X cpt Kähler mfd ($n = \dim X$). L: line bdl. (= divisor)

$$\kappa(X, L) := \limsup_{m \to +\infty} \frac{\log h^0(X, L^m)}{\log m} \in \{-\infty, 0, 1, \cdots, n\}$$

(Kodaira-Iitaka 次元).

ラフにいうと, $h^0(X, L^{\otimes m}) \sim O(m^\kappa)$.

$\kappa(X, L) = -\infty \iff h^0(X, L^{\otimes m}) = 0 \quad \forall m \geq 0$.

◦ L: big (巨大) $\iff \kappa(X, L) = n$.

X: general type $\overset{\text{def}}{\iff} \kappa(X, K_X) = n$

(X, D): (log) general type $\quad \kappa(X, K_X + D) = n$.

例) $n = 1$

$\kappa(K_X)$	$-\infty$	0	1 (general type)
X	$\mathbb{CP}^1$	楕円曲線	$g \geq 2$.
K_Xの曲率	< 0	$= 0$	> 0.
$\pi_1(X)$	1 (Abel)	$\mathbb{Z}^2$ (Abel)	自由群を含む
d_{Kob}	$\equiv 0$	$\equiv 0$	距離.

Campana の Special variety は.

$\pi_1(X)$ が Abel とか $d_{Kob} \equiv 0$ とかと関連するもの.

$\kappa(K_X)$ ではとらえられないもの.

$n=2$. (bimero で分類).

$\kappa(K_X)$	$-\infty$	0	1	2.
$\pi_1(X)$ Almost abel ($\exists H \lhd \pi_1(X)$, H Abel 有限)	なる or ならない	なる.	なる or ならない.	?

$\|$

Special.

<u>うれしい点</u>

$\forall$ cpt Kähler mfd $f: X \dashrightarrow (Y, \Delta)$.

$\cup$

"special" と F

"log general" に分解できる.

<u>Def</u> X: Special $\overset{\text{def}}{\iff}$ $\forall p \in \{1, \cdots, n\}$, $\forall \mathcal{L} \subset \Omega_X^p$ line bdl $(*)$

$\kappa(X, \mathcal{L}) < p$.

<u>Rem</u> ① Bogomolov-Sommese vanishing.

L が $(*)$ をみたす $\Longrightarrow \kappa(X, \mathcal{L}) \leq p$.

② Special は

- Bimeromorphic で不変.
- X special, $X \dashrightarrow Y$ dominate $\Longrightarrow Y$ も special.
- X, X' special $\Longrightarrow X \times X'$ もそう.
- $X \to Y$ finite étale cover, X: special $\iff Y$: special

<u>例</u> Fano ($-K_X$ ample) $\Longrightarrow$ Special

もっと強く. 有理連結 $\Longrightarrow$ Special

(Rationally connected)

$\overset{\text{def}}{\Longleftrightarrow}$ $\forall x, y \in X$ general

$\exists f : \mathbb{CP}^1 \longrightarrow X$ (有理曲線) s.t. $x, y \in f(\mathbb{CP}^1)$.

proof) ($\Longrightarrow$) 有理連結 $\Longrightarrow$ $H^0(X, \Omega_X^p) = 0$.

$\Downarrow$ (Demailly 本)

$\pi_1(X) = 1$.

とくに torus は special.

例2. $c_1(K_X) = 0 \Longrightarrow$ Special

もっと強く. $\kappa(X, K_X) = 0 \Longrightarrow$ Special

(Iitaka Cnm Conj のとけているものを使う).

proof) $c_1(K) = 0 \Longrightarrow \forall \mathcal{L} \subset \Omega_X^p$, $c_1(\mathcal{L}) \cdot [\omega]^{n-1} \leq 0$

KE 存在 (ω-Kähler)

$\Downarrow$

Ω_X^p poly stable $\Longrightarrow \kappa(X, \mathcal{L}) \leq 0$.

例. X general type $\Longrightarrow$ Not special ($\mathcal{L} = K_X \subseteq \Omega_X^p$ とする).

別の def

Prop

X is non-special $\Longleftrightarrow$

$$\begin{array}{ccc} & \tilde{X} & \\ \text{bim } \pi \swarrow & & \searrow \tilde{f} \text{ fib} \\ X & & Y \end{array}$$

s.t. $\tilde{X}, Y$ cpt Kähler, $\dim Y > 0$.

$\tilde{f}$: fibration + "good" condition (neat prepared)

π : bimeromorphic

&

s.t. $\Delta(\tilde{f})$: $\mathbb{Q}$-div on Y (あとで)

$\kappa(K_Y + \Delta(\tilde{f})) = \dim Y$.

ここで $\Delta(\tilde{f})$ とは.

D : prime div on Y

$f^*D = \sum m_i E_i + R$ となっているとき.

$m_i \in \mathbb{N}$

codim $f(R) \geq 2$ なもの.

