昨日 E ベクトル束, (X,ω) cpt Käh

・Semistable

$\overset{\text{def}}{\iff}$ $0 \neq {}^\forall F \subsetneq E$, $\mu(F) \leq \mu(E)$

Here $\mu(F) = \dfrac{c_1(F)\cdot[\omega]^{n-1}}{\mathrm{rk}\, F}$

・Stableの直和 $\iff$ $^\exists$HE

・Kobayashi-Hitchin対応

・Donaldson-Uhlenbeck-Yau定理

(Higgs束の場合 Simpson 88

(Quasi-projのHiggs束 Mochizuki

Cor E semistable, $r = \mathrm{rk}\,E$,

$$\Longrightarrow \Delta(E)\cdot[\omega]^{n-2} = \left(2r\,c_2(E) - (r-1)\,c_1(E)^2\right)[\omega]^{n-2} \geqq 0$$

Bogomolov-Gieseker 不等式

Harder-Narasimhan-filtration

$\forall E$ ベクトル束

$\exists\ 0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$

s.t. $G_i = E_i/E_{i-1}$ μ semistable

$$\underbrace{\mu(G_1)}_{\mathrm{Max}} > \mu(G_2) > \cdots > \mu(G_\ell) = \mu^{\min}(E)$$

$$\left(\inf\{\mu(Q) \mid E \twoheadrightarrow Q\}\right)$$

Thm 6.6

K_X ($:= \det \Omega_X$ 正則余接束) が positive

$$\Longrightarrow \left(2(n+1)c_2(T_X) - n\,c_1(T_X)^2\right) c_1(K_X)^{n-2} \geq 0$$

Miyaoka–Yau

そのために

Def 6.1 (E, θ) がHiggs束

$\underset{\text{def}}{\Longleftrightarrow}$ ・E ベクトル束

・$\theta : E \longrightarrow E \otimes \Omega_X$ 準同型で

$$\theta\wedge\theta : E \xrightarrow{\theta} E\otimes\Omega_X \xrightarrow{\theta\times id} E\otimes\underbrace{\Omega_X\otimes\Omega_X} \xrightarrow{id\times\wedge} E\otimes\Omega_X^2$$ として

$\theta\wedge\theta = 0$ となる.

・$F\subset E$ が θ 不変

$\overset{def}{\Longleftrightarrow}$ $\theta(F)\subset F\otimes\Omega_X$

(E,θ) が stable

$\Longleftrightarrow$ ${}^\forall 0\neq F\subset E$, $\mathrm{rk}F<\mathrm{rk}E$,

かつ F が θ 不変 について,

$$\mu(F)<\mu(E)$$

例) $E = \Omega_X \oplus \mathcal{O}_X$

$E = \Omega_X \oplus \mathcal{O}_X \to E \otimes \Omega_X = (\Omega_X \oplus \mathcal{O}_X) \otimes \Omega_X$

$(a, b) \mapsto (0, a) = (0, 1) \otimes a$

Higgs束

Thm 6.3 [Sim88]

(E, Θ) stable Higgs束

$\Longrightarrow (2r c_2(E) - (r-1) c_1(E)^2)[\omega]^{n-2} \geq 0.$

(証明, HEっぽい計量を (harmonic metric) つくってけいさんする)

Prop 6.

$H = E$

$\Theta : H$

(a, b)

E semistable

Prop 6.5 $E \subset \Omega_X$ 部分束,

$H = E \oplus \mathcal{O}_X$, Higgs束

$\theta : H \longrightarrow H \otimes \Omega_X$ とする

$(a, b) \longmapsto (0, a)$

E semistable & $\mu(E) > 0 \Longrightarrow (E, \theta)$ stable.

Pf of Thm 6.6 $\omega \in c_1(K_X)$ Kähler form

K_X positive $\xRightarrow{\text{(Yau77)}}$ KE $\xRightarrow{\text{Prop 4.20}}$ T_X $\exists$ HE

$\xRightarrow{\text{(さっき)}}$ T_X semistable

$\xRightarrow{\text{(双対)}}$ Ω_X^1 semistable, $\mu(\Omega_X^1) = \dfrac{c_1(X)[\omega]^{n-1}}{n} > 0$

$\xRightarrow{\text{Prop 6.5}}$ $(\Omega_X^1 \oplus \mathcal{O}_X, \theta)$ stable

$\Longrightarrow r = \mathrm{rank}(\Omega_X \oplus \mathcal{O}_X) = n+1$ より,

Thm 6.3

$$0 \leq \Big(2(n+1)\,\boxed{c_2(\Omega_X \oplus \mathcal{O}_X)} = c_2(\Omega_X^1) = c_2(X)$$

$$- n\,\underbrace{c_1(\Omega_X \oplus \mathcal{O}_X)^2}_{= c_1(\Omega_X^1)^2 = c_1(X)^2}\Big)\, c_1(K_X)^{n-2}$$

