

# OpenAI Math Digest

CATALOGUE 003

## The quasi-Riemann hypothesis

Catalogue synthesis & 3 proof overviews

Independent reading guide · English edition

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AI-generated explanations may contain errors. Check important claims and proofs against the original manuscripts. This booklet collects the website's explanations, not the original papers. Expert review and formal verification have not been performed.

# Contents

The synthesis comes first, followed by all 3 overviews in official catalogue order. The contents and citations are clickable. Wide tables are arranged as entries for print; the claims and source references are retained.

|                                                                                       |    |
|---------------------------------------------------------------------------------------|----|
| 00 · Catalogue synthesis                                                              | 3  |
| 01 · The Quasi-Riemann Hypothesis: A Zero-Free Half-Plane $\operatorname{Re} s > 7/8$ | 10 |
| 02 · The Quasi-Riemann Hypothesis                                                     | 25 |
| 03 · Uniform exclusion of Landau–Siegel zeros                                         | 33 |

Original-manuscript snapshot: 1581d4cbdd69a0faf34b2089614fe2a814316a6a

## Catalogue 003 · Synthesis

[Read on the website ↗](#)

### CATALOGUE 003

# The quasi-Riemann hypothesis

Official catalogue 003 contains two manuscripts claiming zero-free half-planes of fixed width and one treating logarithmic gaps for real zeros through interpolation determinants. The guide distinguishes character families, the strength of the boundaries, and proof inputs.

## 1. Papers in official catalogue order

These summaries give the principal scope; consult the cited original results for the complete hypotheses and statements.

## Catalogue 003 · 2026-10-23 checked source snapshot

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01

Manuscript and overview

### The Quasi-Riemann Hypothesis: A Zero-Free Half-Plane $\text{Re } s > 7/8 \rightarrow$

Short name: The Quasi-Riemann Hypothesis: A Zero-Free Half-Plane  $\text{Re } s > 7/8$ 

30 September 2026 (PDF title page) · 199 pages

Original paper (GitHub) ↗

Principal claim

No zeros for  $\text{Re } s > 7/8$  for all finite-order Hecke L-functions over  $K = \mathbb{Q}(\sqrt{-3})$  and all Dirichlet L-functions, with the principal pole allowed. Also claims polylogarithmic least nonresidues and deterministic square-root extraction.

Theorem 1.1 · p. 4; Corollary 1.2 · p. 7 ↗

Role and available overview

Reflected/Poisson comparison of completed theta sums; an internal 11/12 stage is strengthened using prime compensation and two moment estimates.

Read the proof overview →

#

02

Manuscript and overview

## The Quasi-Riemann Hypothesis →

Short name: The Quasi-Riemann Hypothesis (alternate 11/12 proof)

October 5, 2026 (PDF title page) · 49 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

No zeros for  $\text{Re } s > 11/12$  for the same Hecke/Dirichlet families; claims an absolute effective  $O(x^{11/12} \log x)$  prime-counting error for  $q \leq x$ .

[Theorem 1.1; Corollary 1.2 · p. 2](#) ↗

Role and available overview

Alternative construction through sextic Möbius mean squares, theta transformation, and finite cube-inversion iteration. External explicit-formula constants for the corollary remain unchecked.

[Read the proof overview](#) →

#

03

Manuscript and overview

**Uniform exclusion of Landau–Siegel zeros** →

Short name: Uniform exclusion of Landau–Siegel zeros

October 1, 2026 (PDF p. 1 date footnote) · 9 pages

[Original paper \(GitHub\)](#) ↗

Principal claim

An absolute  $c > 0$  satisfies  $(1 - \beta) \log q \geq c$  for every real zero  $\beta \in (0, 1)$  of every primitive nonprincipal real character of conductor  $q \geq 3$ .

[Theorem 1 · p. 1](#) ↗

Role and available overview

Prime bias from a real zero combines with field-uniform interpolation and Frobenius divisibility to contradict norm bounds for a determinant.

[Read the proof overview](#) →

## 2. Inclusion of conclusions and proof routes

Dashed arrows indicate mathematical implications assuming the stated conclusions. They do not represent theorems actually imported into another manuscript's proof. The first two manuscripts concern every finite-order Hecke character over the same field  $K = \mathbf{Q}(\sqrt{-3})$  and every Dirichlet character, without a height restriction.

### Manuscripts cited in the diagram

- [OpenAI26QR78](#) OpenAI — The Quasi-Riemann Hypothesis: A Zero-Free Half-Plane  $\operatorname{Re} s > 7/8$ . 30 September 2026 (PDF title page). Compare the  $7/8$  half-plane in Theorem 1.1 and its internal  $11/12$  stage.
- [OpenAI26QR1112](#) OpenAI — The Quasi-Riemann Hypothesis. October 5, 2026 (PDF title page). Identify the  $11/12$  half-plane in Theorem 1.1 and the alternative mean-square method.

- [OpenAI26LS](#) OpenAI — Uniform exclusion of Landau–Siegel zeros. October 1, 2026 (PDF p. 1 date footnote). Identify the logarithmic real-zero gap in Theorem 1 and its separate determinant route.

### 1 7/8 manuscript: two representations of one completed sum

Finite-order Hecke/ $K = \mathbf{Q}(\sqrt{-3})$ ; all Dirichlet

$\Re s > 7/8$  :  $L(s) \neq 0$  principal pole at  $s = 1$  allowed

Strengthen an internal 11/12 stage using prime compensation and moments

Inclusion of conclusions (not a proof input):  $7/8 < 11/12$

[\[OpenAI26QR78, Theorem 1.1, p. 4\]](#)

[\[OpenAI26QR1112, Theorem 1.1, p. 2\]](#)

### 2 11/12 manuscript: Möbius mean squares and cube inversion

The same character families, every height

$\Re s > 11/12$  :  $L(s) \neq 0$  principal pole at  $s = 1$  allowed

Extract the exponent from prime sixth-power rows, then use Mellin

Consequence of the conclusion (not an imported input): one may take  $c = (\log 3)/12$

[\[OpenAI26QR1112, Theorem 1.1, p. 2\]](#)

[\[OpenAI26LS, Theorem 1, p. 1\]](#)

### 3 Landau–Siegel manuscript: logarithmic real-zero gaps

$\exists c > 0$  :  $(1 - \beta) \log q \geq c$

$q \geq 3$ , primitive nonprincipal real characters; every real zero  $0 < \beta < 1$

Separate proof route: prime bias plus interpolation determinant norms

Strength of the three conclusions and their separate proof routes

[Open figure ↗](#)   [TeX source ↓](#)

Since  $7/8 < 11/12$ , one has  $\{\Re s > 7/8\} \supset \{\Re s > 11/12\}$ . The 7/8 conclusion therefore includes the 11/12 zero-free region for the same families. Both exclude their boundary line and allow the principal pole at  $s = 1$ . Restricting either fixed-width conclusion to real zeros of primitive nonprincipal real characters gives a logarithmic gap: for example, the 11/12 conclusion implies  $(1 - \beta) \log q \geq (\log 3)/12$ . This explicit value is an editorial consequence of comparing the conclusions, not a constant stated in Theorem 1 of the Landau–Siegel manuscript. A logarithmic gap for real zeros alone does not yield a fixed complex half-plane valid at every height.

[\[OpenAI26QR78, Theorem 1.1, p. 4\]](#)

[\[OpenAI26QR1112, Theorem 1.1, p. 2\]](#)

[\[OpenAI26LS, Theorem 1, p. 1\]](#)

Part I of the 7/8 manuscript already contains an internal 11/12 result, Theorem 3.1. This is not an input from the separately listed 49-page manuscript. The 199-page text compares reflected and Poisson representations of the same completed theta sum to obtain its internal 11/12 result, then adds prime compensation, asymmetric scales, and two moment estimates to reach 7/8. The 49-page text instead controls the mean square of sextically twisted Möbius sums by finite iteration after cube inversion, extracting 11/12 from prime sixth-power rows. Both use cubic theta functions and a quadratic large sieve, but their normalized sums and closing nonvanishing arguments differ.

[OpenAI26QR78, Proof overview; Theorem 3.1, pp. 5–6, 10]

[OpenAI26QR1112, §3; Proposition 5.1, pp. 11–12, 18]

The Landau–Siegel manuscript combines prime bias from a logarithmic derivative with interpolation in a quartic field and determinant divisibility from Frobenius congruences. Its proof text does not invoke theorems from the other two manuscripts or pass through a fixed zero-free half-plane.

[OpenAI26LS, §§4–6, pp. 5–8]

### 3. Reading entries and checked connections

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Reading purpose

The widest claimed half-plane and its two-stage proof

Article and source entry

7/8 main theorem and proof route

Type of relationship

Strengthening from an internal 11/12 stage to 7/8

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Reading purpose

Extracting a boundary from a Möbius mean square

Article and source entry

11/12 finite iteration and Mellin reduction

Type of relationship

An alternative construction; the citation of 7/8 provides context

Reading purpose

The algebraic route excluding real zeros on the logarithmic scale

Article and source entry

Landau–Siegel determinant comparison

Type of relationship

The conclusion follows from the half-planes too, but the proof takes a separate interpolation route

Page 2 of the 11/12 manuscript cites the 7/8 manuscript for the stronger result. No direct theorem input between the three texts has been recorded in the main reductions and closing arguments checked. This is not an exhaustive audit of every citation throughout the 199-page manuscript. An unrecorded connection should not be read as a proof of independence.