$m_D = \inf m_i$ とおく. $A(f) = \sum_{\substack{D \subset Y \\ \text{prime div}}} \left(1 - \frac{1}{m_D}\right) D$.

$$E_i = (z_i = 0) \xrightarrow{f} D = (w_1 = 0)$$
$$(U, z_1, \dots, z_n) \longrightarrow (z_1^{m_i}, z_2, \dots, z_n).$$

例 C (hyper elliptic of) $g = 2$. ($\exists\, h : C \xrightarrow[2:1]{} \mathbb{P}^1$, $\exists\, i$: involution ($i \circ i = id$))

E : elliptic curve (a : 位数 2 の元)

$j : E \to E$, $x \mapsto x + a$ ($j \circ j = id$)

ex) $E = \mathbb{C}/(\mathbb{Z} + i\mathbb{Z})$, $a = 1/2$.

$X = C \times E/(i \times j) \xleftarrow{\text{etale}} C \times E$. とおく. $f : X \to C/i \simeq \mathbb{CP}^1$ 射影

今, $h : C \xrightarrow{2:1} \mathbb{CP}^1$ は 6 点 $(p_1, \dots, p_6)$ で分岐する. ($e_p = 2$)

$\{q_1, q_2\} \longmapsto q$

$\bullet \longmapsto p_i$

$\Longrightarrow$ $D = [p]$, $p \in \mathbb{CP}^1$ について,

$$f^*D = \begin{cases} 2 f^{-1}(p)_{red}, & p = p_1, \dots, p_6 \\ E_1 + E_2 & \text{other} \end{cases}$$

$\Longrightarrow$ $A(f) = \sum_{D : \text{div}} \left(1 - \frac{1}{m_D} D\right) = \boxed{\frac{1}{2} \sum_{i=1}^{6} [p_i]}$.

$\deg(K_{\mathbb{CP}^1} + A(f)) = -2 + \frac{1}{2} 6 = 1$.

$\Longrightarrow$ $(\mathbb{CP}^1, A(f))$: log general type

$\boxed{f : X \longrightarrow \mathbb{CP}^1}$ の fiber elliptic $\Longrightarrow$ special

Thm [Core map].

$\forall X$: cpt Kähler, $\exists$ $C(X)$ sm proj.

$\exists$ $c_X : X \dashrightarrow C(X)$ fibration

s.t. ① c_X : gen fiber special

② Almost hol ← (不確定点が $C(X)$ 全体にうつらない.

③
$$\begin{tikzcd} & \hat{X} \arrow[dl, "\pi"'] \arrow[dr, "\hat{f}"] & \\ X \arrow[rr, dashed, "c_X"] & & C(X) \end{tikzcd}$$
とれて $\boxed{\kappa(K_{C(X)} + \Delta(f)) = \dim C(X)}$ log general

special を 割っている →

④ $\forall x \in X$ very general

$\forall Z \subset X$ special

$Z \cap c_X^{-1}(c_X(x)) \neq \emptyset \Longrightarrow \boxed{Z \subset c_X^{-1}(c_X(X))}$

例 $\mathbb{C}^n \to X \Rightarrow$ special

とくに. Oka $\Rightarrow$ special

(Kobayashi - Ochiai).

X : special だと良いこと.

- Alb fibration
- $\pi_1(X) \longrightarrow GL(r, \mathbb{C})$ の像が almost Abel.

CW 15 h-principle

Def X cpx sp

X が h principle を みたす (BOP)

$\Longleftrightarrow$ $\forall S$: Stein mfd, $f : S \longrightarrow X$: conti

$\exists F : S \to X$ hol, $F \underset{\text{homotopic}}{\sim} f$

Oka $\Longrightarrow$ h principle　　CW15 proj でいえる.

　　$\Longrightarrow$ special

例. X: 可縮 $\Longrightarrow$ h-principle

[CW15] X Brody hyperbolic ($\forall \mathbb{C} \to X$: const)

X: 可縮 $\Longleftrightarrow$ h principle

例. $\mathbb{D}$: h-principle をみたす.

C: $g \geq 2$ のリーマン面は h principle をみたさない

$$\overset{h:\mathrm{OK}}{\mathbb{D}} \xrightarrow[\text{étale}]{} \overset{h:\times}{C}$$

Thm CW15

① X: proj mfd, h-principle $\Longrightarrow$ special

② X irr proj & h-principle

$\Longrightarrow$ $\forall Y$: Brody hyperbolic Kähler

$\forall f : X \to Y$: const.

Thm (Jouanolou's trick)

X: proj ~~mfd~~ var, $\exists M$ ~~affine~~ Stein, $\tau : M \to X$: hol.

s.t. ① $M \sim X$ ホモトピー同値.

② τ の fiber $\cong \mathbb{C}^r$

③ τ は $\mathbb{C}^n$ fiber 束.

④ $\exists \sigma : X \longrightarrow M$ C^∞ section

proof) $X = \mathbb{CP}^n$ のとき. $P = \mathbb{CP}^n \times \mathbb{CP}^n$, $D = \{(z,w) \in P \mid \sum_i z_i w_i = 0\}$

$M = P \setminus D$, $\tau : M \longrightarrow X$, $(z,w) \longmapsto z$, $\sigma : \mathbb{CP}^n \longrightarrow M$, $z \longmapsto (z, \bar{z})$

一般は.
$$\begin{tikzcd} M_X \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & M \arrow[d] \\ X \arrow[r, hook] & \mathbb{CP}^N \end{tikzcd}$$
とれる.

