restrict a nonzero section from the previous step to a very general p -fiber. The pulled-back twists disappear, giving relative nonvanishing for D_W . Choose $a > \max\{1, a_0\}$ and a small positive rational λ so that $H = K_S + aP - \lambda A$ is big. Lemma 3.3 uses Fuj to give a nonzero section of a positive multiple of

$$E_- = D_W + aM - \lambda p^* A = K_{W/S} + T + p^* H.$$

After flattening, the possible remaining poles are exceptional over the original smooth W , allowing descent to that fixed space.

Lemma 3.3 · pp. 18–19; (3.6)–(3.7) · p. 20 · Fuj, Theorem 1.1 · pp. 1–2

3. Cancel exactly and preserve ratios

Set

$$\theta = \frac{1-c}{a-c}, \quad \eta = \frac{(1-c)\lambda}{a-1}, \quad E_+ = D_W + cM + \eta p^* A.$$

Then $0 < \theta < 1$ and

$$D_W + M = (1-\theta)E_+ + \theta E_-.$$

Fix $0 \neq e \in H^0(W, qE_-)$. For sufficiently divisible n , multiplication by $e^{n\theta/q}$ gives

$$H^0(W, n(1-\theta)E_+) \hookrightarrow H^0(W, n(D_W + M)).$$

Away from the zeros of e , the common factor cancels from every ratio. The image dimension of the earlier systems is retained. Semiampneness of P is not required for this step.

Proposition 3.4 · (3.8) and the following multiplication · p. 20

Multiplication by a fixed section preserves image dimension, allowing the two positivity inputs in Section 2 to produce the main lower bound.

Ol, Proposition 3.4 · §6 · pp. 19–20, 40

4. Proof of Corollary 6.2 — Compare with the logarithmic base

Proof target : Corollary 6.2 · pp. 40–41 See the hypotheses and conclusion.

One needs both the inclusion of the prescribed boundary on a model and its transfer to the invariant $\kappa(f \mid D_X)$.

Support condition for reduced boundaries

$$\text{Supp}(f^* D_Y) \subseteq \text{Supp}(D_X)$$

$$f : X \rightarrow Y, \quad \Delta = D_X$$

Old and exceptional source components have coefficient one
[Lem. 6.1, p. 40](#)

Use infinite multiplicities on a neat model

$$q : Y' \rightarrow Y, \quad E_Y = (q^{-1} \text{Supp } D_Y)_{\text{red}}$$

$$B(f', \Delta') \geq E_Y$$

Pull back logarithmic forms and use the neat-model identity
[\(2.2\), p. 5; Lem. 6.1, p. 40](#)
[\[Cam\] Cor. 4.11, p. 49](#)

Pass to the invariant base term

$$\kappa(f \mid D_X) \geq \kappa(Y, K_Y + D_Y)$$

Substitute $\Delta = D_X$ into orbifold subadditivity
[Thm. 1.1, p. 3; Cor. 6.2, pp. 40–41](#)

Apply the main theorem to the logarithmic pair

$$\kappa(X, K_X + D_X) \geq \kappa(F, K_F + D_F) + \kappa(Y, K_Y + D_Y)$$

Logarithmic subadditivity: coefficient one yields an invariant base comparison, followed by the main theorem

[Open figure ↗](#) [TeX source ↓](#)

1. Pull back logarithmic systems on a neat model

In Lemma 6.1 take a neat model $f' : (X', \Delta') \rightarrow Y'$, with $q : Y' \rightarrow Y$, and put $E_Y = (q^{-1} \text{Supp } D_Y)_{\text{red}}$. Every source prime dominating a component of E_Y has coefficient one, so $B(f', \Delta') \geq E_Y$. Pullback of logarithmic volume forms preserves section ratios. The neat-model identity from [Cam](#) gives

$$\kappa(f \mid D_X) \geq \kappa(Y, K_Y + D_Y).$$

[Lemma 6.1 · p. 40; \(2.2\) · p. 5](#) [· Cam, Corollary 4.11 · p. 49](#)

2. Substitute into the main theorem and pass to open varieties

Apply Theorem 1.1 with $\Delta = D_X$. For a dominant morphism $U \rightarrow V$ of smooth quasi-projective varieties with geometrically connected general fiber, resolution of the graph and boundaries yields compatible compactifications satisfying Corollary 6.2. Thus $\bar{\kappa}(U) \geq \bar{\kappa}(U_v) + \bar{\kappa}(V)$ for very general v . Zero boundaries give ordinary subadditivity over $\mathbb{C}$.

Proof of Corollary 6.2 and the following discussion · pp. 40–41

This lower bound is used in WV’s parameter-field construction and in the corollary turning RA’s independent upper bound into equality.

WV, Theorem 2.1 · p. 6 · RA, Corollary 1.2 / §7.6 · pp. 2, 50

For compact class-C manifolds, the same neat-model boundary comparison and substitution into Theorem 1.1 apply without adding projectivity of the base or total space. This gives the compact class-C conclusion of Corollary 6.2.

5. Proof of Corollary 6.3 — Geometric generic fibers and field extension

Proof target : Corollary 6.3 · p. 41 See the hypotheses and conclusion.

Section spaces on the geometric generic fiber transfer the inequality from very general complex fibers to algebraically closed fields of characteristic zero.

Descend characteristic-zero data to a finitely generated field

$K \subset k, \quad K/\mathbb{Q}$ finitely generated, $K \hookrightarrow \mathbb{C}$
 $f_K : (X_K, D_{X_K}) \longrightarrow (Y_K, D_{Y_K})$

Retain smoothness, boundary conditions, and connected fibers
Cor. 6.2; proof of Cor. 6.3, p. 41

Apply logarithmic subadditivity over the complex numbers

$$\kappa(X_{\mathbb{C}}, K + D_{X_{\mathbb{C}}}) \geq \kappa(F_{\mathbb{C}}, K + D_{F_{\mathbb{C}}}) + \kappa(Y_{\mathbb{C}}, K + D_{Y_{\mathbb{C}}})$$

Use base change, image dimension in each degree, and countably many exceptions
Proof of Cor. 6.3, p. 41

Pass from very general to geometric generic fibers

$$\kappa(F_{\text{very general}}, K + D) = \kappa(F_{\bar{\eta}}, K + D)$$

Preserve nonvanishing and complete-linear-system image dimensions
Cor. 6.3, p. 41

Return to the original field through section-space invariance

$$H^0(V, mL) \otimes_K K' \simeq H^0(V_{K'}, mL_{K'})$$

$$\kappa(X, K_X + D_X) \geq \kappa(F_{\bar{\eta}}, K + D_F) + \kappa(Y, K_Y + D_Y)$$

Set $D_X = D_Y = 0$ for ordinary subadditivity.

Passage to characteristic zero: degreewise base change and descent to a finitely generated field

[Open figure ↗](#) [TeX source ↓](#)

1. Compare sections and image dimensions degree by degree

For a complex projective family, apply cohomology and base change in every divisible degree, and choose opens on which the relative rational maps have constant fiberwise image dimension. Outside a countable union of exceptional sets, the very general and geometric generic fibers have the same Iitaka dimension. For a field extension $K \subseteq K'$,

$$H^0(V, \mathcal{O}_V(mL)) \otimes_K K' \simeq H^0(V_{K'}, \mathcal{O}_{V_{K'}}(mL_{K'}))$$

preserves nonvanishing and the image dimension of the complete linear system.

[Proof of Corollary 6.3 · p. 41](#)

2. Descend to a finitely generated field and apply the complex inequality

Descend f , its projective embeddings, and boundaries to a subfield $K \subset k$ finitely generated over $\mathbb{Q}$, and choose $K \hookrightarrow \mathbb{C}$. Smoothness, geometric integrality, and boundary conditions descend, as does $f_*\mathcal{O}_X = \mathcal{O}_Y$ by faithful flatness. Apply the complex logarithmic inequality and identify all three Iitaka dimensions with those over k by field-extension invariance.

Proof of Corollary 6.3 · p. 41

Ordinary and logarithmic subadditivity are now available over algebraically closed fields of characteristic zero. PH's ordinary subadditivity is a separate proof.

Ol, Corollary 6.3 · p. 41 · PH, Theorem 1.1 · p. 2

6. Additional results supplied to whole-fiber variation

Section 7 is an additional route following the main theorem. Lemma 7.1 concerns a smooth geometrically integral projective pair (J, D) over a field K of characteristic zero, with reduced SNC D and $\kappa(J_{\bar{K}}, K_{J_{\bar{K}}} + D_{\bar{K}}) = 0$. Taking a root of a form in the least nonvanishing degree p gives a cover with an appropriate pole boundary $D_{\tilde{N}}$ and

$$H^0(\tilde{N}, m(K_{\tilde{N}} + D_{\tilde{N}})) = K\Omega^m \quad (m \geq 1).$$

Corollary 7.2 realizes this **entire highest piece** in a pure integral variation and identifies the normalization of its rational extension with M_W . This strengthens the character-line conclusion of Proposition 2.7 for general rational boundaries, but requires reducedness. In Remark 7.3, five points of coefficient $2/5$ on $\mathbb{P}^1$ give a root cover with six-dimensional highest space.

§7 · Lemma 7.1 · Corollary 7.2 · Remark 7.3 · pp. 43–46

Proposition 9.1 of WV uses these results together with Proposition 2.7 and Lemma 7.4 for an additional relative Iitaka construction. The separately checked inputs to its main route are the lower bound from Corollary 6.2 and adjoint positivity from Theorem 3.1. Theorem 6.28 of BFMT used to identify the moduli divisor in §7, the full valuation comparison of Lemma 7.4, and the complete arguments of WV §§8–9 remain unchecked here. BFMT's separate semiamplicity theorem should not be substituted for this identification.

WV, Theorem 2.1 · p. 6; Theorem 3.8 / Proposition 3.9 · p. 21; Proposition 9.1 · pp. 69–70 ·
Ol, Corollary 7.2 · pp. 45–46

Lemma 7.4 compares the minimum over actual source components with the lct allowing valuations on higher models, on the same SNC preparation. WV Proposition 9.1 uses this statement to normalize the initial boundary so that section comparison and the entire highest line are handled on the same preparation. The input statement and this use were compared; the unverified scope above concerns the full valuation and model-change arguments.

OI, Lemma 7.4 · pp. 46–47 · WV, proof of Proposition 9.1 · p. 70

7. Which papers supply which steps

Input

FF Theorem 1.1(ii), (iv), p. 2

Use and role in this manuscript

§2.6.4–5, pp. 13–15: canonical extension of the logarithmic highest part and passage to a pure weight grade

Scope checked here

Statement and application compared; the full semistable construction is unchecked

Input

Vil Theorem 2.19, p. 16

Use and role in this manuscript

Proposition 4.5, pp. 25–26: compactification of the class-C period quotient

Scope checked here

Statement and application compared

Input

BC Theorem 1.1 / Corollary 1.3, pp. 1–2

Use and role in this manuscript

Proposition 4.5 and proof of Theorem 3.1, pp. 26, 30–31: projective model and $K_S + D_S$ big

Scope checked here

Statements and stated application hypotheses compared

Input

BT Theorem 1.1, p. 2

Use and role in this manuscript

Lemma 4.6, p. 27: expected dimension on the Zariski closure of a line-map fiber

Scope checked here

Statement and application compared; the full rank-loss argument is unchecked

Input

KP Corollaries 6.19, 7.3, pp. 28–29; Theorem 8.1 / Corollary 8.3, p. 40

Use and role in this manuscript

Lemmas 5.2, 5.4, pp. 34, 36: projective stable family and bigness of its relative log canonical divisor

Scope checked here

Statements, maximal variation, and klt general-fiber conditions compared

Input

Fuj Theorem 1.1, pp. 1–2

Use and role in this manuscript

Lemma 3.3, pp. 18–19: nonvanishing after a big base twist

Scope checked here

Statement and class-C lc-pair/projective-base hypotheses compared

Input

Cam Corollary 4.11, p. 49

Use and role in this manuscript

(2.2), p. 5; Lemma 6.1, p. 40: invariant base term on a neat model

Scope checked here

Statement and application compared; the full neat-model theory is unchecked

Within catalogue 033, [OI](#) Corollary 6.2 supplies [WV](#) Theorem 2.1, and [OI](#) Theorem 3.1 supplies [WV](#) Theorem 3.8. In [RA](#), Corollary 6.2 is used only for the additivity conclusion of Corollary 1.2, not for the upper inequality in Theorem 1.1. [PH](#) is presented as an alternative ordinary proof. Connections to [LA](#) and the Kähler manuscripts in catalogue 034 should remain attached to individual results; the path [OI](#)→[LA](#)→[Schnell](#) is not a direct [OI](#)→[Schnell](#) dependency.

[WV](#) · pp. 6, 21, 69–70 · [RA](#) · pp. 2, 50 · [OI](#) · pp. 5, 41

8. Return to the sources

- [Theorem 1.1](#) · pp. 2–3 — Read the main hypotheses and conclusion alongside the proof passages below
- [Proposition 3.4](#) · pp. 19–20 — Check the codimension-two boundary condition and cancellation of the twist by a fixed section
- [Corollary 6.2](#) · pp. 40–41 — Follow the coefficient-one boundary condition into the logarithmic base bound
- [Corollary 6.3](#) · p. 41 — Transfer both ordinary and logarithmic conclusions using the geometric generic fiber and field extension

The manuscript is by OpenAI, dated September 26, 2026, with 55 pages including references. The reference commit is `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`. Start with the assembly in §6, pp. 39–40, then return to the fixed-reference section comparison in Proposition 2.7, pp. 10–17, and cancellation in Proposition 3.4, pp. 19–20. The two positivity inputs are proved in §4, pp. 20–31, and §5, pp. 31–38.

The passages read here cover the model convention, relative Iitaka reduction, order comparison and descent in §2; effectivity and cancellation in §3; the period quotient, the application portion of rank loss, and boundary section counting in §4; the stable-family statement and main construction steps, finite descent and pole tests in §5; and the main proof and Corollaries 6.2–6.3 in §6. For §7, the statements and proof routes of Lemma 7.1 and Corollary 7.2 were read. Statements and applications of the external inputs in the table were compared.

The complete rational-adjoint construction, all of Lemma 4.6 including finite monodromy, technical connections in semistable reduction and canonical extension, the full parameter-space construction for stable-family comparison, Lemma 7.4 and Appendix A, the complete proofs of the core consequences in Corollaries 6.4–6.5, and the external proofs themselves have not been independently verified. This draft explains the manuscript's claims and argument; it does not certify the correctness of the full proof. Unrecorded relations between papers remain uninvestigated.

[Complete manuscript](#)

Logarithmic Kodaira dimension and whole-fiber variation

From a field detected by logarithmic Iitaka systems to a field of definition of the whole fiber

For a projective fibration over a base of nonnegative logarithmic Kodaira dimension, the manuscript adds whole-fiber variation to the Kodaira-dimension lower bound. Its argument constructs a parameter field from logarithmic sections, establishes birational constancy of a minimal-index root cover, fixes a finite normalization of the relative Iitaka base using markings, and descends a finite cover.

Manuscript author: OpenAI · 2026-09-26 · 75 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

These are the manuscript's claims, with its numbering, hypotheses, and conclusions retained. Fix $\Omega = \overline{\mathbb{C}(V)}$ and let F be the geometric generic fiber. The variation is

$$\mathrm{Var}(f) = \min_{\substack{\mathbb{C} \subseteq L \subseteq \Omega \\ L \text{ algebraically closed}}} \{\mathrm{trdeg}_{\mathbb{C}} L : F \text{ is birational to } (F_L)_{\Omega} \text{ for some } F_L/L\}.$$

It measures the function field of the entire F . Variation of a period image, a canonical model, or the relative Iitaka base alone is a different invariant.

[Equation \(1.1\) · p.2](#)

Theorem 1.1 — Logarithmic variation

Let $f : U \rightarrow V$ be a projective surjection with connected fibers between smooth connected complex quasi-projective varieties. If $\bar{\kappa}(V) \geq 0$, then

$$\bar{\kappa}(U) \geq \kappa(F) + \max\{\bar{\kappa}(V), \mathrm{Var}(f)\}.$$

The morphism need not be smooth. No abundance or good-minimal-model hypothesis is imposed on F . When $\kappa(F) = -\infty$, the assertion uses the manuscript's convention $(-\infty) + a = -\infty$.

[Theorem 1.1 · p.2](#)

Corollary 1.2 — Projective Iitaka–Viehweg inequality

Let $f : X \rightarrow Y$ be a surjection with connected fibers between smooth connected complex projective varieties, and assume $\kappa(Y) \geq 0$. Then

$$\kappa(X) \geq \kappa(F) + \max\{\kappa(Y), \text{Var}(f)\}.$$

[Corollary 1.2 · p.2](#)

Corollary 1.3 — Variation and birational isotriviality

Let $f : U \rightarrow V$ be a **smooth projective** surjection with connected fibers between smooth connected complex quasi-projective varieties. Assume every closed fiber is non-uniruled. Then

$$\begin{aligned} \bar{\kappa}(V) = -\infty &\implies \text{Var}(f) < \dim V, \\ \bar{\kappa}(V) \geq 0 &\implies \text{Var}(f) \leq \bar{\kappa}(V). \end{aligned}$$

If V is Campana-special, $\text{Var}(f) = 0$. In particular, $\bar{\kappa}(V) = 0$ implies birational isotriviality. Here isotriviality means variation zero; no trivialization of the family by isomorphisms is asserted. This corollary additionally uses [LA](#) and [Taji](#) and is not an input to [Theorem 1.1](#).

[Corollary 1.3 and its proof · p.5](#)

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Logarithmic Kodaira dimension and whole-fiber variation](#). External inputs use the following keys.

- [OI OpenAI](#) — Orbifold and logarithmic Iitaka subadditivity: Inputs, additional results, or alternatives within the catalogue; see the use and relation type in the table below. 2026-09-26.
- [Fuj Osamu Fujino](#) — Notes on the weak positivity theorems: Weak positivity for producing sections with a negative ample twist. version 0.54, 2015-06-30.
- [Del Pierre Deligne](#) — Théorie de Hodge II: Finite characters; original-source checking of invariant cycles remains open. 1971, printed p.47 = PDF p.44.

- [DHP](#) Jean-Pierre Demailly; Christopher D. Hacon; Mihai Păun — Extension theorems, non-vanishing and the existence of good minimal models: Extension used to contract divisors carrying poles. arXiv:1012.0493v2.
- [PT](#) Mihai Păun; Shigeharu Takayama — Positivity of twisted relative pluricanonical bundles and their direct images: Relative Bergman weights and uniform estimates. arXiv:1409.5504v1.
- [BCHM](#) Caucher Birkar; Paolo Cascini; Christopher D. Hacon; James McKernan — Existence of minimal models for varieties of log general type: Existence of big klt adjoint models. arXiv:math/0610203v2, 81 pages.
- [KP](#) Sándor J. Kovács; Zolt Patakfalvi — Projectivity of the moduli space of stable log-varieties and subadditivity of log-Kodaira dimension: Positivity of projective stable families and variation bounds for marked models. 62-page author manuscript; hash pinned.
- [Han](#) Masaki Hanamura — Structure of birational automorphism groups, I: non-uniruled varieties: Making the cocycle constant; the original theorem statement remains unchecked. 1988, DOI 10.1007/BF01394338.
- [SGA1](#) Alexander Grothendieck et al. — Revêtements étales et groupe fondamental (SGA 1): Purity from codimension-one unramifiedness; the original statement remains unchecked. 2003 re-edition; arXiv:math/0206203v2.
- [Lan](#) Aaron Landesman — Invariance of the tame fundamental group under base change between algebraically closed fields: Descent of finite étale covers, morphisms, and actions across base extension. arXiv:2005.09690v3.
- [BDPP](#) Sébastien Boucksom; Jean-Pierre Demailly; Mihai Păun; Thomas Peternell — The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension: Pseudoeffectivity and movable curves, or the comparison with non-uniruledness. arXiv:math/0405285v1.
- [LA](#) OpenAI — Log abundance in characteristic zero: Good minimal models used only for the consequence Corollary 1.3. 2026-09-24.
- [Taj](#) Behrouz Taji — Birational geometry of smooth families of varieties admitting good minimal models: Variation and special-base consequences for smooth families with good minimal models. arXiv:2005.01025v4.
- [RA](#) OpenAI — The reverse logarithmic Kodaira inequality and additivity: Inputs, additional results, or alternatives within the catalogue; see the use and relation type in the table below. 2026-09-26.
- [PH](#) OpenAI — Projective Hodge lines and ordinary Iitaka subadditivity: Inputs, additional results, or alternatives within the catalogue; see the use and relation type in the table below. 2026-09-27.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Overview of Theorem 1.1

Proof target : [Theorem 1.1](#) · p. 2 See the hypotheses and conclusion.