[OpenAI26QR1112, §1, comparison with the 7/8 manuscript, p. 2]

## 4. Sources and checking scope

The official subject is Number theory, covering catalogues 001–031. The heading and number follow the 6 October 2026 edition of the OpenAI Research Catalog, printed/PDF p.2, and the fixed repository version. The manuscripts, in official order, are dated 30 September 2026 (199 pages), 5 October 2026 (49 pages), and 1 October 2026 (9 pages). Their dates do not by themselves establish supersession.

OpenAI Research Catalog · Catalogue 003; Number theory · p. 2

Corollary 1.2 of the 7/8 text concerns polylogarithmic least nonresidues and square-root extraction. Corollary 1.2 of the 11/12 text concerns prime counting in progressions: identical numbering does not identify the results.

Each article records which proof passages and direct inputs were read. The full recursions, detector, and tail calculations in the 199-page text, and all cusp transformations, common-factor calculations, and external prime-counting constants in the 49-page text remain independently unverified. The full proof route of the Landau–Siegel text was read, without recursively checking the proofs of standard external theorems. These articles describe the authors' claims and do not certify their complete correctness.

# The Quasi-Riemann Hypothesis: A Zero-Free Half-Plane $\Re s > 7/8$

Two representations of a completed cubic-theta sum give a continuation width common to the character family.

The manuscript compares a direct bound from cubic-theta reflection with a Poisson representation containing the Mellin transform of a reciprocal Hecke L-function. It first obtains the boundary  $11/12$  from an uncompensated sum, then improves it to  $7/8$  using prime compensation and two moment estimates. This article presents the manuscript's claims and proof route; it does not independently verify the entire proof, including its long moment inductions.

Manuscript author: OpenAI · 30 September 2026 (PDF title page) · 199 pages, including references

[Read the manuscript on GitHub ↗](#)

## 1. Main results

### Theorem 1.1 — Zero exclusion for finite-order Hecke and Dirichlet families

Let  $F = \mathbf{Q}(\sqrt{-3})$ . For every finite-order Hecke character  $\eta$  modulo any nonzero integral ideal,  $L_F(s, \eta)$  has no zero in

$$\Re s > \frac{7}{8}.$$

The same holds for  $L(s, \chi)$  for every Dirichlet character  $\chi$  modulo any positive integer, including  $\zeta(s)$ , corresponding to the character modulo 1. A principal character's pole at  $s = 1$  is allowed.

A Hecke character here is a ray class group character extended by zero on ideals not coprime to its modulus. The character need not be primitive, and there is no conductor bound or restriction on the imaginary part. The Hecke base field is the fixed field  $F$  above. The boundary  $\Re s = 7/8$  is excluded from the assertion. In particular, real zeros in  $(7/8, 1)$  are excluded; the theorem does not place all nontrivial zeros on the critical line.

Theorem 1.1 · p. 4

**Corollary 1.2 — Least quadratic nonresidues and deterministic square roots**

There are absolute constants  $A, C > 0$  such that, for every odd prime  $p$ , the least positive quadratic nonresidue  $n(p)$  satisfies

$$n(p) \leq C(\log p)^A.$$

Consequently  $n(p) \ll_{\delta} p^{\delta}$  for every  $\delta > 0$ . Given an odd prime  $p$  and  $a \in \mathbf{F}_p$  in binary representation, a deterministic algorithm, polynomial in  $\log p$ , returns a square root of  $a$  or reports that none exists. The manuscript's proof permits  $A = 32$ .

Corollary 1.2 · p. 7

## References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [The Quasi-Riemann Hypothesis: A Zero-Free Half-Plane  \$\operatorname{Re} s > 7/8\$](#) . External inputs use the following keys.

- [DR24](#) Alexander Dunn and Maksym Radziwiłł — Bias in cubic Gauss sums: Patterson's conjecture: Cubic-theta coefficients and transformations at fixed cusps. [arXiv:2109.07463v3](#), 14 May 2024.
- [GL13](#) Leo Goldmakher and Benoît Louvel — A quadratic large sieve inequality over number fields: Large sieve for quadratic Hecke families. [arXiv:1112.1642v2](#), 31 May 2012.
- [TZ19](#) Jesse Thorner and Asif Zaman — A unified and improved Chebotarev density theorem: Prime asymptotic in a fixed ray class and normalization. [arXiv:1803.02823v3](#), 27 October 2018.
- [BIMS17](#) Vishwas Bhargava, Gábor Ivanyos, Rajat Mittal, and Nitin Saxena — Irreducibility and deterministic  $r$ -th root finding over finite fields: Least nonresidues under weak GRH. ISSAC 2017 author manuscript, 8 PDF pages; no printed pagination.
- [Gal18](#) Steven D. Galbraith — Mathematics of Public Key Cryptography: Tonelli-Shanks after obtaining a nonresidue. Version 2.0, 31 October 2018.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

The BIMS17 author manuscript has no printed page numbers; its cited pages indicate PDF positions. In the other versions used here, the recorded printed pages agree with the PDF positions.

## 2. Proof of Theorem 1.1 — Two comparisons and transfer to the full families

Proof target : [Theorem 1.1 · p. 4](#)

**Proof route.** Let  $\beta_*$  be the supremum of  $1/2$  and the real parts of zeros in  $1/2 \leq \Re \rho \leq 1$ , taken over all primitive finite-order Hecke characters. Poles are excluded. Part I proves  $\beta_* \leq 11/12$ . Part II works under that bound and contradicts  $\Delta = \beta_* - 7/8 > 0$ . The common continuation principle concluding both stages is explained in the next section. Finite Euler factors and quadratic factorization then transfer the result to imprimitive Hecke characters and Dirichlet characters.

**1 Retain the completed index**

$$F = \mathbf{Q}(\sqrt{-3}), \quad cn^3 \text{ (} c \text{ squarefree)}$$

Characters act on the whole product and retain zeros on nonunits.

Compare reflected/additive energy with zero detection in Poisson rows.  
[Proposition 6.3 · p. 49](#)   [Proposition 11.2 · pp. 83–84](#)

**2 Balanced sum: the first boundary**

$$C_I(s) = s - 2/3, \quad X = Y = Z^{1/2}$$

$$|J_{I,\eta}| \ll_{\eta} Z^{1/4+\omega}, \quad |J_{I,\eta} - f_{I,\eta}| \ll_{\eta} Z^{C_I(\beta_*)-1/4800}$$

Internal conclusion:  $\beta_* \leq 11/12$ .

Use marked reflection and the Gram bound; normalize by fixed-ray prime sums.  
[Equation \(12.5\) · p. 86](#)  
[Proposition 15.3 · p. 107](#)  
[Equation \(16.3\) · p. 108](#)  
[Section 20.1 · pp. 187–188](#)

**3 Prime compensation and unequal scales**

$$0 < \Delta = \beta_* - 7/8 \leq 1/24, \quad \kappa = 2\beta_* - 1$$

$$X = Z^{17/48}, \quad Y = Z^{23/48}, \quad J_{II,\eta} = I_{\eta,\text{modified}}/(c_S A_T)$$

$$C_{II}(s) = s - 11/16, \quad A_T(Z) \asymp (\log Z)^{-K}$$

Count rows using inverse second and plain fourth moments; allocate losses.  
[Proposition 19.2 · p. 183](#)   [Proposition 20.3 · pp. 194–197](#)

**4 Two bounds for the same sum and signal**

$$|J_{II,\eta}| \ll_{\eta} Z^{3/16+\omega}, \quad 0 < \omega < \Delta$$

$$|J_{II,\eta} - f_{II,\eta}| \ll_{\eta} Z^{C_{II}(\beta_*)-\sigma}, \quad \sigma > 0$$

The margins  $\omega, \sigma$  are common; constants, heights and thresholds may depend on  $\eta$ .

Contradict the supremum with a common width; transfer by Euler factors and quadratic factorization.  
[Proposition 2.1 · pp. 8–9](#)   [Proposition 11.3 · pp. 84–85](#)

**5 Common reciprocal continuation and transfer**

$$\beta_* \leq 7/8 \implies L_F(s, \eta) \neq 0, \quad L(s, \chi) \neq 0 \quad (\Re s > 7/8)$$

Includes imprimitive characters and all heights; allows the principal pole at  $s = 1$ .

The two stages and family transfer in Theorem 1.1

[Open figure ↗](#)   [TeX source ↓](#)

## 1. Retaining completed indices through reflection and the first half-plane

The base theta sum retains the index  $cn^3$ , with  $c$  squarefree and  $n$  unrestricted; they need not be coprime. Characters act on the whole product, retaining zero values on nonunits even when a local exponent is divisible by six. Proposition 5.1 derives the masked reflection from theta coefficients at three fixed cusps. The quadratic large sieve controls the quadratic character arising after reflection and yields a mean-square bound for completed rows.

[Proposition 5.1 · pp. 24–27](#); [Equation \(5.26\) · p. 36](#)

In Part I,  $X = Y = Z^{1/2}$ . Cauchy–Schwarz combines reflected row energy with the mean square of an additive polynomial supported at distinct reduced fractions. Proposition 6.3 gives  $I_\eta \ll_{\eta,\epsilon} Z^{1/4+\epsilon}$ . Nonzero frequencies in the Poisson representation take the form  $ua^6$ , with  $u$  sixth-power-free. The inverse polynomial supplied by the zero detector and the sextic large sieve of Lemma 9.1 count nonprincipal rows; residues of the principal row  $u = 1$  are extracted separately. For the normalization with  $C_I(s) = s - 2/3$ , this gives

$$|J_{I,\eta}| \ll_\eta Z^{1/4+\omega}, \quad |J_{I,\eta} - f_{I,\eta}| \ll_\eta Z^{\beta_* - 2/3 - 1/4800}.$$

If  $\beta_* > 11/12$ , one chooses a common  $0 < \omega < \beta_* - 11/12$  and applies Proposition 2.1 to obtain Theorem 3.1. This is **Theorem 3.1 within the 199-page manuscript**, not an input from the separate 49-page paper in catalogue 003.