Thm I. Jinnouchi–Zhang 25 (Jin2)

K_X "singular" positive (big)

$$\Longrightarrow \Big(2(n+1)c_2(X) - n c_1(X)^2\Big)\,\underbrace{\langle c_1(K_X)^{n-2}\rangle}_{\text{non-pluripolar product}} \geq 0$$

(2) Sketch

⓪ $\langle c_1(K_X)^{n-1}\rangle$ stabilityをていぎする

(1-1) X_{min} (Xのminimal model) について $T_{X_{min}}$は $\langle c_1(K_{X_{min}})^{n-1}\rangle$ semi stable

(1-2) T_X は $\langle c_1(K_X)^{n-1}\rangle$ sem-stable

(2) Bogomolov-Gieseker不等式を確立する

"K_X semipositiveの場合"

$$K_X \equiv 0 \quad (c_1(K_X)=0)$$

$$\xrightarrow[\text{Yau}]{} \exists \mathrm{KE} \Rightarrow T_X \text{ semistable}$$

$$\underset{(\mathrm{B.G})}{\Rightarrow} c_2(T_X)[\omega]^{n-2} \geq 0$$

(nef) ≒ semipositive

Thm 6.8 (Many people)

K_X nef $\Longrightarrow$ $\forall \omega_1, \dots, \omega_{n-1}$ Kähler form

$\forall \Omega_X \twoheadrightarrow Q \neq 0$ に対して

$$c_1(Q)[\omega_1]\cdots[\omega_{n-1}] \geqq 0$$

・ $-K_X (= \det T_X)$ が nef

$\Longrightarrow$ $\forall \omega_1, \dots, \omega_{n-1}$, Kähler form

$\forall T_X \twoheadrightarrow Q \neq 0$,

$$c_1(Q)[\omega_1]\cdots[\omega_{n-1}] \geqq 0$$

Thm 6.10 K_X nef

$\Rightarrow c_2(X)[\omega]^{n-2} \geqq 0$

$\forall \omega$ Kähler form

Pr.

Lem 6.11 (レポート)

$r_1, \dots, r_\ell \in \mathbb{Z}_{\geq 1}$, $n \in \mathbb{Z}_{\geq 2}$

$\mu_1, \dots, \mu_\ell, d \in \mathbb{R}$, $d > 0$

Assume ① $\mu_1 > \mu_2 > \dots > \mu_\ell \geq 0$.

$\sum_{i=1}^{\ell} r_i \mu_i = d$, $\boxed{\sum_{i=1}^{\ell} r_i = n}$

このとき

① $d - \frac{1}{d}\sum_{i=1}^{\ell} r_i \mu_i^2 \geqq 0$

② "=" 成立

$\Rightarrow \ell = 2,\ \mu_1 = d \quad r_1 = 1$

$\qquad\qquad\ \ \mu_2 = 0 \quad r_2 = n-1$

Pf of Thm 6.10 $\forall \mathcal{E} \subset T_X$

slope を $\mu(\mathcal{E}) = \dfrac{c_1(\mathcal{E}) \cdot \overset{(\text{nef})}{\boxed{c_1(-K_X)}}\, [H]^{n-2}}{\mathrm{rk}\,\mathcal{E}}$ でさだめる

HN filtration

$0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = T_X$

s.t. $\underset{E_i/E_{i-1}}{\overset{\parallel}{G_i}}$: rank r_i semistable slope μ_i

2) ・ $\mu_1 > \mu_2 > \cdots > \mu_\ell = \inf\{\mu(Q) \mid E \twoheadrightarrow Q\}$

$\boxed{\text{Thm 6.8}} \Rightarrow \geq 0$

さらに $\displaystyle\sum_{i=1}^{\ell} r_i \mu_i = \sum_{i=1}^{\ell} c_1(G_i)\, c_1(-K_X)[w]^{n-2}$

$= c_1(-K_X)\cdot c_1(-K_X)[w]^{n-2}$

$= d$

$d>0$を仮定（$d=0$のときは資料参照

さらにHodge index Thm (5.15)

$$c_1(G_i)^2[w]^{n-2} \leq \frac{1}{c_1(-K_X)^2[w]^{n-2}}\left(c_1(G_i)c_1(-K_X)[w]^{n-2}\right)^2 = \frac{1}{d} r_i^2 \mu_i^2$$

$$\boxed{2c_2(T_X)}[\omega]^{n-2}$$

$$= \Big(2\sum_{i=1}^{\ell} \underbrace{c_2(G_i)}_{\text{semistable}} - \sum_{i=1}^{\ell} c_1(G_i)^2\Big)[\omega]^{n-2} + \underbrace{c_1(T_X)^2[\omega]^{n-2}}_{= d}$$

$$\geq -\sum_{i=1}^{\ell} \frac{1}{r_i} c_1(G_i)^2[\omega]^{n-2} + d$$

$$\geq -\sum_{i=1}^{\ell} \frac{1}{d} r_i \mu_i^2 + d \underset{(\text{レポート})}{\geq} 0$$

