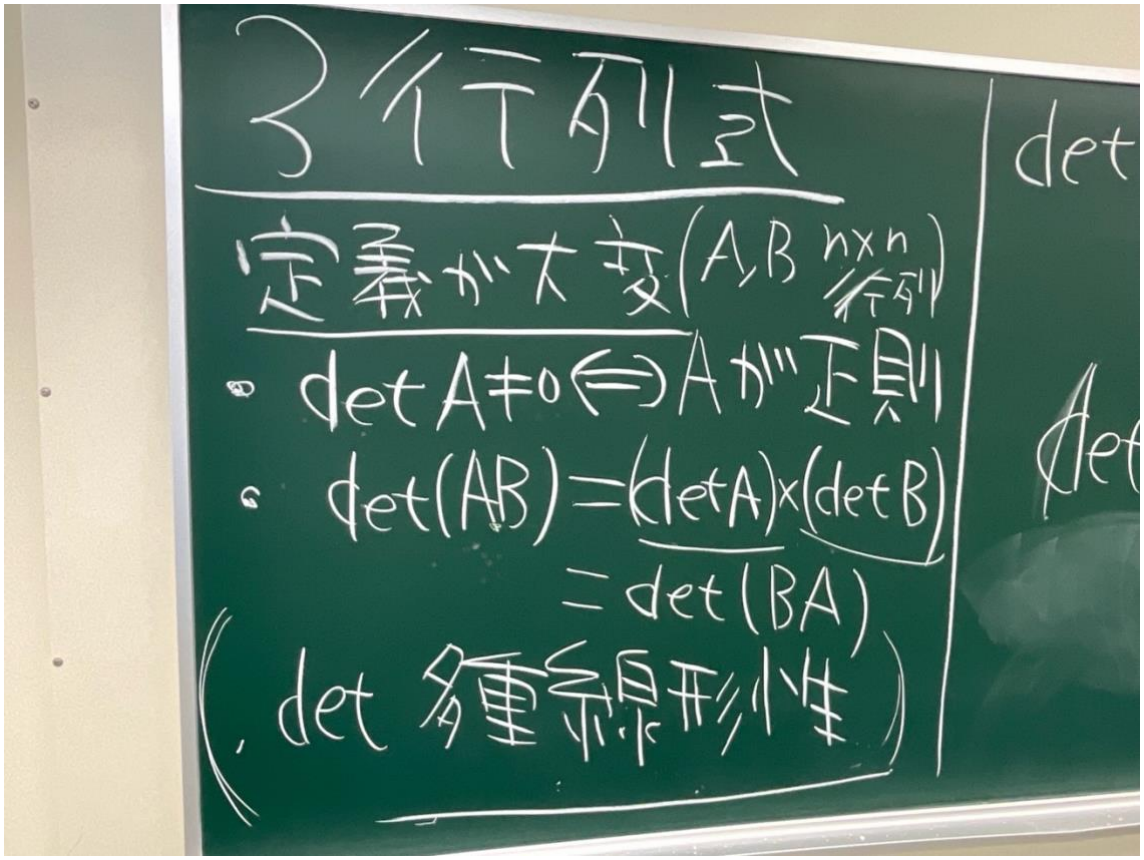
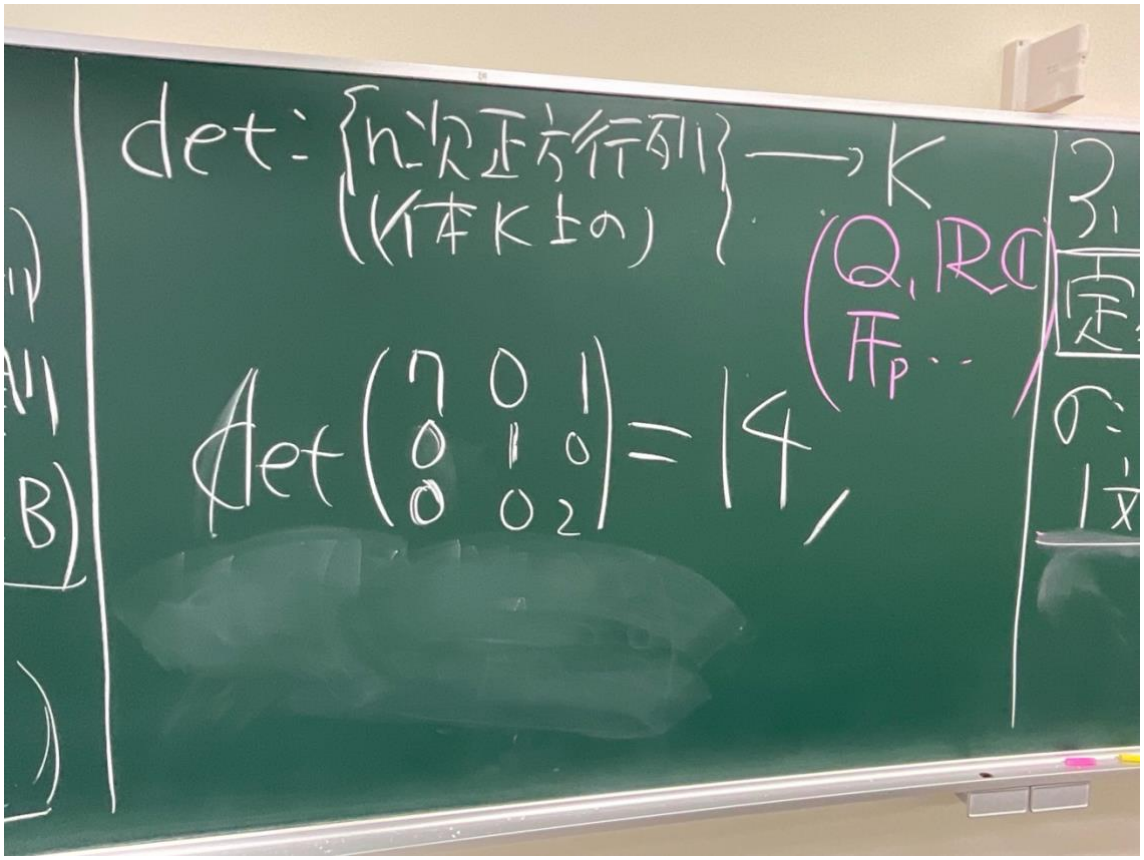
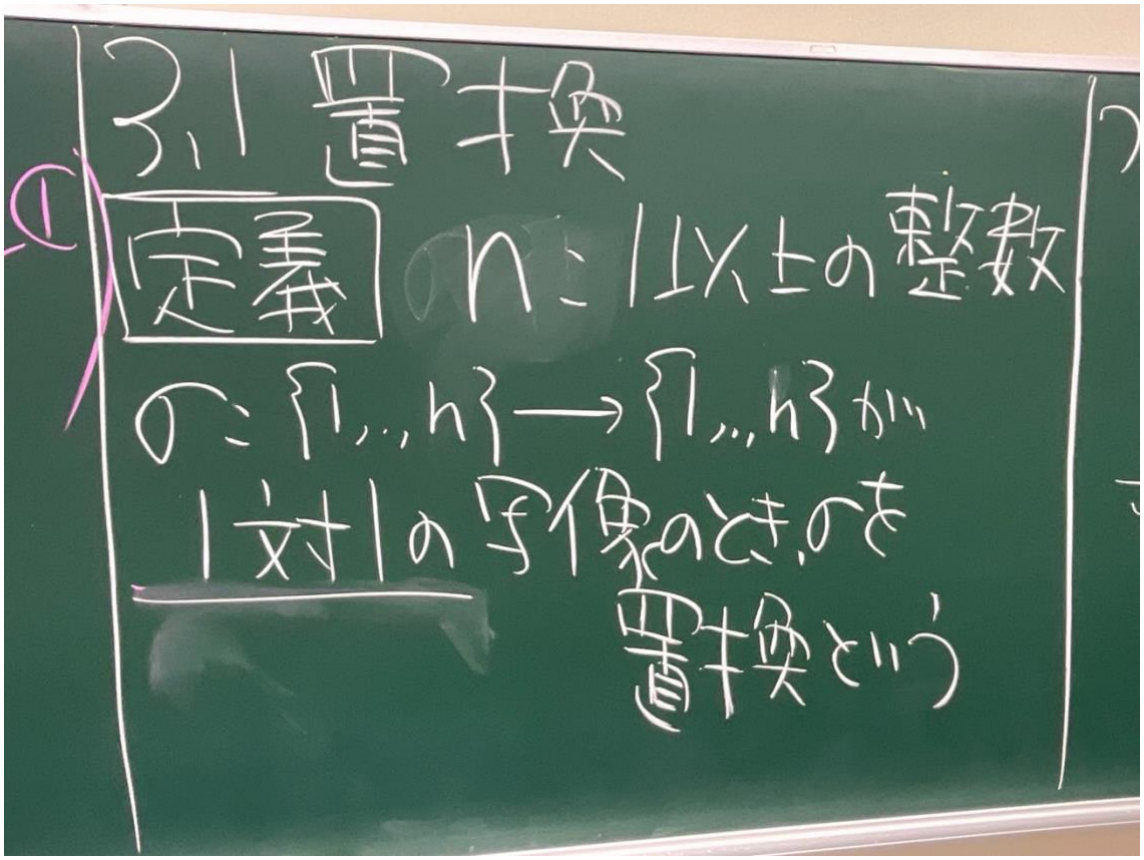
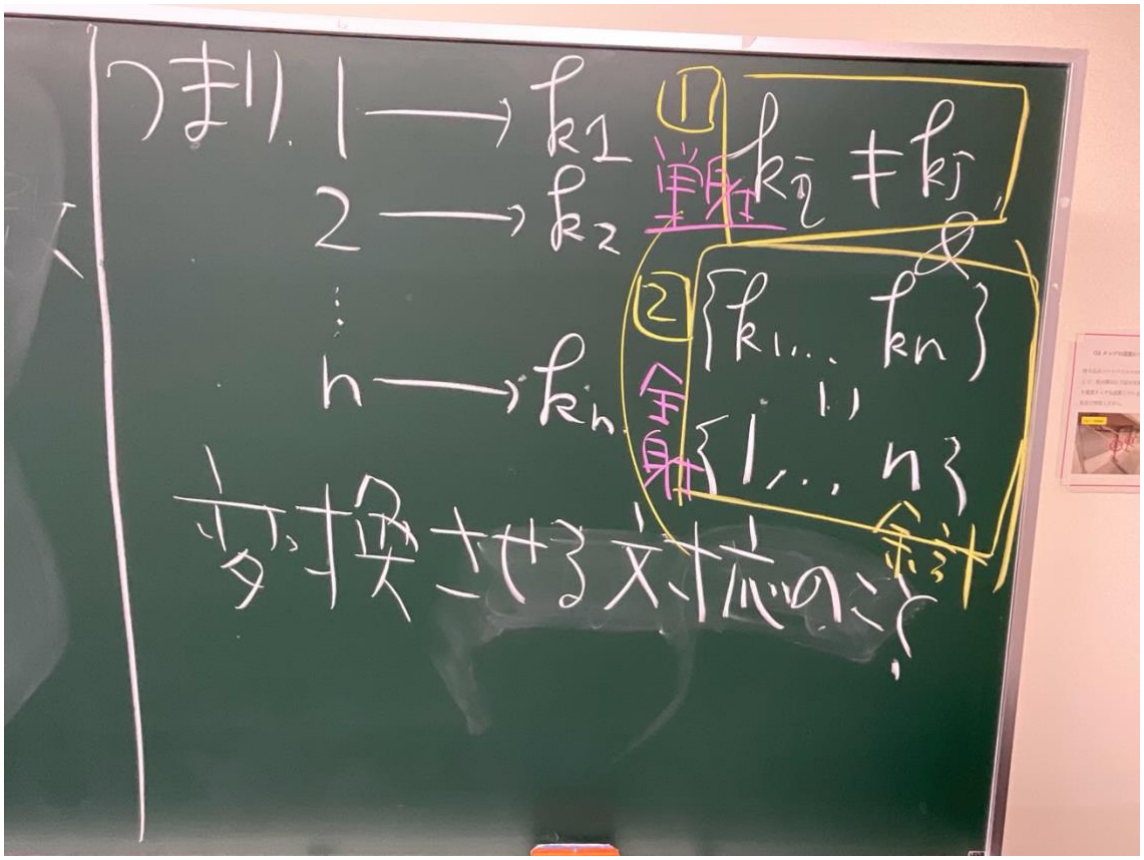
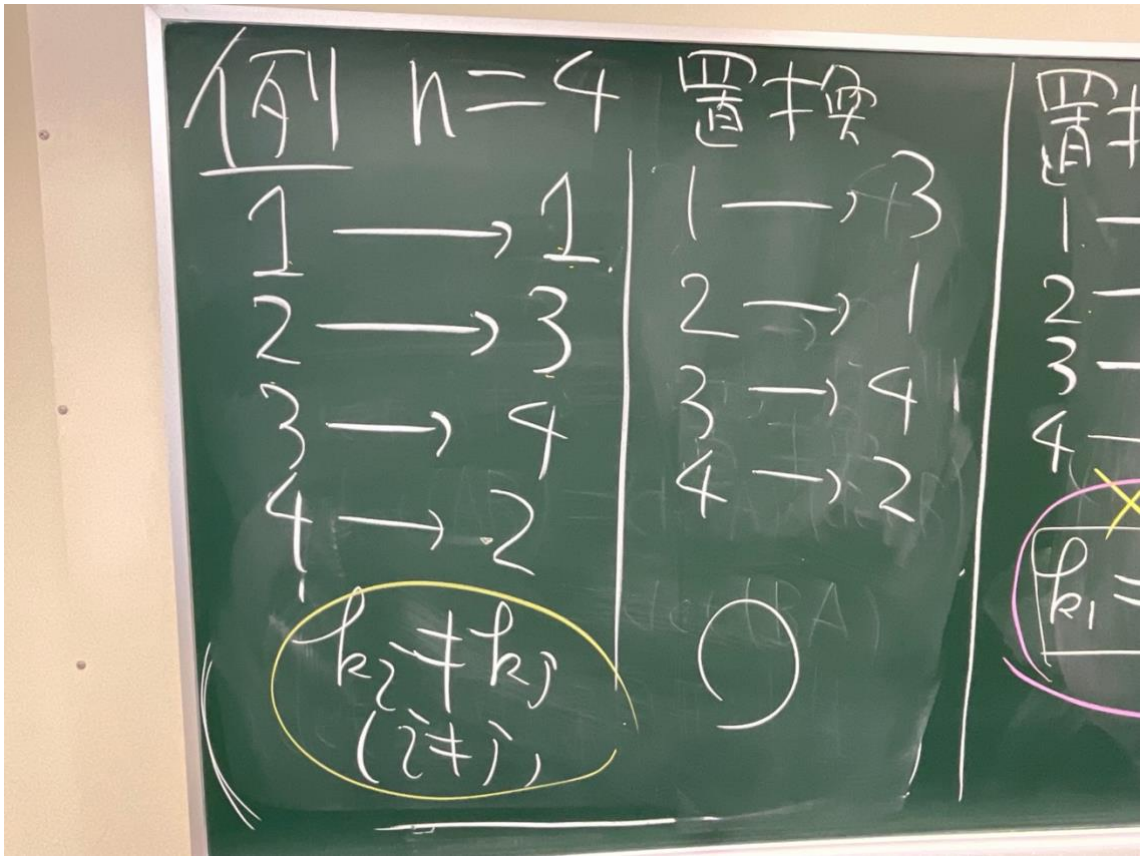
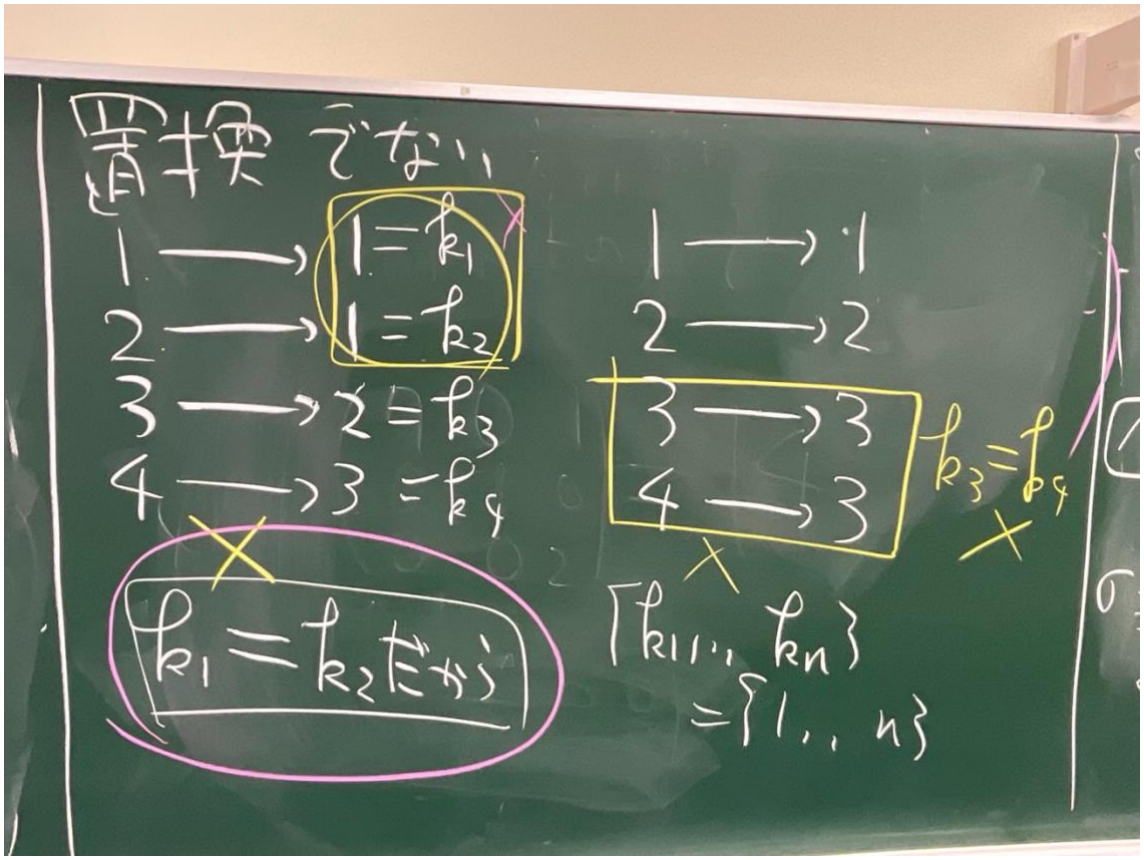