Proposition 6.3 · p. 49; Lemma 9.1 · p. 61; Proposition 11.2 · pp. 83–84; Theorem 3.1, proof · p. 85

## 2. The compensated low bound and principal normalization

Part II has  $0 < \Delta \leq 1/24$  and  $\kappa = 2\beta_* - 1 = 3/4 + 2\Delta \leq 5/6$ . It fixes

$$h = \frac{13}{16}, \quad \ell = \frac{1}{6}, \quad l_x = \frac{17}{48}, \quad l_y = \frac{23}{48}, \quad X = Z^{l_x}, \quad Y = Z^{l_y}.$$

Primes are chosen from disjoint windows. At each prime  $\mathfrak{p}$ , the operation subtracts a rescaled term from a term marked by  $\mathfrak{p} \mid cn^3$ . Defining this finite expression first ensures that the low and high estimates concern the same object. Proposition 15.3 combines marked reflected energy with an additive Gram bound to give  $I_{\eta,\text{modified}} \ll Z^{3/16+\epsilon}$ .

Equation (12.5) · p. 86; Proposition 15.3 · p. 107

Contour moves use the full holomorphic correction in Equation (16.3); a quotient by an individual Euler factor is not used across a zero of that factor. The prime sums in the principal residue have positive main terms by prime counting in a **fixed** ray class. With an even number  $K$  of windows, the normalizer  $A_T(Z) \asymp (\log Z)^{-K}$  is nonzero for sufficiently large  $Z$ . With  $c_S > 0$ , put

$$J_{II,\eta}(Z) = \frac{I_{\eta,\text{modified}}(Z)}{c_S A_T(Z)}, \quad C_{II}(s) = s - \frac{11}{16}.$$

Absorbing the logarithmic cost of  $A_T^{-1}$  into an arbitrarily small positive power yields

$|J_{II,\eta}| \ll_{\eta,\epsilon} Z^{3/16+\epsilon}$ . The principal row, with this same normalization, supplies the Mellin signal of  $1/L_F^S(s, \eta)$ . No prime asymptotic uniform in a moving conductor is assumed.

Equation (16.3) · p. 108; Section 20.1 · pp. 187–188

### 3. Two witnesses, polynomial moments, and an exponent margin

The zero detector supplies a large inverse polynomial with Möbius coefficients and a large plain polynomial with one common row character and twist height. Lemma 17.1 estimates the second moment of the inverse polynomial with selected prime factors; Lemma 18.1 estimates the second moment of a product of two plain polynomials with selected prime factors. Applying the latter to two copies of the same plain witness gives its **fourth power** in the denominator of the row count.

Here the inverse column length is  $Z^r$ , the row norm cap is  $Z^m$ , and  $z$  is the sum of the selected prime-slot log-lengths. The plain column lengths are  $Z^{n_1}, Z^{n_2}$ ; the effective width is  $M = m + q$ , where  $Z^q$  bounds the norm of the union of prime supports of the displayed moving character factors. This counts their support before cancellation and retains all zero extensions.

The inverse lengths satisfy  $r + 2z \leq m - c_1$  and  $2r + 8z \leq 3m - c_2$ , with  $c_1, c_2 > 0$ . For positive prime length  $z$ , the plain moment requires  $n_1 + n_2 + 6\kappa z \leq M$ . Its additional zero bound when  $\kappa < 1$ , namely  $\beta_* \leq (1 + \kappa)/2$ , holds by equality for  $\kappa = 2\beta_* - 1$ . Replacing this dynamic value prematurely by its upper bound  $5/6$  loses the later capacity allocation.

Lemma 17.1 · pp. 114–115; Lemma 18.1 · pp. 152–154

The two moments have separate inductions. The inverse argument uses two finite Poisson transforms to shorten rows and terminate at marked reflected energy. The plain argument decreases effective width and treats rows inducing a character in a fixed finite group  $\Theta$  separately. With positive prime factors, the estimate excludes that family; with no prime factors it excludes only principal inducing characters. In the induction, common main terms on exceptional rows cancel in a difference of two products with the same profiles, character, mask, and product of scales. This cancellation does not apply to arbitrary uncentered products. The large physical rows ramify outside  $S$ , placing their inducing characters outside  $\Theta$ , and the selected prime factors have the required coefficient class.

Sections 17.5–17.6 · pp. 121, 142; Section 18.3 · p. 159; Section 18.8 · pp. 178–180; Section 19.2 · pp. 182–183

Proposition 19.2 converts the moments into row counts, and Lemma 20.1 converts the counts and prime amplitudes into a high exponent. The endpoint calculation of Lemma 20.2 and the treatment of all frequency ranges give

$$E_{\text{actual}}(h) - \Delta \leq -\frac{49}{440640} - \frac{51}{64}\Delta + \epsilon_{\text{total}}.$$

The comparison scale is  $C(\beta_*) = C(7/8) + \Delta$ . The required inequality is the negativity of this right side; the argument does not assert  $E_{\text{actual}}(h) < 0$  for every  $\Delta$ . The floor bin, small rows, large rows, and the endpoint  $\delta = 5/6$  have separate estimates.

Lemma 20.1 · pp. 189–190; Lemma 20.2 · pp. 192–193; Sections 20.3–20.5 · pp. 191–194

#### 4. A common continuation width and transfer to Dirichlet characters

Proposition 20.3 chooses the geometry, losses, slot mesh and number, bin width, and positive margins in terms of  $\Delta$  **before fixing the target**. After fixing  $\eta$  and its finite excluded set, it fixes finite height and scale orders  $A_\eta, B_\eta$ , then chooses  $T_1 = Z^{\tau_\eta}$  and an external tail order  $N_\eta$ .

Lemma 11.1 converts

$$|J - f| \ll_{\eta, N} Z^{C(\beta_*)-m} (1 + T_1)^{A_\eta} + Z^{B_\eta} T_1^{-N}$$

into  $|J - f| \ll_\eta Z^{C(\beta_*)-m/2}$ . The number  $m > 0$  is common to the targets;  $\tau_\eta, N_\eta$ , constants, and lower thresholds may depend on the target. Neither  $J$  nor  $f$  depends on  $T_1$ . Proposition 2.1 now gives  $\beta_* \leq 7/8$ .

Lemma 11.1 · pp. 82–83; Proposition 20.3 · pp. 194–197

Imprimitive Hecke characters are handled by the nonvanishing of finitely many factors  $1 - \eta(\mathfrak{p})(N\mathfrak{p})^{-s}$  for  $\Re s > 0$ . For a Dirichlet character  $\chi$ , take  $\eta(\mathfrak{a}) = \chi(N\mathfrak{a})$ . Deleting Euler factors at primes dividing  $3q$  gives

$$L_F^{S_F}(s, \eta) = L^{S_Q}(s, \chi) L^{S_Q}(s, \chi\chi_{-3}).$$

For  $0 < \Re s < 1$  and for  $s = 1 + it$ ,  $t \neq 0$ , no pole on the right can cancel a zero. At  $s = 1$ , if one primitive inducing character is principal, the other is  $\chi_{-3}$ , and  $L(1, \chi_{-3}) = \pi/(3\sqrt{3}) > 0$ . This separate treatment preserves the principal pole and proves the strict half-plane assertion for all Dirichlet characters and  $\zeta$ , at every height.

Proposition 11.3 · pp. 84–85

### 3. Proof of Proposition 2.1 — Continuing reciprocals by Mellin inversion

Proof target : Proposition 2.1 · pp. 8–9

### Proposition 2.1 — Continuation from a common signal

Let  $1/2 < \sigma_0 < 1$ ,  $\Delta_0 = \beta_* - \sigma_0 > 0$ , and  $C(s) = s + c$  for a fixed real  $c$ . Suppose there are target-independent numbers  $0 < \omega < \Delta_0$  and  $\sigma > 0$ , and, for each primitive finite-order Hecke character  $\eta$ , a finite set  $S$ , a holomorphic  $H_\eta$  on  $\Re s > \sigma_0$  with  $|H_\eta - 1| \leq 1/2$ , and a function  $J_\eta$  for sufficiently large real  $Z$ . Define

$$f_\eta(Z) = \frac{1}{2\pi i} \int_{\Re s=2} Z^{C(s)} e^{(s-5/6)^2} \frac{H_\eta(s)}{L_F^S(s, \eta)} ds.$$

If  $|J_\eta| \ll_\eta Z^{C(\sigma_0)+\omega}$  and  $|J_\eta - f_\eta| \ll_\eta Z^{C(\beta_*)-\sigma}$ , these assumptions contradict  $\Delta_0 > 0$ .

Constants, lower thresholds,  $S$ , and  $H_\eta$  may depend on the target.

Proposition 2.1 · pp. 8–9

**Proof route.** The two bounds reduce the growth exponent of  $f_\eta$ , making its Mellin transform converge in a common half-plane. Fourier inversion on  $\Re s = 2$  identifies the transform with the reciprocal expression without shifting a contour through zeros.