When $\kappa(F) = -\infty$, the inequality is automatic under the stated convention. Below, $d = \kappa(F) \geq 0$: combine the base term supplied by OI with the variation term supplied by descent of the whole fiber.

Proof route. The main theorem has three stages: construct the parameter field, establish constancy of the Kodaira-zero fibre, and descend the original whole fibre. The empty-boundary corollary and the smooth-family consequence using LA and Taji are treated separately after the main proof.

For $d = \kappa(F) \geq 0$, make the field $b = \overline{\mathbb{C}(T_0)}$ detected by section systems into a birational definition field of the original whole geometric generic fiber F . The Kodaira-zero relative Iitaka fiber J is distinct from F .

Compare the logarithmic litaka system with the original base

$$d = \kappa(F) \geq 0, \quad \bar{\kappa}(V) \geq 0, \quad X' \longrightarrow W \longrightarrow Z$$

Use the logarithmic lower bound and proportional forms
[\[OI\] Cor. 6.2, pp. 40–41](#); [\[WV\] Prop. 2.10, pp. 12–14](#)

Embed the parameter field into the original base field

$$\mathbb{C}(T_0) \subseteq \mathbb{C}(Y), \quad \bar{\kappa}(U) = d + \dim T_0$$

Restricted adjoint comparison, contraction of poles and equivariant descent
[\[OI\] Thm. 3.1, pp. 17–18](#); [\[WV\] Prop. 3.12, p. 23](#)
[\[WV\] Cor. 3.16, p. 25](#); [Thm. 4.1 / Prop. 4.10, pp. 26, 42–43](#)

First birational constancy along the litaka fibers

$M|_{W_z} \sim_{\mathbb{Q}} 0 \implies$ flat top line with finite character
 From constancy of the root cover to constancy of J

Retain coordinates, actions and covers beyond constancy of J
[\[WV\] Prop. 6.2, pp. 47–51](#); [Lem. 7.1 / Prop. 7.2, pp. 52–56](#)

Fix the litaka-base coordinates and the whole function field

$b = \overline{\mathbb{C}(T_0)}$, $C_* \rightarrow P_0$, $V' \rightarrow P_0$
 Markings, constant cocycles, and no ramification on moving divisors

Descend the cover and action; identify the original geometric generic fiber
[\[WV\] Prop. 7.3 / §7.4, pp. 57–59](#)

Insert whole-fiber variation into the lower bound

$$\mathrm{Var}(f) \leq \dim T_0, \quad \bar{\kappa}(U) \geq d + \max\{\bar{\kappa}(V), \mathrm{Var}(f)\}$$

Overview linking the parameter field, first constancy, markings and descent of finite covers

[Open figure ↗](#) [TeX source ↓](#)

1. Obtain the candidate field from section systems

OI's logarithmic lower bound yields $\mathbb{C}(T_0) \subseteq \mathbb{C}(Y)$ and $\bar{\kappa}(U) = d + \dim T_0$. The next section develops this construction.

Proposition 2.10, Remark 2.11 · pp.12–14 · Ol, Corollary 6.2 · pp.40–41

2. Turn Hodge-line triviality into birational constancy

Apply Ol's adjoint comparison to the restricted root-cover variation to make its entire top line flat with finite character. Contraction of poles and local flows make J constant; article Section 4 develops this step.

Propositions 3.9, 3.12, Corollary 3.16, Proposition 4.10 · pp.21–25, 42–43 · Ol, Theorem 3.1 · pp.17–18

3. Return to the whole fiber while retaining coordinates and actions

Markings fix litaka-base coordinates and a finite normalization. Constant cocycles and inertia at moving divisors allow descent of covers and actions and identification of the original function field. Article Section 5 obtains $\text{Var}(f) \leq \dim T_0$.

Proposition 6.2, Propositions 7.2–7.3, §7.4 · pp.47–59

Combine this bound with the section-system dimension identity to obtain Theorem 1.1. Corollary 1.2 sets the boundaries to zero; Corollary 1.3 is a separate consequence using LA and Taji. The diagrams for the [parameter field](#), [first constancy](#), and [whole-fibre descent](#) expand these three stages.

3. Proposition 2.10 — Constructing the parameter field

Proof target : [Proposition 2.10](#) · pp. 12–14

Assume $d = \kappa(F) \geq 0$. The first task is to compare the absolute logarithmic litaka base Z and the original base Y inside the same function field, producing a subfield of $\mathbb{C}(Y)$ of transcendence degree $\bar{\kappa}(U) - d$. No constancy assertion is used at this stage.

Proposition 2.10

Assume Theorem 1.1 and $d = \kappa(F) \geq 0$. Choose a compatible SNC compactification $(X, D_X) \rightarrow (Y, D_Y)$ with the reduced inverse boundary. In the models of Proposition 2.7, let $q : X' \rightarrow Z$ be the absolute logarithmic litaka map, $q = h \circ x$, and $X' \xrightarrow{x} W \xrightarrow{g} Y$. The maps x, g, h have connected fibers and $W \rightarrow Y \times Z$ is generically finite onto its image. Put $r = \dim W_z$ and $p_1 = \dim Y - r$.

The models can be chosen with smooth connected projective varieties S, T_0 , a birational morphism $\rho : S \rightarrow Y$, a map $g_S : W \rightarrow S$ satisfying $g = \rho \circ g_S$, and compatible morphisms from S, Z to T_0 , such that

$W \longrightarrow (S \times_{T_0} Z)_{\text{main}}$ is dominant and generically finite.

$$\begin{aligned} \dim(W/Y) &= d, & \dim T_0 &= p_1, \\ \dim Z &= d + \dim T_0 \end{aligned}$$

The generic fibers of $S \rightarrow T_0$ and $Z \rightarrow T_0$ are geometrically integral. With $D_S = (\rho^* D_Y)_{\text{red}}$ made SNC, very general t satisfies $\kappa(S_t, K_{S_t} + D_S|_{S_t}) \geq 0$. In particular, $\mathbb{C}(T_0) \subseteq \mathbb{C}(S) = \mathbb{C}(Y)$ holds **inside the original field tower**.

Proposition 2.10 · pp. 12–14

01 Logarithmic lower bound and a base pluriform

$$d = \kappa(F) \geq 0, \quad \bar{\kappa}(U) \geq d + \bar{\kappa}(V),$$

$$0 \neq \xi \in H^0(Y, m(K_Y + D_Y)).$$

Retain the reduced inverse boundary; take the relative algebraic closure of the compositum.

[\[OI\] Cor. 6.2, pp.40–41](#) [Lemma 2.6; Prop. 2.7, pp.8–10](#)

02 Combine the absolute litaka map with the original fibration

$$X' \xrightarrow{x} W, \quad W \xrightarrow{g} Y, \quad W \xrightarrow{h} Z, \quad q = hx.$$

$\dim Z = \bar{\kappa}(U)$, $W \rightarrow Y \times Z$ is generically finite onto its image.

Contract $g^*\xi$ with $p_1 = \dim Y - \dim W_z$ base directions.
Subadditivity for x_z forces both nonnegative terms to vanish.

[Lemmas 2.8–2.9, pp.10–12](#)

03 Force proportionality on litaka fibers

$$\kappa(J) = \kappa(J, K_J + D'|_J) = 0,$$

$$\kappa(W_z, K_{W_z} + D_W^0|_{W_z}) = 0, \quad D_W^0 = (g^*D_Y)_{\text{red}}.$$

Constant ratios of minors give the rank of the image family.
Relative algebraic closedness in $\mathbb{C}(Y) \subseteq \mathbb{C}(S) \subseteq \mathbb{C}(X)$ gives $\mathbb{C}(S) = \mathbb{C}(Y)$.

[Prop. 2.10, \(2.10\)–\(2.12\), pp.12–14](#)

04 Recover a subfield of the original base from the image family

$$\mathbb{C}(T_0) \subseteq \mathbb{C}(Y), \quad \bar{\kappa}(U) = d + \dim T_0.$$

Remaining goal: F descends to $b = \overline{\mathbb{C}(T_0)}$.

Logarithmic subadditivity and the family of images produce an embedded parameter field

[Open figure ↗](#) [TeX source ↓](#)

1. Combine the two maps while retaining coefficient-one boundary

Choose SNC compactifications $(X, D_X) \rightarrow (Y, D_Y)$ with $f^{-1}(V) = U$. Applying [OI Corollary 6.2](#) gives $\bar{\kappa}(U) \geq d + \bar{\kappa}(V)$. Resolve the absolute litaka map of $K_X + D_X$ as $q : X' \rightarrow Z$. The relative algebraic closure of $\mathbb{C}(Y)\mathbb{C}(Z)$ in $\mathbb{C}(X)$ defines $X' \xrightarrow{x} W \xrightarrow{h} Z$ and $g : W \rightarrow Y$. The map $W \rightarrow Y \times Z$ is generically finite onto its image, and the relevant generic fibers are geometrically integral.

[Lemma 2.6, Proposition 2.7 · pp.8–10](#) · [OI, Corollary 6.2 · pp.40–41](#)

2. Contract logarithmic forms to compute the rank of the image family

Fix $0 \neq \xi \in H^0(Y, m(K_Y + D_Y))$. Set $r = \dim W_z$ and $p_1 = \dim Y - r$. Contracting $g^*\xi$ against p_1 directions from Z produces a nonzero logarithmic pluricanonical form on W_z . Since X'_z has logarithmic Kodaira dimension zero, subadditivity for x_z gives

$$\kappa(W_z, K_{W_z} + D_W^0|_{W_z}) = 0, \quad \kappa(J) = \kappa(J, K_J + D'|_J) = 0, \quad D_W^0 = (g^*D_Y)_{\text{red}}.$$

Here J is the geometric generic fiber of x . The contracted sections in a fixed degree are proportional. Consequently the ratios of maximal minors of the normal deformation map $T_z Z \rightarrow N_{g(W_z)/Y, y}$ do not depend on y . Its constant kernel gives rank p_1 for the differential of the Hilbert parameter map.

Lemmas 2.8–2.9, proof of Proposition 2.10 · pp.10–13

3. Embed the parameter field in the original base field

Take the relative algebraic closure in $\mathbb{C}(Z)$ of the parameter field of the image family $g(W_z)$, and call its smooth model T_0 . The resolved universal image family S maps generically finitely to Y . In the tower

$$\mathbb{C}(Y) \subseteq \mathbb{C}(S) \subseteq \mathbb{C}(W) \subseteq \mathbb{C}(X),$$

the first extension is algebraic, while $\mathbb{C}(Y)$ is relatively algebraically closed in $\mathbb{C}(X)$. Thus $\mathbb{C}(S) = \mathbb{C}(Y)$, an identification stronger than a dimension count. Together with $\dim(W/Y) = d$, it gives

$$\mathbb{C}(T_0) \subseteq \mathbb{C}(Y), \quad \bar{\kappa}(U) = \dim Z = d + \dim T_0.$$

It remains to descend the entire fiber to $b = \overline{\mathbb{C}(T_0)} \subseteq \Omega$.

Proposition 2.10, Remark 2.11 · pp.12–14

Fix the resulting $b = \overline{\mathbb{C}(T_0)}$ as the candidate definition field. Constancy of J , followed by markings and descent of finite covers, must still recover the original F .

WV, §§3–7 · pp. 14–59

For the last conclusion of Proposition 2.10, pull the fixed nonzero base section ξ to S and restrict it nontrivially to very general S_t . Adjunction gives $\kappa(S_t, K_{S_t} + D_S|_{S_t}) \geq 0$, retaining base nonvanishing for the later marked-model comparison.

4. Theorem 4.1 / Proposition 4.10 — From the root cover to first constancy

Proof target : Theorem 4.1 · p. 26

The least ordinary pluricanonical index makes the **entire** highest Hodge piece rank one, rather than just an eigenspace. Manuscript Section 4 supplies the analytic passage from its flatness to birational constancy.

Theorem 4.1

Let $\pi^\circ : P^\circ \rightarrow C^\circ$ be a smooth projective morphism with connected fibers over a smooth connected complex algebraic curve. Assume that its very general fiber D has $\kappa(D) = 0$ and $h^0(D, K_D) = 1$, and put $n = \dim D$. Suppose the **entire top Hodge bundle** $F^n(R^n\pi_*^\circ \mathbb{C} \otimes \mathcal{O}_{C^\circ})$ is a flat line with finite monodromy character.

Then a finite extension $L/\mathbb{C}(C^\circ)$ and a smooth connected projective variety $D_0/\mathbb{C}$ exist such that the generic fiber over L is birational to $(D_0)_L$. For a finite group Γ acting over the base, the identification can be chosen Γ -equivariant. A finite fiberwise birational action is also allowed after passage to an equivariant smooth projective model.

Theorem 4.1 · p. 26

Proposition 4.10

Consider the maps $x : X' \rightarrow W$, $h : W \rightarrow Z$ fixed in Propositions 2.7–2.10 and the family of their Kodaira-dimension-zero geometric generic fibers J . For very general $z \in Z$, the geometric generic fiber of this family over W_z is birational to the base change of a smooth projective variety over $\mathbb{C}$. Relatively, after extending $\mathbb{C}(Z)$ to its algebraic closure, the geometric generic fiber of x over the resulting h -fiber is birationally constant over that algebraically closed field.

Proposition 4.10 · p. 43

01 Take the root cover at the least index

$$p = \min\{m > 0 : H^0(J, mK_J) \neq 0\}, \quad \eta^{\otimes p} = \pi^* \omega_J.$$

$h^0(\tilde{J}, mK_{\tilde{J}}) = 1 \ (m > 0) :$ the entire highest Hodge piece is a line.

The Hodge integrability threshold gives the parabolic order.
Multiply by the relative form and transfer ratios to the fixed $X_{*,z}$.

[Lemma 3.1; Props. 3.3, 3.5, 3.7, pp.14–21](#)

02 Compare section systems using actual relative orders

$$B_P = 1 - \lambda_P + t_P, \quad M \sim_{\mathbb{Q}} \sum_P (\lambda_P - 1)P, \quad B \geq D_W^0,$$

$$\kappa(W_z, K_{W_z} + B_z + M_z) \leq 0.$$

Apply comparison to the restricted integral variation. Weak positivity gives
 $\kappa(D + cM_z) \geq \dim R$, contradicting the upper bound if $\dim R > 0$.

[\[OI\] Thm. 3.1, pp.17–18](#) [\[Fuj\] Thm. 1.1, pp.1–2](#)

[Props. 3.9, 3.12; Lemma 3.11, pp.21–24](#)

03 Interpolate the adjoint comparison on the restriction

$$D = K_{W_z} + B_z \sim_{\mathbb{Q}} D_{\text{eff}}, \quad M_z \sim_{\mathbb{Q}} a^* A,$$

$$D + M_z = c^{-1}(D + cM_z) + (1 - c^{-1})D,$$

$$\dim R = 0 \implies M_z \sim_{\mathbb{Q}} 0, \quad \kappa(D) = 0.$$

Vanishing semipositive curvature on curves gives a line subsystem; apply finite character on the restriction.

[Cors. 3.13–3.16, pp.24–25](#) [\[Del\] Cor. 4.2.8\(iii\)\(b\), PDF p.44](#)

04 Obtain a flat entire top line with finite character

$$\deg M_z|_C = 0 \implies \text{constant projective top line; finite character.}$$

Extension, Bergman weights, and minimization with a prescribed leading term make poles fixed. Contract on a big adjoint model, spread flows, and take the equivariant quotient.

[\[DHP\] Thm. 4.1, pp.20–21](#) [\[PT\] Thm. 4.2.7, p.36](#)

[\[BCHM\] Thm. 1.2\(2\), p.5](#) [Thm. 4.1; Prop. 4.10, pp.26–43](#)

05 Contract poles to obtain the first constancy statement

$$\sigma_S(K_D) > 0 \text{ for pole divisors, } K_D + \varepsilon H \text{ big klt.}$$

J is geometrically birationally constant along h -fibers.

The minimal root cover, restricted Hodge-line triviality, and analytic contraction give the first constancy statement

[Open figure ↗](#) [TeX source ↓](#)

1. Relate actual relative-form orders to the Hodge line

The connected cyclic cover obtained from the least $p > 0$ and $0 \neq \omega_J \in H^0(J, pK_J)$ has a resolution $\tilde{J}$ with $h^0(\tilde{J}, mK_{\tilde{J}}) = 1$ for every $m > 0$. Its root form spans the entire top-form space. For a base divisor P , distinguish the minimum t_P on actual source components from the threshold λ_P on all higher models:

$$t_P = \min_{Q \subset X', Q \rightarrow P} \frac{r_Q + d_Q}{m_Q}, \quad \lambda_P = \inf_{Q \rightarrow P} \frac{1 + r_Q}{m_Q}, \quad B_P = 1 - \lambda_P + t_P.$$

Here $m_Q = \text{ord}_Q(x^*P)$, $r_Q = p^{-1} \text{ord}_Q(\omega_J)$ is measured in the actual relative canonical bundle, and $d_Q = \text{coeff}_Q(D')$. The integrability exponent of the Hodge norm identifies the parabolic order of its line M as $\lambda_P - 1$. Thus $T = \sum t_P P \sim_{\mathbb{Q}} B + M$, $0 \leq B \leq 1$, and $B \geq D_W^0$.

Lemma 3.1, Propositions 3.3 and 3.5 · pp.14–19

2. Apply adjoint comparison anew to the restricted variation

Multiplication by the relative generator transfers sections to the fixed $X_{*,z}$ while preserving their ratios, giving

$$\kappa(W_z, K_{W_z} + B|_{W_z} + M|_{W_z}) \leq 0.$$

Meanwhile $D = K_{W_z} + B|_{W_z}$ is rationally effective. Apply [OI Theorem 3.1](#) to the **restricted integral variation** to obtain $M|_{W_z} \sim_{\mathbb{Q}} a^*A$ with $K_R + cA$ big. Weak positivity from [Fuj](#) and section transfer give $\kappa(D + cM) \geq \dim R$. For $c \geq 1$,

$$D + M = c^{-1}(D + cM) + (1 - c^{-1})D$$

transfers that lower bound to $D + M$, forcing R to be a point. Hence $M|_{W_z} \sim_{\mathbb{Q}} 0$ and $\kappa(W_z, D) = 0$. Curvature on test curves and [Del's](#) finite-character theorem, applied on the restriction, give a flat entire top line with finite monodromy.

[Propositions 3.7, 3.9 and 3.12](#), [Lemma 3.11](#), [Corollaries 3.13–3.16](#) · pp.20–25 · [OI, Theorem 3.1](#) · pp.17–18 · [Fuj, Theorem 1.1](#) · pp.1–2 · [Del, Corollary 4.2.8\(iii\)\(b\)](#) · printed p.47 / PDF p.44

3. Contract poles of the lifted vector field

Over a curve, kill the finite character and lift the invariant class to a global form Θ . Its kernel defines a rational vector field with $d\pi(v) = \partial_t$ and $\iota_v \Theta = 0$. Poles may remain along the zero divisor of the relative top form. [Lemma 4.4](#) uses [DHP](#) extension, while [Lemmas 4.6–4.7](#) use [PT](#) relative Bergman weights. These lead to positive limiting fixed order $\sigma_S(K_D) > 0$ at every pole divisor S . The weight is constructed before the final fiber is selected, and the regularization, degree, and cutoff limits have a prescribed order.

[Sections 4.1–4.5](#) · pp.26–40 · [DHP, Theorem 4.1](#) · pp.20–21 · [PT, Theorem 4.2.7 and Remark 4.3.1](#) · pp.36–37

Choose one small $\varepsilon > 0$ for all the finitely many poles. The big klt adjoint model of $K_D + \varepsilon H$, supplied by BCHM, contracts their positive fixed orders. Normality extends the resulting regular vector field across codimension two. Local flows and algebraic spreading of graphs give birational constancy after a finite field extension. Applying this equivariantly to the root cover and taking invariant function fields makes J constant in the h -fiber directions. Descending the original whole F still requires the next step.