3 行列式
定義が大変 (A, B n×n 行列)
・detA≠0 ⇔ Aが正則
・det(AB) = (detA)×(detB)
= det(BA)
(, det 多重線形性)
det

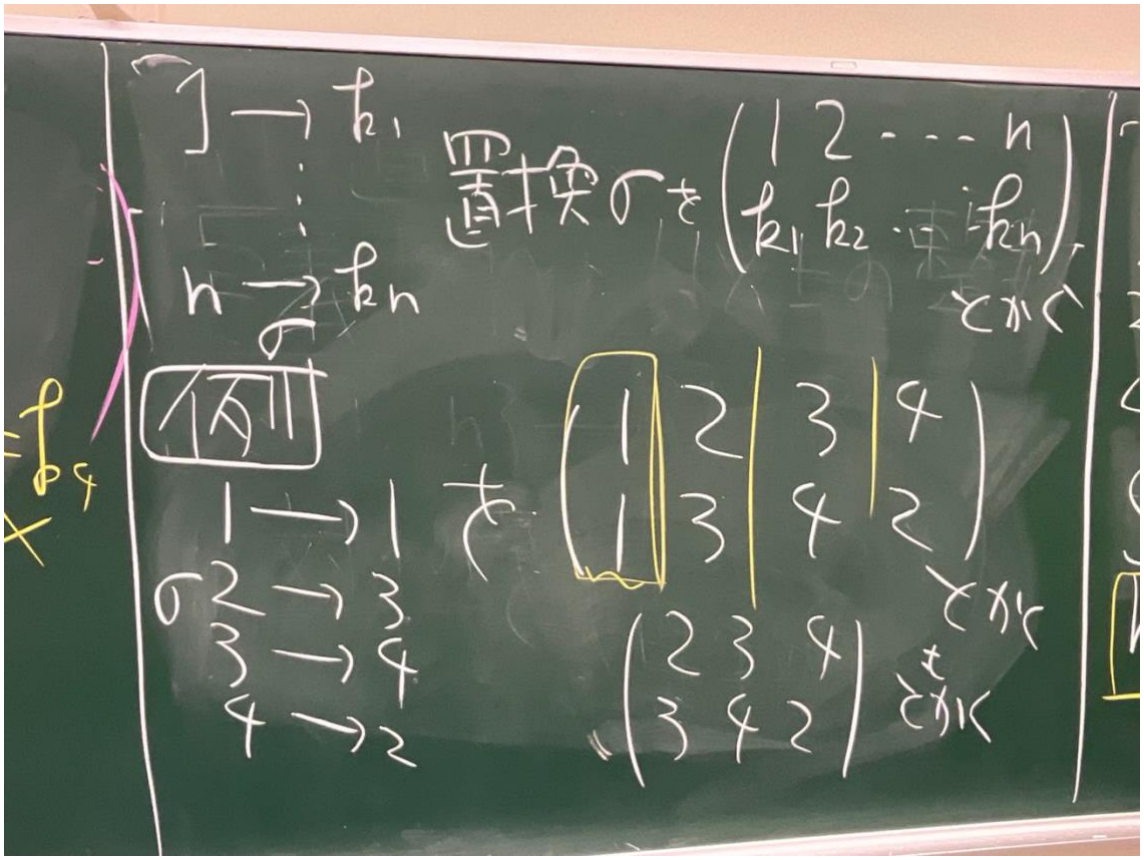

det: {n次正方行列 (成分K上の)} → K
(Q, R, C, 𝔽p ...)
det(7 0 1 / 0 1 0 / 0 0 2) = 14,

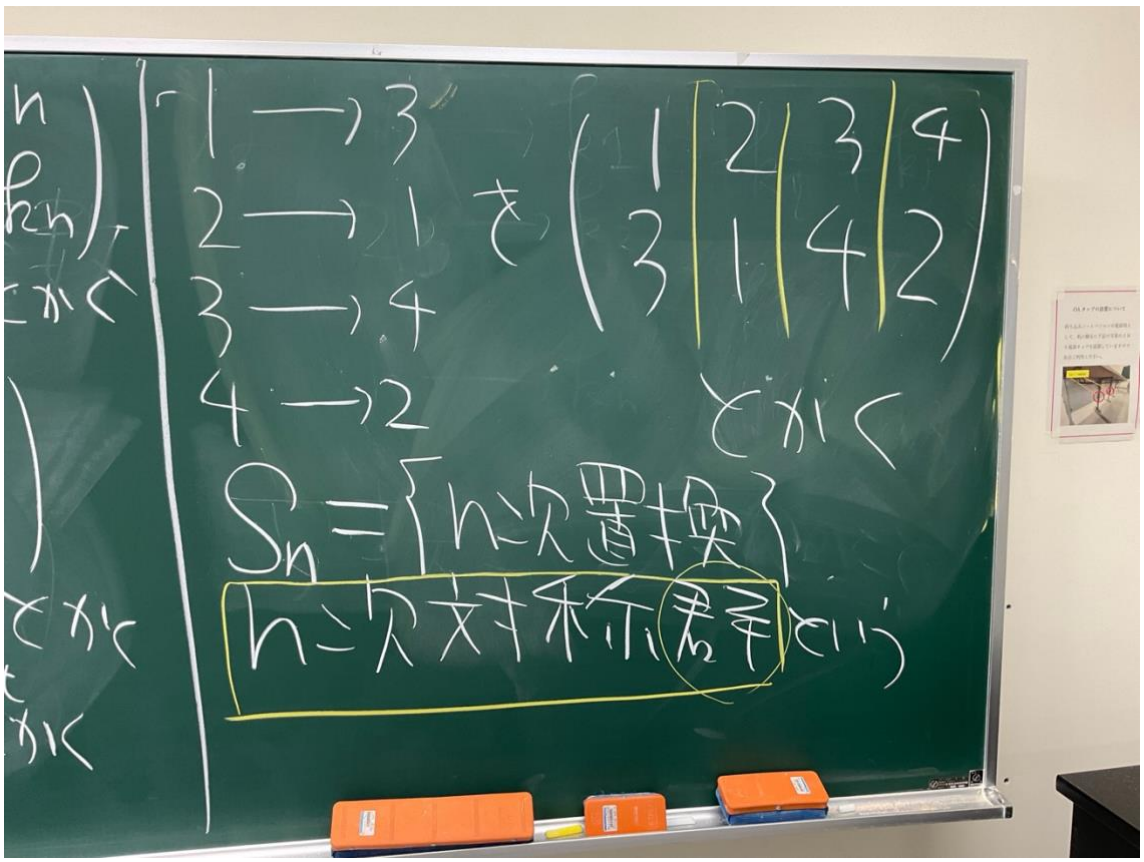

3.1 置換
定義 nは1以上の整数
の {1,..,n} → {1,..,n} から
1対1の写像のとき,σを
置換という

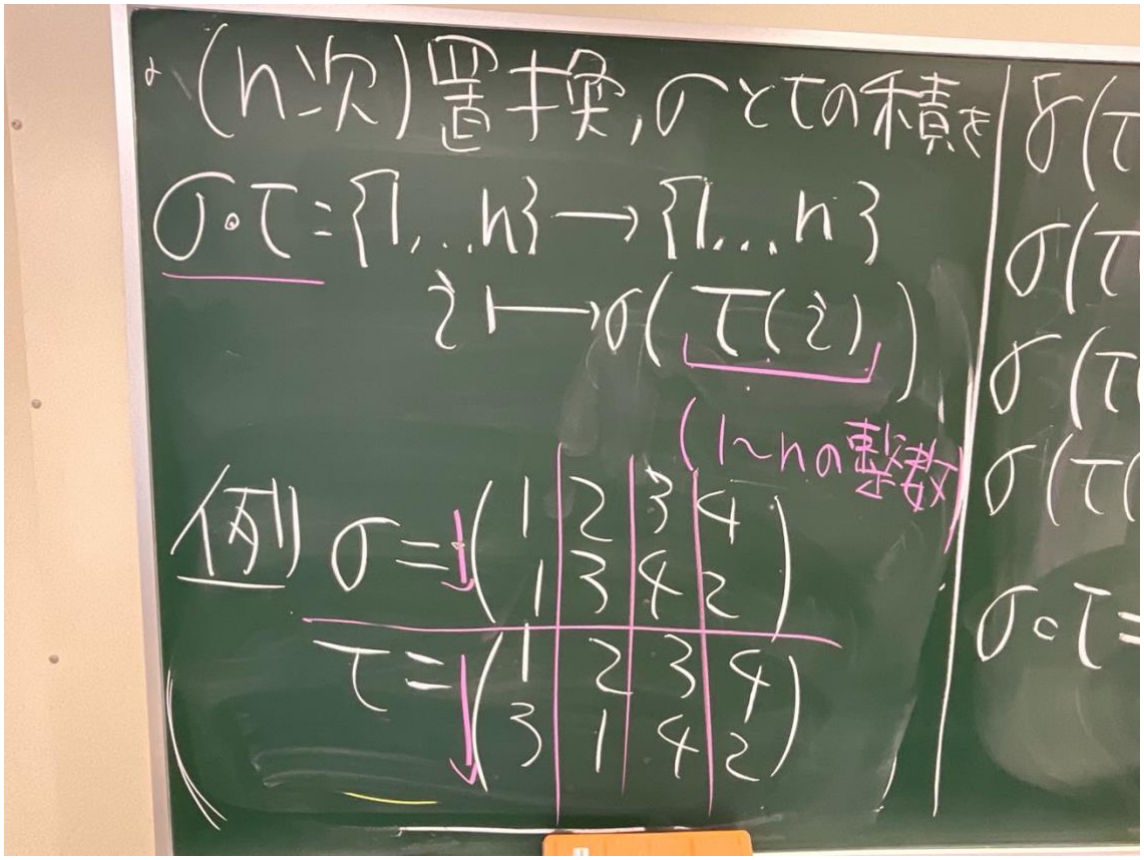

つまり 1 → k1
2 → k2
n → kn
異なる
全単射
{k1,... kn}
=
{1,.. n}
置換させる対応のこと

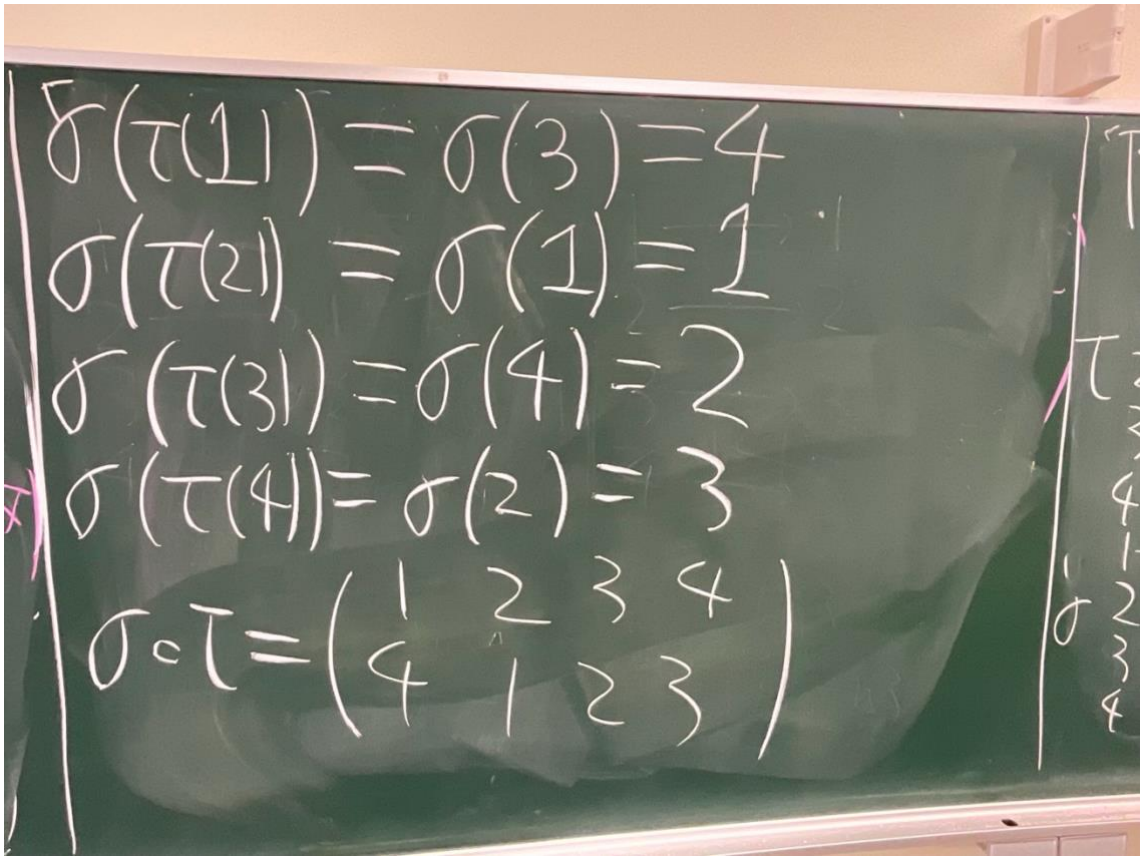

例 n=4
1 → 1
2 → 3
3 → 4
4 → 2
$k_2 \neq k_3$
(≠)
置換
1 → 3
2 → 1
3 → 4
4 → 2
○

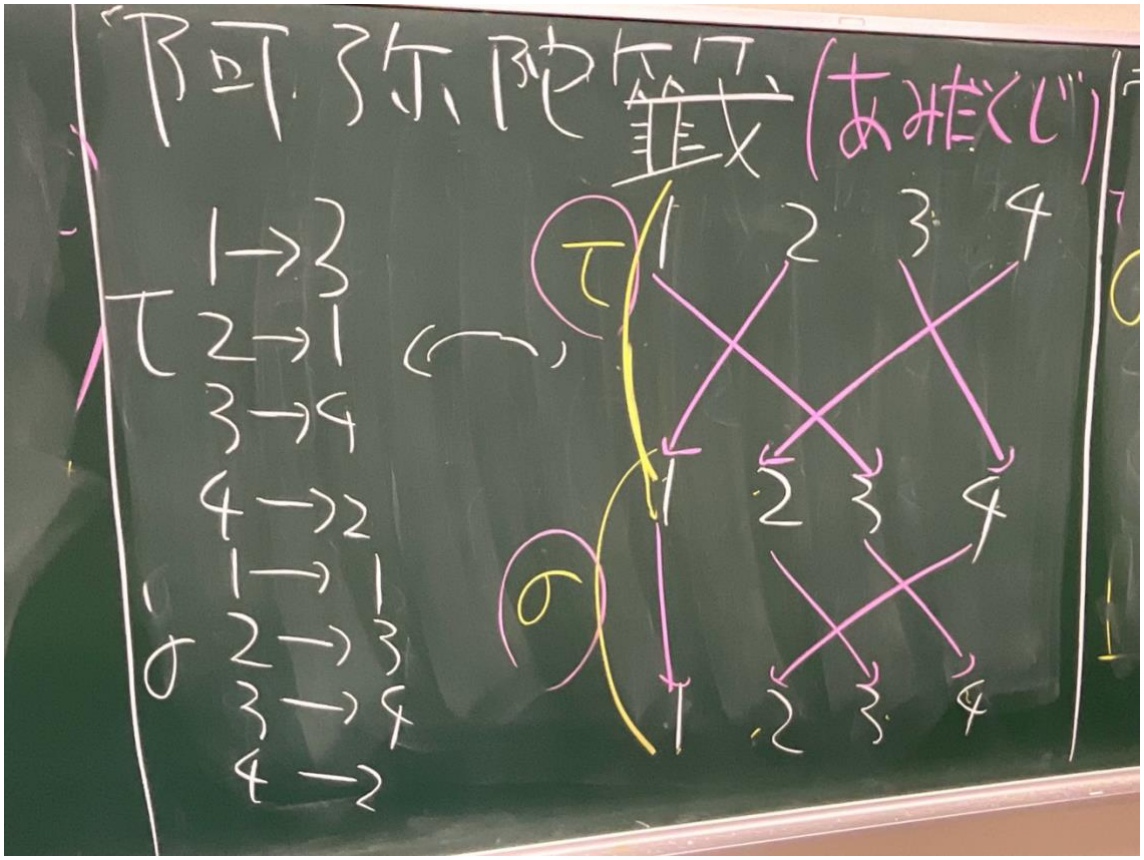

置換でない
1 → 1 = k_1
2 → 1 = k_2
3 → 2 = k_3
4 → 3 = k_4
$k_1 = k_2$だから
1 → 1
2 → 2
3 → 3
4 → 3
$k_3 = k_4$
$\{k_1, \ldots, k_n\}$
$= \{1, \ldots, n\}$