**1 Two estimates for the signal**

$\Delta_0 = \beta_* - \sigma_0 > 0$ ,  $C(s) = s + c$   
 $|J_\eta| \ll_\eta Z^{C(\sigma_0)+\omega}$ ,  $|J_\eta - f_\eta| \ll_\eta Z^{C(\beta_*)-\sigma}$   
 $0 < \omega < \Delta_0$ ,  $\sigma > 0$  are common to all targets.

Use the triangle inequality and shift right into absolute convergence.  
[Proposition 2.1 · pp. 8–9](#)

**2 Make the Mellin integral converge at both ends**

$\epsilon_* = \min\{\Delta_0 - \omega, \sigma\} > 0$   
 $f_\eta(Z) \ll_\eta Z^{C(\beta_*)-\epsilon_*} (Z \rightarrow \infty)$ ,  $f_\eta(Z) \ll_{\eta,B} Z^{B+c} (Z \downarrow 0)$   
 $F_\eta(s) = \int_0^\infty f_\eta(Z) Z^{-C(s)} \frac{dZ}{Z}$  is holomorphic for  $\Re s > \beta_* - \epsilon_*$ .

Set  $Z = e^u$ , apply Fourier inversion, then the identity theorem.  
[Proposition 2.1 · pp. 8–9](#)

**3 Identify the reciprocal**

$F_\eta(s) = e^{(s-5/6)^2} H_\eta(s) / L_F^S(s, \eta) \quad (\Re s > 1)$   
 $|H_\eta - 1| \leq 1/2 \implies |H_\eta| \geq 1/2$   
Thus  $1/L_F^S$  continues holomorphically to  $\Re s > \beta_* - \epsilon_*$ .

Deleted factors are nonzero; a principal pole becomes a reciprocal zero.  
[Proposition 2.1 · pp. 8–9](#)

**4 Contradict the definition of the supremum**

For the common  $\epsilon_*$ , some target has a zero with  $\Re \rho > \beta_* - \epsilon_*$ .  
Its reciprocal pole contradicts continuation; attainment of  $\beta_*$  is unnecessary.

The common continuation width in Proposition 2.1

[Open figure ↗](#)   [TeX source ↓](#)

## 1. Endpoint estimates and a holomorphic Mellin transform

Put  $\epsilon_* = \min\{\Delta_0 - \omega, \sigma\} > 0$ . The triangle inequality gives  $|f_\eta(Z)| \ll_\eta Z^{C(\beta_*)-\epsilon_*}$ . For  $Z \downarrow 0$ , shifting the contour right to any fixed  $B > 2$  gives  $f_\eta(Z) \ll_{\eta,B} Z^{B+c}$ . Hence

$$F_\eta(s) = \int_0^\infty f_\eta(Z) Z^{-C(s)} \frac{dZ}{Z}$$

converges locally uniformly and is holomorphic on  $\Re s > \beta_* - \epsilon_*$ .

[Proposition 2.1 · pp. 8–9](#)

## 2. Identification with the reciprocal and contradiction to the supremum

Writing  $Z = e^u$ , Fourier inversion first gives  $F_\eta(s) = e^{(s-5/6)^2} H_\eta(s) / L_F^S(s, \eta)$  on  $\Re s = 2$ ; the identity theorem extends the equality to  $\Re s > 1$ . Because  $|H_\eta| \geq 1/2$ , the function  $e^{-(s-5/6)^2} F_\eta / H_\eta$  provides a holomorphic continuation of the reciprocal. The number  $\epsilon_*$  is common to all targets. The supremum need not be attained: its definition supplies a target with a zero having real part greater than  $\beta_* - \epsilon_*$ , contradicting the pole of its reciprocal. A principal pole of the original L-function becomes a zero of its reciprocal and causes no obstruction.

Proposition 2.1 · pp. 8–9

## 4. Proof of Corollary 1.2 — From a zero strip to nonresidue search

Proof target : Corollary 1.2 · p. 7

**Proof route.** Theorem 1.1 and the Dirichlet functional equation put the nontrivial zeros in a closed strip. A slightly larger open strip meets the weak-GRH input, yielding a logarithmic-power bound for the least nonresidue. Deterministic nonresidue search then replaces the random step in square-root extraction.



The three conclusions of Corollary 1.2

[Open figure ↗](#)   [TeX source ↓](#)

## 1. An open-strip margin and the two nonresidue bounds

Conjugate primitive Dirichlet characters belong to the same family. The functional equation therefore places nontrivial zeros in  $1/8 \leq \Re \rho \leq 7/8$ , contained in  $1/16 < \Re \rho < 15/16$ . This meets BIMS17's weak-GRH hypothesis with  $\epsilon = 7/16$ . The exponent in Theorem 6.7 is  $4/(1 - 2\epsilon) = 32$ , giving  $n(p) \ll (\log p)^{32}$  with an absolute constant for all odd primes; finitely many small primes are absorbed into that constant. Since a fixed logarithmic power is  $O_{\delta}(p^{\delta})$  for every  $\delta > 0$ , the second conclusion follows.

[\[BIMS17, Conjecture 6.3, p. 7; Theorem 6.7, p. 8\]](#)

## 2. Deterministic square roots and running time

Handle  $a = 0$  first and test a nonzero  $a$  by Euler's criterion. For a square, inspect  $2, 3, \dots$  by their Legendre symbols until the first nonresidue is found. The logarithmic-power bound makes the number of trials polynomial; the stopping rule requires no knowledge of  $C$ . With this nonresidue, the remaining Tonelli–Shanks steps are deterministic and polynomial-time. Writing  $p - 1 = 2^e u$  with  $u$  odd only requires removing factors of two, not factoring  $u$ .

[Gal18, Algorithm 3 and Lemma 2.9.5, p. 59]

## 5. Which papers supply which steps

---

Source and result

DR24, §§5.1–5.2 and Appendix A

Used at

Proposition 5.1, especially p.27

Role and required hypotheses

Cubic-theta coefficients and transformation rules at three cusps; the GRH-conditional prime asymptotic is not used

Checking scope

Input formulas and cusps 1, 10, and 19 checked; the full reflection calculation was not independently verified

---

Source and result

GL13, Definition 1 and Theorem 1.1

Used at

§5.4, Equation (5.26)

Role and required hypotheses

Quadratic large sieve on squarefree ideals; the manuscript checks primitive conductors, fixed ray sectors, and reciprocity conditions

Checking scope

Input and application explanation on p.36 compared; all local family calculations were not recomputed

Source and result

TZ19, Theorem 1.1

Used at

Lemma 13.1 and §20.1

Role and required hypotheses

Prime asymptotic for the identity Frobenius class of a fixed abelian extension; no uniform error in a moving conductor is needed

Checking scope

Fixed-field decay of the exceptional-zero contribution and use in the positive normalizer checked

---

Source and result

BIMS17, Conjecture 6.3 and Theorem 6.7

Used at

Corollary 1.2

Role and required hypotheses

Weak GRH with a fixed strip width bounds the least nonresidue

Checking scope

Open-strip hypothesis, exponent 32, and application compared

---

Source and result

Gal18, Algorithm 3 and Lemma 2.9.5

Used at

Corollary 1.2

Role and required hypotheses

The deterministic part of Tonelli–Shanks after nonresidue search

Checking scope

Algorithm and the location of randomness in the complexity proof checked

[DR24, Sections 5.1–5.2, pp. 17–19; Appendix A, p. 78]

[GL13, Definition 1, pp. 1–2; Theorem 1.1, p. 2]

[TZ19, Theorem 1.1, p. 2]

The sextic large sieve in §9 and the moments in §§17–18 are proved within this manuscript. Connections with earlier sieve and recursion methods do not establish direct dependencies on the other two papers in catalogue 003. No use of those two papers was identified in the main-theorem reductions, conclusions, and bibliography read here. An exhaustive citation audit of all 199 pages was not performed.

## 6. Return to the sources

- Theorem 1.1 · p. 4 — Check the families, strict boundary, and principal pole
- Proposition 2.1 · pp. 8–9 — Follow the common exponent margin into reciprocal continuation
- Proposition 6.3 · p. 49; Lemma 9.1 · p. 61; Proposition 11.2 · pp. 83–84; Theorem 3.1, proof · p. 85 — Read the first low/high bounds and internal 11/12 conclusion
- Equation (12.5) · p. 86; Proposition 15.3 · p. 107 — Read the finite compensation and low bound for the same physical sum
- Lemma 17.1 · pp. 114–115; Lemma 18.1 · pp. 152–154 — Compare length, character-family, and uniformity conditions for the moments
- Lemma 20.1 · pp. 189–190; Lemma 20.2 · pp. 192–193; Sections 20.3–20.5 · pp. 191–194 — Follow row counts through the endpoint and frequency estimates
- Lemma 11.1 · pp. 82–83; Proposition 20.3 · pp. 194–197 — Separate common real parameters from target-dependent height choices
- Proposition 11.3 · pp. 84–85 — Check imprimitive transfer and the separate treatment of poles at one
- Corollary 1.2 · p. 7 — Follow all nonresidue and deterministic-algorithm conclusions

Editorial checking covered the hypotheses, families, and poles; the proof of Proposition 2.1; the first low bound and conclusion; the compensated low bound; principal normalization; moment statements and the beginnings and ends of their inductions; the passage from row counts to the final exponent; the order of choices; quadratic transfer; and every conclusion of Corollary 1.2. The external inputs above were checked at the level of their statements and application sites, without recursively verifying their full proofs.

**Core verification remains incomplete.** All finite Poisson transforms, local masks, and uniformity steps in §§17–18, the complete transformation calculation in Proposition 5.1, and every detector, contour, and tail estimate in §§8–10 have not been independently reverified. The uniform

exponents and finite height orders supplied by those inputs were followed at the level of the manuscript's statements and use conditions. This article does not certify the claimed zero exclusion.