∴ 0 まめ式

Thm 6.12 K_X nef

$$\Rightarrow \left(3c_2(X) - c_1(X)^2\right)[\omega]^{n-2} \geq 0$$

Lem 6.13 (レポート)

Lem 6.11の設定でさらに

$n \geq 3$をかていする

② $\ell, r_1 \geq 2$

$$\Rightarrow -\frac{r_1^2\mu_1^2}{d(r_1+1)} - \frac{1}{d}\sum_{i=2}^{\ell} r_i\mu_i^2 + d \geq \frac{2}{3}d$$

Pf of Thm 6.12

$$\mu(\mathcal{E}) = \frac{c_1(\mathcal{E})c_1(K_X)[w]^{n-2}}{\mathrm{rk}\,\mathcal{E}}$$ という slope (=7.1.2)

HN filtration $0 = E_0 \subset E_1 \subset \cdots \subset E_\ell = E$,

G_i: rk r_i, slope μ_i, semistable

$\mu_1 > \mu_2 > \cdots > \mu_\ell \geq 0$ (Thm 6.8)

かわる点　$G_1 \subset \Omega_X^1$ なので、

($G_1 \oplus G_X$ が stable-Higgs束 (右上)
の構造をもつ)

$$\Rightarrow c_2(G_1)[w]^{n-2} \geq \frac{r_1}{2(r_1+1)} c_1(G_1)^2 [w]^{n-2}$$

BG

$\psi := c_1(K_X)^2[w]^{n-2}$、$r_1 \geq 2$ なので

$$2c_2(\Omega_X^1)[w]^{n-2}$$

$$\stackrel{(右上)}{=} \left(2c_2(G_1) - c_1(G_1)^2\right)[w]^{n-2}$$

$$+ \left(\sum_{i=2}^{\ell} 2c_2(G_i) - c_1(G_i)^2\right)[w]^{n-2}$$

$$S + \underbrace{c_1(K_X)^2[w]^{n-2}}_{\psi}$$

$$\underset{BG}{\geq} \frac{-1}{r_1+1} c_1(G_1)^2[\omega]^{n-2} - \sum_{i=2}^{\ell} \frac{1}{r_i} c_1(G_i)^2[\omega]^{n-2} + d$$

$$\underset{\text{Hodge}}{\geq} -\frac{r_1^2\mu_1^2}{r_1+1} - \frac{1}{d}\sum_{i=2}^{\ell} r_i\mu_i^2 + d$$

$$\underset{(\text{しぱーん})}{\geq} \frac{2}{3}d = \frac{2}{3} c_1(K_X)^2[\omega]^{n-2}. \quad \square$$

(Th 6.8)

Conj Nonvanishing Conj

K_X pseudo-effective

($\exists h$ 特異Hermite計量, $\sqrt{-1}F_h \geq 0$)

(2,8章)

ならば $\exists m \in \mathbb{N}_{>0}$, $H^0(X, K_X^{\otimes m}) \neq 0$

Thm Hashizume 18

$\forall X \subset \mathbb{P}^n$ submfd, nonvanishing 予想がとけたとする

すると $\forall (X, \Delta)$ log canonical pair について極小モデルが存在。

Thm 6.14 K_X nef, $\dim X = 3$

[Miy87] なら nonvanishing が成立

$\boxed{\text{pf}}$ $\kappa(K_X) \overset{\text{def}}{=} \limsup_{m \to \infty} \dfrac{\log \dim H^0(X, K_X^{\otimes m})}{\log m} \in \{-\infty, 0, \dots, \dim X\}$

小平次元

$\kappa(K_X) \geqq 0 \iff \exists m \in \mathbb{N}_0,\ H^0(X, K_X^{\otimes m}) \neq 0$

$\cdots 3 d = \cdots 3\, c_1(K_X)^2 [u]^{n-2}$, □

K_X nef $\Longrightarrow c_2(X)[u]^{n-2} \geqq 0$

$\xrightarrow[(K_X \text{ nef})]{\text{Thm 6.12}} \underline{c_2(X) c_1(K_X) \geqq 0}$

$\xrightarrow{\text{R.R}} \chi(X, \mathcal{O}_X) = -\frac{1}{24}\int_X c_1(K_X) c_2(X) \leqq 0$

$\Longrightarrow 0 \geq \chi(X, \mathcal{O}_X)$

$= \underbrace{h^0(\mathcal{O}_X)}_{1} - h^1(\mathcal{O}_X) + \underbrace{h^2(\mathcal{O}_X)}_{\geqq 0} - h^3(\mathcal{O}_X)$