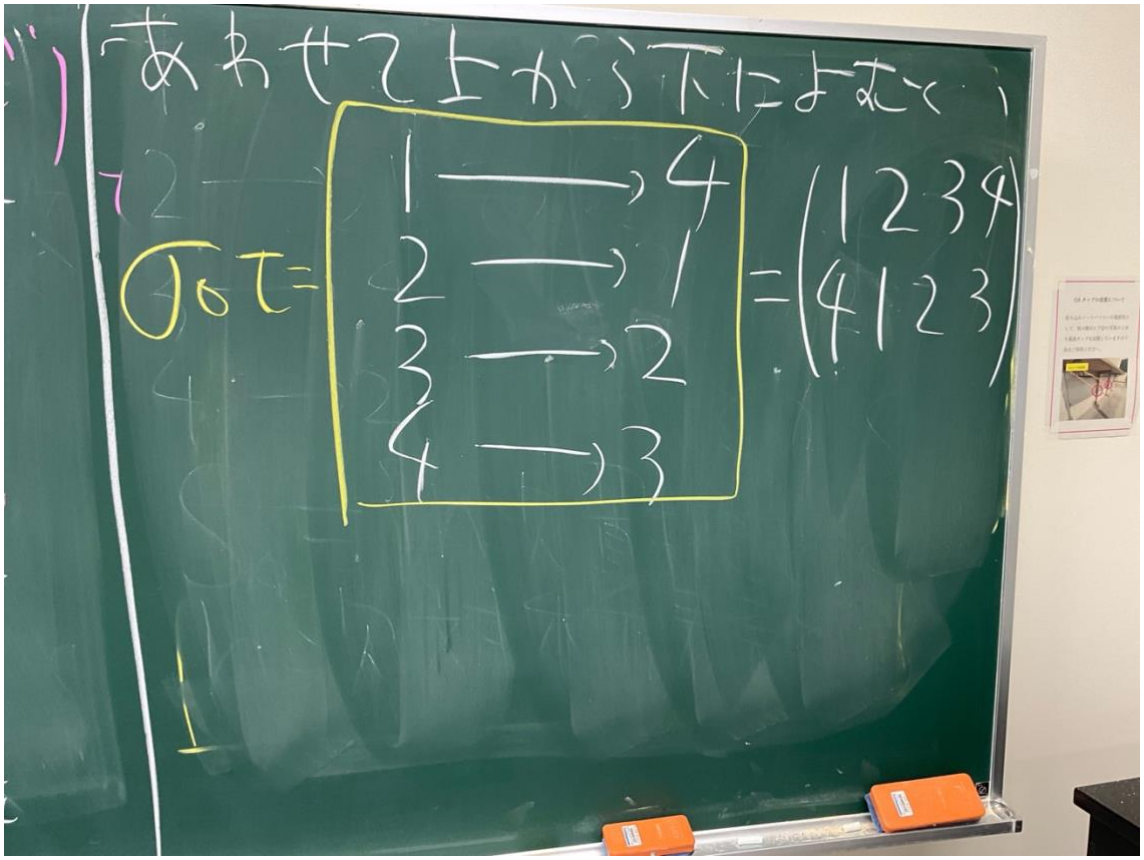

1 → k1
n → kn
置換のを
とかく
例
1 → 1
2 → 3
3 → 4
4 → 2
を
とかく
とかく

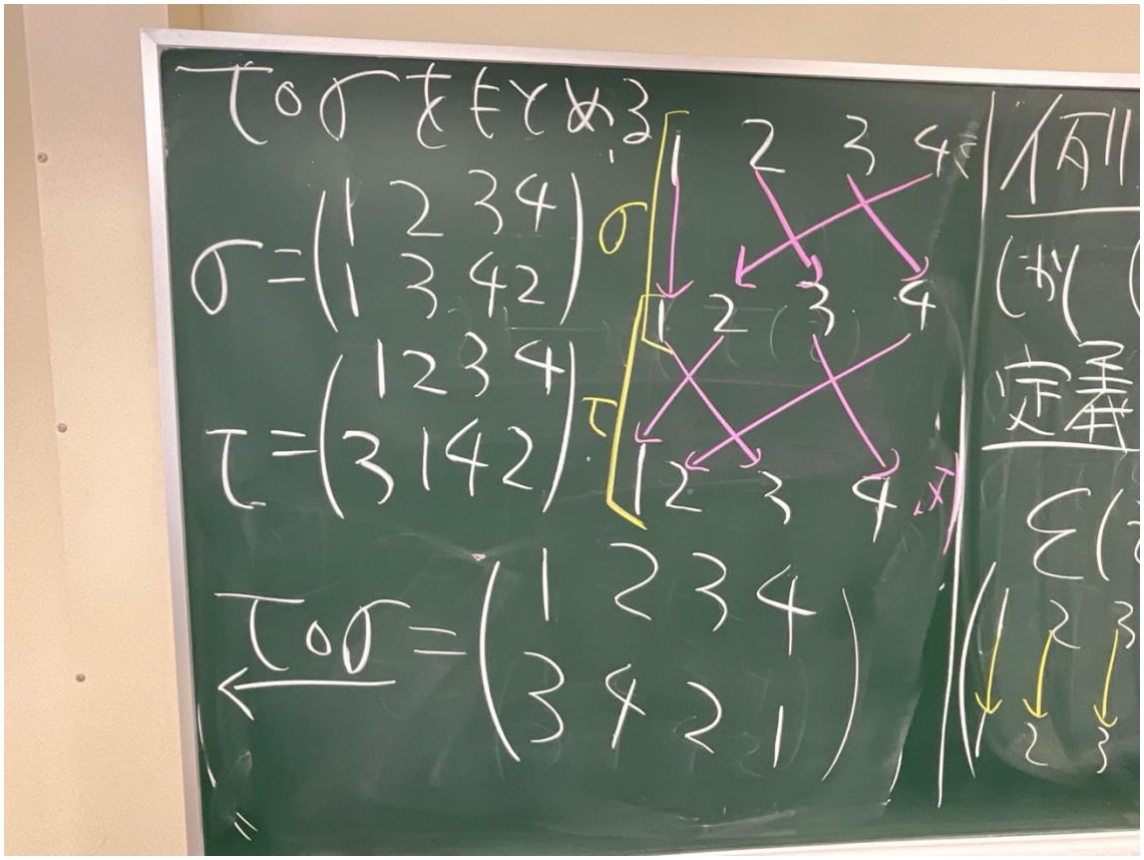

1 → 3
2 → 1
3 → 4
4 → 2
を
とかく
Sn = {n次置換}
n次対称群という

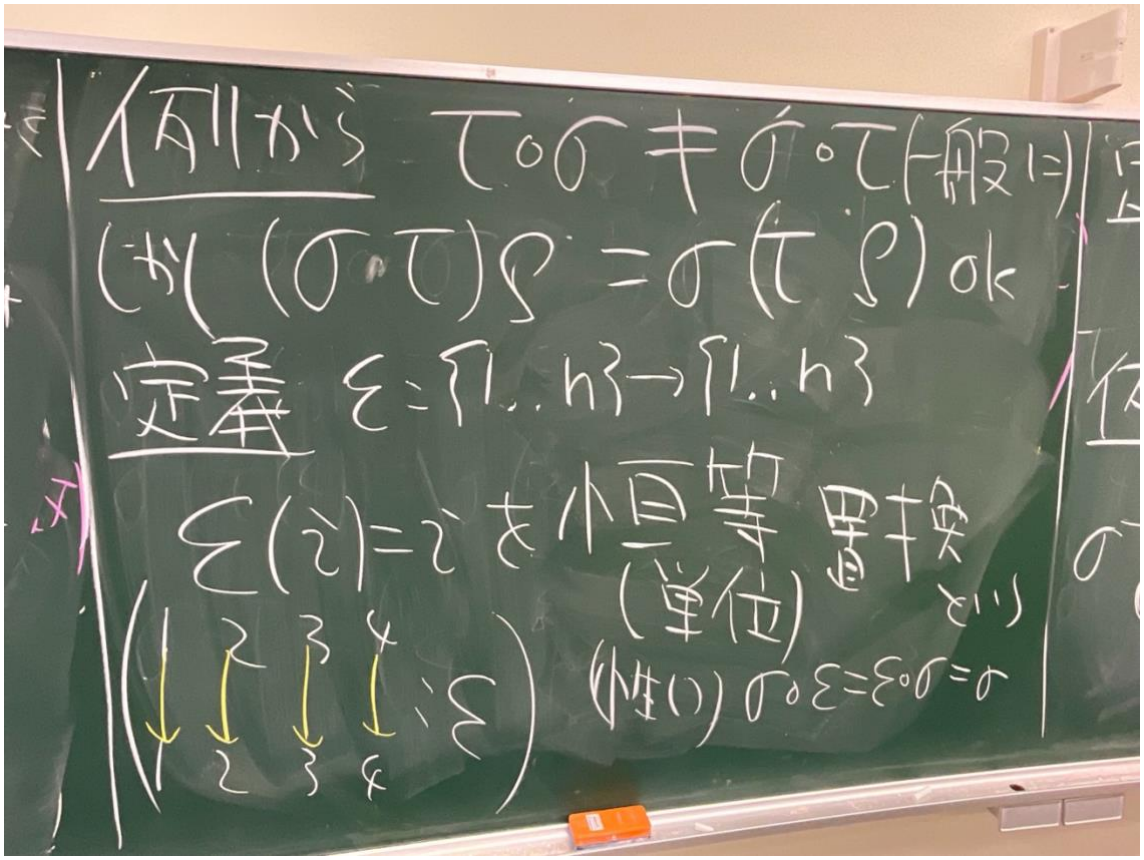

(n次)置換,のとての和を
σ∘τ:{1,...n}→{1...n}
i↦σ(τ(i))
(1~nの整数)
例 σ=(1 2 3 4 / 1 3 4 2)
τ=(1 2 3 4 / 3 1 4 2)