# The Quasi-Riemann Hypothesis

## Sextic mean squares, cubic theta transformation, and finite iteration after cube inversion

The manuscript claims a zero-free half-plane for finite-order Hecke L-functions over  $K = \mathbf{Q}(\sqrt{-3})$ , with a boundary independent of conductor and height. It embeds a Möbius sum in a sextically twisted family and passes through cubic theta coefficients to a quadratic large sieve. This article describes the argument and its direct inputs; it does not certify an independent verification of the complete proof.

Manuscript author: OpenAI · October 5, 2026 (PDF title page) · 49 pages, including references

[Read the manuscript on GitHub ↗](#)

## 1. Main results

### Theorem 1.1 — A zero-free half-plane of fixed width

For every finite-order Hecke character  $\nu$  over  $K = \mathbf{Q}(\sqrt{-3})$ ,  $L_K(s, \nu)$  has no zeros in

$$\operatorname{Re} s > \frac{11}{12}.$$

Consequently every Dirichlet L-function  $L(s, \chi)$ , including  $\zeta(s)$ , has no zeros in the same half-plane. There is no restriction on the imaginary part; the boundary line is excluded. For a principal character,  $s = 1$  is a pole: nonvanishing here does not assert holomorphy.

[Theorem 1.1 · p. 2](#)

### Corollary 1.2 — A uniform prime-number estimate in progressions

Let  $\pi(x; q, a) = \#\{p \leq x : p \equiv a \pmod{q}\}$  and let  $\varphi$  denote Euler's totient. For every  $x \geq 2$ , the manuscript claims

$$\sup_{\substack{1 \leq q \leq x \\ (a,q)=1}} \left| \pi(x; q, a) - \frac{1}{\varphi(q)} \int_2^x \frac{dt}{\log t} \right| \ll x^{11/12} \log x,$$

with an absolute, effective implied constant.

Corollary 1.2 · p. 2

## References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [The Quasi-Riemann Hypothesis \(alternate 11/12 proof\)](#). External inputs use the following keys.

- [GL13](#) Leo Goldmakher and Benoît Louvel — A quadratic large sieve inequality over number fields: Quadratic Hecke-family large sieve, applied to squarefree rows and columns in Lemma 6.5. [arXiv:1112.1642v2](#), 31 May 2012 (published 2013).
- [DR24](#) Alexander Dunn and Maksym Radziwiłł — Bias in cubic Gauss sums: Patterson’s conjecture: Cubic theta coefficients and cube completion; the GRH-conditional prime asymptotic is not an input. [arXiv:2109.07463v3](#), 14 May 2024; preprint pagination.
- [D00](#) Harold Davenport; revised by Hugh L. Montgomery — Multiplicative Number Theory: Explicit formula cited for Corollary 1.2; external text and constants remain unchecked. Third edition, Springer, 2000; external chapter text not checked.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

## 2. Proof of Theorem 1.1 — From a mean square to Mellin nonvanishing

Proof target : [Theorem 1.1](#) · p. 2

**Proof route.** Poisson summation converts Möbius coefficients into cubic Gauss sums. A theta transformation controls a cube-completed sum; finite iteration removes the extra cubes. Prime sixth-power rows then extract the exponent  $11/12$ , and a Mellin transform excludes zeros. Write  $N$  for the absolute norm,  $\mathcal{S}$  for a fixed set containing the primes above  $\mathbf{2}, \mathbf{3}$  and the conductor of  $\nu$ ,  $\chi_n(u) = (u/n)_6$ , and  $W$  for a smooth weight of fixed compact support.

**1 Convert the sextic family to cubic Gauss coefficients**

$$A_u(D) = \sum \mu(n) \nu(n) \chi_n(u) W(Nn/D)$$

$E(\mathcal{H}, X, F)$ : retain common factors in the auxiliary variable  $f$

After Poisson and Gauss identities, transform theta in fixed ray classes  
 Lemma 4.1; Proposition 4.5 · pp. 13–18   Proposition 6.2; Lemmas 6.3–6.5 · pp. 24–26  
[\[DR24, \(5.6\)–\(5.8\), p. 17\]](#)   [\[GL13, Definition 1; Theorem 1.1, pp. 1–2\]](#)

**2 Cube completion and the quadratic large sieve**

$-1 - 2 \equiv 3 \pmod{6}$  the transformed character is quadratic

$$\sum_k |T(X; k, f)|^2 \ll D^\varepsilon \|W\|_{C^J}^2 (\mathcal{H} + \mathcal{H}^2 N f / X)$$

Bound short cube divisors directly; use two Poisson transforms for the rest  
 Proposition 5.2 · p. 19   Proposition 5.1; Lemma 5.3; Proposition 5.4 · pp. 18–22

**3 Invert cube completion and contract remaining rows**

$$\Sigma = XF, \quad \mathcal{H} \leq \Sigma D^{-\kappa}$$

$$\mathcal{H}' < D^{-2\kappa} \mathcal{H}, \quad \mathcal{H}' / \Sigma' \leq \mathcal{H} / \Sigma$$

$$E(\mathcal{H}, X, F) \ll D^\varepsilon \Sigma \|W\|_{C^J}^2$$

Return to the original mean square and average about  $Y / \log Y$  rows  
 Proposition 3.1; (3.2)–(3.3) · pp. 11–12

**4 Extract the exponent from prime sixth-power rows**

$$\sum_{0 < Nu \leq D^{1+\vartheta}} |A_u(D)|^2 \ll D^{2+\vartheta+\varepsilon}$$

$$Y = D^{(1+\vartheta)/6}, \quad A_{p^6} = A_1 + O_W(D/Y)$$

$$A_1(D) \ll_{\nu, S, W, \varepsilon} D^{11/12+\varepsilon}$$

Obtain a holomorphic Mellin integral for each character; constants may depend on  $\nu$   
 §3, proof of Theorem 1.1 · p. 12

**5 A Mellin weight adapted to a hypothetical zero**

$$\Re \rho > 11/12, \quad W(y) = y^{-\rho} \phi(y), \quad \phi \geq 0, \quad \phi \neq 0$$

$$L_K^S(s, \nu) M_W(s) = \widehat{W}(s)$$

$$s = \rho: \quad 0 = \widehat{W}(\rho) = \int \phi(y) dy / y > 0$$

Finite Euler factors are nonzero;  $L(1, \chi_{-3}) > 0$  excludes pole–zero cancellation  
 §3, proof of Theorem 1.1 · p. 12

**6 Descend to every Dirichlet L-function**

$$L_K(s, \chi \circ N) = L(s, \chi) L(s, \chi \chi_{-3})$$

No zeros in  $\Re s > 11/12$ ; retain the principal pole at  $s = 1$

Mean square, finite iteration, and Mellin nonvanishing

[Open figure ↗](#)   [TeX source ↓](#)

## 1. The theta transformation produces a quadratic large-sieve family

Set  $A_u(D) = \sum_{(n,S)=1} \mu(n)\nu(n)\chi_n(u)W(Nn/D)$ . Poisson summation in  $u$ , followed by the Gauss–Jacobi identities in Lemma 4.1, converts  $\mu(n)\gamma_{-1}(n)$  into  $a_\xi(n) = \overline{\alpha(n)}\gamma_2(n)\xi(n)$ , with  $\alpha(n) = n/|n|$  and  $\xi$  a character of a fixed ray class group. Extracting common divisors introduces an auxiliary variable  $f$ . The exact family is

$$E(\mathcal{H}, X, F) = \frac{1}{XF} \sum_{Nf \asymp F}^* \sum_{0 < Nk \leq \mathcal{H}} \left| \sum_n^* a_\xi(n) \chi_n(k) \chi_n(f)^4 W(Nn/X) \right|^2.$$

Stars restrict to squarefree primary elements coprime to  $S$ . Proposition 4.5 transfers a bound for this family back to the original mean square.

Lemma 4.1; Proposition 4.5 · pp. 13–18

Cubic theta coefficients have squarefree-times-cube support  $nb^3$ . Adding the cube indices produces  $T(X; k, f)$ , to which theta automorphy applies. At each prime in the squarefree row, the transformed exponent is  $-1 - 2 \equiv 3 \pmod{6}$ , hence quadratic. Lemma 6.3 fixes cusp coefficient sequences within each ray class before GL13 is applied. Lemma 6.5 checks squarefree rows and columns, finite reciprocity corrections, and primitive conductors. Separating repeated prime factors then gives

$$\sum_{0 < Nk \ll \mathcal{H}} |T(X; k, f)|^2 \ll D^\varepsilon \|W\|_{C^J}^2 \left( \mathcal{H} + \frac{\mathcal{H}^2 N f}{X} \right).$$

DR24 supplies theta coefficients and the transformation framework; its GRH-conditional prime asymptotic is not used. The general sextic-twist transformation and its uniformity belong to the present manuscript's Proposition 6.2 and Lemma 6.3.