Proof of Theorem 4.1, Corollary 4.9, Proposition 4.10 · pp.40–43 · BCHM, Theorem 1.2(2) · p.5

This fixes the Kodaira-zero fiber J . The next descent retains litaka-base coordinates and group actions to fix the entire geometric generic fiber.

WV, §6 introduction · §7.4 · pp. 46, 58–59

5. Proposition 7.3 — Descent retaining the whole litaka base

Proof target : Proposition 7.3 · pp. 57–59

Put $b = \overline{\mathbb{C}(T_0)}$ and denote the geometric generic fibers of W, S, Z over T_0 by H_0, I, P_0 . Write $p_I : H_0 \rightarrow I$ and $a_0 : H_0 \rightarrow P_0$ for the induced maps. The map $H_0 \rightarrow I \times_b P_0$ is generically finite and $\dim P_0 = d$. Constancy of J must now be made compatible with the coordinates of P_0 .

Proposition 7.3

In the setting of §§6–7, fix $b = \overline{\mathbb{C}(T_0)} \subseteq \Omega$. After the marking of Proposition 6.2 and finite parameter extension, let V'/b be the fixed finite normalization of the original litaka base and I/b the parameter variety; put $k = b(V')$, $L_0 = b(I)$, and $K = L_0(V')$. The Kodaira-dimension-zero fiber J/K has total field $K(J)$ from the original fibration. Use the finite Galois splitting with constant cocycle of Lemma 7.1 and the unramifiedness at moving divisors **after every finite parameter extension** from Proposition 7.2.

For an algebraic closure Λ of L_0 , there is a finitely generated field F_b/b containing k and an isomorphism

$$\begin{aligned} & \text{Frac}(\Lambda \otimes_b F_b) \\ & \simeq \text{Frac}(\Lambda \otimes_{L_0} K(J)). \end{aligned}$$

On identifying $\Lambda = \Omega$, the right-hand side is the function field of the **whole original geometric generic fiber**. Thus $\text{Var}(f) \leq \dim T_0$.

Proposition 7.3 · pp. 57–59

01 Add markings recognized by their coefficients

$$H_0 \longrightarrow I \times_b P_0, \quad \dim P_0 = d, \quad b = \overline{\mathbb{C}(T_0)},$$

$$(x_0 = 0), (x_j = 0), (x_0 + x_j = 0) : \quad d + \text{Var}(\text{marked model}) \leq \kappa(K_{H_0} + \Delta) \leq d.$$

Force marked-model variation zero; sum markings fix the remaining scalars.
Use the first constancy statement to choose a reference fiber J_0/k .

[\[KP\] Thm. 9.9, p.50](#) [Props. 5.3, 6.2, pp.45–51](#)

02 Fix the map, finite normalization, and old-boundary images

$$x_j/x_0 = c_j f_j, \quad f_j \in b(C_*), \quad c_j \in b,$$

$$V' \longrightarrow P_0 \text{ finite}, \quad k = b(V'), \quad K = k(I), \quad J_0/k.$$

Average the abelian part of the birational cocycle. Fixed old-boundary images
and equality in the order test produce an actual Q after every finite parameter extension.

[Lemmas 6.4, 7.1, pp.51–54](#) [\[Han\] Thms. 2.1–2.2](#)

Original [Han] statements not compared.

03 Combine a constant cocycle with a multiplicity-one minimizer

$$\tilde{K}/K \text{ Galois}, \quad \rho_g = z_g s_g, \quad (z_g, g|_{k'}) \text{ faithful},$$

$$P \text{ moving} : \quad B_P = 0, \quad m_Q = 1, \quad (1 + r_Q)/m_Q = \lambda_P.$$

The constant family's special fiber uniquely minimizes the threshold.
Equal inertia and faithful cocycle pairs force unramifiedness.

[Prop. 7.2, \(7.12\)–\(7.16\), pp.55–56](#)

04 Remove ramification at moving divisors

$$m_{Q'} = e_Q m_Q / e, \quad e_Q = e, \quad I_{Q'} = I_{P'} = 1.$$

Remove fixed branching and apply purity. Descend the finite étale cover, action,
and coefficient field together, then take invariants retaining the k -coordinates.

[\[Lan\] Thm. 1.1, p.2](#) [Prop. 7.3; Section 7.4, pp.57–59](#)

[\[SGA1\] X, Thm. 3.1](#) Original statement not compared.

05 Descend the function field of the whole fiber

$$k = b(V') \subseteq F_b = E_1(J_0)^{G_1},$$

$$\text{Frac}(\Omega \otimes_b F_b) \simeq \text{Frac}(\Omega \otimes_{\mathbb{C}(Y)} \mathbb{C}(X)),$$

$$\text{Var}(f) \leq \dim T_0 \implies \bar{\kappa}(U) \geq d + \max\{\bar{\kappa}(V), \text{Var}(f)\}.$$

Markings fix a finite normalization, and unramifiedness at moving divisors permits descent of the entire function field

[Open figure ↗](#) [TeX source ↓](#)

1. Recover the finite normalization using coordinate and sum markings

For an embedding of P_0 with coordinates $x_0, \dots, x_N$, label the pullbacks of $x_0 = 0$, $x_j = 0$, and $x_0 + x_j = 0$ by distinct small coefficients. Slightly decrease the old boundary to make the generic pair klt, and add further general markings to make it big. The added line bundles restrict trivially to a_0 -fibers, preserving the upper bound $\kappa(H_0, K_{H_0} + \Delta) \leq d$. Proposition 5.3, using KP, gives the lower bound $d + \text{Var}(\text{marked model})$. The marked model is therefore constant after a finite parameter extension. Let C_* be the underlying normal projective variety of this model pair over b .

Proposition 5.3, Proposition 6.2 · pp.45–50 · KP, Theorem 9.9 · p.50

Coordinate divisors determine $x_j/x_0 = c_j f_j$ with $f_j \in b(C_*)$; the sum divisors force $c_j \in b$. The morphism $C_* \rightarrow P_0$ itself descends, fixing the normalization $V' \rightarrow P_0$ in its function field. Images of positive old-boundary components become fixed b -divisors as well. With $k = b(V')$ and $K = k(I)$, the preceding constancy statement supplies a reference variety J_0/k . The markings are auxiliary identifiers and are discarded from the final fiber.

Proposition 6.2, especially (6.12) · pp.47–51

2. Remove the nonconstant cocycle and the inertia at moving divisors

Lemma 7.1 uses Han's structure theorem for birational automorphism groups. The abelian identity component and the unique divisibility of $A_1(\overline{K})/A_1(\overline{k})$ allow averaging over a finite quotient to remove the nonconstant cocycle as a coboundary. The resulting finite Galois splitting has a constant cocycle with faithful pairs $(z_g, g|_{k'})$.

Lemma 7.1 · pp.52–54

Comparison with the original Han statements remains pending.

At a moving divisor P , the old boundary coefficient is $B_P = 0$. Equality in the order inequalities produces an **actual source component** Q with $m_Q = 1$ attaining λ_P . On the split constant family the special fiber is the unique minimizing divisor. The identities

$$m_{Q'} = \frac{e_Q m_Q}{e}, \quad \frac{1 + r_{Q'}}{m_{Q'}} = e \frac{1 + r_Q}{m_Q} - (e - 1)$$

give $e_Q = e$. Total and base inertia coincide, and faithfulness of the constant cocycle makes inertia trivial. Repeating the argument after every finite parameter extension is needed to control all moving divisors over the algebraic closure.

Corollary 3.4 · p.18, Lemma 6.4 · pp.51–52, Proposition 7.2 · pp.55–56

3. Descend the cover, its action, and its coefficient field together

Remove the fixed branch divisors and singular locus to obtain $\mathcal{O} \subset V'$. The manuscript invokes purity to make the splitting cover finite étale over $\mathcal{O}$. In characteristic zero, Lan's base-change invariance descends covers and their morphisms, retaining both the group action and the map carrying k' . For the field E_1 of a connected component and its stabilizer G_1 , set

$$F_b = E_1(J_0)^{G_1}, \quad k = b(V') \subseteq F_b.$$

Commutation of finite-group invariants with field extension and identification of the original field tower give

$$\mathrm{Frac}(\Omega \otimes_b F_b) \simeq \mathrm{Frac}(\Omega \otimes_{\mathbb{C}(Y)} \mathbb{C}(X)).$$

The right side is the function field of the entire original geometric generic fiber. Hence $\mathrm{Var}(f) \leq \mathrm{trdeg}_{\mathbb{C}} b = \dim T_0$. Combine this with the dimension identity and subadditivity from article Section 3 to obtain Theorem 1.1.

Proposition 7.3 and Section 7.4 · pp.57–59 · Lan, Theorem 1.1 and Remark 1.5 · pp.2–3

For purity, only the receiving argument and its stated hypotheses were checked; the original SGA1 statement was not rechecked.

This completes the lower bound involving variation in Theorem 1.1. Corollary 1.2 uses this output with empty boundaries.

WV, Theorem 1.1 / Corollary 1.2 · p. 2; §7.4 · p. 59

6. Corollary 1.2: take empty boundaries

Proof target : Corollary 1.2 · p. 2 See the hypotheses and conclusion.

For projective X, Y , use empty compactification boundaries. The geometric generic fiber and its field-of-definition invariant are unchanged.

01 Choose empty boundaries in the projective setting

$$f : X \longrightarrow Y, \quad \kappa(Y) \geq 0, \quad D_X = D_Y = 0.$$

Smooth connected complex projective varieties; surjective; connected fibers.

Use $\bar{\kappa}(X) = \kappa(X)$ and $\bar{\kappa}(Y) = \kappa(Y)$; the field invariant is unchanged.

Thm. 1.1; Cor. 1.2 and proof, p.2

02 Specialize with the same whole-fiber variation

$$\kappa(X) \geq \kappa(F) + \max\{\kappa(Y), \text{Var}(f)\}.$$

Empty boundaries specialize the logarithmic inequality to the projective inequality

[Open figure ↗](#) [TeX source ↓](#)

1. Replace logarithmic by ordinary Kodaira dimensions

With $D_X = D_Y = 0$, one has $\bar{\kappa}(X) = \kappa(X)$ and $\bar{\kappa}(Y) = \kappa(Y)$. The hypothesis $\kappa(Y) \geq 0$ is exactly the base hypothesis of Theorem 1.1, so its inequality gives Corollary 1.2 immediately.

[Proof of Corollary 1.2 · p.2](#), [Section 7.4 · p.59](#)

The ordinary Iitaka–Viehweg inequality applies to projective fibrations over a base of nonnegative Kodaira dimension. The additional relative-Iitaka route is not a premise of this proof.

[WV, Corollary 1.2 · p. 2](#); [§9 introduction · p. 69](#)

7. Corollary 1.3: good models for a smooth family

Proof target : [Corollary 1.3 · p. 5](#) See the hypotheses and conclusion.

This route uses smoothness of the family and non-uniruledness of every closed fiber. It does not enter the proof of the main theorem.

01 Use non-uniruledness of every closed fiber

$f : U \rightarrow V$ smooth projective, F_v non-uniruled (v closed).

Smooth connected complex quasi-projective varieties; surjective; connected fibers.

Apply the contrapositive of the projective uniruledness criterion.

[\[BDPP\] Cor. 0.3, p.2](#)

02 Make the canonical divisor pseudo-effective

K_{F_v} pseudo-effective (v closed).

Apply smooth canonical good-model existence to each closed fiber.

[\[LA\] Cor. 11.2, pp.73–74](#) [Cor. 1.3 proof, p.5](#)

03 Obtain a semiample model and an exceptional comparison

$p^*K_{F_v} = q^*K_{M_v} + E$, $E \geq 0$, E q -exceptional.

K_{M_v} semiample.

Compare discrepancies on a common resolution containing any prime exceptional over M_v .

[Cor. 1.3 proof, p.5](#)

04 Check the canonical good model needed by Taji

$a(P, M_v) = a(P, F_v) + \text{coeff}_P E \geq 0$.

Use smoothness and good models of the fibers.

The case $\bar{\kappa}(V) = 0$ also has variation zero. This route is not used in the main proof.

[\[Taj\] Thms. 1.1–1.2 and variation, p.2](#)

05 Bound the same whole-fiber variation

$\bar{\kappa}(V) = -\infty \Rightarrow \text{Var}(f) < \dim V$,

$\bar{\kappa}(V) \geq 0 \Rightarrow \text{Var}(f) \leq \bar{\kappa}(V)$,

V Campana-special $\Rightarrow \text{Var}(f) = 0$.

Non-uniruled closed fibers acquire good models from LA, allowing application of Taji

[Open figure ↗](#) [TeX source ↓](#)

1. Obtain canonical good models from LA's comparison

For every closed fiber F_v , BDPP makes K_{F_v} pseudo-effective. LA Corollary 11.2 supplies a semiample K_{M_v} and a common-resolution identity

$$p^*K_{F_v} = q^*K_{M_v} + E, \quad E \geq 0, \quad E \text{ is } q\text{-exceptional}.$$

For every prime P exceptional over M_v ,

$$a(P, M_v) = a(P, F_v) + \text{coeff}_P E \geq 0.$$

Thus M_v is canonical and gives a good minimal model in the sense used by Taji.

Proof of Corollary 1.3 · p.5 · BDPP, Corollary 0.3 · p.2 · LA, Corollary 11.2 · pp.73–74

2. Apply Taji with whole-fiber variation

For smooth projective families whose fibers have good minimal models, [Taj Theorems 1.1–1.2](#) supply isotriviality over special bases and the two variation bounds according to the sign of the base's logarithmic Kodaira dimension. Taji's definition on p.2 also uses a birational field of definition of the entire fiber. When $\bar{\kappa}(V) = 0$, combine the bound with $\text{Var}(f) \geq 0$.

Corollary 1.3 · p.5 · Taj, Theorems 1.1–1.2 and definition of variation · p.2

For a smooth family with $\kappa(F) \geq 0$ and $\bar{\kappa}(V) \geq 0$, comparing [RA Corollary 1.2's](#) additivity with [Theorem 1.1](#) gives the same upper bound. This alternative does not cover the negative-base branch or all special bases.

Comparison paragraph · p.5 · RA, Corollary 1.2 · p.2

A base of logarithmic Kodaira dimension zero gives birational isotriviality. For the negative-base and special-base conclusions, retain the hypotheses of the LA–Taji route.

WV, Corollary 1.3 · p. 5

8. Additional results in Sections 8–9

The main proof ends in Section 7.4. [Theorem 8.1](#) replaces the Hodge contribution by an arbitrary nef pullback $M = h^*A$, for morphisms $g : W \rightarrow Y$ and $h : W \rightarrow Z$, an effective SNC rational boundary $B \geq (g^*D_Y)_{\text{red}}$, and $\kappa(Y, K_Y + D_Y) \geq 0$. The varieties are smooth complex projective, both morphisms are surjective with connected fibers, D_Y is reduced SNC, the coefficients of B lie in $[0, 1]$, and here $d = \dim(W/Y)$. If $K_Z + aA$ is big for all sufficiently large rational a and $K_W + B + M$ is big over Y , it asserts a second Iitaka base U_2 (called U in the manuscript) satisfying

$$\kappa(W, K_W + B + M) = \dim U_2 = d + \dim T_0, \quad \kappa(Y, K_Y + D_Y) \leq \dim T_0.$$

The final interpolation compares $P_{U_2}(c)$ and $P_{U_2}(a)$ on one fixed model. This reading covered the statement and the final proof passage, not Section 8's intermediate volume and effective finite-group degree estimates.

[Theorem 8.1](#) · pp.59–60, final proof passage · p.69

Section 9 applies this construction to the ordinary relative Iitaka fibration. It cites [OI Proposition 2.7](#), [Lemma 7.1](#), [Corollary 7.2](#), and [Lemma 7.4](#) for section comparison and the reduced-root entire top line. [Corollary 9.4](#) identifies the absolute Iitaka field and the intermediate field, recovering $\mathbb{C}(T_0) = \mathbb{C}(Y) \cap \mathbb{C}(U_2)$ inside $\mathbb{C}(X)$ before applying [Proposition 7.3](#). The additional [OI](#) statements and their use in [Proposition 9.1](#) were compared. Their full proofs and the complete arguments of §§8–9 were not independently verified.

[Proposition 9.1](#), [Corollaries 9.2–9.4](#) · [pp.69–72](#)

9. Which papers supply which steps

Input

[OI Corollary 6.2](#), [pp.40–41](#)

Use and role in this manuscript

[Theorem 2.1](#), [Section 2](#): logarithmic lower bounds and zero-dimensional Iitaka systems

Relation and comparison scope

direct; input statement and applications compared

Input

[OI Theorem 3.1](#), [pp.17–18](#)

Use and role in this manuscript

[Theorem 3.8](#), [Propositions 3.9](#) and [3.12](#): restricted adjoint comparison

Relation and comparison scope

direct; integral ambient and complex-summand hypotheses compared

Input

[Fuj Theorem 1.1](#), [pp.1–2](#)

Use and role in this manuscript

[Lemma 3.11](#), [pp.22–23](#): sections after a big base twist

Relation and comparison scope

direct; lc, Cartier multiple, and projective-base hypotheses compared

Input

[Del Corollary 4.2.8\(iii\)\(b\)](#), printed p.47

Use and role in this manuscript

Corollaries 3.14–3.16: finite character of a rank-one subsystem

Relation and comparison scope

direct; statement and use compared; invariant-cycle citation separately unverified

Input

[DHP Theorem 4.1](#), pp.20–21

Use and role in this manuscript

Lemma 4.4, pp.30–31: extension under a putatively nonzero restriction to the zero divisor

Relation and comparison scope

direct; conditions (19)–(23) compared with the application

Input

[PT Theorem 4.2.7](#) and [Remark 4.3.1](#), pp.36–37

Use and role in this manuscript

Lemma 4.6, pp.34–35: relative Bergman weight and uniform upper bound

Relation and comparison scope

direct; statement and use compared

Input

[BCHM Theorem 1.2\(2\)](#), p.5

Use and role in this manuscript

Section 4.6, p.40, Section 6, p.48: big klt adjoint models

Relation and comparison scope

direct; model-existence statement and uses compared; auxiliary ample-model locators unverified

Input

KP Theorem 9.9, p.50

Use and role in this manuscript

Proposition 5.3, p.46 $\rightarrow$ Proposition 6.2: lower bound for marked-model variation

Relation and comparison scope

direct; auxiliary base divisor $M = 0$ and klt big generic pair compared

Input

Han Theorems 2.1–2.2

Use and role in this manuscript

Lemma 7.1, p.53: birational identity component and cocycle

Relation and comparison scope

direct; receiving citation and argument read; original statements not compared

Input

SGA1 X, Théorème 3.1

Use and role in this manuscript

Proposition 7.3, p.57: purity for a finite normalization unramified in height one

Relation and comparison scope

direct; hypotheses as listed by the receiving paper checked

Input

Lan Theorem 1.1 and Remark 1.5, pp.2–3

Use and role in this manuscript

Proposition 7.3, pp.57–58: descent of finite étale covers and morphisms

Relation and comparison scope

direct; characteristic-zero specialization and use compared

Input

BDPP Corollary 0.3, LA Corollary 11.2, Taj Theorems 1.1–1.2

Use and role in this manuscript

Corollary 1.3, p.5

Relation and comparison scope

consequence; input statements and interfaces compared; not used in the main theorem

Input

PH Lemma 8.4 and Theorem 8.1

Use and role in this manuscript

Remark 3.17, p.25

Relation and comparison scope

alternative; input statements and use compared, including checks recorded in the PH article;
input proofs unverified

Input

OI Proposition 2.7, Lemma 7.1, Corollary 7.2, Lemma 7.4

Use and role in this manuscript

Proposition 9.1, pp.69–70

Relation and comparison scope

consequence (directly used by the additional result); statements and uses compared, full proofs
unverified

Unrecorded relations remain unexamined. Comparing an input's statement with its application is distinct from verifying the input theorem's entire proof.