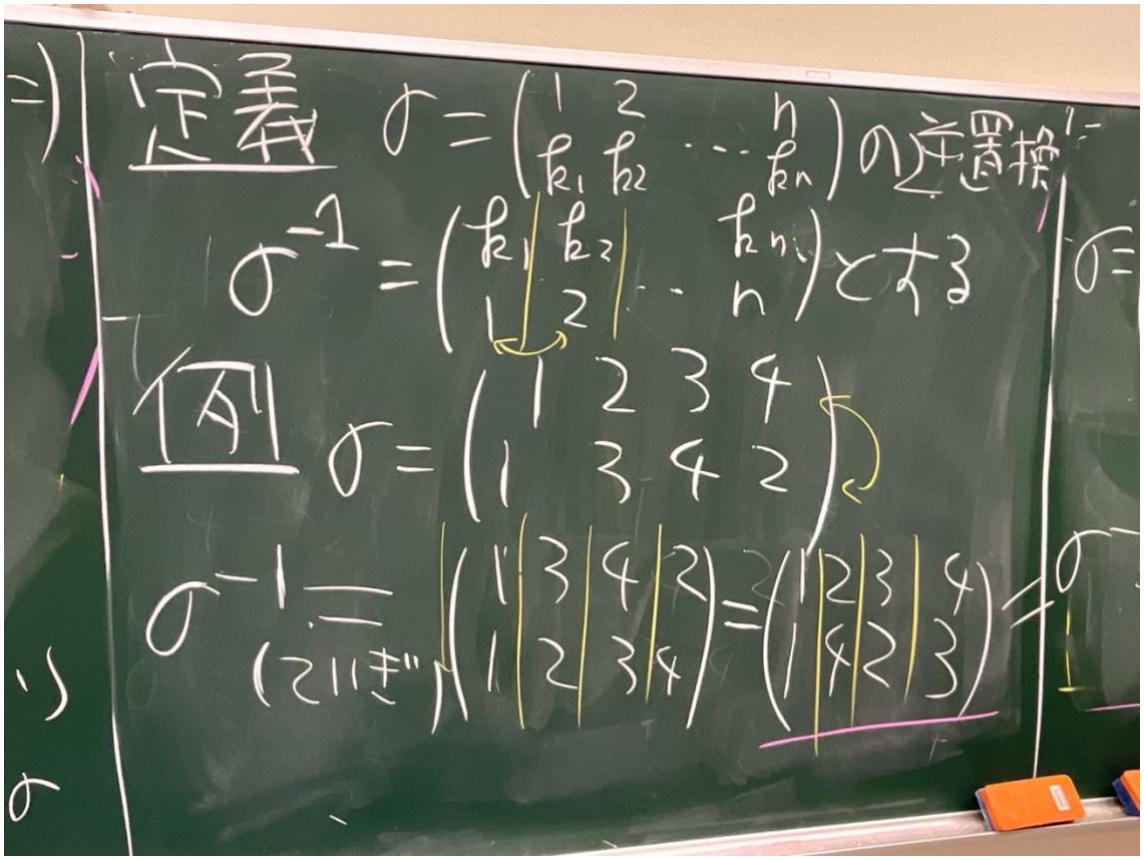

σ(τ(1))=σ(3)=4
σ(τ(2))=σ(1)=1
σ(τ(3))=σ(4)=2
σ(τ(4))=σ(2)=3
σ∘τ=(1 2 3 4 / 4 1 2 3)

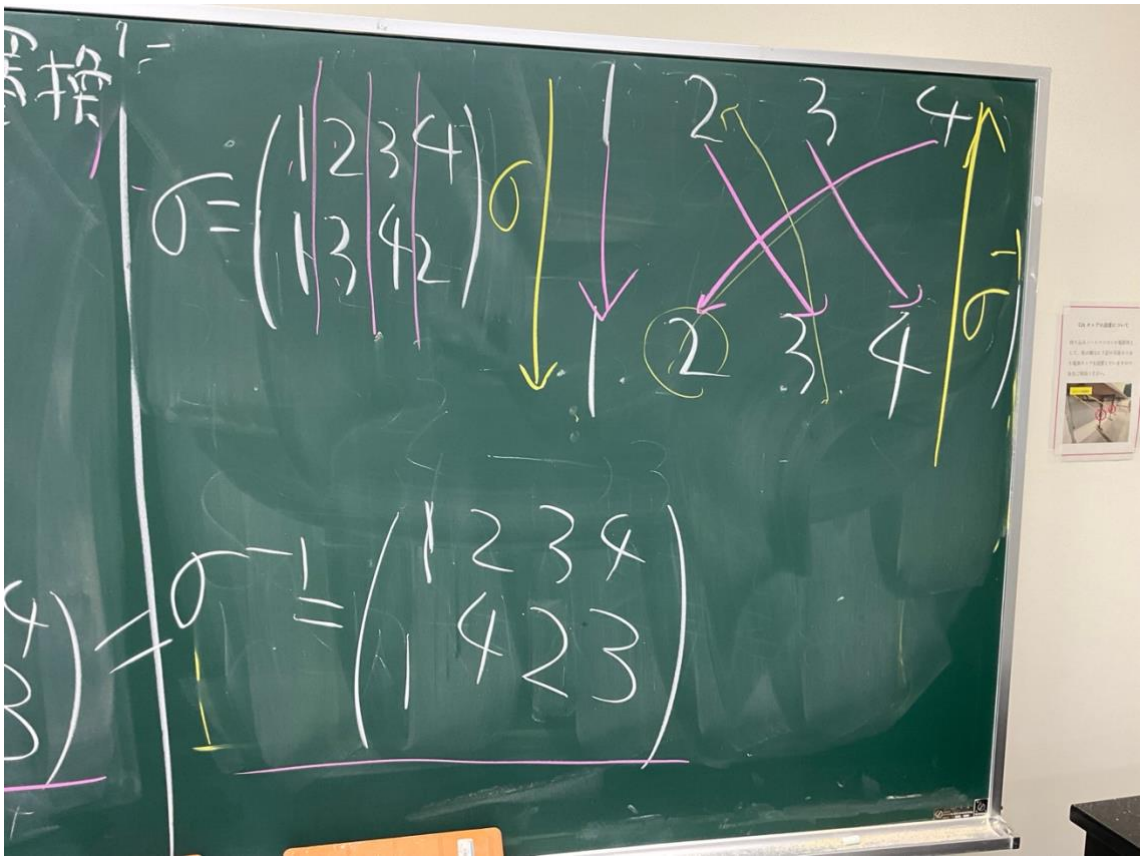

阿弥陀籤(あみだくじ)
τ
1→3
2→1
3→4
4→2
σ
1→1
2→3
3→4
4→2
1 2 3 4
1 2 3 4
1 2 3 4

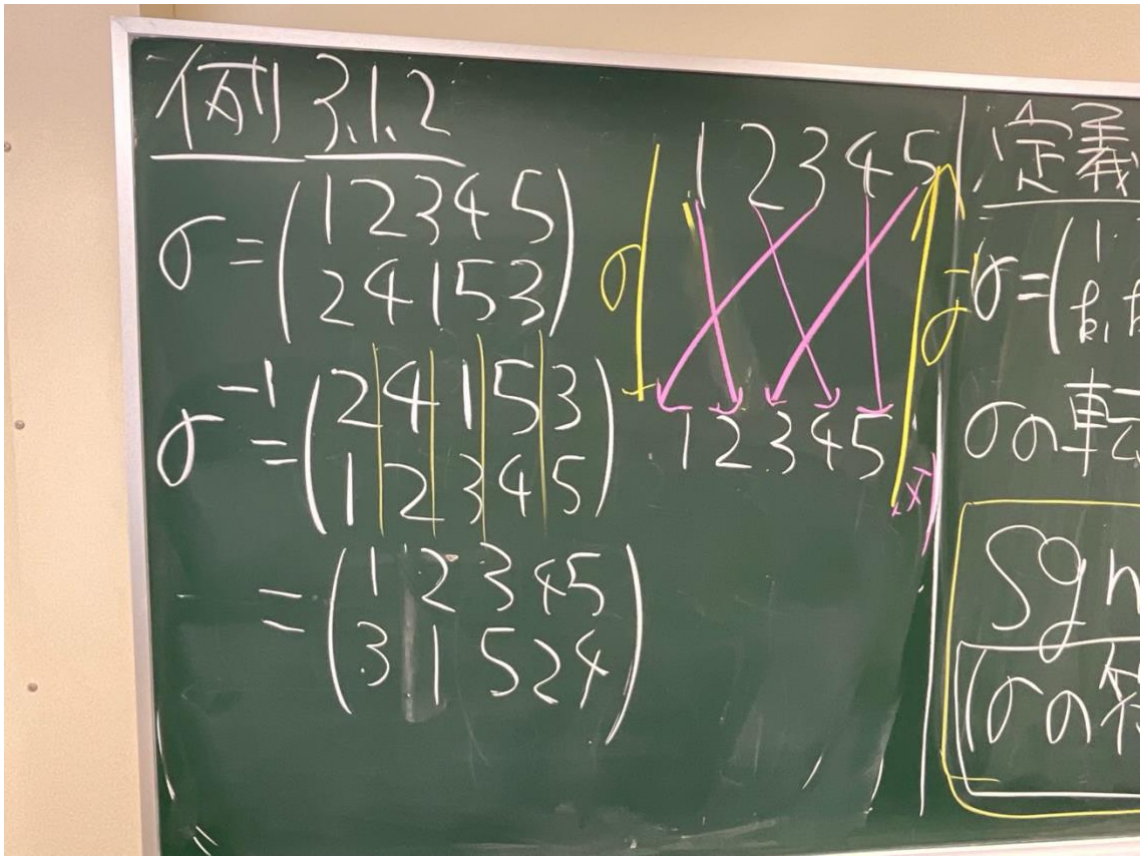

あわせて上から下によむ
σ∘τ=
1→4
2→1
3→2
4→3
=(1234 / 4123)