Proposition 5.2 · p. 19

Proposition 6.2; Lemmas 6.3–6.5 · pp. 24–26

[GL13, Definition 1; Theorem 1.1, pp. 1–2]

[DR24, (5.6)–(5.8), p. 17]

## 2. Finite iteration closes the cube-inversion remainder

Cancellation prevents simply retaining the  $b = 1$  term of the completed sum. Complete multiplicativity allows Möbius inversion. The completed estimate handles inverse divisors up to  $H_c^3 = \min\{X, X^2/\mathcal{H}^2\}$ . Larger cube factors leave a shorter column length  $L_b = X/(Nb)^3$ . Proposition 5.4 enlarges a nonnegative row range between two Poisson transformations. With target  $\Sigma = XF$ , the resulting family satisfies

$$\mathcal{H}' \leq \frac{\mathcal{H}L_b}{\Sigma F}, \quad \frac{\mathcal{H}'}{\Sigma'} \leq \frac{\mathcal{H}}{\Sigma}, \quad \Sigma' = X'F' \leq L_b.$$

If  $\mathcal{H} \leq \Sigma D^{-\kappa}$ , then  $Nb > H_c$  implies  $\mathcal{H}' < D^{-2\kappa}\mathcal{H}$ . Section 5.5 inducts on  $\mathcal{H} \leq D^{j\kappa}$  to obtain  $E \ll D^\varepsilon \Sigma \|W\|_{C^J}^2$ . Support intervals enlarge and derivative orders increase, but the number of iterations depends only on  $\kappa, C_0$ . Section 5.6 takes  $\kappa = \vartheta/2$  to recover the original mean square.

Proposition 5.1; Lemma 5.3; Proposition 5.4 · pp. 18–22

§7, Lemmas 7.1–7.3 · pp. 29–34

### 3. Sixth-power rows give 11/12; a zero gives a contradiction

For fixed  $0 < \vartheta \leq 1/10$ , Proposition 3.1 gives  $\sum_{0 < Nu \leq D^{1+\vartheta}} |A_u(D)|^2 \ll D^{2+\vartheta+\varepsilon}$ . Put  $Y = D^{(1+\vartheta)/6}$ . For primary prime generators with  $Y/2 < Np \leq Y$ , the identity  $\chi_n(p^6) = 1_{p \nmid n}$  gives  $A_{p^6}(D) = A_1(D) + O_W(D/Y)$ . The prime ideal theorem in the fixed field supplies about  $Y/\log Y$  rows, yielding

$$|A_1(D)|^2 \ll D^{11/6+5\vartheta/6+\varepsilon} + D^{5/3-\vartheta/3}.$$

Choosing  $\vartheta$  for each requested exponent loss gives  $A_1(D) \ll_{\nu, S, W, \varepsilon} D^{11/12+\varepsilon}$ . The constant may depend on  $\nu$ . The same exponent for each character, rather than a conductor-uniform mean-square constant, gives the final conductor-independent boundary.

Proposition 3.1; (3.2)–(3.3) · pp. 11–12

If  $\operatorname{Re} \rho > 11/12$  were a zero, choose  $W(y) = y^{-\rho} \phi(y)$  with  $0 \neq \phi \geq 0$  and  $\phi \in C_c^\infty((1, 2))$ . The Mellin integral  $M_W(s) = \int_0^\infty A_1(D) D^{-s} dD/D$  is holomorphic in this half-plane. In  $\operatorname{Re} s > 1$  it satisfies  $L_K^S(s, \nu) M_W(s) = \widehat{W}(s)$ ; meromorphic continuation and the identity theorem extend this relation. Evaluation at  $\rho$  contradicts  $0 = \widehat{W}(\rho) = \int \phi(y) dy/y > 0$ . Removed Euler factors are nonzero for  $\operatorname{Re} s > 0$ .

§3, proof of Theorem 1.1 · p. 12

### 4. Descent to Dirichlet functions and the point $s = 1$

Quadratic base change gives, up to finitely many Euler factors nonzero in  $\operatorname{Re} s > 0$ ,

$$L_K(s, \chi \circ N) = L(s, \chi) L(s, \chi \chi_{-3}).$$

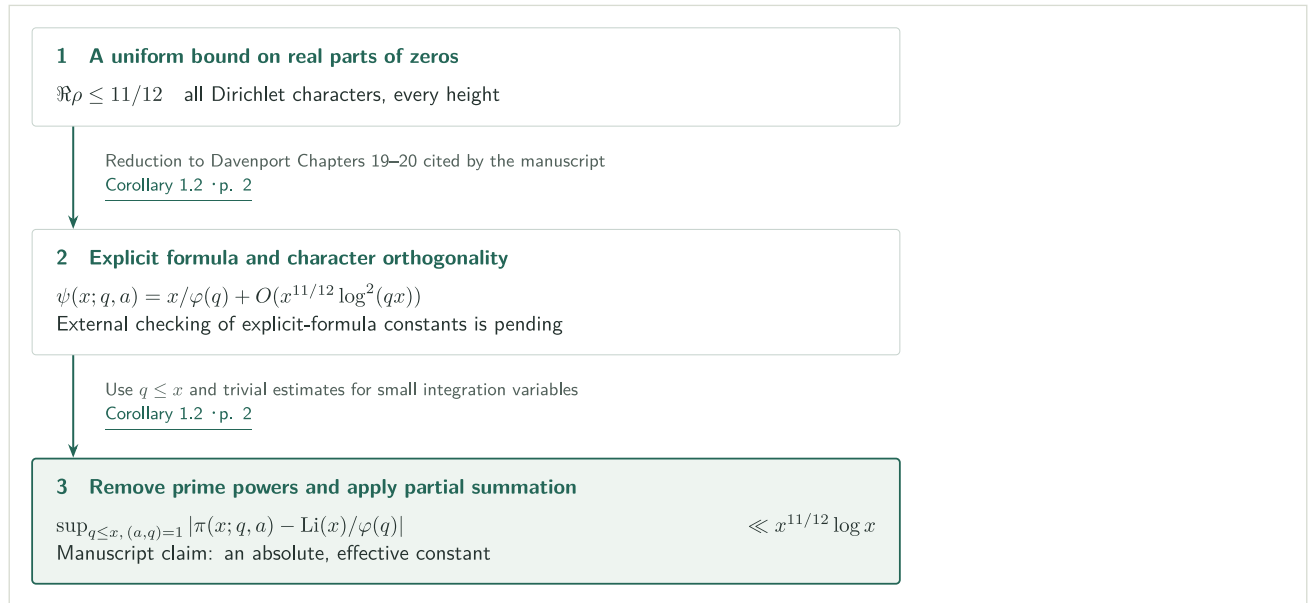
Away from  $s = 1$  no pole can cancel a zero of either factor. At  $s = 1$ , if one factor is principal, the other primitive function is  $L(s, \chi_{-3})$ ; Dirichlet nonvanishing  $L(1, \chi_{-3}) > 0$  excludes pole-zero cancellation. Extra Euler factors for imprimitive characters also do not vanish in this region.

§3, proof of Theorem 1.1 · p. 12

## 3. Proof of Corollary 1.2 — The explicit-formula connection

Proof target : Corollary 1.2 · p. 2

The manuscript cites Chapters 19–20 of Davenport's third edition and supplies no separate proof section for this corollary. The diagram records the reduction. Checking the external explicit formula's edition, pages, and constants remains pending.



From the zero boundary to prime counting in progressions

[Open figure ↗](#)   [TeX source ↓](#)

## 1. Character sums, prime powers, and partial summation

Inserting Theorem 1.1 into a truncated explicit formula gives the target error  $O(x^{11/12} \log^2(qx))$  for  $\psi(x, \chi) = \sum_{n \leq x} \Lambda(n) \chi(n)$ . Character orthogonality returns to  $\psi(x; q, a)$ , with the  $\varphi(q)$  character sum compensated by  $1/\varphi(q)$ . Removing prime powers and partial summation pass to  $\pi$  and save one logarithm. The condition  $q \leq x$ , together with trivial treatment of small integration variables, is needed in obtaining the displayed uniform error. The manuscript's assertion of an absolute, effective constant includes the constants in the explicit formula and zero counting. This external input has not been independently rechecked here.

Corollary 1.2 · p. 2

## 4. Which papers supply which steps

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Source and result

GL13, Definition 1 and Theorem 1.1

Used at

Lemmas 6.5–6.6

Role and hypotheses

Primitive quadratic Hecke families; squarefree rows and columns; fixed ray classes;  $M + N$  bound

Checking scope

External statement and application conditions compared; external proof unchecked

---

Source and result

DR24, (5.6)–(5.8)

Used at

§6.1 and Appendix A.2

Role and hypotheses

Cubic theta coefficients with squarefree-times-cube support

Checking scope

Coefficient statements checked; complete cusp calculations and uniformity not independently recomputed

---

Source and result

DR24, Lemma 5.4

Used at

Comparison with cube completion and inversion

Role and hypotheses

Coefficient identity for primitive cubic twists

Checking scope

Statement checked; the manuscript's general twists do not follow from this lemma alone

Source and result

Davenport, third edition, Chapters 19–20

Used at

Corollary 1.2

Role and hypotheses

Dirichlet explicit formula, prime counting, constant control

Checking scope

Citation in the manuscript checked; external text and page mapping pending

[GL13, Definition 1; Theorem 1.1, pp. 1–2]

[DR24, Lemma 5.4, p. 28]

The 7/8 manuscript is cited on p.2 for a stronger result; it does not supply Proposition 3.1. The 11/12 argument is organized to close within §§3–7 of the present manuscript.

## 5. Return to the sources

- Proposition 3.1; (3.2)–(3.3) · pp. 11–12 — Extract the exponent 11/12 from the mean square.
- Proposition 5.1; Lemma 5.3; Proposition 5.4 · pp. 18–22 — Close cube-inversion remainders by finite iteration.
- Proposition 6.2; Lemmas 6.3–6.5 · pp. 24–26 — The sextic-to-quadratic transform and its uniformity.
- §7, Lemmas 7.1–7.3 · pp. 29–34 — Parameter contraction through two Poisson transformations.
- Corollary 1.2 · p. 2 — The claimed prime-number estimate and its cited input.