10. Return to the sources

- [Theorem 1.1](#) · p. 2 — Read the main hypotheses and conclusion alongside the proof passages below
- [Proposition 2.10](#) · pp. 12–14 — Follow the embedding of the parameter field into the original base field and the dimension identity

- [Theorem 4.1](#) · p. 26 — Follow the flat entire top line through contraction of poles and equivariant flows to birational constancy
- [Proposition 7.3](#) · pp. 57–59 — Retain the fixed Itaka-base coordinates while descending the cover and action to the whole fiber
- [Corollary 1.2](#) · p. 2 — Set the boundary to zero to obtain the projective Itaka–Viehweg inequality
- [Corollary 1.3](#) · p. 5 — Check how good models and Taji also supply the negative-base and special-base consequences
- Embedded parameter field: Proposition 2.10, equation (2.12), pp.12–14.
- Actual minimum versus all-valuation threshold: Propositions 3.3 and 3.5, pp.17–19, returning in Lemma 6.4's multiplicity-one minimizer.
- First constancy statement: Proposition 3.12, Theorem 4.1, Lemmas 4.6–4.7, Proposition 4.8, pp.23–43.
- Second constancy statement and the whole field: Proposition 6.2, Lemma 6.4, Propositions 7.2–7.3, pp.47–59.

This reading covered the main statements and corollaries, Section 2's parameter field, Section 3's root cover, orders, and restricted triviality, the principal analytic constancy argument of Section 4, and the marked models and descent in Sections 5–7. The external interfaces compared are listed in the table. This is an account of the argument's route, not an independent verification of every inference.

Remaining work includes the full regularization and L^2 estimates of Section 4, external Hodge-extension and boundary-growth inputs, Hanamura's original statements, the original SGA1 purity statement, Section 8's intermediate proofs, and full proofs of Section 9's additional [OI](#) inputs. The full proofs of external inputs, expert review, and formal verification have not been completed. The separate scope for Sections 8–9 is stated above.

The catalogue-033 manuscripts, including [WV](#) and [OI](#), use commit

`fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`. [LA](#) uses `adc7f1241b42e322a6451854ab7e4b4c146bf78a`, matching the existing catalogue-034 article. Page locators refer to the respective PDFs; Deligne's printed pagination is additionally indicated.

For the additional route, the statements of [OI](#) Proposition 2.7, Lemma 7.1, Corollary 7.2 and Lemma 7.4 were compared with their use in Proposition 9.1. Complete valuation comparisons, all BFMT interfaces and the full proofs of §§8–9 remain unverified.

OI, Proposition 2.7 · Lemma 7.1 · Corollary 7.2 · Lemma 7.4 · pp.10–17, 43–47 ·
WV, Proposition 9.1 · pp.69–70

The reverse logarithmic Kodaira inequality and additivity

Sections retaining a prescribed zero, and removal of the period twist

For projective pairs whose total space and all boundary strata are smooth over the open base, the manuscript proves an upper bound for logarithmic Kodaira dimension. Its central construction, Proposition 1.3, turns a total-space pluriform into a pluriform on the original base and transfers vanishing on a specified fiber to vanishing at the specified base point. We first follow the independent upper-bound argument and then treat additivity, which additionally uses the lower bound from OI.

Manuscript author: OpenAI · 2026-09-26 · 51 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

Theorem 1.1 — Reverse logarithmic Kodaira inequality

Let X, Y be smooth connected projective complex varieties and let $f : X \rightarrow Y$ be surjective with connected fibers. Let $E \subset X$ and $D \subset Y$ be reduced SNC divisors, either of which may be zero, satisfying

$$\text{Supp}(f^*D) \subseteq \text{Supp}(E).$$

Set $V = Y \setminus \text{Supp } D$. Assume that X , and every irreducible component of every nonempty intersection of components of E , restricted to $f^{-1}(V)$, are smooth over V . For very general $y \in V$, put $F = X_y$, $E_F = E|_F$, and write

$$L_X = K_X + E, \quad L_Y = K_Y + D, \quad L_F = K_F + E_F.$$

Then

$$\kappa(X, L_X) \leq \kappa(Y, L_Y) + \kappa(F, L_F).$$

If either term on the right is $-\infty$, then $H^0(X, mL_X) = 0$ for every integer $m > 0$. A point has Kodaira dimension 0, and $(-\infty) + a = -\infty$, including $a = -\infty$. The boundary condition concerns supports and does not discard the multiplicities of f^*D . No smoothness over D , abundance, semiampleness, or good minimal model is assumed.

Theorem 1.1 and (1.1)–(1.2) · p. 2

Corollary 1.2 — Logarithmic additivity

For the same pairs, morphism, and stratum-smoothness assumptions as in Theorem 1.1, with the same $-\infty$ convention,

$$\kappa(X, K_X + E) = \kappa(Y, K_Y + D) + \kappa(F, K_F + E_F).$$

This combines the upper bound with [OI, Corollary 6.2]. The proof of the upper bound in Theorem 1.1 does not use OI.

Corollary 1.2 · p. 2; §7.6 · p. 50 · OI, Corollary 6.2 · pp. 40–41

Proposition 1.3 — A section on the original base with a prescribed zero

Suppose f, E, D satisfy the hypotheses of Theorem 1.1 and $0 \neq s \in H^0(X, mL_X)$ for some integer $m > 0$. Then $\kappa(Y, L_Y) \geq 0$. If, moreover, $\kappa(Y, L_Y) = 0$ and $s|_{X_y} = 0$ for some $y \in V$, there are an integer $a > 0$ and a section

$$0 \neq \tau \in H^0(Y, aL_Y), \quad \tau(y) = 0.$$

The point y may be any such point of the original V : it need not lie in the smooth locus of the cyclic-cover family constructed from s .

Proposition 1.3 · p. 3

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [The reverse logarithmic Kodaira inequality and additivity](#). External inputs use the following keys.

- OI OpenAI — Orbifold and logarithmic Iitaka subadditivity: Inputs, additional results, or alternatives within the catalogue; see the use and relation type in the table below. 2026-09-26.
- FF14 Osamu Fujino; Taro Fujisawa — Variations of mixed Hodge structure and semipositivity theorems: Local freeness of the Hodge and weight filtrations. Author manuscript 2014-03-17, vmhs-sp108.pdf.

- [FF17](#) Osamu Fujino; Taro Fujisawa — A short remark on [2]: The strictness correction to FF14. 2017-07-07.
- [BBT](#) Benjamin Bakker; Yohan Brunebarbe; Jacob Tsimmerman — o-minimal GAGA and a conjecture of Griffiths: Algebraicity of integral period images and positivity of Hodge lines. arXiv:1811.12230v3, 2022-10-23.
- [FG](#) Osamu Fujino; Yoshinori Gongyo — On the moduli b-divisors of lc-trivial fibrations: b-nefness for lc-trivial fibrations allowing negative boundary coefficients. Annales de l’Institut Fourier 64 (2014), 1721–1735; published PDF with cover.
- [CP](#) Frédéric Campana; Mihai Păun — Foliations with positive slopes and birational stability of orbifold cotangent bundles: Orbifold cotangent degree estimates and relative pseudoeffectivity. arXiv:1508.02456v4, 2017-06-14 (RA bibliography cites Publ. Math. IHÉS 129 (2019)).
- [BDPP](#) Sébastien Boucksom; Jean-Pierre Demailly; Mihai Păun; Thomas Peternell — The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension: Pseudoeffectivity and movable curves, or the comparison with non-uniruledness. arXiv:math/0405285v1.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Proof of Theorem 1.1 — Reduction by Kodaira dimension

Proof target : [Theorem 1.1](#) · p. 2 See the [hypotheses and conclusion](#).

Proof route. First derive the upper bound with Proposition 1.3 as input. Its proof is then expanded into [constructing sections from a Hodge vector](#) and [removing the twist for a positive-dimensional period quotient](#). OI enters only in the final [additivity corollary](#).

Write $\kappa_X, \kappa_Y, \kappa_F$ for the logarithmic Kodaira dimensions of the respective pairs. Given Proposition 1.3, the upper bound follows from restriction of sections and the logarithmic Iitaka fibration of the base.

Theorem 1.1 Compare sections in each Kodaira-dimension case

$$\kappa_X = \kappa(X, L_X), \quad \kappa_Y = \kappa(Y, L_Y), \quad \kappa_F = \kappa(F, L_F).$$

A Fiber of negative-infinite dimension

$$\kappa_F = -\infty.$$

A nonzero total section restricts nontrivially to a very general fiber.

§2.2, p. 5

Restrict to a very general fiber

$$H^0(X, mL_X) = 0 \quad (m > 0).$$

B Base of negative-infinite dimension

$$\kappa_Y = -\infty.$$

A nonzero total section forces $\kappa_Y \geq 0$.

Prop. 1.3, p. 3 §2.2, p. 5

Contradict base nonvanishing

$$H^0(X, mL_X) = 0 \quad (m > 0).$$

C Base of Kodaira dimension zero

$$\kappa_Y = 0, \quad h^0(Y, aL_Y) \leq 1.$$

Choose $y \in V$ outside all base zero divisors.

If $s|_{X_y} = 0$, a base section vanishes at the original y , contradicting its choice.

Prop. 1.3, p. 3

§2.3, (2.3)–(2.5), pp. 5–6

Injective restriction in every degree

$$H^0(X, mL_X) \hookrightarrow H^0(F, mL_F).$$

$$\kappa_X \leq \kappa_F.$$

D Base of positive Kodaira dimension

$$k = \kappa_Y > 0, \quad q : Y' \rightarrow T, \quad \dim T = k.$$

$$\kappa(G, K_G + D'|_G) = 0.$$

The induced pairs retain stratum smoothness. Bound the image jointly with $q \circ f'$.

§2.4, pp. 6–7

Apply the zero-base case on litaka fibers

$$H = (q \circ f')^{-1}(t), \quad H \rightarrow G.$$

$$\begin{aligned} \kappa_X &\leq k + \kappa(H, K_H + E'|_H) \\ &\leq k + \kappa_F. \end{aligned}$$

END The same upper bound in all cases

$$\kappa_X \leq \kappa_Y + \kappa_F, \quad (-\infty) + a = -\infty.$$

Upper bound: the two negative-infinity cases, restriction injectivity over a zero-dimensional litaka image, and the positive-base reduction

Open figure ↗ TeX source ↓

1. Treat both negative-infinity cases as vanishing statements

If $\kappa_F = -\infty$, the nonvanishing locus of a hypothetical nonzero total-space section has image containing a dense open subset of the base. Restriction and adjunction would give

$H^0(F, mL_F) \neq 0$ on a very general fiber, a contradiction. If $\kappa_Y = -\infty$, Proposition 1.3 instead contradicts the existence of a total-space section by producing base nonvanishing. Both arguments exclude sections in every positive degree.

§2.2 · p. 5

2. Obtain restriction injectivity in every degree at one fiber

When $\kappa_Y = 0$, one has $h^0(Y, aL_Y) \leq 1$. Each degree therefore contributes at most one zero divisor of a nonzero section. Choose y outside their countable union, also making all fiber plurigenera attain their very general values. A nonzero s in the restriction kernel would, by Proposition 1.3, produce a nonzero base section vanishing at y , contradicting this choice. Hence

$$H^0(X, mL_X) \hookrightarrow H^0(F, mL_F) \quad (m > 0), \quad \kappa_X \leq \kappa_F.$$

The essential point is that Proposition 1.3 applies at the specified point of the **original** V , independently of the auxiliary open set attached to each section. Finite generation of the section rings is unnecessary.

§2.3 and (2.3)–(2.5) · pp. 5–6

3. Reduce positive base dimension to zero on an Iitaka fiber

For $k = \kappa_Y > 0$, resolve the logarithmic Iitaka map to $q : Y' \rightarrow T$ with $\dim T = k$. Use strict-transform boundaries plus reduced exceptional divisors and make a compatible resolution $f' : X' \rightarrow Y'$; logarithmic section spaces are unchanged. A very general fiber G of q has $\kappa(G, K_G + D'|_G) = 0$. The induced morphism from $H = (q \circ f')^{-1}(t)$ to G retains the required smoothness with its boundary strata, so the preceding case applies:

$$\kappa(H, K_H + E'|_H) \leq \kappa_F, \quad \kappa_X \leq \dim T + \kappa(H, K_H + E'|_H) \leq k + \kappa_F.$$

The last upper bound is the image-dimension estimate for a total-space linear system combined with $q \circ f'$. It includes the case where G is a point. If H has Kodaira dimension $-\infty$, restriction excludes all total-space sections instead.

§2.1 and (2.1) · p. 5; §2.4 · pp. 6–7

Section 5 combines this independent upper bound with OI's logarithmic lower bound. Its key input is Proposition 1.3, developed next.

RA, §2 · §7.6 · pp. 5–7, 50

3. Proof of Proposition 1.3 — From a Hodge vector to a base section

Proof target : Proposition 1.3 · p. 3 See the hypotheses and conclusion.

Fix a nonzero section $\mathbf{s}$. The constructed Hodge vector need not generate the whole highest Hodge bundle. The argument therefore retains its coefficients and the compact flags determined by its projective direction. Below, $\mathcal{E}_0$ denotes the highest Hodge bundle, distinct from the boundary E .

START Fix one total-space pluriform

$$0 \neq s \in H^0(X, mL_X), \quad V = Y \setminus \text{Supp } D.$$

Take the root class and a nonzero pure weight quotient. Semistable comparison and weight subbundles extend each coefficient regularly.

Prop. 3.1, Lem. 3.2, pp. 8–13 [FF14] §4.9, pp. 24–25

[FF17] pp. 1–2

01 Keep the original coefficient lattice in a pure weight quotient

$$u : \mathcal{O}_Y(-L_Y) \dashrightarrow \mathcal{E}^d, \quad \theta^j(u) : \mathcal{O}_Y(-L_Y) \dashrightarrow \mathcal{E}^{d-j} \otimes \Omega_Y^1(\log D)^{\otimes j}.$$

Use the original source frame after blowing up. Divide out the root's scalar factor before weight projection; homogeneous operations preserve the zero.

Lem. 3.3, (3.15), pp. 13–14

02 Preserve the specified zero in the original frame

$$u_* : \mathcal{O}_Y(-aL_Y) \dashrightarrow \mathcal{E}_0, \quad y \in V.$$

$$s|_{X_y} = 0 \implies \text{ord}_\tau(u_*) \geq ae/m > 0 \quad (t = \tau^e, m \mid e).$$

Algebraize the integral Lie-period image and add the compact marks.

Prop. 4.1, pp. 14–15 Lem. 4.7, pp. 19–20

[BBT] Thm. 1.1, p. 1

03 Retain the full Lie period and compact flags

$$Y \dashrightarrow S, \quad \mathcal{E}_0 \text{ comes from a variation on } S.$$

Constant-frame coefficients are regular.
Positive exceptional order retains the zero.

§7.1, p. 43

Use the exact section comparison and the numerical removal in the next diagram.

Prop. 6.3, pp. 40–43

Prop. 7.7, p. 49

A S a point: select a nonzero coordinate

$$0 \neq \tau \in H^0(Y, aL_Y).$$

$$s|_{X_y} = 0 \implies \tau(y) = 0.$$

B $\dim S > 0$: remove the twist

$$L_W \text{ big} \implies$$

$$\kappa(Y, L_Y) \geq \dim W \geq \dim S > 0.$$

END Base nonvanishing and a prescribed zero when the base has Kodaira dimension zero

$$\kappa(Y, L_Y) \geq 0; \quad \kappa(Y, L_Y) = 0 \implies S = \text{pt.}$$

Prop. 7.1, p. 43 Prop. 1.3: proof, p. 50

Section construction: preserve the original coefficient lattice and zero test, then separate point and positive-dimensional period quotients

Open figure ↗ TeX source ↓

1. Keep new cyclic-cover discriminants out of the base pole divisor

Taking an m th root of s produces a logarithmic mixed Hodge class; projection to a suitable pure weight quotient remains nonzero. The coefficients of the resulting $u : \mathcal{O}_Y(-L_Y) \dashrightarrow \mathcal{E}^d$ and its Higgs iterates are measured in the original $\Omega_Y^1(\log D)$ frames. Proposition 3.1 proves their regularity in the unipotent Hodge extension on each prescribed higher model.

In transverse-curve tests, the cotangent tensor frames remain frames of an external pullback bundle. They are not replaced by differentials on the testing curve and then divided back. Thus an extra cyclic-cover discriminant inside the original V introduces no new allowed poles. Regular projection to a pure weight quotient uses the subbundle property of Lemma 3.2, with the semistable logarithmic de Rham/canonical-extension comparison from FF14 and the strictness supplement FF17.

Proposition 3.1 and Lemma 3.2 · pp. 8–13 · FF14, §4.9 and Lemma 4.10, Step 1 · pp. 24–31 · FF17 · pp. 1–2

2. Retain the original source frame when blowing up the specified point

If $s|_{X_y} = 0$, its coefficient near the smooth relative SNC fiber belongs to $\mathfrak{m}_y \mathcal{O}_X$. Blow up y , take a transverse parameter t at the exceptional divisor, and set $t = \tau^e$ with $m \mid e$. In the original L_X frame, the root r satisfies

$$(r/\tau^{e/m})^m = s/\tau^e.$$

The right-hand side is regular, and normality makes $r/\tau^{e/m}$ regular. The weight-subbundle property in Lemma 3.2 allows this scalar factor to be divided out before projection. After a homogeneous tensor operation of degree a , the vanishing order is still at least $ae/m > 0$ in the pulled-back original $\mathcal{O}_Y(-aL_Y)$ frame. Replacing it by the new logarithmic canonical frame on the blowup would lose the prescribed-point information.

Lemma 3.3 and (3.15) · pp. 13–14

3. Extract a scalar coordinate for a point quotient; prove positivity otherwise

Proposition 4.1 constructs a homogeneous replacement $u_* : \mathcal{O}_Y(-aL_Y) \dashrightarrow \mathcal{E}_0$ and a connected quotient $\mathcal{S}$ retaining the full Lie period and compact flags. [BBT, Theorem 1.1] algebraizes the period image of the full Lie variation, in which an integral lattice has been retained. The quotient is not obtained merely from the determinant of the highest Hodge bundle.

If $\mathcal{S}$ is a point, the descended variation is constant. In a constant frame, each coordinate of u_* is a rational section of aL_Y . Divisorial regularity and normality make it global; choose a nonzero coordinate as τ . The positive exceptional order from the preceding step is the vanishing order at

the original y , so $s|_{X_y} = 0$ implies $\tau(y) = 0$. When $\dim S > 0$, the next section gives $\kappa_Y > 0$. Consequently only the point case can occur for $\kappa_Y = 0$, proving both conclusions of Proposition 1.3.

Proposition 4.1 · pp. 14–15; Lemma 4.7 · pp. 19–20; §7.1 and Proposition 7.1 · p. 43; §7.5 · pp. 49–50 ·
BBT, Theorem 1.1 · p. 1

A base section vanishing at the specified point makes restriction to one fiber injective in every degree when the base has Kodaira dimension zero. The next section handles the positive-dimensional period quotient.

RA, §2.3 · proof of Proposition 1.3 · pp. 5–6, 50

4. Proposition 7.7 — A positive-dimensional period quotient and twist removal

Proof target : Proposition 7.7 · pp. 49–50

Assume $\dim S > 0$. Write $Y \xrightarrow{x} W \xrightarrow{h} S$ on suitably prepared models; the section comparison always returns to a fixed original pair. The notation $\geq_{\text{pe}}$ means that the difference is pseudoeffective.

Proposition 7.7

Use the data of §§4–7 attached to the nonzero total-space pluriform in Proposition 1.3. Here S is the quotient recording the full Lie period and compact flag, $Y \xrightarrow{x} W \xrightarrow{h} S$ is the relative Iitaka reduction, and L_W represents the complete section spaces of the fixed original pair (Y, D) , as in Step 1 below. If $\dim S > 0$, then

$$L_W \text{ is big, } \quad \kappa(Y, K_Y + D) \geq \dim W > 0.$$

The numerical twist-removal result used here is Proposition 7.2 below.