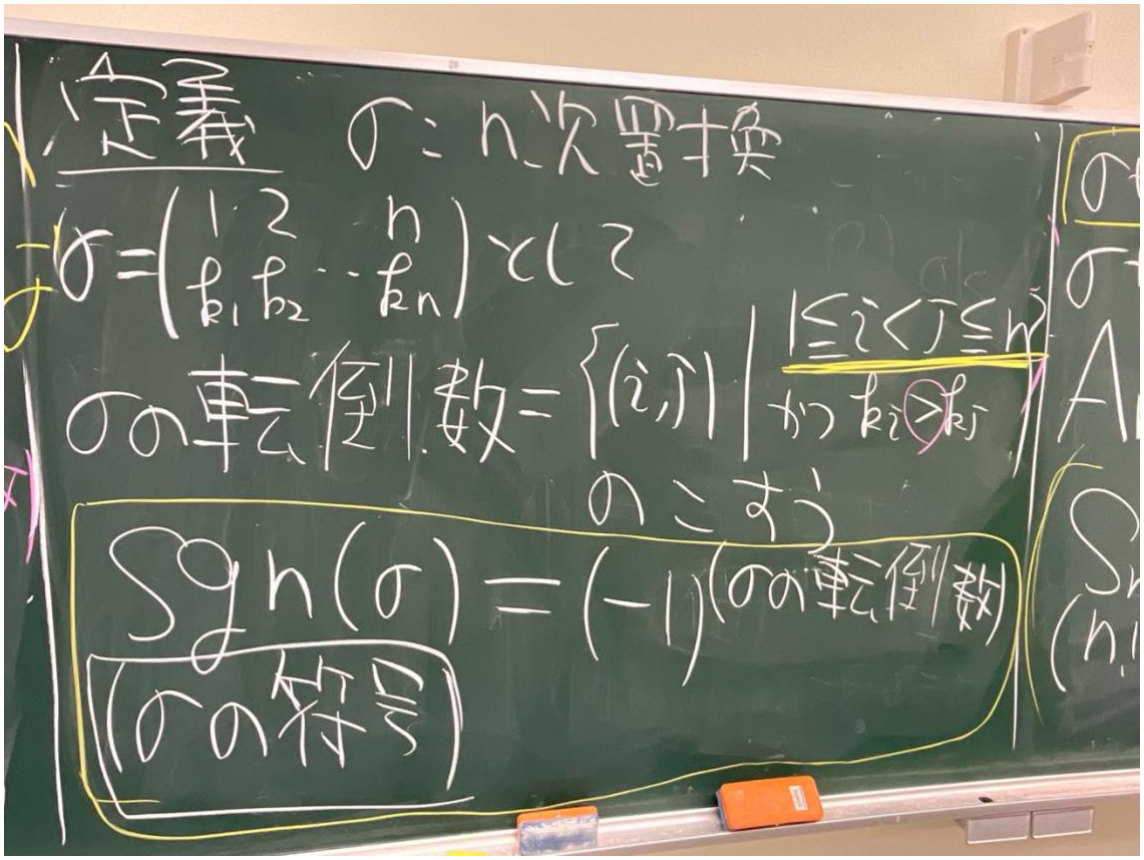

τ∘σ をもとめる
σ = (1 2 3 4 / 1 3 4 2)
τ = (1 2 3 4 / 3 1 4 2)
τ∘σ = (1 2 3 4 / 3 4 2 1)
σ
τ
1 2 3 4
1 2 3 4
1 2 3 4
1列
(例)
定義

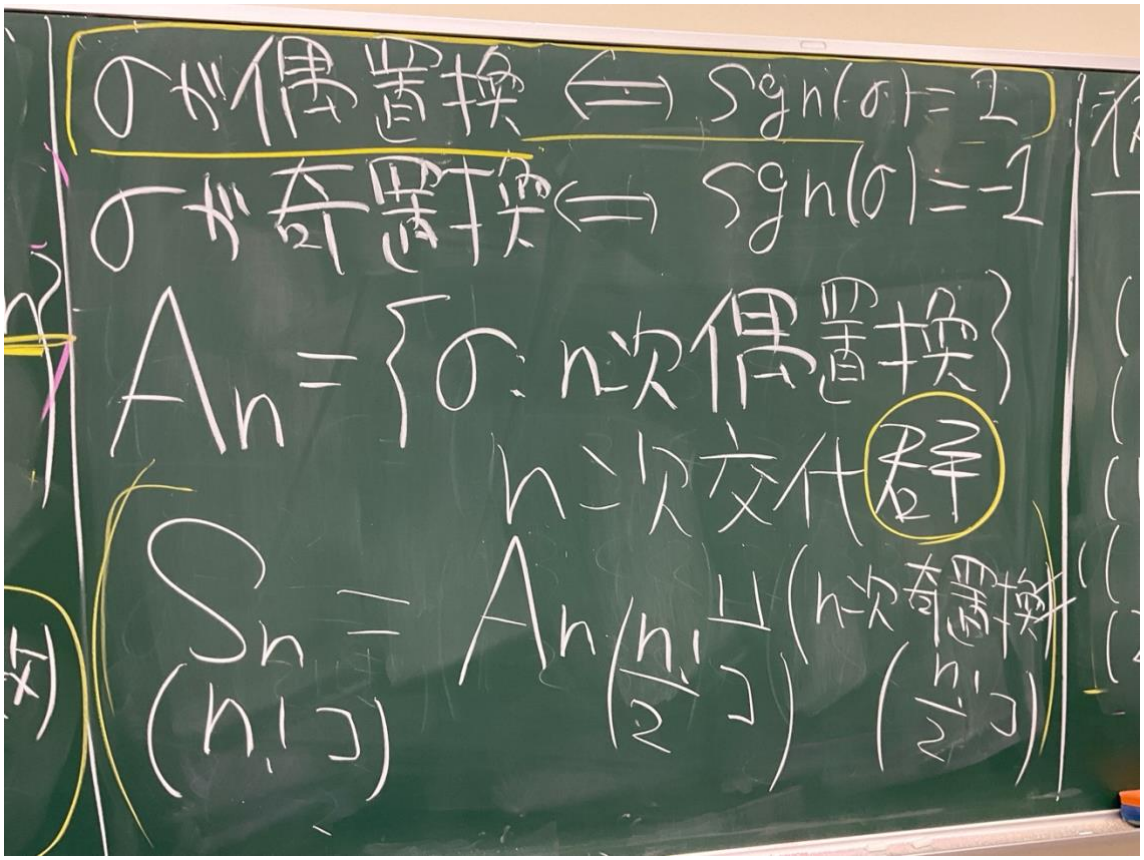

1列から τ∘σ や σ∘τ (一般に)
(例) (σ∘τ)ρ = σ(τρ) ok
定義 ε: {1..n} → {1..n}
ε(i) = i を恒等置換 (単位) 置換 という
(注1) σ∘ε = ε∘σ = σ

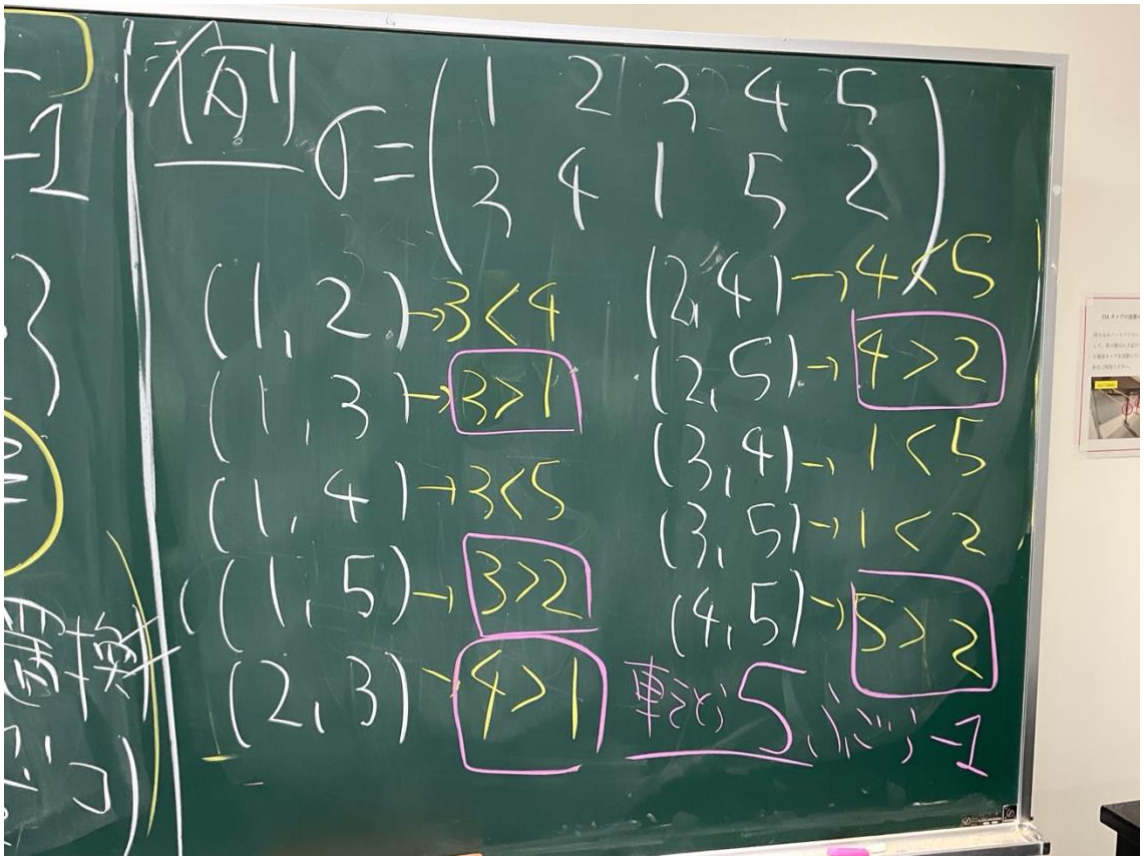

定義
の逆置換
とする
例
(逆に)