The passages read cover exponent extraction and the Mellin contradiction in §3, the exact family, cube inversion, transfer conditions, and finite induction in §§4–5, the quadratic large-sieve application in §6, and the main transfer steps in §7. GL13's family conditions and theorem and DR24's theta coefficients and cube identity were compared with the external sources. The complete Appendix A.2 transformation, all of Lemma 6.3's uniformity proof, every common-factor term in §7, and the external explicit-formula constants for Corollary 1.2 were not independently verified. This is neither a certification of the full claims nor a completed peer review or numerical verification over all conductors.

003 / 03

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CATALOGUE 003 · PAPER 03 / 3 Overview

# Uniform exclusion of Landau–Siegel zeros

Turn prime bias forced by a real zero into incompatible divisibility and size bounds for an interpolation determinant

A zero too close to **1** forces sufficient logarithmic mass on primes with  $\chi(p) = -1$ . The manuscript constructs a nonzero determinant from conjugates in a quartic field and compares divisibility at those primes with an Archimedean upper bound. The key points are interpolation constants independent of the field and a row selection that concentrates degree in one conjugate.

Manuscript author: OpenAI · October 1, 2026 (PDF p. 1 date footnote) · 9 pages, including references

[Read the manuscript on GitHub ↗](#)

## 1. Main results

### Theorem 1 — A uniform zero gap on the logarithmic scale

The manuscript claims that an absolute constant  $c > 0$  exists such that, for every primitive nonprincipal real Dirichlet character  $\chi$  of conductor  $q \geq 3$  and every real zero  $\beta \in (0, 1)$  of  $L(s, \chi)$ ,

$$(1 - \beta) \log q \geq c.$$

This excludes real zeros in a conductor-dependent interval of width  $c / \log q$ ; it does not assert a complex zero-free half-plane of fixed width. The theorem specifies neither a numerical value nor effectivity for  $c$ .

[Theorem 1 · p. 1](#)

## References on the arrows

Unprefixed theorem, lemma, section and equation numbers refer to [Uniform exclusion of Landau–Siegel zeros](#). External inputs use the following keys.

- [D80](#) Harold Davenport; revised by Hugh L. Montgomery — [Multiplicative Number Theory: Bound the logarithmic derivative for primitive characters by a zero sum; retain the real zero in Lemma 2](#). Second edition, Springer, 1980; publisher preview of §14, printed pp.88–89.

Each citation gives the result and pages. When printed pagination differs from the PDF position, both are shown. GitHub previews do not automatically jump to the cited page.

## 2. Proof of Theorem 1 — Weighted row selection and a norm contradiction

Proof target : [Theorem 1](#) · p. 1

**Proof route.** Assume a counterexample sequence with  $\delta = (1 - \beta) \log q \rightarrow 0$ . Connect the prime-sum lower bound from a logarithmic derivative to a separately constructed determinant. A row order adapted to Frobenius congruences makes the leading coefficient 1 in the norm lower bound compete with  $3/4$  in the upper bound. Lemma 3, explained in the next section, supplies uniform interpolation.

**1 A zero counterexample forces prime bias**

$$\ell = \log q, \quad \delta = (1 - \beta)\ell \rightarrow 0$$

$$\sum_{\substack{H < p \leq U \\ p \nmid 2q, \chi(p) = -1}} \frac{\log p}{p} \geq \log U - C\ell - C\delta(\log U)^2/\ell - C_H$$

Prepare interpolation independently, then combine the two inputs

[Lemma 2; \(2.3\)–\(2.4\) · pp. 2–3](#) [\[D80, §14, \(2\), print p. 89 / PDF p. 2\]](#)

[Lemma 3 · p. 3](#) [§4; Lemma 6 · pp. 5–6](#)

**2 Select conjugate rows in weighted order**

$$\theta_n = n_1 + n_2\sqrt{d} + n_3\sqrt{2} + n_4\sqrt{2d}$$

$$\Delta = \det(\theta_n^{\alpha_1} \sigma(\theta_n)^{\alpha_2} \sigma\tau(\theta_n)^{\alpha_3}) \neq 0$$

$$1 \leq H \leq N, \quad M = N^4 = U^3, \quad S_2/S_1 \ll H^{-1}$$

Fix  $H$ ; take  $N$  sufficiently large in terms of  $H$

For  $p > H$ , corrections lie in the earlier span and preserve the determinant

[Lemma 7; \(5.2\)–\(5.6\) · p. 7](#)

**3 Frobenius congruence gives divisibility**

$$\theta^p \equiv g_p(\theta) \pmod{pR}, \quad g_p \in \{\sigma, \sigma\tau\}$$

$$w(\alpha - pme_1 + me_j) = w(\alpha) - m(p - H)$$

$$\Delta \in p^{E_p}R, \quad E_p = \sum_{\alpha} \lfloor \alpha_1/p \rfloor$$

Sum logarithmic prime mass and apply Hadamard at all complex embeddings

[\(5.1\), \(5.7\) · pp. 7–8](#)

**4 Compare bounds for the nonzero integer norm**

$$\frac{1}{4} \log |\mathbf{N}\Delta| \geq S_1 \log U - \text{errors}$$

$$\frac{1}{4} \log |\mathbf{N}\Delta| \leq (S_1 + S_2)(\frac{3}{4} \log U + \frac{1}{2} \ell + \log 8) + \frac{M}{2} \log M$$

Use  $L(1, \chi) \neq 0$  to force  $q \rightarrow \infty$  and remove the fixed exception  $d = 2$

[§6; \(6.1\)–\(6.2\) · p. 8](#)

**5 The order of choices gives an absolute lower bound**

$$H \gg 1 \rightarrow N = \lceil q^\gamma \rceil, \quad \gamma \gg 1 \rightarrow q \rightarrow \infty$$

$$1 \leq 13/16 + 1/16 = 7/8 \quad \text{contradiction}$$

$$\exists c > 0: (1 - \beta) \log q \geq c$$

Prime bias and the weighted determinant contradiction

[Open figure ↗](#) [TeX source ↓](#)

**1. A counterexample sequence and primes with negative character value**

Put  $\ell = \log q$  and  $\delta = (1 - \beta)\ell$ . Primitivity allows the Hadamard product and functional equation to express the logarithmic derivative as a zero sum. For real  $s > 1$ , every  $\operatorname{Re}(s - \rho)^{-1}$  is nonnegative. Retain only  $\beta$  and add  $-\zeta'/\zeta(s) = 1/(s - 1) + O(1)$ :

$$-\frac{\zeta'}{\zeta}(s) - \frac{L'}{L}(s, \chi) \leq C\ell + \frac{1}{s-1} - \frac{1}{s-\beta}.$$

Taking  $s = 1 + 1/\log X$  and comparing the nonnegative Euler coefficients  $\Lambda(n)(1 + \chi(n))$  proves Lemma 2's bound  $\sum_{p \leq X, \chi(p)=1} \log p/p \ll \ell + \delta(\log X)^2/\ell$ . Mertens' estimate then gives, for fixed  $H \geq 2$ ,

$$\sum_{\substack{H < p \leq X \\ p \nmid 2q, \chi(p)=-1}} \frac{\log p}{p} \geq \log X - C\ell - C \frac{\delta(\log X)^2}{\ell} - C_H.$$

The zero-sum input was compared with D80, §14, (2). Lemma 2 derives the required prime sum within the manuscript.

Lemma 2; (2.3)–(2.4) · pp. 2–3

[D80, §14, (2), print p. 89 / PDF p. 2]

## 2. Field-uniform interpolation selects a nonzero determinant

Let  $D$  be the fundamental discriminant corresponding to  $\chi$  and write  $\mathbf{Q}(\sqrt{D}) = \mathbf{Q}(\sqrt{d})$  with  $d$  squarefree. When  $d \neq 2$ , put  $a = \sqrt{d}$ ,  $b = \sqrt{2}$ ,  $K = \mathbf{Q}(a, b)$ , and  $R = \mathbf{Z}[a, b]$ . Let  $\sigma$  change the sign of  $a$  and  $\tau$  that of  $b$ . For  $n \in E_N = \{0, \dots, N-1\}^4$ , set  $\theta_n = n_1 + n_2a + n_3b + n_4ab$ . The map

$$An = (\theta_n, \sigma(\theta_n), \sigma\tau(\theta_n))$$

has kernel  $\mathbf{C}(ab, b, -a, -1)$ , whose coordinates are rationally independent.

Take integers  $1 \leq H \leq N$ . With  $U = N^{4/3}$  and  $M = N^4 = U^3$ , Lemma 3 and Corollary 4 say that monomial evaluation rows of separate degree at most  $32H^{2/3}U, 32H^{-1/3}U, 32H^{-1/3}U$  span the full column space. Scan all rows in increasing  $w(\alpha) = \alpha_1 + H\alpha_2 + H\alpha_3$ , retaining a row precisely when it increases rank. This gives  $\Delta \in R \setminus \{0\}$ . For fixed  $H$  and  $N$  sufficiently large in terms of  $H$  alone, Lemma 6 bounds  $S_1 = \sum \alpha_1$  and  $S_2 = \sum(\alpha_2 + \alpha_3)$  by

$$S_1 \geq c_0 M H^{2/3} U, \quad S_2 \leq C_0 M H^{-1/3} U,$$

so  $S_2/S_1 \ll H^{-1}$ . The auxiliary interpolation coefficients do not enter  $\Delta$ , removing any need to bound their field-dependent size.