Proposition 7.7 · pp. 49–50

Proposition 7.2

Let $h : W \rightarrow S$ be a surjective morphism of smooth projective varieties with $\dim S > 0$. Let L be a rational divisor on W and H a nef rational divisor on S . Assume that L is big on the generic h -fiber, $L - h^*K_S$ is pseudoeffective, and $K_S + bH$ is big for some rational $b \geq 0$. Suppose there are a dominant generically finite morphism $\pi : \widehat{W} \rightarrow W$ from a smooth projective variety, a nef rational divisor P on $\widehat{W}$, and a rational $Q > 0$ such that, with $\tilde{H} = \pi^*h^*H$,

$$\begin{aligned}\nu(P + \tilde{H}) &= \nu(P), \\ Q\pi^*L &\geq_{\text{pe}} P.\end{aligned}$$

Then L is big.

Proposition 7.2 · pp. 44–45

START Start from a positive-dimensional marked quotient

$$\dim S > 0, \quad Y \xrightarrow{x} W \xrightarrow{h} S, \quad u_* = \omega x^* u_W.$$

Factor by the generic-fiber generator and extract original source valuations.
The minimum detects all poles and preserves section ratios.

[Lem. 6.1–6.2, pp. 37–40](#) [Prop. 6.3, pp. 40–41](#)

01 Identify complete section spaces with the original base

$$L_W = K_W + \sum_A t_A A, \quad t_A = \min_{I \mapsto A} \frac{a^{-1} \operatorname{ord}_I \omega + d_I}{m_I}.$$

$$H^0(W, \ell L_W) \simeq H^0(Y_0, \ell L_{Y_0}) \quad (\ell \text{ sufficiently divisible}), \quad L_W \text{ big over } S.$$

Allow negative coefficients in Δ and verify rank one for the discrepancy sheaf.

[Prop. 6.3, \(6.12\)–\(6.13\), pp. 41–42](#)

[\[FG\] Def. 3.2, Thm. 3.6, PDF pp. 5, 7](#)

02 Obtain a nef moduli part from an auxiliary lc-trivial subpair

$$L_W = K_W + B_W + M_W, \quad 0 \leq B_W \leq 1, \quad M_W \text{ nef.}$$

Remove D_0 by boundary rank loss; compare curvature-null directions of the tails.

[Prop. 5.3, 5.5, pp. 30–34](#) [Prop. 5.6, pp. 34–36](#)

03 Compare positivity from periods and Higgs tails

$$K_S + bH \text{ big}, \quad P = -\sum_i (i+1) \det G_i \text{ nef}, \quad \tilde{H} = \pi^* h^* H.$$

$$\nu(P + \tilde{H}) = \nu(P).$$

Transfer fixed Higgs maps to orbifold cotangent tensors. Movable-curve estimates
force threshold zero; perturbation also gives relative pseudoeffectivity.

[Lem. 7.4–7.6, pp. 46–49](#)

[\[CP\] Thm. 1.3, 3.4, pp. 3, 12–14](#) [\[BDPP\] Thm. 2.2, p. 6](#)

04 Obtain a comparison uniform under boundary perturbation

$$Q\pi^* L_W \geq_{\text{pe}} P \quad (Q > 0), \quad L_W - h^* K_S \geq_{\text{pe}} 0.$$

First $L_W + bh^* H$ is big. Nef subtraction removes the twist; the complete
section-space comparison returns sections to the original base.

[Prop. 7.2, Lem. 7.3, pp. 44–45](#) [Prop. 7.7, p. 49](#)

05 Subtract the period twist

$$(1 + tQ)\pi^* L_W \geq_{\text{pe}} A' + tP - b\tilde{H} \text{ big} \quad (t \gg 0).$$

$$L_W \text{ big} \implies \kappa(Y, L_Y) \geq \dim W \geq \dim S > 0.$$

Positive-dimensional quotient: the exact section lattice, nef moduli and Higgs-tail lines, and numerical removal of the period twist

Open figure ↗ [TeX source](#) ↓

1. Construct a divisor representing the complete section spaces

The coefficient system is constant on the generic fiber over S , so a nonzero logarithmic section exists there and its Iitaka dimension is nonnegative. Form its relative Iitaka fibration. The unique degree- a generator ω on the generic x -fiber J factors $u_* = \omega x^* u_W$. For a prime A on W , define

$$t_A = \min_{I \rightarrow A} \frac{a^{-1} \operatorname{ord}_I(\omega) + d_I}{m_I}, \quad m_I = \operatorname{ord}_I(x^* A), \quad d_I = \operatorname{coeff}_I D_Y, \quad L_W = K_W + \sum_A t_A A.$$

Here D_Y is the reduced boundary on this model. If a prime on the original source is contracted to codimension at least two on a temporary base, its restricted valuation is first extracted on the base. The minimum rule then yields, in all sufficiently divisible degrees ℓ , an isomorphism

$$H^0(W, \ell L_W) \xrightarrow{\sigma \mapsto \omega^{\ell/a} x^* \sigma} H^0(Y_0, \ell L_{Y_0})$$

preserving section ratios. The fixed model Y_0 has the same section spaces as the original (Y, D) . Moreover, L_W is big over S .

Lemmas 6.1–6.2 and (6.5)–(6.6); Proposition 6.3 and (6.8) · pp. 37–41

2. Check the auxiliary subpair's rank condition before applying b-nefness

The manuscript sets $\Delta = -a^{-1} \operatorname{div}_{aK_{Y/W}}(\omega) + x^* \sum t_A A$, obtaining $K_Y + \Delta \sim_{\mathbb{Q}} x^* L_W$; negative coefficients of Δ are allowed. The logarithmic zero divisor Z_J on the generic fiber has Iitaka dimension zero. Sections of the discrepancy sheaf are bounded above by sections of $\mathcal{O}_J(NZ_J)$ and below by the constant section, giving

$$\operatorname{rank} x_* \mathcal{O}_Y([\mathbf{A}^*(Y, \Delta)]) = 1.$$

[FG, Definition 3.2 / Theorem 3.6] now applies. On a suitable model,

$$L_W = K_W + B_W + M_W, \quad 0 \leq B_W \leq 1, \quad M_W \text{ nef.}$$

The lower bound on B_W comes from testing the discriminant threshold at the same prime attaining the minimum for t_A . The external input supplies b-nefness, without requiring semiamplessness of the moduli part.

Proposition 6.3 and (6.9)–(6.15) · pp. 41–43 ·

FG, Definition 3.2 and Theorem 3.6 · PDF pp. 5, 7 (printed pp. 1724, 1726)

3. Account for the period directions in the Higgs-tail numerical dimension

On each component of the boundary D_0 with nonidentity Lie monodromy, the period line satisfies $\nu(H|_{D_i}) < \nu(H)$. Comparing growth orders of sections removes this boundary and makes $K_S + bH$ big. Let G_i be generated by all contractions of the i th Higgs iterate of u_W . On a unipotent cover followed by resolution, $\pi : \widehat{W} \rightarrow W$, put

$$P = - \sum_i (i+1) \det G_i, \quad \tilde{H} = \pi^* h^* H.$$

Proposition 5.6 proves that P is nef and $\nu(P + \tilde{H}) = \nu(P)$. A curvature-null direction fixes the chosen line and its compact flags; an orbit calculation and the representation-theoretic Lemma 5.7 then show that it fixes the whole highest Hodge space. The chosen vector is not assumed to span that space.

Proposition 5.3 · pp. 30–33; Lemma 5.4 and Proposition 5.5 · pp. 33–34; Proposition 5.6 and (5.11)–(5.18) · pp. 34–36

4. Obtain a fixed comparison from estimates uniform in the boundary

Lemma 7.4 uses the minimizing prime for t_A and the discriminant estimate together to transfer Higgs determinants into orbifold cotangent tensors. Represent $M_W + \epsilon A$ by a general effective Λ_ϵ and apply [CP, Theorem 1.3] whenever $K_W + B_W + \Lambda_\epsilon \sim_{\mathbb{Q}} L_W + \epsilon A$ is pseudoeffective. [BDPP, Theorem 2.2] converts movable-curve degree inequalities into

$$C(L_W + \epsilon A) + RL_W \geq_{\text{pe}} P_W,$$

with $C \geq 0$ and $R > 0$ independent of ϵ . A positive pseudoeffective threshold t_0 would satisfy $t_0 \leq Ct_0/(C+R) < t_0$, so $t_0 = 0$. Taking limits and using $\pi^* P_W \geq_{\text{pe}} P$ gives $Q\pi^* L_W \geq_{\text{pe}} P$ for the fixed $Q = C + R > 0$. Applying [CP, Theorem 3.4] to the same perturbed pairs and pushing down from good models also gives $L_W - h^* K_S \geq_{\text{pe}} 0$.

Lemmas 7.4–7.6 and (7.5)–(7.11) · pp. 46–49 · CP, Theorems 1.3, 3.4 and Remarks 3.3, 3.6 · pp. 3, 12–14 · BDPP, Theorem 2.2 · p. 6

5. Subtract the numerical twist and return to the original base

Relative bigness, $L_W - h^* K_S \geq_{\text{pe}} 0$, and bigness of $K_S + bH$ first make $L_W + bh^* H$ big. After pullback, choose an ample lower bound A' and add the fixed comparison from the preceding step:

$$(1 + tQ)\pi^* L_W \geq_{\text{pe}} A' + tP - b\tilde{H}.$$

In Lemma 7.3, the equality $\nu(P + \tilde{H}) = \nu(P)$ makes the loss term in the nef Morse inequality have smaller degree in t than the supply term. The right-hand side is therefore big for $t \gg 0$, and Proposition 7.2 makes L_W itself big. The **complete section-space isomorphism** from Step 1 returns these systems to the original base and gives $\kappa_Y \geq \dim W \geq \dim S > 0$. No equality between numerical and Iitaka dimensions of an arbitrary pseudoeffective divisor is assumed.

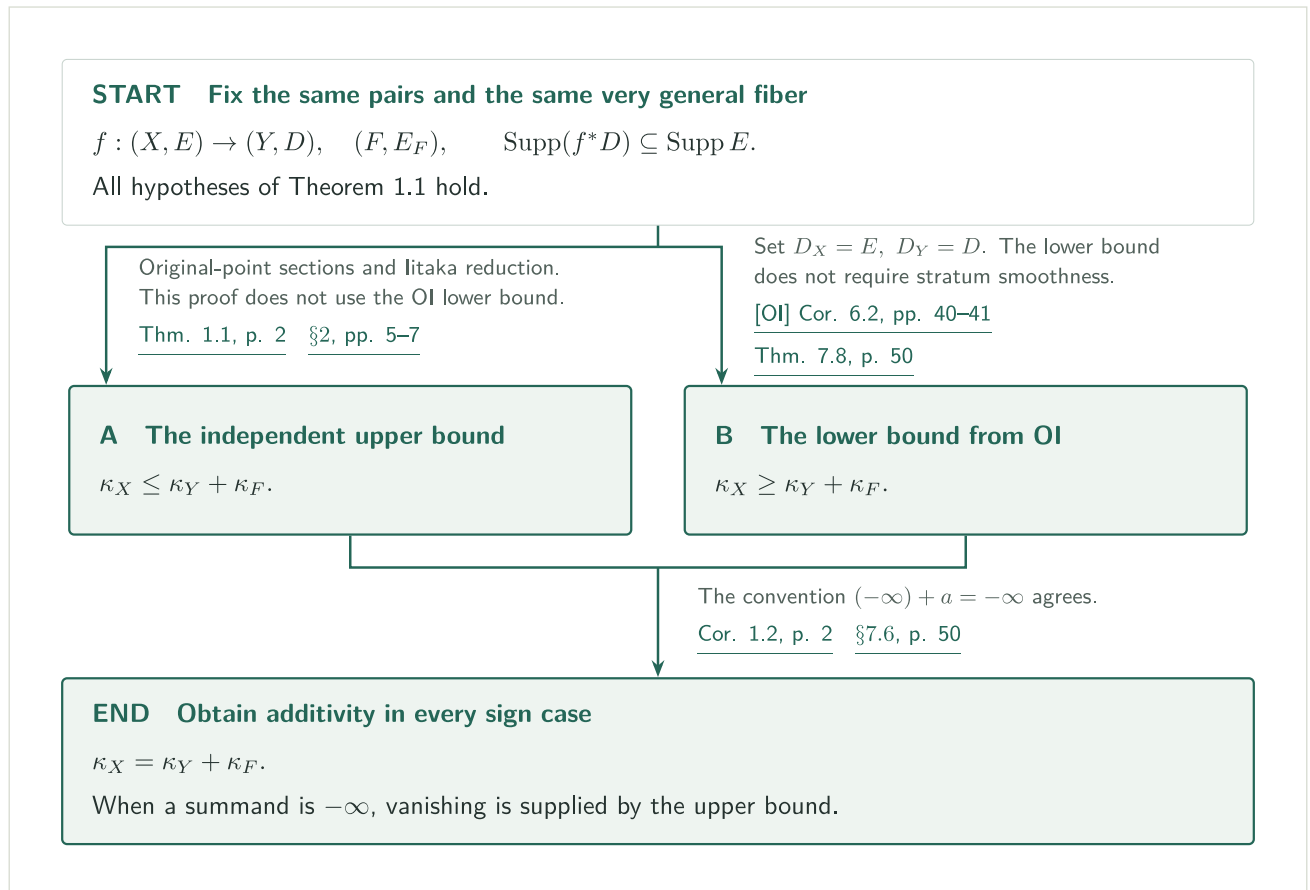
Proposition 7.2 and Lemma 7.3 · pp. 44–45; Proposition 7.7 · p. 49; proof of Proposition 1.3 · p. 50

Positive base Kodaira dimension excludes this branch when the base has Kodaira dimension zero. This completes Proposition 1.3 and returns to the upper-bound argument in Section 2.

5. Proof of Corollary 1.2 — Add the lower bound

Proof target : Corollary 1.2 · p. 2 See the hypotheses and conclusion.

Only here does the other catalogue-033 manuscript OI enter. The notation $\kappa_X, \kappa_Y, \kappa_F$ and the $-\infty$ convention are unchanged from Section 2.



Additivity: combine the independent upper bound with OI logarithmic subadditivity for the same pairs and the same fiber

[Open figure ↗](#) [TeX source ↓](#)

1. Match the boundary condition and the very general fiber

Apply [OI, Corollary 6.2] with $D_X = E$ and $D_Y = D$. Smooth connected projective complex varieties, connected fibers, reduced SNC boundaries, and support inclusion are all supplied by Theorem 1.1. The lower bound does not require stratum smoothness, and its very general pair is exactly (F, E_F) . The manuscript restates it as Theorem 7.8.

Theorem 7.8 and §7.6 · p. 50 · OI, Corollary 6.2 · pp. 40–41

2. Keep the vanishing supplied by the upper bound in the negative-infinity case

When both summands are finite, the two inequalities give equality. If either is $-\infty$, the lower bound is automatic; the content of equality is the vanishing $H^0(X, mL_X) = 0$ supplied by Theorem 1.1. This vanishing is not extracted from the proof in OI.

Theorem 1.1 · p. 2; §2.2 · p. 5; §7.6 · p. 50 · OI, proof of Corollary 6.2 · p. 41

For smooth families with nonnegative fiber and base Kodaira dimensions, comparison with WV Theorem 1.1 gives an alternative upper bound on variation.

WV, comparison after Corollary 1.3 · p. 5

6. Which papers supply which steps

Input

OI, Corollary 6.2 · pp. 40–41

Use in this manuscript

Corollary 1.2 and Theorem 7.8 · pp. 2, 50

Role and checking scope

Input to a consequence: lower bound for the same boundaries and fiber. Compared the statement, short corollary proof, and application; the complete OI proof is not verified.

Input

FF14, §4.9 and Lemma 4.10, Step 1 · pp. 24–31; FF17 · pp. 1–2

Use in this manuscript

Lemma 3.2 · pp. 12–13

Role and checking scope

Hodge extension and regular weight projection. Compared the statements and the local-freeness/strictness interface; the full proofs back to Steenbrink and Deligne are not traced.

Input

BBT, Theorem 1.1 · p. 1

Use in this manuscript

Lemma 4.7 · pp. 19–20

Role and checking scope

Algebraizes the full integral Lie-period image. Compared the input statement and object to which it is applied; compact-mark construction and finite-monodromy descent are not fully independently checked.

Input

FG, Definition 3.2 and Theorem 3.6 · PDF pp. 5, 7

Use in this manuscript

Proposition 6.3 · pp. 41–43

Role and checking scope

b-nefness after checking rank one for the auxiliary subpair, allowing negative coefficients. Compared the definition and theorem with (6.12)–(6.13); input proof not verified.

Input

CP, Theorems 1.3, 3.4 · pp. 3, 12–14

Use in this manuscript

Lemmas 7.5–7.6 · pp. 47–49

Role and checking scope

Orbifold cotangent quotient degrees and relative pseudoeffectivity. Compared perturbed pairs, good models, and pushdown at the application; complete input proofs not verified.

Input

BDPP, Theorem 2.2 · p. 6

Use in this manuscript

Lemma 7.5 and (7.10) · p. 48

Role and checking scope

Passes from movable-curve degrees to pseudoeffectivity. Compared the projective statement and its application; complete input proof not verified.

AS, Theorem A · pp. 1–2 is cited for the connection-form argument in §5.1 and Lemma 5.1 · pp. 26–28. The manuscript supplies the needed semisimple argument itself; we distinguish this as a relation of proof method. Theorem A and the manuscript's stated use were compared, without tracing the full proof in AS. Other relations within catalogue 033 remain unexamined or unrecorded; a missing arrow does not mean absence of dependence.

7. Return to the sources

- Theorem 1.1 · p. 2 — Read the main hypotheses and conclusion alongside the proof passages below
- Proposition 1.3 · p. 3 — Read the section construction that retains the zero at a prescribed point of the original base
- Proposition 7.7 · pp. 49–50 — Combine numerical twist removal with the complete section comparison to recover positive base Kodaira dimension
- Corollary 1.2 · p. 2 — Locate the added lower bound and the treatment of negative infinity
- §2 · pp. 5–7: all sign cases of the upper bound, using Proposition 1.3 alone.
- Proposition 3.1 and Lemmas 3.2–3.3 · pp. 8–14: the original coefficient lattice and prescribed zero.
- Proposition 4.1 and Lemma 4.7 · pp. 14–15, 19–20; Propositions 5.3, 5.5–5.6 · pp. 30–36: the marked quotient, boundary rank loss, and Higgs tails.
- §6 · pp. 37–43: relative Iitaka, extraction of source valuations, exact section comparison, and lc-trivial conditions.

- §7 · pp. 43–50: constant coordinates, twist removal, return to the original base, and addition of the lower bound.

Checked in this draft: the main hypotheses and conclusions and the selected proof passages above, concentrating on §§2–3, the valuation and rank conditions in §6, and the threshold and numerical removal in §7. In §§4–5, the quotient/descent passages, boundary-drop and Higgs-tail statements, their later uses, and selected proof paragraphs were read. External statements and applications were compared as specified in the table.

Not checked: an independent verification of the full homogeneous-replacement/compact-flag construction in §4.1–4.2, all metric estimates in §4.4, or all boundary-degeneration and representation-theoretic calculations in §5. Complete external proofs, recursive verification of their references, an exhaustive dependency audit, expert review, and formal verification have not been performed. The diagrams organize the manuscript's argument and do not certify its full correctness.

The manuscript and [OI](#) are pinned to commit `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`, version 2026-09-26. Page numbers refer to the linked PDFs; [FG](#) includes a cover, so PDF and printed page numbers are distinguished.

033 / 04

[Read on the website ↗](#)

CATALOGUE 033 · PAPER 04 / 5 Overview

Projective Hodge lines and ordinary Iitaka subadditivity

From rank loss of the highest Hodge line to ordinary Iitaka subadditivity

The manuscript claims ordinary Iitaka subadditivity by combining adjoint positivity for an entire rank-one highest Hodge piece with a canonical bundle formula and weak positivity. Its central mechanism separates the dimension of the full period image from the rank of the highest-line map, removes boundary and finite-quotient ramification, and constructs sections without assuming semiample.

Manuscript author: OpenAI · 2026-09-27 · 40 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

The following statements report the manuscript's claims in their original order and with their hypotheses.

Theorem 1.1 — Ordinary Iitaka subadditivity

Let k be an algebraically closed field of characteristic 0, and let $f : X \rightarrow Z$ be a surjective projective morphism with connected fibres between smooth connected projective k -varieties. For its geometric generic fibre F ,

$$\kappa(X) \geq \kappa(F) + \kappa(Z).$$

Use $(-\infty) + a = -\infty$ and $\kappa(\text{point}) = 0$.