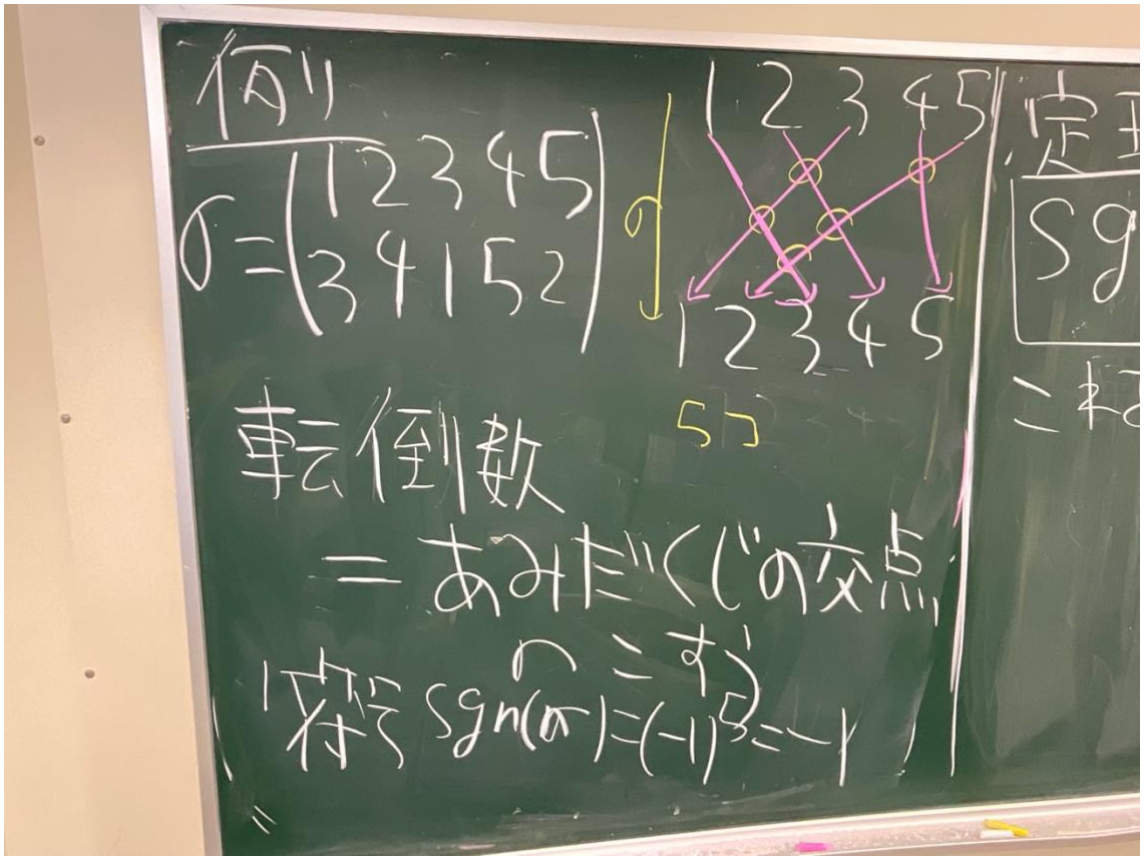

置換
の

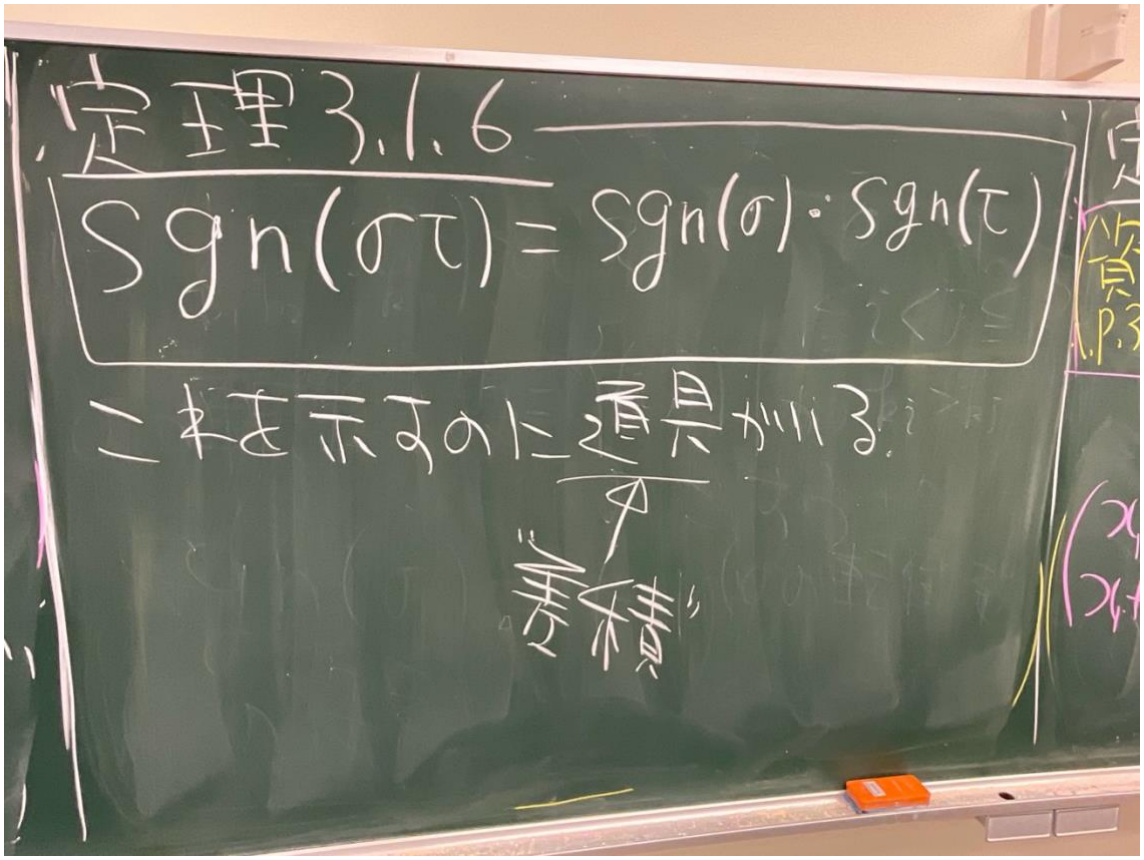

例 3.1.2
$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \end{pmatrix}$
$\sigma^{-1} = \begin{pmatrix} 2 & 4 & 1 & 5 & 3 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$
$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 5 & 2 & 4 \end{pmatrix}$
1 2 3 4 5
1 2 3 4 5
定義
Sgn
(σの符

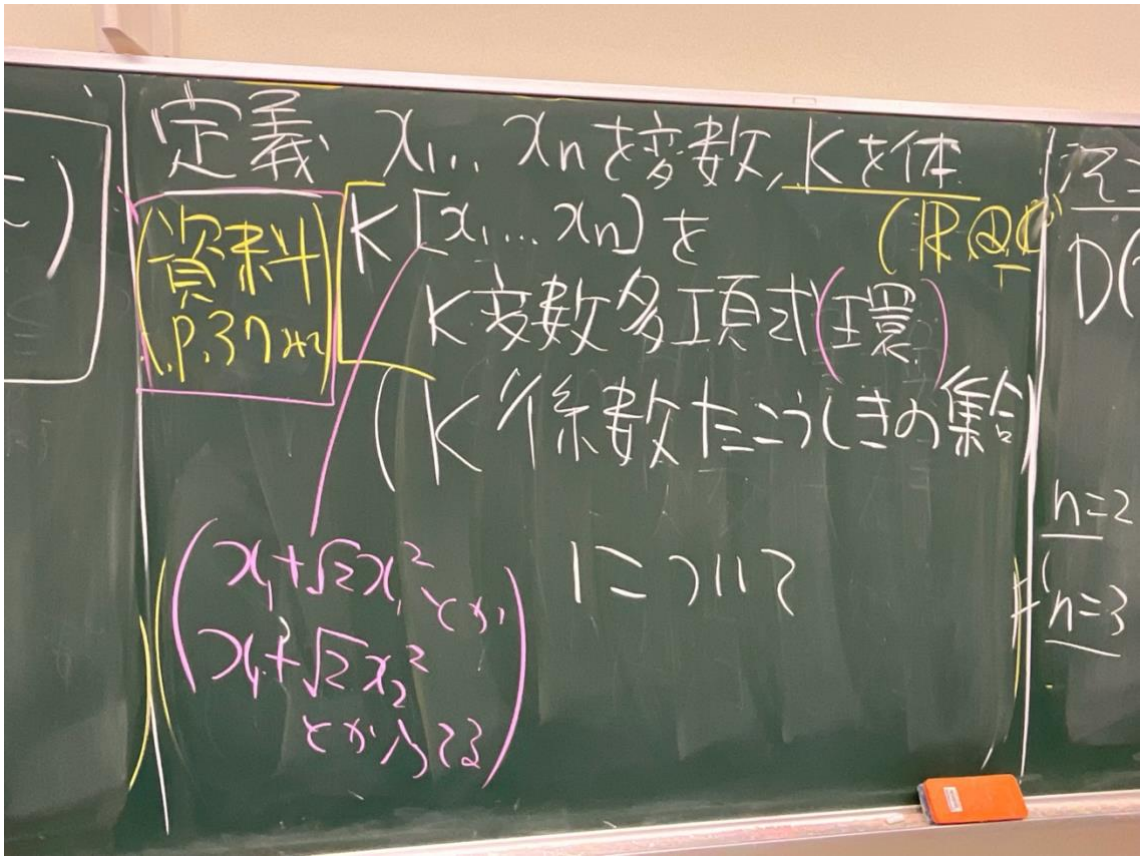

定義 σ:n次置換
$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ k_1 & k_2 & \cdots & k_n \end{pmatrix}$ として
σの転倒数 = {(i,j) | 1≦i<j≦n かつ $k_i > k_j$ のこと}
$\mathrm{Sgn}(\sigma) = (-1)^{(\sigma\text{の転倒数})}$
(σの符号)

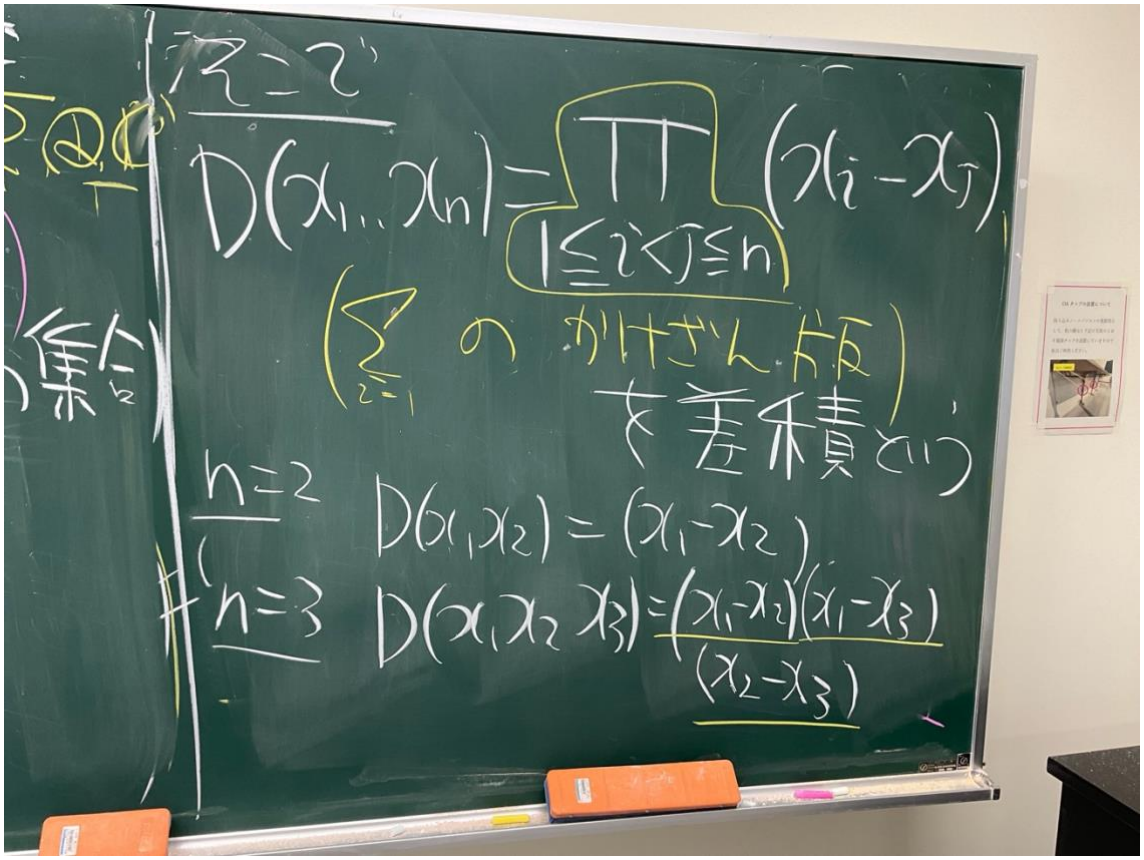

σが偶置換 ⟺ sgn(σ)=1
σが奇置換 ⟺ sgn(σ)=-1
An={σ:n次偶置換}
n次交代群
Sn
(n!コ)
An
(n!/2コ)
(n次奇置換)
(n!/2コ)

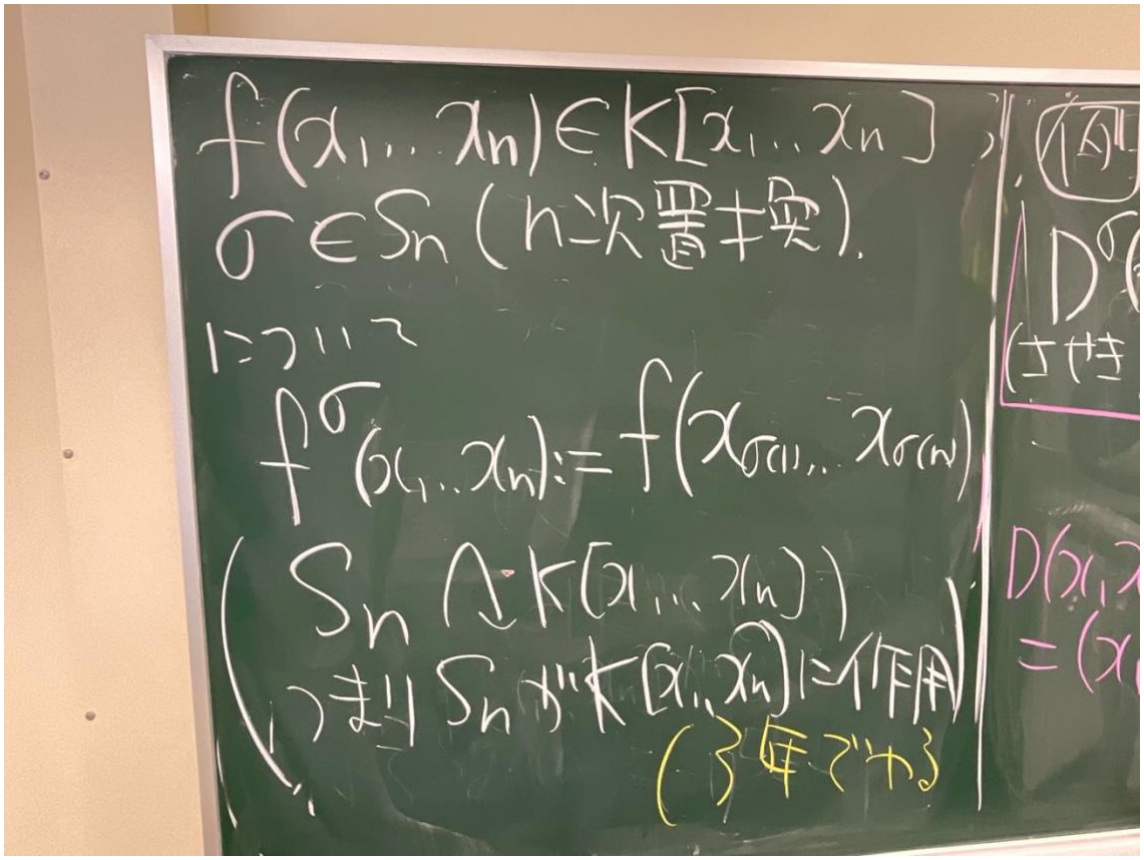

例 σ=(1 2 3 4 5 / 3 4 1 5 2)
(1,2)→3<4
(1,3)→3>1
(1,4)→3<5
(1,5)→3>2
(2,3)→4>1
(2,4)→4<5
(2,5)→4>2
(3,4)→1<5
(3,5)→1<2
(4,5)→5>2
転倒数5(小さい)