§4; Lemma 6 · pp. 5–6

## 3. Weight-decreasing row operations extract prime powers

If  $p > H$ ,  $p \nmid 2q$ , and  $\chi(p) = -1$ , then  $\theta^p = g_p(\theta)$  in  $R/pR$ , where  $g_p \in \{\sigma, \sigma\tau\}$ . Write  $\alpha_1 = pk + r$  and replace the first factor by  $\theta_n^r(\theta_n^p - g_p(\theta_n))^k$ . Each correction row has weight lower by  $m(p - H) > 0$ , hence lies in the  $K$ -span of earlier retained rows. The determinant is unchanged. Meanwhile each replacement entry belongs to  $p^k R$ , giving

$$\Delta \in p^{E_p} R, \quad E_p = \sum_{\alpha \in P} \left\lfloor \frac{\alpha_1}{p} \right\rfloor.$$

The row-operation matrix need not be integral: divisibility follows directly from the replacement entries. Taking the nonzero integer norm and combining distinct primes, Lemma 2 and Chebyshev's estimate yield

$$\frac{1}{4} \log |\mathbf{N}\Delta| \geq S_1 \left( \log U - C\ell - C \frac{\delta(\log U)^2}{\ell} - C_H \right) - CMU.$$

For every complex embedding,  $|\nu(\theta_n)| \leq 8N\sqrt{q}$ , so Hadamard's inequality gives

$$\frac{1}{4} \log |\mathbf{N}\Delta| \leq \frac{M}{2} \log M + (S_1 + S_2) (\log N + \frac{1}{2}\ell + \log 8).$$

The strict margin is  $\log N = \frac{3}{4} \log U$ .

Lemma 7; (5.2)–(5.6) · p. 7

(5.1), (5.7) · pp. 7–8

#### 4. Choose $H$ , then $\gamma$ , then take the counterexample limit

Negating the theorem gives a sequence with  $\delta \rightarrow 0$ . Bounded conductors supply only finitely many characters, and  $L(1, \chi) \neq 0$  plus analyticity prevent their zeros from accumulating at 1. Thus a subsequence has  $q \rightarrow \infty$ , also removing the fixed exceptional field  $\mathbf{Q}(\sqrt{2})$ .

First fix an absolute large  $H$  so that  $S_2/S_1 \leq 1/12$ . Next set  $N = \lceil q^\gamma \rceil$  and choose a fixed large  $\gamma$  to control the term containing  $\ell/\log U$ . Only then let  $q \rightarrow \infty$ . Now  $\delta \log U/\ell \rightarrow 0$  and  $C_H/\log U \rightarrow 0$ ; Lemma 6 also makes  $MU/(S_1 \log U)$  and  $M \log M/(S_1 \log U)$  vanish. Equation (6.2) tends to the contradiction  $1 \leq 13/16 + 1/16 = 7/8$ . This ordering establishes existence of an absolute positive lower bound, rather than a separate constant for each conductor.

§6; (6.1)–(6.2) · p. 8

### 3. Proof of Lemma 3 — Uniform interpolation on a projected integer box

Proof target : Lemma 3 · p. 3

### Lemma 3 — Surjectivity with separate degree bounds

Let  $A : \mathbf{C}^4 \rightarrow \mathbf{C}^3$  be surjective, with  $\ker A = \mathbf{C}c$ , and suppose the four coordinates of  $c$  are linearly independent over  $\mathbf{Q}$ . If integers  $N \geq 1$  and  $t_1, t_2, t_3$  satisfy

$$t_j \geq 3(N-1), \quad \prod_{j=1}^3 (t_j - 3(N-1) + 1) > (4N-3)^4,$$

then evaluation  $B \mapsto (B(An))_{n \in E_N}$  maps polynomials with  $\deg_{z_j} B \leq t_j$  onto  $\mathbf{C}^{E_N}$ .

Lemma 3 · p. 3

#### 1 Assume a nontrivial evaluation relation

$$\sum_{n \in E_N} v_n B(An) = 0, \quad v \neq 0$$

$$S_0 = \{n \in E_N : v_n \neq 0\}, \quad P = \text{conv } S_0$$

The degree product exceeds the lattice count; choose a nearest nonvanishing  $m$   
[Lemma 3 · p. 3](#)   [§3, proof of Lemma 3; Corollary 4 · pp. 3–5](#)

#### 2 Dimension counting and a nearest point select a face

$$R(A(4P \cap \mathbf{Z}^4)) = 0, \quad R \neq 0$$

$$R(Am) \neq 0, \quad y \in 4P, \quad h = m - y$$

$$u \in G, \quad y - u \in 3P$$

Use strict distance decrease and Lagrange interpolation from rational kernel independence  
[§3, proof of Lemma 3; Corollary 4 · pp. 3–5](#)

#### 3 Kill points off the face and select one vertex on it

$$n \in S_0 \setminus G : R(A(m + n - u)) = 0$$

$$A|_{\mathcal{H}} : \mathcal{H} \xrightarrow{\sim} \mathbf{C}^3, \quad Q(An) = 1_{n=u} \quad (n \in S_0 \cap G)$$

Fit the total degree  $\leq 3(N-1)$  of  $Q$  into the reserved degrees  
[§3, proof of Lemma 3; Corollary 4 · pp. 3–5](#)

#### 4 Reach a contradiction within the degree bounds

$$B(z) = Q(z)R(z + A(m - u)), \quad \deg_{z_j} B \leq t_j$$

$$0 = v_u R(Am) \neq 0$$

The degree criterion is independent of the size of the coefficients of  $A$

Dimension counting, nearest points, and interpolation on a supporting face

[Open figure ↗](#)   [TeX source ↓](#)

## 1. Push an evaluation relation onto a supporting face

Assume failure of surjectivity, giving a nonzero relation  $\sum v_n B(An) = 0$ . Put  $S_0 = \{n \in E_N : v_n \neq 0\}$  and  $P = \text{conv } S_0$ . The product condition supplies a nonzero  $R$ , of separate degrees at most  $t_j - 3(N - 1)$ , vanishing at  $A(4P \cap \mathbf{Z}^4)$ . Choose an integer point  $m$  with  $R(Am) \neq 0$  nearest to  $4P$ , let  $y$  be its nearest point, and put  $h = m - y$ . A vertex  $u$  on the supporting face  $G$  can be selected with  $y - u \in 3P$ . For  $n \in S_0 \setminus G$ , the point  $m + n - u$  is strictly closer to  $4P$ , so minimality forces  $R(A(m + n - u)) = 0$ .

## 2. Select a single vertex on the remaining face

The integer vertices of  $G$  lie in a rational affine hyperplane  $\mathcal{H}$ . Rational independence of the kernel coordinates makes  $A|_{\mathcal{H}}$  an affine isomorphism onto  $\mathbf{C}^3$ . Lagrange interpolation in three coordinates of its inverse gives  $Q$ , of total degree at most  $3(N - 1)$ , selecting  $u$  among points of  $S_0 \cap G$ . Then  $B(z) = Q(z)R(z + A(m - u))$  preserves the allowed separate degrees and reduces the relation to  $0 = v_u R(Am) \neq 0$ . The degree criterion does not involve the size of the coefficients of  $A$ , allowing the elongated degree box needed when  $H$  is large.

§3, proof of Lemma 3; Corollary 4 · pp. 3–5

## 4. Which papers supply which steps

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Source and result

D80, §14, (2)

Used at

Lemma 2, (2.3)–(2.4)

Role and hypotheses

Primitive nonprincipal character, real  $s > 1$ ; nonnegative zero sum permits retaining one real zero

Checking scope

Publisher preview, printed p.89 / PDF p.2, compared with manuscript p.3

Source and result

Lemma 3 and Corollary 4 of this manuscript

Used at

Row selection in §4

Role and hypotheses

Rational independence of four kernel coordinates; field-independent separate degrees

Checking scope

Proof and application on pp.3–6 read

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Source and result

Lemma 7 of this manuscript

Used at

(5.6)–(5.7)

Role and hypotheses

Weight decrease for  $p > H$  and divisibility of replacement entries

Checking scope

Proof on pp.7–8 read

[D80, §14, (2), print p. 89 / PDF p. 2]

Bost, Fischler, Laurent, and Philippon are cited for context. The main proof does not apply their general theorems: the manuscript establishes its required algebraic interpolation and divisibility estimates. No theorem from the 7/8 or 11/12 manuscripts appears as an input in the text.

## 5. Return to the sources

- Lemma 2; (2.3)–(2.4) · pp. 2–3 — Convert the hypothetical zero into a prime-sum lower bound.
- §3, proof of Lemma 3; Corollary 4 · pp. 3–5 — Interpolation degree bounds independent of the map and field.
- §4; Lemma 6 · pp. 5–6 — Weighted selection and concentration of degree.
- Lemma 7; (5.2)–(5.6) · p. 7 — Extract prime powers while preserving the determinant.

- §6; (6.1)–(6.2) · p. 8 — Order the limits to obtain absolute constants and remove the exceptional field.

The complete proof text on pp.2–8 was read, comparing the theorem's quantifiers, Lemma 3's hypotheses, weighted row selection, Frobenius congruence, norm bounds, and ordering of parameter choices. The direct analytic input was checked against D80, §14, p.89. The full proof of Davenport's Hadamard-product formula, standard Mertens and Chebyshev estimates, and external proofs of  $L(1, \chi) \neq 0$  were not recursively verified. This does not constitute an independent recomputation of every auxiliary calculation or a certification of the complete claim.