[Theorem 1.1 · p. 2](#)

Theorem 1.2 — Projective adjoint positivity

Let Y be a smooth connected projective complex variety, $U \subset Y$ a dense Zariski open subset, and $\mathbb{V}$ a rationally polarized integral pure variation of Hodge structure on U .

Assume that its **entire highest nonzero Hodge filtration step is a line**: $F^{p+1} = 0$ and $\text{rank } F^p = 1$. Let $M \in \text{Pic}(Y) \otimes \mathbb{Q}$. Assume that, after a smooth projective birational modification if needed, a smooth projective alteration $\tau : \hat{Y} \rightarrow Y$ and $c \in \mathbb{Q}_{>0}$ exist such that the pulled-back variation has unipotent monodromy along an SNC boundary and its extended highest Hodge line satisfies

$$\tau^*M \sim_{\mathbb{Q}} cJ_{\hat{Y}}.$$

After further smooth projective birational modification, there are a surjective morphism $p : Y \rightarrow S$ with connected fibres onto a smooth projective variety and a nef rational line L such that

$$M \sim_{\mathbb{Q}} p^*L, \quad K_S + jL \text{ is big for every sufficiently large integer } j.$$

If S is a point, the conclusion is $M \sim_{\mathbb{Q}} 0$.

Theorem 1.2 · p. 2

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Projective Hodge lines and ordinary Iitaka subadditivity](#). External inputs use the following keys.

- [BBT](#) Benjamin Bakker; Yohan Brunebarbe; Jacob Tsimerman — o-minimal GAGA and a conjecture of Griffiths: Algebraicity of integral period images and positivity of Hodge lines. arXiv:1811.12230v3 (2022-10-23).
- [BC](#) Yohan Brunebarbe; Benoît Cadorel — Hyperbolicity of varieties supporting a variation of Hodge structure: Log-general type from an immersive period map. arXiv:1707.01327v1.
- [FM](#) Osamu Fujino; Shigefumi Mori — A canonical bundle formula: Residual divisors and all-degree direct images in the canonical bundle formula. JDG 56 (2000), author-hosted journal PDF.
- [Amb](#) Florin Ambro — Shokurov's boundary property: Moduli b-divisors and the extended Hodge eigensheaf. arXiv:math/0210271v1 (2002-10-17).
- [Fuj](#) Osamu Fujino — Notes on the weak positivity theorems: Weak positivity for producing sections with a negative ample twist. 2015-06-30, version 0.54.
- [Has](#) Kenta Hashizume — Log Iitaka conjecture for abundant log canonical fibrations: Addition for general-type fibers of effective klt pairs. arXiv:1902.10923v2 (2019-05-01).

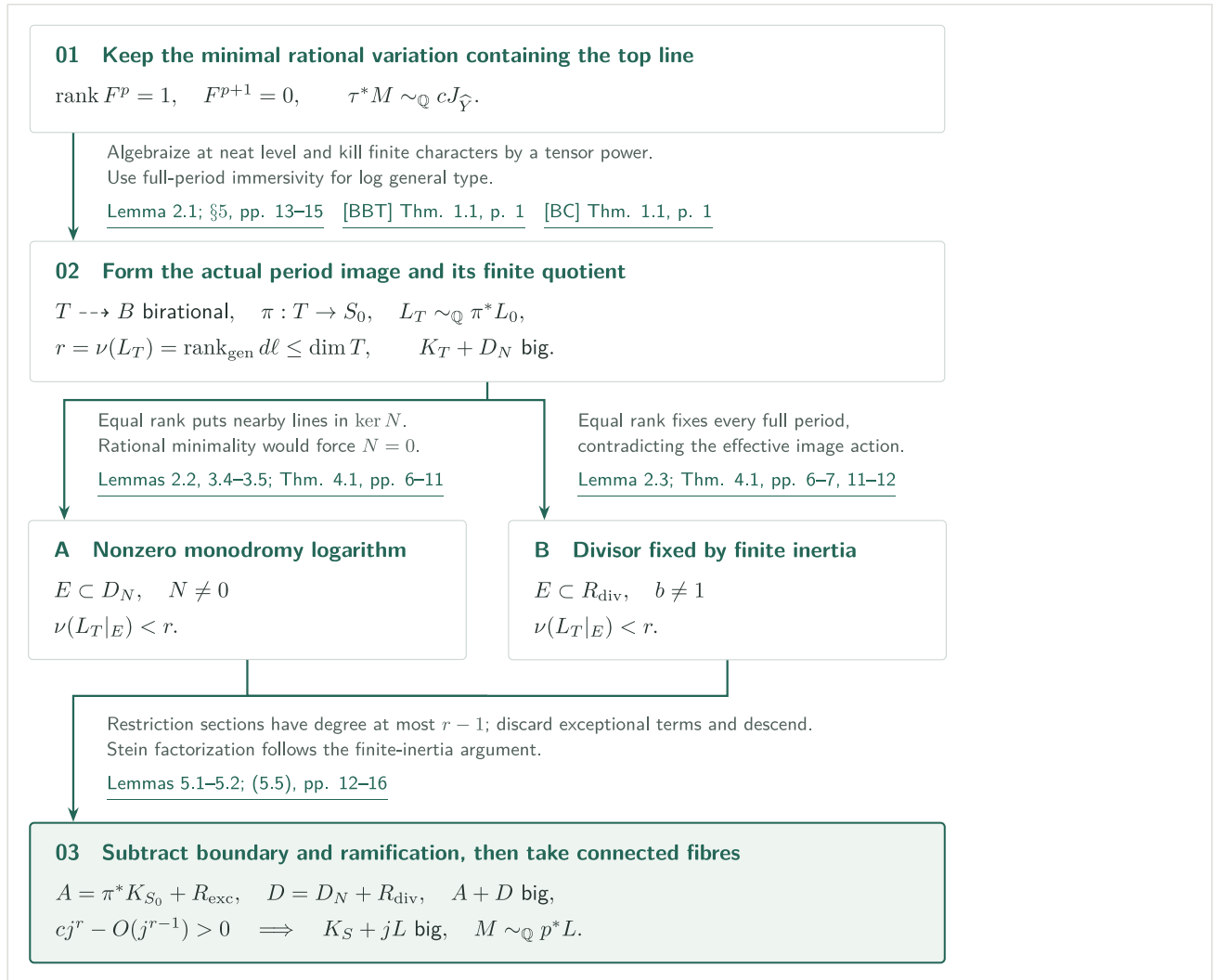
Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Theorem 1.2 — Removing boundary and ramification from the period image

Proof target : [Theorem 1.2 · p. 2](#) See the [hypotheses and conclusion](#).

Proof route. Two preparatory arguments meet in Theorem 1.1. This section establishes the adjoint positivity of Theorem 1.2; the [canonical bundle formula](#) supplies the [section comparison](#). The final [two-base argument and interpolation](#) combine them to prove ordinary Iitaka subadditivity.

Let T be a smooth projective model of the actual full period image at neat level, J its extended highest line, and $L_T = cJ$. Write D_N for the boundary with nonzero monodromy logarithm, R_{div} for divisorial ramification over the finite quotient, and R_{exc} for the Jacobian components exceptional over the quotient base. The number $r = \nu(J)$ measures the highest-line map and need not equal $\dim T$.



Rational minimality and two rank drops yield adjoint bigness

[Open figure ↗](#) [TeX source ↓](#)

1. Retain the finite action on the actual period image

Replace the variation by its smallest rational subvariation containing the highest line. A generic Hodge structure is then rationally generated by that line, allowing one to eliminate rational normal monodromy subgroups fixing it. Algebraize the period image at neat level and retain the function field of its normalization when taking the finite quotient. Tensor powers kill the finite stabilizer characters on the line, permitting rational line-bundle descent. The identity with the original M is obtained by comparing extensions on a common alteration and applying a norm; it is not inferred solely from the open locus.

[Lemmas 2.1–2.3 · pp. 5–7; §5, \(5.1\) · pp. 13–15](#) · [BBT, Theorem 1.1 · p. 1](#)

If the period image is a point, the extended line is constant. Descending its identification on a common alteration by the norm gives $M \sim_{\mathbb{Q}} 0$, which is the point-case conclusion of Theorem 1.2.

2. Treat the nilpotent and finite obstructions separately

The nef line satisfies $\nu(J) = \text{rank}_{\text{gen}} d\ell$. If its numerical dimension did not drop on a divisor E , the tangential limiting map would have rank r . Lemma 3.4 would then put the nearby line image inside that limiting image. For $N \neq 0$, it lies in $\mathbb{P}(\ker N)$, contradicting rational minimality. For finite inertia it lies in a single eigenspace; Lemma 2.3 then forces every full period to be fixed. The final contradiction uses the fact that $\mathbb{C}(T)$ is the **function field of the actual period image**. Passing first to an arbitrary finite cover or Stein factor would lose this argument.

Proposition 3.2 · pp. 7–8; Lemmas 3.4–3.5 · pp. 9–10; Theorem 4.1 · pp. 11–12 ·
BBT, Lemma 6.17 · pp. 43–44

3. Remove both divisors by comparing degrees of section growth

After extending the variation across components with $N = 0$, generic immersivity of the full period map makes $K_T + D_N$ big. Apply Lemma 5.1 with

$$A = K_T - R_{\text{div}} = \pi^* K_{S_0} + R_{\text{exc}}, \quad D = D_N + R_{\text{div}}, \quad J = L_T.$$

An ample perturbation yields interior section growth of degree r in j , whereas restrictions to the removed divisors have degree at most $r - 1$. Thus $A + jL_T$ is big. Removing the exceptional part and descending through the generically finite map makes $K_{S_0} + jL_0$ big. Connected-fibre factorization is performed only afterwards; its effective Jacobian preserves bigness of $K_S + jL$.

Lemmas 5.1–5.2 · pp. 12–13; (5.2)–(5.5) · pp. 15–16 · BC, Theorem 1.1 · p. 1

The identity $M = p^*L$ and adjoint bigness on the base enter the weak-positivity and interpolation argument of Section 4. Semiamplessness of M is not concluded here.

PH, §7.3 · pp. 25–26

3. Proposition 6.4 — Canonical bundle data and section comparison

Proof target : Proposition 6.4 · pp. 20–21

Put $d = \kappa(F) \geq 0$ and write the relative Iitaka factorization as $\tilde{X} \xrightarrow{g} Y \xrightarrow{q} Z$. Then $\dim(Y/Z) = d$, and the geometric generic fibre C of g has Kodaira dimension 0. The construction must retain both the line required by Theorem 1.2 and its comparison with the original pluricanonical sections.

Proposition 6.4

Work over $\mathbb{C}$. Let $f : X \rightarrow Z$ be a surjective morphism with connected fibers between smooth connected projective varieties, and let its geometric generic fiber F satisfy $d = \kappa(F) \geq 0$. There exist smooth projective $\tilde{X}, Y$, a birational morphism $\mu : \tilde{X} \rightarrow X$, and surjective maps with connected fibers $\tilde{X} \xrightarrow{g} Y \xrightarrow{q} Z$ such that $qg = f\mu$, $\dim Y - \dim Z = d$, and the geometric generic fiber of g has Kodaira dimension zero.

They carry the data B, M, R of Theorem 6.1 (pp. 16–18): $B \geq 0$ has SNC support, (Y, B) is klt, and M is the nef trace of the moduli b-divisor descending to Y , with

$$\begin{aligned} K_{\tilde{X}} &\sim_{\mathbb{Q}} g^*(K_Y + B + M) + R, \\ g_*\mathcal{O}_{\tilde{X}}(\lfloor kR^+ \rfloor) &= \mathcal{O}_Y \quad (k \in \mathbb{Z}_{>0}). \end{aligned}$$

The image of R^- in Y has codimension at least two, and R^- is **exceptional over the original X** . After a generically finite semistable base change, M is realized as a positive rational multiple of the extended Hodge line of a polarizable rational variation with entire top piece of rank one, as in (6.4). For $D = K_Y + B + M$,

$$\kappa(X) \geq \kappa(Y, D), \quad D|_{Y_{\bar{\eta}}} \text{ is big,}$$

where $\bar{\eta}$ is a geometric generic point of Z . The same bigness holds on very general smooth q -fibers.

Proposition 6.4 · pp. 20–21

01 Take the relative litaka map and its Kodaira-zero fibre

$$\tilde{X} \xrightarrow{g} Y \xrightarrow{q} Z, \quad d = \dim(Y/Z) = \kappa(F), \quad \kappa(C) = 0.$$

Minimality gives a degree- b cover with a unique top form.
Keep the rational variation rather than only its character eigenspace.

[Lemma 6.3; Prop. 6.4, pp. 19–21](#)

02 Spread out the least-index root cover

$$b = \min\{m > 0 : H^0(C, mK_C) \neq 0\}, \quad \xi^b = s/\omega^b, \\ \kappa(W) = 0, \quad h^0(W, K_W) = 1, \quad \mathbb{V} = R^n h_* \mathbb{Q}, \quad \text{rank } F^n = 1.$$

Compare the bundle formula and Hodge eigensheaf on extended alterations.
The auxiliary boundary $-\text{div}(s)/b$ can be negative.

[Thm. 6.1; \(6.4\), pp. 16–20](#)

[\[FM\] Thm. 4.5, PDF pp. 11–12](#) [\[Amb\] Lemma 5.2; Prop. 5.5, pp. 15, 17–18](#)

03 Retain the extended Hodge identity and the residual divisor

$$D = K_Y + B + M, \quad B \geq 0, (Y, B) \text{ klt}, \quad M \text{ nef}, \quad \tau^* M \sim_{\mathbb{Q}} aJ, \\ K_{\tilde{X}} \sim_{\mathbb{Q}} g^* D + R, \quad g_* \mathcal{O}(\lfloor kR^+ \rfloor) = \mathcal{O}_Y, \quad R^- \text{ exceptional over } X.$$

Multiply by the section of R^+ and use exceptionality of R^- .
On the generic fibre use all-degree direct images and flat base change.

[Lemma 6.5; Lemma 6.6, pp. 21–22](#)

04 Compare sections on the total space and the generic fibre

$$H^0(Y, mD) \hookrightarrow H^0(X, mK_X), \quad \kappa(X) \geq \kappa(Y, D), \\ d \leq \kappa(Y_{\bar{\eta}}, D|_{Y_{\bar{\eta}}}) \leq \dim Y_{\bar{\eta}} = d \implies D|_{Y_{\bar{\eta}}} \text{ big}.$$

The least-index root cover and residual divisor supply the line and section comparison

[Open figure ↗](#) [TeX source ↓](#)

1. Use the least index to obtain the entire rank-one top piece

Let $b = \min\{m > 0 : H^0(C, mK_C) \neq 0\}$. Its unique section s defines the root cover by $\xi^b = s/\omega^b$. Lemma 6.3 proves $\kappa(W) = 0$ and $h^0(W, K_W) = 1$ for a resolution W . Spreading out the construction makes the entire top piece of $R^n h_* \mathbb{Q}$ rank one. Even when the cyclic character is not rational, the full rational variation is retained.

[Lemma 6.3, \(6.4\) · pp. 19–20](#)

2. Separate the negative auxiliary boundary from the residual divisor

With $B \geq 0$, (Y, B) klt, and M nef, the formula gives

$$K_{\tilde{X}} \sim_{\mathbb{Q}} g^*(K_Y + B + M) + R, \quad g_* \mathcal{O}_{\tilde{X}}(\lfloor kR^+ \rfloor) = \mathcal{O}_Y \quad (k > 0).$$

The image of R^- has codimension at least two on the base, and R^- is also exceptional over the original X . The latter property requires flattening; it does not follow just from the image dimension. On the generic fibre the auxiliary sub-pair has boundary $-\operatorname{div}(s)/b$, which need not be effective. The argument uses nefness and the extended Hodge-line identification, without importing moduli abundance that requires an effective generic boundary.

Theorem 6.1, Remark 6.2 · pp. 16–19 · FM, Theorem 4.5 · PDF pp. 11–12 ·
Amb, Lemma 5.2(4)–(5), Proposition 5.5 · pp. 15, 17–18

3. Compare sections on the total space and on the generic fibre

Set $D = K_Y + B + M$. Exceptionality of R^- identifies the sections of $K_{\tilde{X}} + R^-$ with the original pluricanonical sections. Multiplication by the section of R^+ injects $H^0(Y, mD)$ into that space. On the geometric generic q -fibre, flat base change and the all-degree direct-image condition give

$$\kappa(X) \geq \kappa(Y, D), \quad d = \kappa(F) \leq \kappa(Y_{\bar{\eta}}, D|_{Y_{\bar{\eta}}}) \leq \dim Y_{\bar{\eta}} = d.$$

Thus $D|_{Y_{\bar{\eta}}}$ is big. Subsequent birational modifications use the positive part of the crepant boundary and an effective exceptional correction to preserve these comparisons.

Proposition 6.4, Lemmas 6.5–6.6 · pp. 20–22

Section 4 uses both q to the original base and p to the period base. The section injection is retained to transfer the final lower bound back to X .

PH, Proposition 6.4 · §7.3 · pp. 20–21, 25–26

4. Theorem 1.1 — Two bases and an interpolation identity

Proof target : Theorem 1.1 · p. 2 See the hypotheses and conclusion.

If $\kappa(F) = -\infty$ or $\kappa(Z) = -\infty$, the inequality in Theorem 1.1 is automatic under its convention. The argument below treats the case where both are nonnegative.

Assume $\kappa(Z) \geq 0$. The identity $M = p^*L$ from Theorem 1.2 does not supply a factorization between $p : Y \rightarrow S$ and $q : Y \rightarrow Z$. If S is a point, apply log-general-type-fibre subadditivity directly to q . The remaining argument treats $\dim S > 0$.

01 Keep the two bases distinct

$$q : Y \rightarrow Z, \quad p : Y \rightarrow S, \quad D = K_Y + B + M, \quad M \sim_{\mathbb{Q}} p^* L, \\ \kappa(Z) \geq 0, \quad D|_{Y_{\bar{\eta}}} \text{ big}, \quad K_S + jL \text{ big } (j \gg 0).$$

Contract forms from Z to W_s . The general fibre of $Y_s \rightarrow W_s$ is of log general type, so apply the established addition theorem.

[Lemma 7.3, pp. 23–24](#) [\[Has\] Thm. 2.11, p. 5](#)

02 Use the product map for nonvanishing on a period fibre

$$Y \rightarrow W \rightarrow S \times Z, \quad \kappa(W_s) \geq 0, \\ \kappa(Y_s, K_{Y_s} + B|_{Y_s}) \geq 0.$$

Apply weak positivity to a positive-rank log pluricanonical direct image. Multiply symmetric powers and push down exceptional poles.

[Lemma 7.4; \(7.6\), pp. 24–26](#) [\[Fuj\] Thm. 1.1, pp. 1–2](#)

03 Construct an effective divisor using weak positivity

$$P_0 = K_S + jL - H \text{ big}, \quad E_1 = K_Y + B + jM - p^* H \sim_{\mathbb{Q}} E \geq 0.$$

Choose v by openness of bigness and a general Γ using ample A_S . Then E_2 is q -generically big and $(Y, B + \Gamma)$ is klt.

[\(7.5\)–\(7.7\), p. 26](#)

04 Interpolate coefficients and return to an effective klt pair

$$0 < v < 1, \quad \lambda = \frac{1-v}{j-v}, \quad A_S = vL + \frac{\lambda}{1-\lambda} H, \\ E_2 = K_Y + B + p^* A_S \sim_{\mathbb{Q}} K_Y + B + \Gamma, \quad D \sim_{\mathbb{Q}} \lambda E_1 + (1-\lambda) E_2.$$

Apply addition to q and multiply by sections of E_1 . Scalar extension preserves all three Kodaira dimensions.