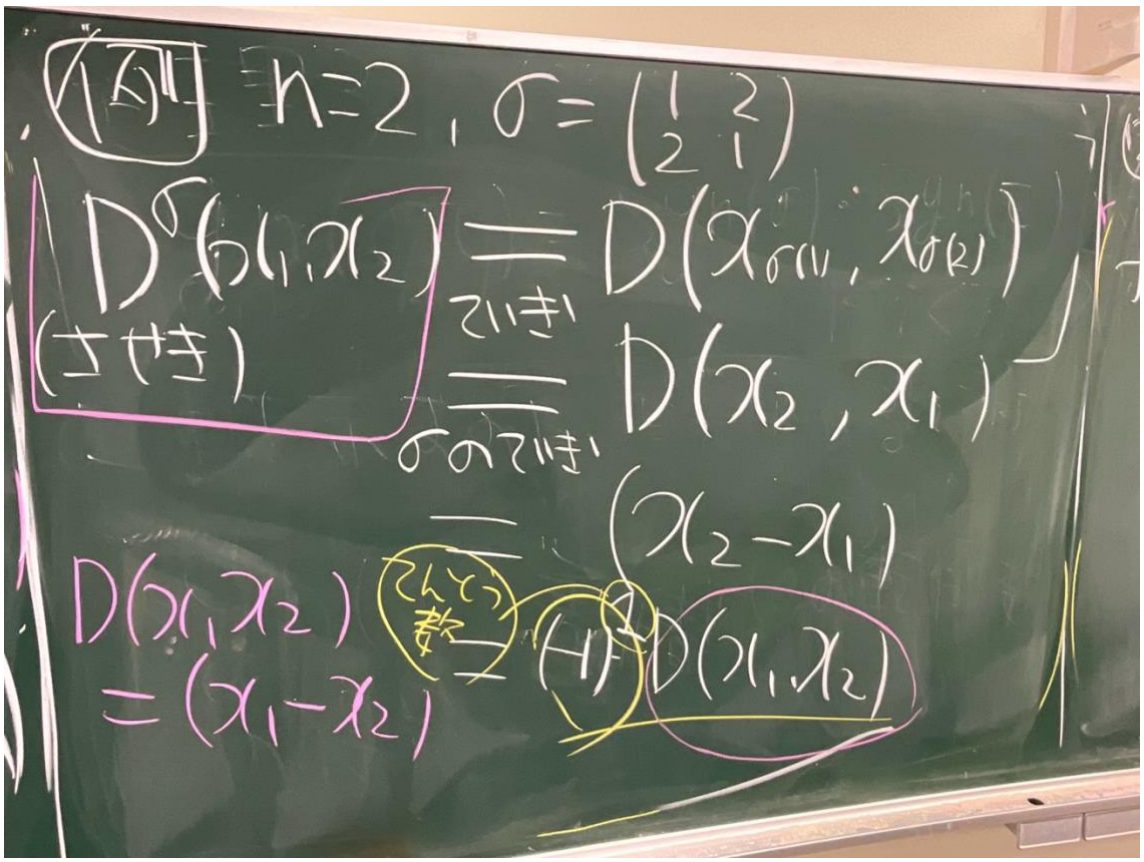

例1
$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 1 & 5 & 2 \end{pmatrix}$
1 2 3 4 5
1 2 3 4 5
5コ
転倒数
＝あみだくじの交点の個数
符号 $\mathrm{sgn}(\sigma) = (-1)^{5} = -1$

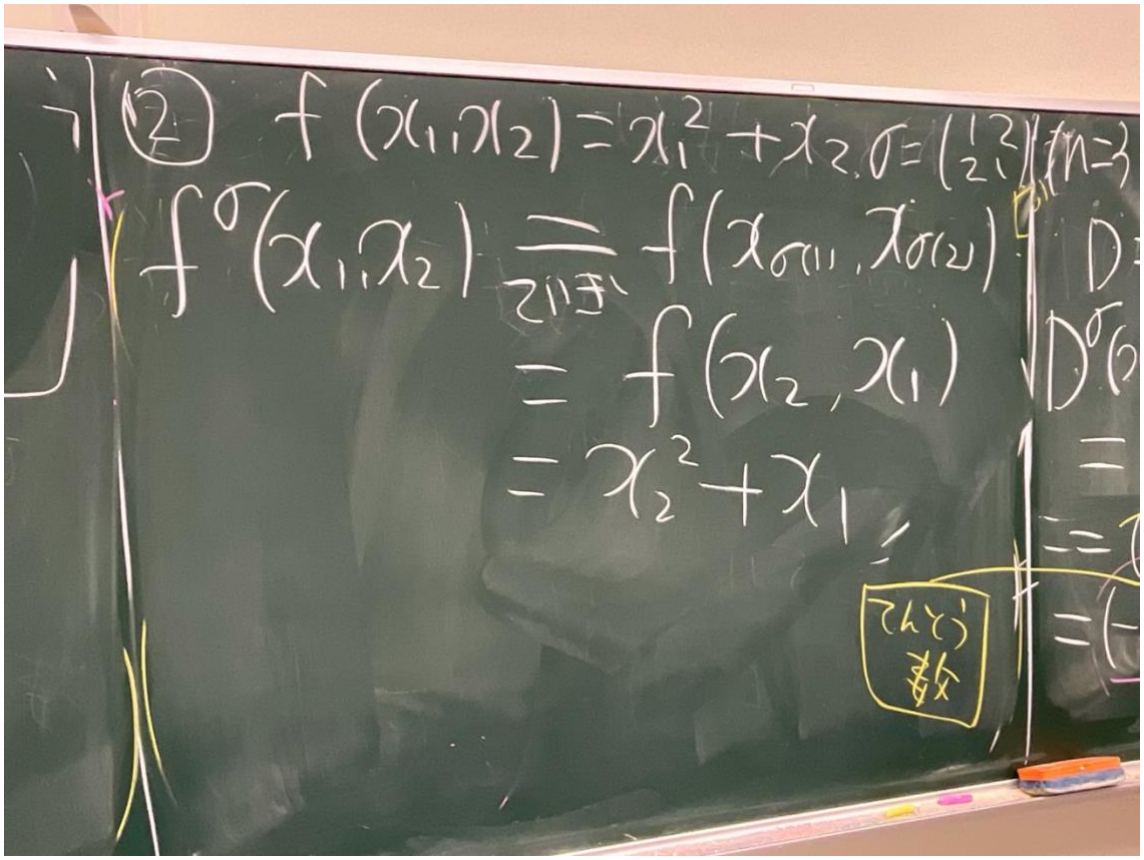

定理3.1.6
$\mathrm{sgn}(\sigma\tau) = \mathrm{sgn}(\sigma) \cdot \mathrm{sgn}(\tau)$
これを示すのに道具がいる
多項式

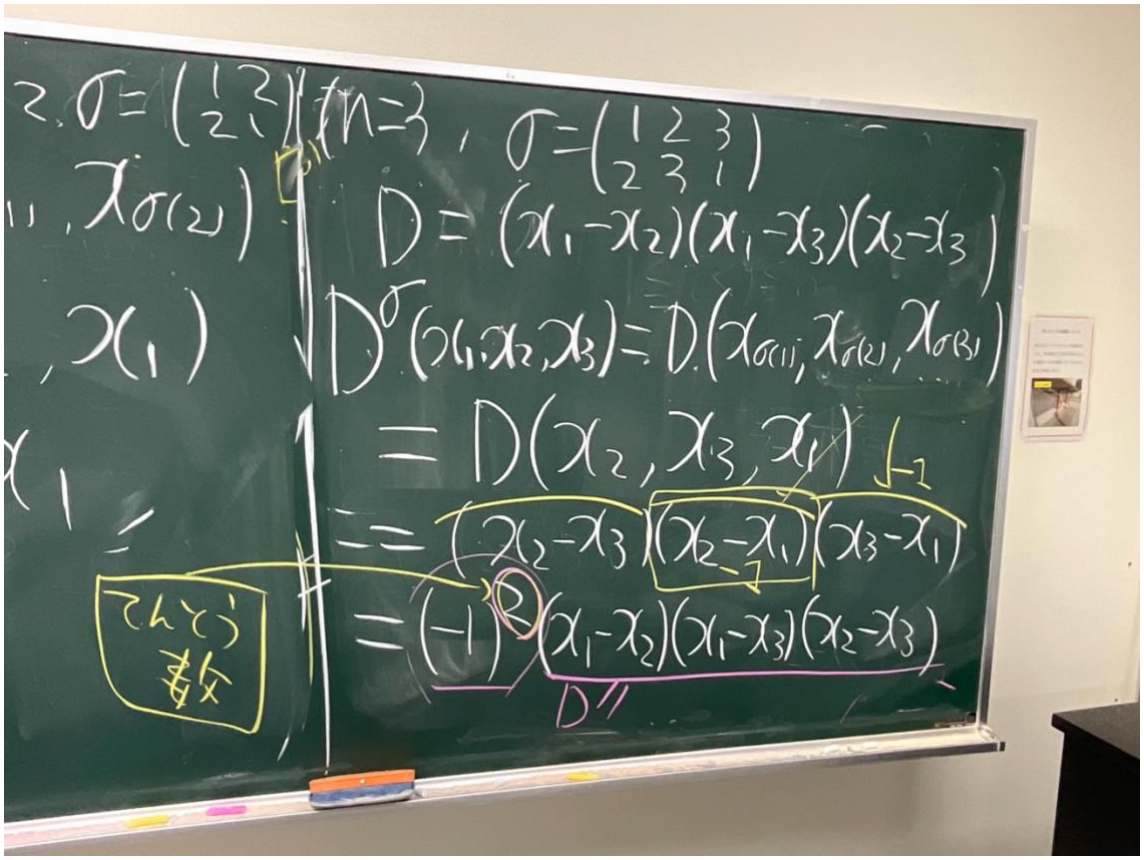

定義 $x_1, \dots, x_n$ を変数, K を体
資料 p.37 ~
$K[x_1 \dots x_n]$ を
K 係数多項式(環)
(K 係数たこうしきの集合)
$x_1 + \sqrt{2}x_1^2$ とか
$x_1 + \sqrt{2}x_2^2$
とか入る
について
$n=2$
$n=3$

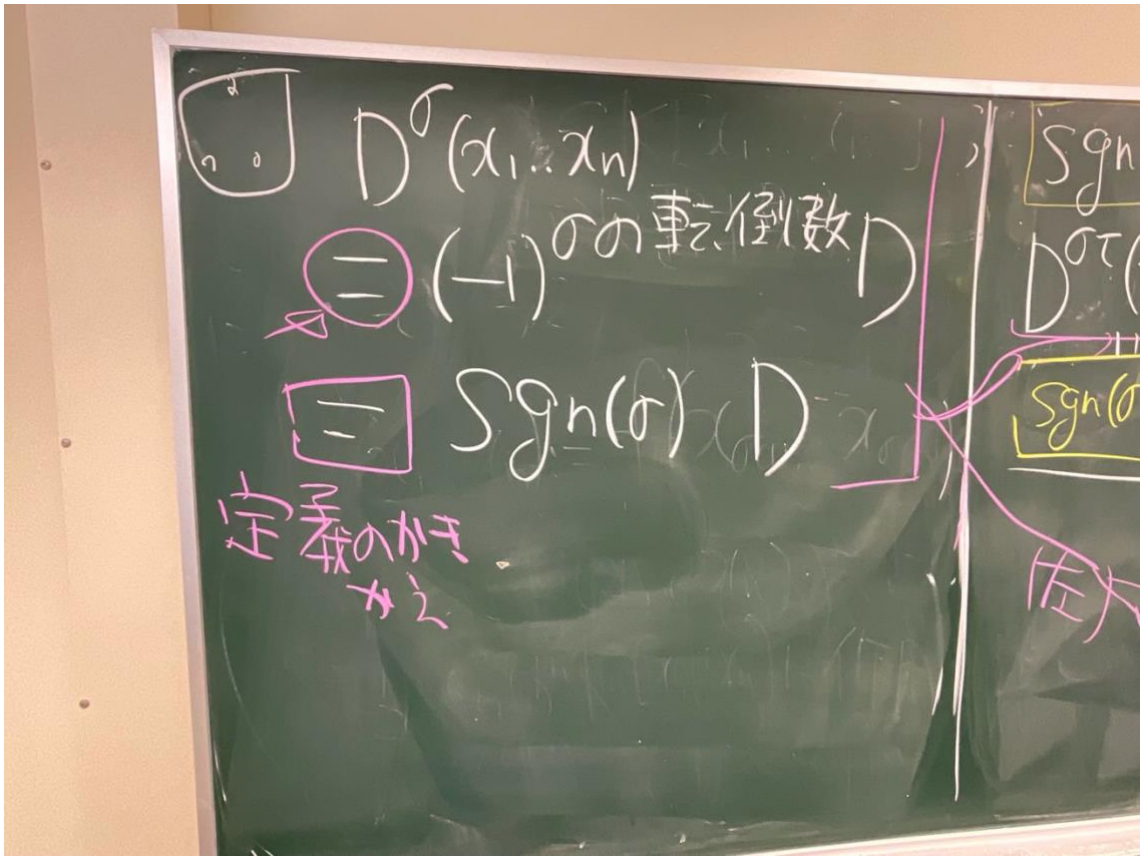

そこで
$D(x_1 \dots x_n) = \prod_{1 \le i < j \le n} (x_i - x_j)$
($\le$ の かけざん 版)
を差積という
$n=2$ $D(x_1, x_2) = (x_1 - x_2)$
$n=3$ $D(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$