Def X : n dim cpx str. J : Xの概複素構造

$\overline{X} = (X_{\mathbb{R}}, -J)$ opposite cpx str

例. $X = \mathbb{C}^n$. $(z_1, \cdots, z_n)$: coord
$\overline{X} = \mathbb{C}^n$, $(\bar{z}_1, \cdots, \bar{z}_n)$: coord.

$(T_{\mathbb{R}} X_{\mathbb{R}} \otimes \mathbb{C} = T'X \oplus T''X$
$(X_{\mathbb{R}}, J)$ $(X_{\mathbb{R}}, -J)$

$h : X$ Hermite 計量 $\rightsquigarrow \omega_h$
$\rightsquigarrow \bar{h} : \overline{X}$ 〃 $\rightsquigarrow \omega_{\bar{h}}$

$\bar{h}_{ij} = h_{ji}$

$\Longrightarrow \omega_{\bar{h}} = -\omega_h$ (h Kähler $\Longleftrightarrow$ $\bar{h}$ Kähler)

Lem (X, ω_h) : cpt Kähler

s.t. ① $\zeta : \overline{X} \longrightarrow X$: C^∞ map, $\zeta \sim id_X$

② $c : X \dashrightarrow Y$, Y cpt Kähler, c : meromorphic map

$$\forall d, \int_{\overline{X}} (\omega_{\bar{h}})^{n-d} \wedge (c \circ \zeta)^* \omega_Y^d = (-1)^d \int_X \omega_h^{n-d} \wedge c^* \omega_Y^d$$

X と $\overline{X}$ の向きは $(-1)^n$ される.

とくに c が dominate $\Longrightarrow$ $c \circ \zeta$ は meromorphic でない.

例 $s = id_X = c$

$$\begin{tikzcd} X \arrow[r, "s", "C^\infty"'] \arrow[rr, bend right, "\text{not hol.}"'] & X \arrow[r, "c", "\text{hol}"'] & X \end{tikzcd}$$

X : h principle $\Longrightarrow$ $\forall Y$ Brody Kähler
$f : X \to Y$: const.

背理法

proof. $C \subset X$ curve s.t. $f|_C : C \to Y$ not const

$\Longrightarrow$ $\exists M \to \bar{C}$, M affine $\mathbb{C}^n$ bdl.
① ~ ④ を満たす.

$$\begin{tikzcd} M \arrow[d, "\tau"', "\text{hol}"] \arrow[rrr] \arrow[rrr, bend left, "\exists h"] & & & X \arrow[r, "f", "\text{hol}"'] & Y \\ \bar{C} \arrow[r, "id_C", "C^\infty"'] & C \arrow[rr, hook, "\iota"'] & & X & \end{tikzcd}$$

$\Longrightarrow$ $\exists h$: hol $M \to X$, $h \sim id \circ \tau$.

$$\begin{tikzcd} M \arrow[d, "\tau"'] \arrow[dr, "f \circ h"] \arrow[d, phantom, shift left=4, "\circlearrowleft"] & \\ \bar{C} \arrow[r, "\exists \varphi"'] & Y \end{tikzcd}$$

$\forall c \in \bar{C}$
$f \circ h(\tau^{-1}(c)) = pt$
$\hookrightarrow$ $\mathbb{C}^n \to Y$ は1点
Brody,

$$\begin{tikzcd} \bar{C} \arrow[r, "h \circ \sigma"] \arrow[rr, bend right, "\varphi : \text{hol}"'] \arrow[rr, phantom, bend right=15, "\circlearrowleft"] & X \arrow[r, "f", "② \ \text{hol}"'] & Y \end{tikzcd}$$

$\varphi \sim f \circ \iota$ より. $0 < \int_{\bar{C}} \varphi^* \omega_Y = \int_{\bar{C}} (f \circ \iota)^* \omega_Y$
$= (-1) \int_C f^* \omega_Y < 0$.

よって Lem に矛盾 □

proof of h-principle $\Longrightarrow$ special.

背理法 X not special

$\Longrightarrow$ $c_X : X \dashrightarrow C(X)$ core map.

$\Longrightarrow$ $\exists M \longrightarrow \bar{X}$ affine

$\Longrightarrow$ ([illegible])

$$\begin{tikzcd} M \arrow[d, "\text{hol}"'] \arrow[dr, "\exists h \text{ hol}"] \arrow[dr, phantom, "\sim"] & & \\ \bar{X} \arrow[r] & X \arrow[r, dashed] & C(X) \end{tikzcd}$$

$\Longrightarrow$ $\exists \varphi : \bar{X} \dashrightarrow C(X)$ meromorphic.

$\varphi \sim \mathrm{id} \circ c_X$ 矛盾.