[\[Has\] Thm. 2.11, p. 5](#) [§7.3; Lemma 7.5, pp. 26–27](#)

END Apply addition, transfer sections, and extend scalars

$$\kappa(X) \geq \kappa(Y, D) \geq \kappa(Y, E_2) \geq \kappa(F) + \kappa(Z), \\ H^0(V, nK_V) \otimes_K K' \simeq H^0(V_{K'}, nK_{V_{K'}}).$$

The product map, weak positivity, exact interpolation, and scalar extension

[Open figure ↗](#) [TeX source ↓](#)

1. Obtain nonvanishing on a period fibre through the product map

Resolve the Stein factorization of the image of (p, q) as $Y \rightarrow W \rightarrow S \times Z$. A general W_s is generically finite onto its image in Z , while $W \rightarrow Z$ is dominant. Contracting top forms with the exterior base-direction factor turns a pluricanonical section on Z into a nonzero section on W_s . On the general fibre of $Y_s \rightarrow W_s$, the line M becomes trivial, so bigness of D becomes log-canonical bigness. A first use of log-general-type-fibre subadditivity gives log nonvanishing on the geometric generic p -fibre.

[Lemma 7.3 · pp. 23–24](#) · [Has, Theorem 2.11 · p. 5](#)

2. Use weak positivity to construct an effective divisor with a negative ample twist

Choose $j > 1$ and an ample rational divisor H with $K_S + jL - H$ big. The preceding nonvanishing makes a relative log pluricanonical direct image have positive rank, allowing weak positivity. Multiplying sections of symmetric powers and pushing down exceptional poles after flattening yields

$$E_1 := K_Y + B + jM - p^*H \sim_{\mathbb{Q}} \text{an effective divisor.}$$

The extension and pushdown in Lemma 7.4 connect generic generation to effectivity on the whole total space.

Theorem 7.2, Lemma 7.4, (7.6) · pp. 23–26 · Fuj, Theorem 1.1 · pp. 1–2

3. Balance coefficients and reduce to an effective klt pair

Openness of the big cone allows $0 < v < 1$ with $(K_Y + B + vM)|_{Y_{\bar{\eta}}}$ big. Set

$$\lambda = \frac{1-v}{j-v}, \quad A_S = vL + \frac{\lambda}{1-\lambda}H, \quad E_2 = K_Y + B + p^*A_S, \quad D \sim_{\mathbb{Q}} \lambda E_1 + (1-\lambda)E_2.$$

These choices simultaneously balance the coefficient of M and cancel the ample twist. Since A_S is ample, a sufficiently divisible general section gives $\Gamma \sim_{\mathbb{Q}} p^*A_S$ with $(Y, B + \Gamma)$ klt. The divisor E_2 is its log canonical class and is big on the generic q -fibre. Apply subadditivity a second time and multiply by sections of E_1 to obtain

$$\kappa(X) \geq \kappa(Y, D) \geq \kappa(Y, E_2) \geq d + \kappa(Z).$$

Finally descend the data to a finitely generated field $K/\mathbb{Q}$ and choose $K \hookrightarrow \mathbb{C}$. Flat base change for every pluricanonical space, including those of the geometric generic fibre, transfers the result to any algebraically closed characteristic-zero field.

§7.3, (7.5)–(7.7) · pp. 25–26; Lemma 7.5, §7.4 · p. 27

This yields ordinary subadditivity, Theorem 1.1. WV’s alternative adjoint comparison concerns the separate results of §8; this main theorem is not substituted as a required WV input.

PH, Theorem 1.1 · p. 2; Theorem 8.1 · p. 28 · WV, Remark 3.17 · p. 25

5. Which papers supply which steps

Input

BBT arXiv:1811.12230v3, Theorem 1.1 / Lemma 6.17 (pp. 1, 43–44)

Use and role

Period image in §5; nefness and numerical rank in §3

Checking scope

Statements compared with uses

Input

BC arXiv:1707.01327v1, Theorem 1.1 (p. 1)

Use and role

Bigness of $K_T + D_N$ in §5 p.15

Checking scope

Statement, immersivity, and choice of boundary compared

Input

FM JDG 56 (2000), Theorem 4.5 (PDF pp. 11–12; journal pp. 177–178)

Use and role

Residual divisor and sections in Theorem 6.1

Checking scope

Statement and boundary-zero use compared; complete model construction unchecked

Input

Amb arXiv:math/0210271v1, Theorem 4.4, Lemma 5.2(4)–(5), Proposition 5.5 (pp. 12–15, 17–18)

Use and role

Hodge eigensheaf and pullback in Theorem 6.1 and (6.4)

Checking scope

Relevant statements compared; full cited package and base-change proofs unchecked

Input

Fuj 2015-06-30, v0.54, Theorem 1.1 (pp. 1–2)

Use and role

Theorem 7.2 → effectivity in Lemma 7.4

Checking scope

Statement and application to the projective klt pair compared

Input

Has arXiv:1902.10923v2, Theorem 2.11 (p. 5)

Use and role

Addition in Lemma 7.3 and §7.3

Checking scope

Statement compared; no abundance of the base canonical class required

These are direct inputs to the main route. The uses of the fixed-part theorem, monodromy normality, and nilpotent-orbit and extension theory were read in PH, but their external source statements have not all been checked. Within catalogue 033, WV, Remark 3.17 · p. 25 cites Lemma 8.4 / Theorem 8.1 as an alternative adjoint comparison. WV explicitly uses its direct comparison in its own proof, so this relationship is `alternative`. The ordinary-subadditivity comparison with OI is likewise treated as a separate proof route; no direct PH→OI edge is asserted. Other relationships remain unexamined.

6. Return to the sources

- Theorem 1.2 · p. 2 — Distinguish the highest-line hypothesis, nef descent, and adjoint bigness
- Proposition 6.4 · pp. 20–21 — Read the canonical bundle reduction preserving exceptionality of the residual divisor and the Hodge identification
- Theorem 1.1 · p. 2 — Read the main hypotheses and conclusion alongside the proof passages below
- §§2–5 · pp. 5–16: rational minimality, tangential limits, the two rank losses, quotient descent, and volume estimates.

- §6 · pp. 16–22: the least-index root cover, the Hodge identification across the boundary, and residual section comparisons.
- §7 · pp. 22–27: the product map, effectivity from weak positivity, interpolation, and scalar extension.
- §§8–9 · pp. 27–39: alternative boundary estimates by integral tensor replacement and Thom forms, separate from the main route.

The reading covers Theorems 1.1–1.2, Lemmas 2.1–2.3 / 3.4, Theorem 4.1, descent and volume estimates in §5, and the central proof passages of §§6–7. Section 5 specifies which input statements and applications were compared. Only the statements of Theorem 8.1 / Lemma 8.4 and the corresponding WV comparison were read. The boundary mixed-Hodge identification, multivariable extensions, every auxiliary proof, the alternative proofs in §§8–9, and full external proofs have not been independently verified. This is an AI-generated account of the authors' argument, not certification of the manuscript or expert peer review. The source is the 40-page version of 2026-09-27, pinned to commit `fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb`.

B-semiampleness for compact log-smooth Kähler fibrations

From threshold orders to global generation through a family of product decompositions

The manuscript claims stabilization and semiampleness of the moduli b-line for compact log-smooth Kähler fibrations, allowing boundary coefficients equal to one. Its central steps identify the extension order of a local root eigenline with the log canonical threshold, then descend global generation of a specified holomorphic line bundle from a product family over a compact auxiliary base.

Manuscript author: OpenAI · 2026-09-10 · 38 pages, including references

[Read the manuscript on GitHub ↗](#)

1. Main results

The following is the manuscript's claim. Spaces and maps are complex analytic. Equality in $\text{Pic}(X)_{\mathbb{Q}} = \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ means an isomorphism of holomorphic line bundles after a common positive integral multiple.

Let $f : Y \rightarrow X$ be a surjective holomorphic map with connected fibres between smooth compact connected Kähler manifolds. Let Δ be an effective rational SNC divisor with coefficients in $[0, 1]$, and assume

$$K_Y + \Delta \sim_{\mathbb{Q}} f^*L, \quad L \in \text{Pic}(X)_{\mathbb{Q}}.$$

For a smooth compact Kähler modification $\mu : X' \rightarrow X$, resolve the main component to obtain $f' : Y' \rightarrow X'$ and $h : Y' \rightarrow Y$, and set $\Delta' = h^*\Delta - K_{Y'/Y}$. For a prime divisor P on X' , define

$$t_P = \inf_{E \rightarrow P} \frac{a(E; Y', \Delta')}{\text{ord}_E((f')^*P)}, \quad B_{X'} = \sum_P (1 - t_P)P, \quad M_{X'} = \mu^*L - K_{X'} - B_{X'}.$$

The infimum ranges over all divisorial valuations dominating P with positive denominator; a is log discrepancy. Exceptional coefficients of Δ' may be negative. This is the threshold discriminant, distinct from the inf-multiplicity divisor of an orbifold base.

§1.1, (1.1)–(1.3) · pp. 2–3

Theorem 1.1 — Stabilization and global generation

Under these assumptions there is a smooth compact Kähler modification $\mu_0 : S \rightarrow X$ with the following properties.

(a) For every smooth compact Kähler modification $\nu : S_1 \rightarrow S$,

$$M_{S_1} = \nu^* M_S \quad \text{in } \text{Pic}(S_1)_{\mathbb{Q}}.$$

(b) For some positive integer m , the actual holomorphic line bundle representing mM_S is globally generated.

The statement includes a point base and relative dimension zero. No projectivity of f, X, Y is assumed.

Theorem 1.1 · p. 3

References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to B-semiampleness for compact log-smooth Kähler fibrations. External inputs use the following keys.

- [FF](#) Osamu Fujino; Taro Fujisawa — Variation of mixed Hodge structure and its applications: Hodge extensions and relative logarithmic top forms. arXiv:2304.00672v3 (2025-03-11).
- [MWWZ](#) Shin-ichi Matsumura; Juanyong Wang; Xiaojun Wu; Qimin Zhang — Compact Kähler manifolds with nef anti-canonical bundle: Finite étale product decompositions of klt log Calabi–Yau fibers. arXiv:2506.23218v1 (2025-06-29).
- [Toma](#) Matei Toma — Bounded sets of sheaves on Kähler manifolds: Compactness of Douady components in a fixed Kähler space. arXiv:1103.5835v3.
- [BFMT](#) Benjamin Bakker; Stefano Filipazzi; Mirko Mauri; Jacob Tsimerman — Baily–Borel compactifications of period images and the b-semiampleness conjecture: Hodge/moduli identification and projective semiampleness; the results used and checking limits are distinguished below. arXiv:2508.19215v2 (2025-12-18).
- [BB](#) Walter L. Baily, Jr.; Armand Borel — Compactification of arithmetic quotients of bounded symmetric domains: Algebraicity of arithmetic quotients; the external original remains unchecked. Annals of Mathematics 84 (1966), 442–528.

- [Fujiki Akira Fujiki](#) — Countability of the Douady space of a complex space: Countability of Douady components; the external original remains unchecked. Japanese Journal of Mathematics 5 (1979), 431–447.
- [Vil Mads Bach Villadsen](#) — Hodge modules and Kähler morphisms: Compactification of period quotients or finite covers; see the checked scope below. arXiv:2401.09544v1 (2024-01-17).

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

2. Overview of Theorem 1.1

Proof target : Theorem 1.1 · p. 3 See the hypotheses and conclusion.

Proof route. First, [compare thresholds with extension orders](#) to obtain stabilization in Theorem 1.1(a). Part (b) then [organizes product decompositions into a family](#), obtains semiampness for each factor, and [extends the specified isomorphism before descending global generation](#).

Retain the actual moduli line stabilized in (a) and obtain global generation at every point in (b). The two main parts are the threshold calculation and the product/norm comparison on an auxiliary base P .

Compare root-eigenline extension orders with thresholds

$$\text{ord}_{\text{Hdg}} \tau = t_P - 1, \quad B_S = \sum_P (1 - t_P) P$$

Mixed Hodge extension, residues and orders on higher models
[\[BS\] Thm. 4.4 / Cor. 4.5, pp. 15–17](#); [\[FF\] Thm. 1.1, p. 2](#)

Identify and stabilize the actual moduli line

$$M_{S_1} = \nu^* M_S \quad \text{in } \text{Pic}(S_1)_{\mathbb{Q}}$$

Theorem 1.1(a) and the specified line retained below

Choose one family using finite covers, Hilbert–Douady and Baire
[\[BS\] Prop. 5.6, pp. 22–25](#); [\[MWWZ\] Cor. 1.3, pp. 2–3](#)
[\[Toma\] Cor. 5.3, PDF p. 19](#)

Realize fiberwise products as a family on a compact auxiliary base

$$r : P \rightarrow S, \quad (J, D_J) = (Q, C) \times \prod_i T_i$$

The map r need not be generically finite.

Use algebraic b-semiample and arithmetic compactifications separately
[\[BS\] Prop. 6.1 / Prop. 7.5, pp. 25–28, 32–34](#)
[\[BFMT\] Thms. 1.5, 5.2, 5.5, pp. 4, 30–32](#)

Obtain semiample extensions for the separate factors

H_Q projective comparison factor, H_i torus / symplectic factors

Two-sided norm bounds remove flat twists; take norms after Stein factorization
[\[BS\] Lemmas 8.1, 8.3 / §9, pp. 34–37](#)

Extend the isomorphism and descend generation at every point

$$r^* \mathcal{O}_S(NM_S) \simeq H_Q \otimes \bigotimes_i H_i$$

M_S semiample Theorem 1.1(b)

Overview from threshold stabilization through product factors to extension and descent

Open figure ↗ TeX source ↓

1. Fix the stabilized line by a local calculation

Compare the root-eigenline extension order with the lct and obtain pullback identities on every higher model. Track the specified holomorphic line, not only its numerical class.

Theorem 4.4, Corollary 4.5 · pp.15–17

2. Realize a product family and handle the factors separately

On a compact auxiliary base, separate the projective factor from the torus and symplectic factors. The former uses b-semiampleness for a projective comparison family; the latter use arithmetic period compactifications. Retain the same specified line and product norm.

Proposition 5.6, Proposition 6.1, Proposition 7.5 · pp.22–34

3. Extend the specified isomorphism and descend generation everywhere

Two-sided norm bounds remove poles from both the isomorphism and its inverse, leaving neither boundary nor flat twists. Norms after the Stein factorization of a proper surjection give generation by one common multiple.

Lemmas 8.1, 8.3, §9 · pp.34–37

Together (a) and (b) prove Theorem 1.1. The detailed diagrams below expand [stabilization](#), [family construction](#), [semiampleness of the factors](#), and [descent of the isomorphism and sections](#).

3. Theorem 1.1(a) — Read thresholds as extension orders

Proof target : Theorem 1.1 · p. 3 See the [hypotheses and conclusion](#).

Let $p : W \rightarrow S$ be a model with SNC boundary, let D_W be the crepant boundary, and put $Q_S = \mu_0^* L - K_S$. Fix $\mathcal{O}_W(m(K_{W/S} + D_W)) \simeq p^* \mathcal{O}_S(mQ_S)$ for sufficiently divisible m . The cover obtained by taking an m th root of a local generator retains every component, even if disconnected.

1 Take a local root and retain every sheet

$$K_{W/S} + D_W \sim_{\mathbb{Q}} p^* Q_S, \quad F^d \mathbb{V}_{\chi} = \mathcal{O}_{\tau}$$

The specified character eigenline; not the entire highest space

Use the maximal coefficient-one horizontal intersection and iterated residue.
Prop. 3.4, Lem. 3.5, pp. 10–12

2 Pass to a pure line by the deepest residue

$$F^d \operatorname{Gr}_{d+k}^W \mathbb{V}_{\chi} \simeq \text{klt log-volume line on a depth-}k \text{ stratum}$$

Retain the local frame and its degree- m identification

Test extension in both directions on disk models, including the Jacobian of du .
Lem. 4.2–4.3, Thm. 4.4, pp. 13–16

Mixed Hodge extension and highest direct image: [FF] Thm. 1.1, p. 2

3 Identify threshold and extension order

$$t_P = \min_E \frac{1 - \delta_E}{a_E}, \quad \operatorname{ord}_{\operatorname{Hdg}} \tau = t_P - 1$$

$$a_E = \operatorname{ord}_E(t), \quad \delta_E = \operatorname{coeff}_E D_W$$

Kill boundary orders on power charts; cancel relative canonical and threshold changes.
Thm. 4.4, Cor. 4.5, pp. 15–17

4 Correct the boundary and descend to every higher model

$$-K_{S_1/S} - B_{S_1} + \nu^* B_S = 0$$

$$M_{S_1} = \nu^* M_S \quad \text{in } \operatorname{Pic}(S_1)_{\mathbb{Q}}$$

Theorem 1.1(a), including centres inside the good open set

The root eigenline, residues, threshold orders, and stabilization on higher models

[Open figure ↗](#) [TeX source ↓](#)

1. Isolate the relevant line in the mixed Hodge structure

For the character χ of the tautological root τ , the line $F^d \mathbb{V}_{\chi} = \mathcal{O}_{\tau}$ has rank one. This does not assert rank one for the highest space of the entire variation. Let k be the largest number of horizontal coefficient-one components meeting on a good fibre. The line lies in $\operatorname{Gr}_{d+k}^W$, and after removing the Tate twist, iterated residue identifies it with the pure eigenline of a klt log-volume on a depth- k stratum. The local frame and its degree- m identification are retained. If $k = 0$, no residue operation is needed.

Proposition 3.4, Lemma 3.5 · pp. 10–12

2. Determine the extension order in both directions

Above $P = (t = 0)$, put $a_E = \text{ord}_E(t)$ and $\delta_E = \text{coeff}_E D_W$. The minimum on an SNC resolution equals the threshold defined using all divisorial valuations. The manuscript uses normalized root models over a disk and mixed Hodge extensions to prove

$$t_P = \min_E \frac{1 - \delta_E}{a_E}, \quad \text{ord}_{\text{Hdg}} \tau = \min_E \frac{l_E + 1 - a_E}{a_E} = t_P - 1, \quad l_E = -\delta_E.$$

The Hodge order is normalized by dividing by the ramification index of the base change. After $t = u^N$, test $u^{-Nb} \tau$. Transforming the total form introduces the Jacobian of du . Extension holds when every $\alpha_E = l_E + 1 - a_E(1 + b)$ is nonnegative; a negative value produces a pole on a retained component over E . Thus the calculation establishes necessity as well as sufficiency. The direct-image identification from FF applies to disk models whose strata dominate the base and are Kähler. Dualizing the lower compact-support extension gives the upper highest-line extension in ordinary cohomology.

Lemmas 4.1–4.3, Theorem 4.4 · pp. 12–16 · FF, Theorem 1.1 · p. 2

3. Glue the boundary correction as an actual line bundle

If $B_S = \sum b_i(t_i = 0)$, the corrected frame $\prod_i w_i^{N b_i} \pi^* \tau$ has order zero on a power chart $t_i = w_i^{N_i}$. A tensor power removes roots-of-units ambiguities, and Hartogs extension handles crossings. The prescribed identification extends to $\pi^* \mathcal{O}_S(am M_S) \simeq E_X^{\otimes am}$. On a higher model, retain $Q_{S_1} = \nu^* Q_S - K_{S_1/S}$ in the same calculation. Relative canonical orders cancel the change of thresholds, yielding

$$-K_{S_1/S} - B_{S_1} + \nu^* B_S = 0.$$

This includes modifications with centres inside the good open set and proves (a).

Theorem 4.4, Corollary 4.5 · pp. 15–17

Retain the stabilized moduli line and its specified log-volume identification; this fixes the object compared by the ensuing product decomposition.

BS, Corollaries 4.5–4.6 · pp. 16–18

4. Proposition 5.6 — From fiberwise splittings to a family

Proof target : Proposition 5.6 · pp. 22–25

The deepest stratum supplies a family with klt log Calabi–Yau good fibres. Write G_m for its degree- m log-plurivolume line. Beyond a decomposition of each fibre, the proof needs a single family carrying both a line identification and its norm.

Proposition 5.6

In the setting of Theorem 1.1, use the **marked klt family** extracted from the deepest coefficient-one stratum by Corollary 4.6 and Lemma 5.1 (pp. 17–19). Fix its good open $U \subset S$, a sufficiently divisible positive integer m , and the specified degree- m log-volume line G_m with its residue norm.