$\boxed{\text{ていぎ}}$ $\boxed{\dim_{\mathbb{C}} H^0(X, \mathcal{O}_X)}$

$\geq 1 - h^1(\mathcal{O}_X) - h^3(\mathcal{O}_X)$

$\Longrightarrow h^1(\mathcal{O}_X) \neq 0$ or $h^3(\mathcal{O}_X) \neq 0$

$h^3(\cdots$

$h^1(\mathcal{O}_X \cdots$

$\Longrightarrow$ Albanese map

$h^3(\mathcal{O}_X) \neq 0 \Longrightarrow h^0(K_X) \neq 0$ ok

Serre duality

$h^1(\mathcal{O}_X) \neq 0$

$\Longrightarrow \exists\, \alpha : X \longrightarrow \mathrm{Alb}(X)$

Albanese map

$\kappa(K_{\alpha(X)}) \geqq 0$

(Ueno '75)

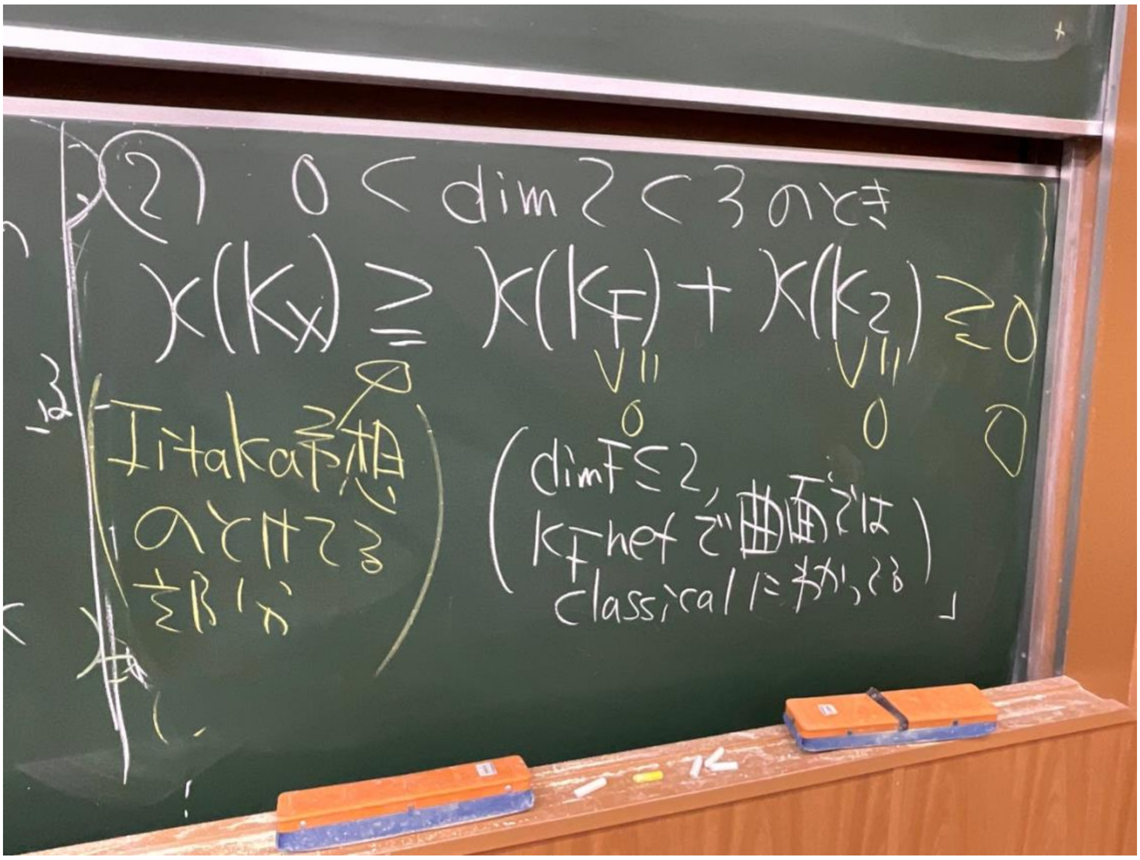
$\Longrightarrow \exists\, f : X \longrightarrow Z$ fibration

s.t. $\kappa(K_Z) \geqq 0$

$1 < \dim Z \leqq 3$

① $\dim Z = \dim X$

$\Longrightarrow \kappa(K_X) = \kappa(K_Z) \geqq 0$ ok


② 0 < dim Z < 3 のとき
κ(K_X) ≧ κ(K_F) + κ(K_Z) ≧ 0
Iitaka予想のとけてる部分
dim F ≦ 2, K_F nef で曲面では classical にわかってる