After a proper surjection $r : P \rightarrow S$ from a smooth compact connected manifold, there is a dense open $P^\circ \subseteq r^{-1}(U)$ on which a finite étale cover of the pulled-back family decomposes as

$$(J, D_J) \simeq (Q, C) \times_{P^\circ} \prod_i T_i.$$

All families are proper and smooth with connected fibers. The projective family (Q, C) consists of effective klt SNC pairs, embeds in $P^\circ \times \mathbb{P}^N$, and is pulled back from universal marked families by a holomorphic map from all of P to a fixed product of projective Hilbert schemes. Each T_i is a family of submanifolds of a fixed compact Kähler manifold, with $n_i = \dim T_i > 0$, integral lattices modulo torsion, flat real polarizations, and highest line $\lambda_i = (\pi_i)_* \omega_{T_i/P^\circ}$. In torus type, $h^{1,0} = n_i$ and $\bigwedge^{n_i} F^1 R^1 \simeq \lambda_i$; in symplectic type, n_i is even, $h^{2,0} = 1$, and $(F^2 R^2)^{\otimes n_i/2} \simeq \lambda_i$. These maps are cup products.

On every fiber, $m(K_Q + C)$ and mK_{T_i} are trivial. For $G_{m,Q} = (\pi_Q)_* \mathcal{O}_Q(m(K_{Q/P^\circ} + C))$, there is a **specified isomorphism of holomorphic lines** with matching norms:

$$r^* G_m \simeq G_{m,Q} \otimes \bigotimes_i \lambda_i^{\otimes m},$$

$$\|s\|_{\text{res}} = c \|s_Q\|_{Q, \text{res}} \prod_i \|\omega_i\|_{\text{Hdg}}^m.$$

For cover degree e , compatible volume conventions give $c = e^{-m/2} > 0$. This norm gives generators and duals of sufficiently divisible powers of $r^*(mM_S)$ two-sided logarithmic growth bounds. The assertions persist on the good open after further smooth compact auxiliary base changes. Neither generic finiteness of r nor a Kähler form on P is required.

Proposition 5.6 · pp. 22–25

1 Extract a klt log-volume family with a section

$$\Sigma \longrightarrow S, \quad m(K_F + D_F) \sim 0, \quad (F, D_F) \text{ klt}$$

Stein factorization of the deepest stratum and a diagonal section

Combine fibrewise splitting with finiteness of subgroups of fixed index.

Lem. 5.1, 5.3–5.4, pp. 18–21

Splitting input: [MWWZ] Cor. 1.3, pp. 2–3

2 Enumerate finite étale product decompositions in families

$$(J, D_J) = (Q, C) \times \prod_i T_i$$

Q is projective; each T_i is a torus or irreducible symplectic factor.

Impose constructible graph/boundary conditions; apply Baire to countably many closures.

Lem. 5.5, Prop. 5.6, pp. 22–25

Compact Douady components: [Toma] Cor. 5.3, p. 19

3 Select a dominating compact parameter space

$$K = B \times \text{Hilb} \times \prod_i \text{Douady}_i \times \text{Douady}_{\text{graph}}$$

$$r : P \longrightarrow S \quad \text{proper, surjective}$$

The auxiliary base P is smooth, compact, and complex.

Carry the log-volume isomorphism and residue-integral comparison with the family.

Prop. 5.6, (5.5)–(5.6), pp. 22–25

4 Retain the specified line and product norm

$$r^* G_m \simeq G_{m,Q} \otimes \bigotimes_i \lambda_i^{\otimes m} \quad \text{on } P^\circ$$

$$\|s\|_{\text{res}} = c \|s_Q\|_{Q,\text{res}} \prod_i \|\omega_i\|_{\text{Hdg}}^m, \quad c > 0$$

No Kählerness, projectivity, or generic finiteness is assumed for P .

Finite covers and compact parameter spaces produce a product family

Open figure ↗ TeX source ↓

1. Enumerate product decompositions as families of finite covers

Stein factorization of the deepest stratum and resolution of a graph produce a family with a diagonal section. The finite étale decomposition supplied by MWWZ separates each klt fibre into a projective factor (Q, C) and torus or irreducible holomorphic symplectic factors T_i . All boundary lies on Q , which may be a point. The section expresses the fundamental group as a semidirect product. Finiteness of subgroups of fixed index in the finitely generated fibre group then spreads each finite étale cover after a finite base cover.

Lemmas 5.1, 5.3–5.4 · pp. 18–21 · MWWZ, Corollary 1.3 · pp. 2–3

2. Use compact parameter spaces and Baire to choose one family

Hilbert schemes record the projective factor; Douady components in fixed compact Kähler spaces record the T_i and the product graph. Graph, étaleness, and boundary conditions are imposed as analytically constructible conditions. Torus/symplectic type, expressed by Hodge numbers and cup-product ranks, persists along a stratum containing one point of the desired type. Countably many compact closures have images covering the good base, so Baire selects a dominating one. Its resolution gives a proper surjection $r : P \rightarrow S$. The smooth compact complex manifold P is not assumed Kähler, projective, or generically finite over S .

Lemma 5.5, Proposition 5.6 · pp. 22–25 · Toma, Corollary 5.3 · PDF p. 19

3. Carry the specified line and norm along the decomposition

Over a dense open set P° , the chosen finite étale cover is a family satisfying

$$(J, D_J) = (Q, C) \times_{P^\circ} \prod_i T_i, \quad r^* G_m \simeq G_{m,Q} \otimes \bigotimes_i \lambda_i^{\otimes m},$$

where λ_i is the highest line of the corresponding factor. The residue-integral norm agrees with the product norm up to a positive constant depending only on the cover degree. This **specified isomorphism**, together with its norm, is passed to the next steps; a comparison only of numerical classes or up to an unspecified flat twist would not suffice.

Corollary 4.6 · pp. 17–18; Proposition 5.6, (5.5)–(5.6) · pp. 22–25

The product decomposition and product norm on the same auxiliary base P enter the separate treatments of the projective and torus/symplectic factors.

BS, Proposition 5.6 · §§6–7 · pp. 22–34

5. Proposition 6.1 / Proposition 7.5 — Semiample extensions of the factors

Proof target : Proposition 6.1 · pp. 25–28

Different inputs treat the projective pair (Q, C) and the possibly nonprojective T_i . In this section, P also denotes the auxiliary base after finite covers, graph resolutions, and a common tensor power.

Proposition 6.1

Use $P^\circ \subset P$, the projective factor (Q, C) , and its line $G_{m,Q}$ from Proposition 5.6. After a proper modification $\pi : P' \rightarrow P$ with P' smooth compact and shrinking the good open, there are a positive integer a , a **globally generated** holomorphic line H_Q , and a specified isomorphism

$$H_Q|_{\pi^{-1}(P^\circ)} \simeq (\pi^* G_{m,Q})^{\otimes a}|_{\pi^{-1}(P^\circ)}.$$

Transport the a -th power of $N_{m,Q}(s) = (\int_{Q_x} |s|^{2/m})^{m/2}$ to the left. Along every holomorphic disk whose punctured disk lies in the good open, a local generator and its dual have norms bounded by powers of $1 + |\log |t||$, after a finite power substitution if needed.

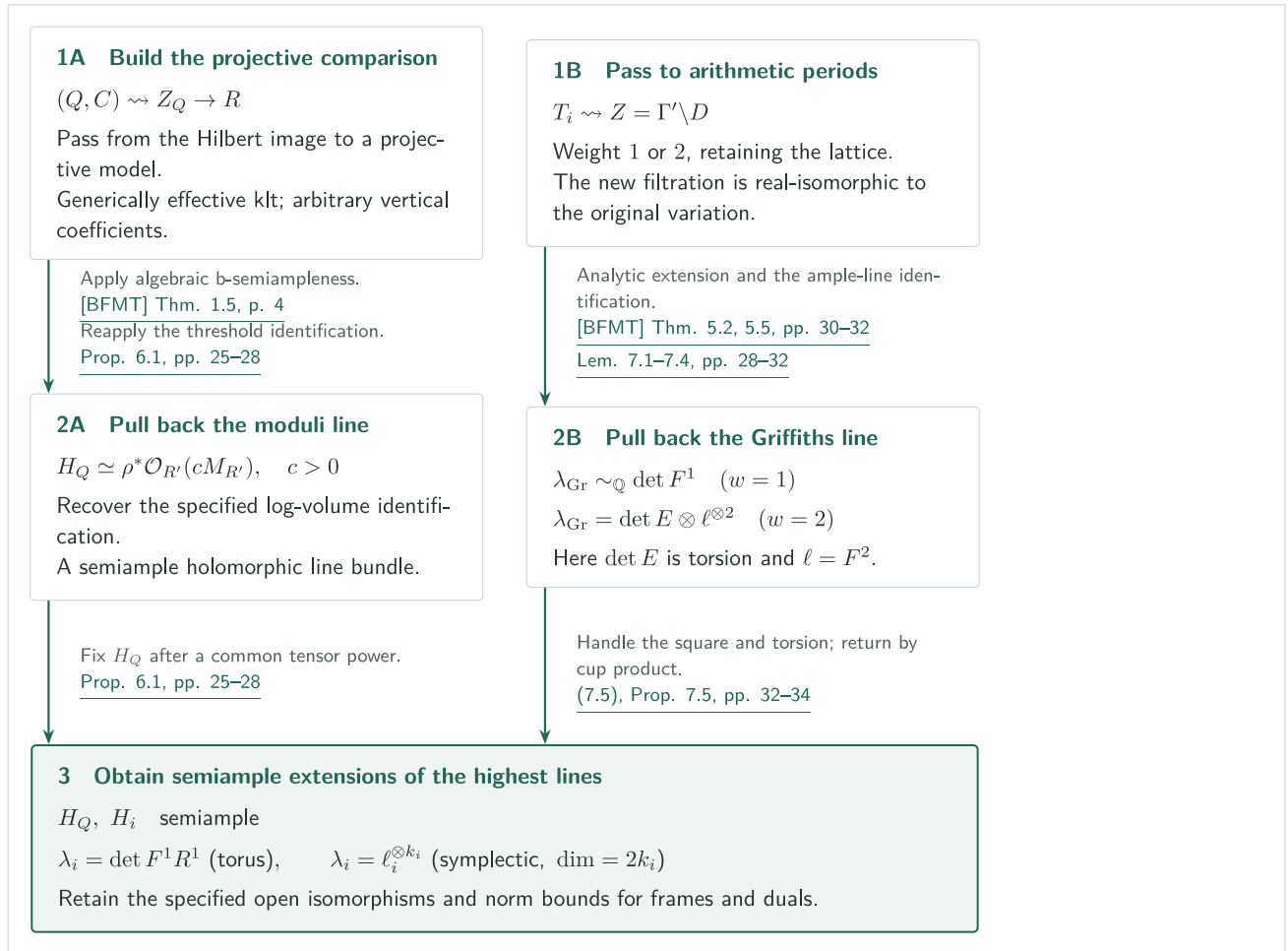
Proposition 6.1 · pp. 25–28

Proposition 7.5

For $r : P \rightarrow S$ and $T_i \rightarrow P^\circ$ in Proposition 5.6, put $\mathcal{H}_i = F^{n_i}(R^{n_i}(t_i)_* \mathbb{C} \otimes \mathcal{O}_{P^\circ}) = \lambda_i$. After a proper generically finite surjection $\pi : P' \rightarrow P$ obtained by extending a finite étale cover of the good open and resolving, there are semiample holomorphic lines H_i on smooth compact P' with **specified** restrictions $\pi^* \mathcal{H}_i$.

The boundary can be taken SNC. Along any disk whose punctured disk lies in the good open, a generator of H_i and its inverse have powers-of-logarithm growth bounds in the volume norm (7.4) and its dual. The bounds persist for all positive powers, including those used in the product-volume identification.

Proposition 7.5 · pp. 33–34



Separate projective comparison from the period maps of torus and symplectic factors

[Open figure ↗](#) [TeX source ↓](#)

1. Apply algebraic b-semiampleness to the projective factor

The compact image of P in a Hilbert scheme is projective by Chow. The universal family over a projective model R yields a projective lc-trivial fibration with generically effective klt boundary. Vertical coefficients in the auxiliary boundary defined by a rational frame may be negative. Pull back the line supplied by algebraic b-semiampleness in BFMT, then reapply the manuscript's local threshold calculation to recover the specified log-volume identification and two-sided norm estimates. This does not require a global rational frame on the original nonprojective space.

Proposition 6.1 · pp. 25–28 · BFMT, Theorem 1.5, Definition 6.18, Theorem 6.28 · pp. 4, 42, 44

2. Replace the real polarization and pass to arithmetic period quotients

Since the T_i occur inside a fixed compact Kähler space, its Kähler class supplies flat real polarizations. Lemma 7.1 uses a nearby rational monodromy-invariant form and a commuting real automorphism to change the filtration while retaining the lattice. It does not make the original fibres rationally polarized, but a real isomorphism compares the holomorphic Hodge

bundles and their extensions. Weight 1 for tori and weight 2 with $h^{2,0} = 1$ for symplectic factors give maps to neat Siegel or type IV arithmetic quotients. Analytic extension of the period map, with the Griffiths-line identification, pulls back sections of an ample line on the compactification.

Lemmas 7.1–7.4 · pp. 28–32 · BFMT, Theorems 5.2, 5.5 · pp. 30–32

3. Recover the required highest lines from Griffiths lines

For a torus, $\lambda_i \simeq \det F^1 R^1$. For a symplectic factor of dimension $2k_i$, put $\ell_i = F^2 R^2$; cup product gives $\lambda_i \simeq \ell_i^{\otimes k_i}$. The weight-2 Griffiths line is $\det E \otimes \ell_i^{\otimes 2}$. Its **square** and the finite-order factor $\det E$ must be handled before concluding semiample-ness of ℓ_i . A common finite cover treats all monodromies; compatibility of extensions with tensor and cup products yields semiample extensions H_Q, H_i of the specified lines.

Lemma 7.3, §7.4, Proposition 7.5, (7.5) · pp. 30, 32–34

Tensor the semiample extensions and retain the two-sided norm bounds needed to extend the specified isomorphism across the boundary.

BS, Lemma 8.1 · §9 · pp. 34–37

6. Theorem 1.1(b) — Extend the specified isomorphism and descend sections

Proof target : Theorem 1.1 · p. 3 See the hypotheses and conclusion.

The preceding identifications initially live on P° . First extend them without a residual boundary twist; then descend global generation through a proper surjection.

1 Bound the comparison and its inverse

$$g : r^* \mathcal{O}_S(NM_S)|_{P^\circ} \simeq (H_Q \otimes \bigotimes_i H_i)|_{P^\circ}$$

$$|g|, |g^{-1}| \leq C(1 + |\log |t||)^b \quad \text{on transverse disks}$$

Two-sided bounds kill negative Laurent coefficients; Hartogs handles intersections.
Lem. 8.1, pp. 34–35; §9, pp. 36–37

2 Extend the specified isomorphism across the boundary

$$r^* \mathcal{O}_S(NM_S) \simeq H_Q \otimes \bigotimes_i H_i \quad \text{on } P$$

The right-hand side is semiample; no residual boundary or flat twist remains.

Descend sections along connected fibres; choose one section nonzero on a finite fibre.
Lem. 8.3, pp. 35–36

3 Use Stein factorization and the finite norm

$$P \xrightarrow{h} T \xrightarrow{q} S, \quad h_* \mathcal{O}_P = \mathcal{O}_T, \quad \deg q = d$$

$$q^* A^{\otimes n} \text{ globally generated} \implies A^{\otimes nd} \text{ globally generated}$$

Use one exponent at all points; generic finiteness of $P \rightarrow S$ is unnecessary.
Lem. 8.3; §9, pp. 35–37

4 Obtain global generation of the moduli line

$$A = \mathcal{O}_S(NM_S), \quad \boxed{\mathcal{O}_S(ndNM_S) \text{ globally generated}}$$

Theorem 1.1(b): a conclusion about the actual holomorphic line bundle

Two-sided norm bounds extend the isomorphism; Stein factorization and norms descend generation

[Open figure ↗](#) [TeX source ↓](#)

1. Exclude poles of both the isomorphism and its inverse

Corollary 4.6 and Propositions 6.1 and 7.5 bound norms of local frames and their duals by powers of logarithms. The product-norm identity therefore bounds the coefficient g of the comparison and g^{-1} on transverse disks by logarithmic powers. Negative Laurent coefficients vanish, so both extend holomorphically. Holomorphic dependence in tangential directions and Hartogs extension handle intersections. For a common sufficiently divisible $N > 0$, this gives an actual holomorphic line-bundle isomorphism

$$r^* \mathcal{O}_S(NM_S) \simeq H_Q \otimes \bigotimes_i H_i.$$

The right-hand factors absorb the required common tensor powers. Extending the specified open isomorphism also excludes an invisible flat twist.

[Lemma 8.1 · pp. 34–35; §9 · pp. 36–37](#)

2. Descend generation by norms without generic finiteness

Write the Stein factorization as $P \xrightarrow{h} T \xrightarrow{q} S$ with $\deg q = d$. Since $h_*\mathcal{O}_P = \mathcal{O}_T$, sections descend to T . If $q^*A^{\otimes n}$ is globally generated, for every $s \in S$ choose a single section nonzero at all points of the finite set $q^{-1}(s)$. Its norm is a section of $A^{\otimes nd}$ nonzero at s . Even if q is not flat, the generic product extends by normality. The same n, d work at every point, so A is semiample. Apply this to $A = \mathcal{O}_S(NM_S)$ to obtain (b).

Lemma 8.3 · pp. 35–36; §9 · pp. 36–37

One common multiple is generated at every point of S . Together with pullback stability in (a), this proves semiampleness of the moduli b-line. No direct dependency with another 033 paper is inferred from this conclusion.

BS, Theorem 1.1 · §9 · pp. 3, 36–37

7. Which papers supply which steps

Input

FF, Theorem 1.1, v3, p. 2

Use and role

§3 and Lemma 4.2: mixed Hodge extension and highest direct image for SNC families with Kähler strata

Check performed

Statement and application model compared

Input

MWWZ, Corollary 1.3, v1, pp. 2–3

Use and role

Lemma 5.3: finite étale splitting of effective klt fibres with numerically trivial log canonical class

Check performed

Statement and hypotheses compared

Input

[Toma, Corollary 5.3, v3, PDF p. 19](#)

Use and role

Proposition 5.6: compact Douady components in fixed compact Kähler spaces

Check performed

Statement and use compared

Input

[BFMT, Theorem 1.5 / Definition 6.18 / Theorem 6.28, v2, pp. 4, 42, 44](#)

Use and role

Proposition 6.1: projective comparison and Hodge/threshold moduli

Check performed

Generic effectivity, rank condition, and use compared

Input

[BFMT, Theorems 5.2, 5.5, v2, pp. 30–32](#)

Use and role

Lemma 7.4: analytic pullback of an ample Griffiths extension

Check performed

Statements and local liftability compared

These are direct inputs to auxiliary results in [BS](#). Their complete proofs were not verified. Uses of Baily–Borel algebraicity, Fujiki countability, compactification of finite covers, and general pure Hodge extension theory were read in [BS](#), but the corresponding external passages remain unchecked. The statement on p. 3 that the Campana orbifold Iitaka theorem is not used does not exclude other external dependencies. Direct dependencies involving other Catalog 033 papers or Catalog 034 remain uninvestigated.

8. Return to the sources

- [Theorem 1.1 · p. 3](#) — Read the main hypotheses and conclusion alongside the proof passages below

- [Proposition 5.6 · pp. 22–25](#) — Follow the simultaneous transport of the decomposition, specified line, cup products, and product norm
- [Proposition 6.1 · pp. 25–28](#) — Obtain global generation and two-sided norm bounds for the specified line through the projective comparison factor

The source is OpenAI's September 10, 2026 manuscript, 38 pages, pinned at commit [fd4aeeb2ee4fc729c18d98444fed42fd0529eeeb](#). Follow [§§3–4, pp. 6–18](#) for eigenlines and thresholds, [Proposition 5.6, pp. 22–25](#) for the family construction, and [§§6–9, pp. 25–37](#) for factor semiampleness and descent.

The editorial reading covered the setup and both conclusions of Theorem 1.1, the cited proof passages, and the listed external statements at their applications. It does not certify the complete proof. The construction of mixed Hodge extensions, all details of the Hilbert–Douady parameter spaces, and the change of real polarization with all extension compatibilities were not independently proved. Connections whose external sources remain unchecked are left pending and distinguished from the author's claims.