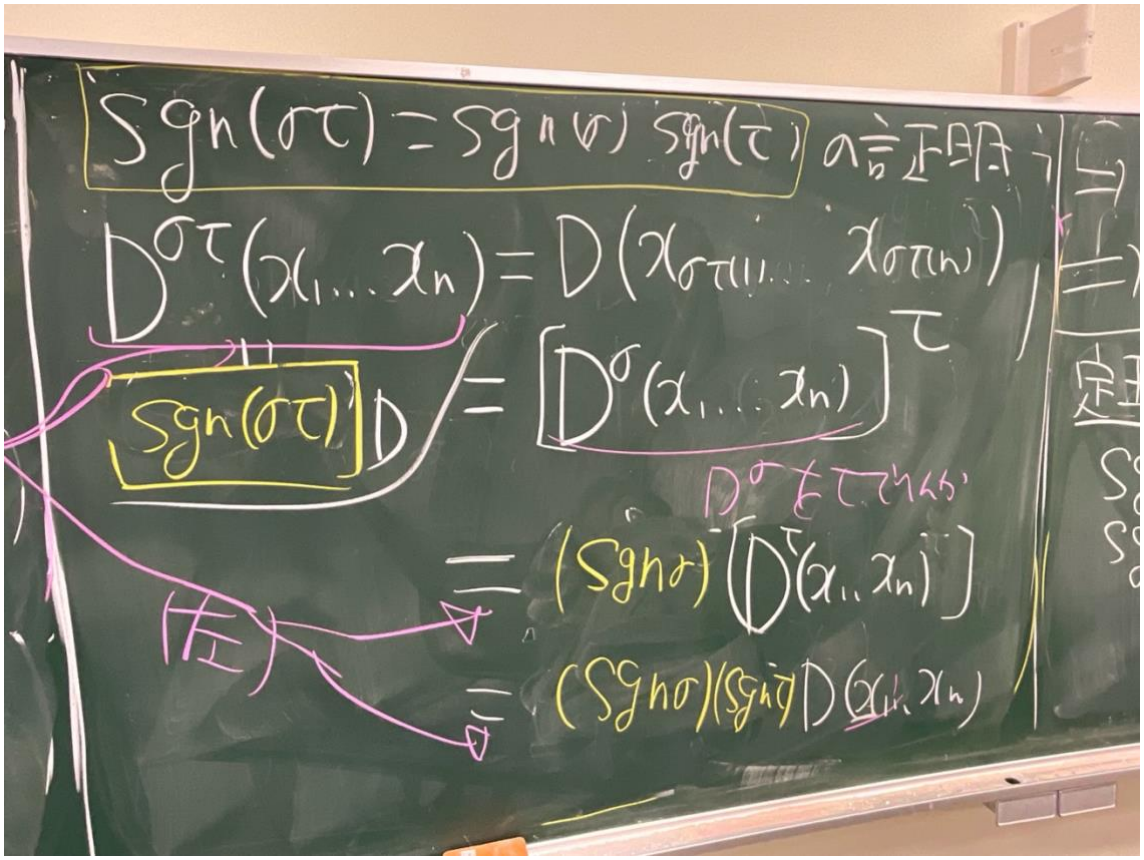

$f(x_1 \cdots x_n) \in K[x_1 \cdots x_n]$
$\sigma \in S_n$ (n次置換).
について
$f^\sigma(x_1 \cdots x_n) := f(x_{\sigma(1)} \cdots x_{\sigma(n)})$
($S_n \curvearrowright K(x_1 \cdots x_n)$
つまり S_n が $K(x_1 \cdots x_n)$ に作用)
(3年でやる)

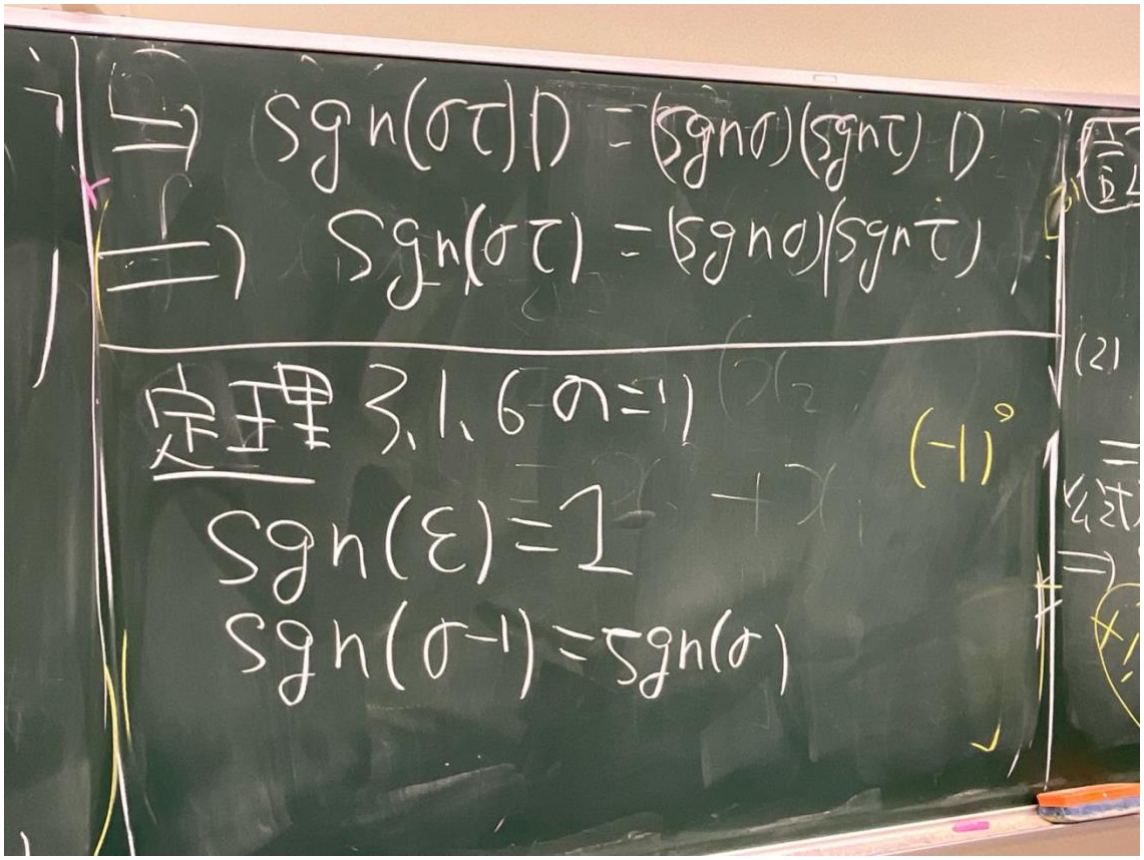

例 $n=2$, $\sigma = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$
$D^\sigma(x_1, x_2)$
(さっき)
$= D(x_{\sigma(1)}, x_{\sigma(2)})$
ていき
$= D(x_2, x_1)$
σのていき
$= (x_2 - x_1)$
$= (-1) D(x_1, x_2)$
$D(x_1, x_2) = (x_1 - x_2)$
でんじつ数

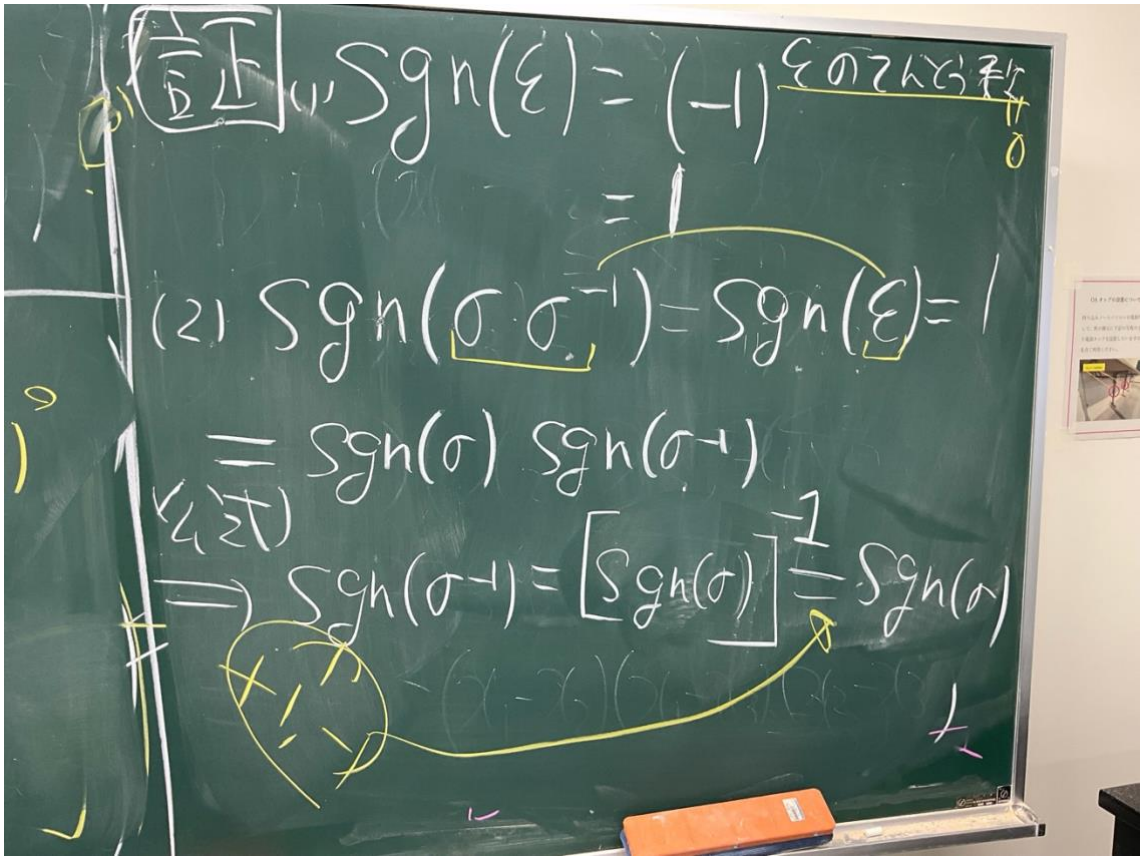

② $f(x_1,x_2)=x_1^2+x_2$. $\sigma=\begin{pmatrix}1&2\\2&1\end{pmatrix}$
$f^\sigma(x_1,x_2) = f(x_{\sigma(1)},x_{\sigma(2)})$
$= f(x_2,x_1)$
$= x_2^2+x_1$
てんとう数


$n=3$, $\sigma=\begin{pmatrix}1&2&3\\2&3&1\end{pmatrix}$
$D=(x_1-x_2)(x_1-x_3)(x_2-x_3)$
$D^\sigma(x_1,x_2,x_3)=D(x_{\sigma(1)},x_{\sigma(2)},x_{\sigma(3)})$
$=D(x_2,x_3,x_1)$
$=(x_2-x_3)(x_2-x_1)(x_3-x_1)$
$=(-1)^2(x_1-x_2)(x_1-x_3)(x_2-x_3)$
D
てんとう数


$D^\sigma(x_1 \cdots x_n)$
$= (-1)^{\sigma\text{の転倒数}} D$
$= \mathrm{Sgn}(\sigma) D$
定義のかき換え
Sgn
$D^{\sigma\tau}$
Sgn(σ
(左)


$\mathrm{Sgn}(\sigma\tau) = \mathrm{Sgn}(\sigma)\,\mathrm{Sgn}(\tau)$ の証明
$D^{\sigma\tau}(x_1 \cdots x_n) = D(x_{\sigma\tau(1)} \cdots x_{\sigma\tau(n)})$
$= [D^\sigma(x_1 \cdots x_n)]^\tau$
$\mathrm{Sgn}(\sigma\tau) D$
D^σ をてでくくる
$= (\mathrm{Sgn}\sigma)[D^\tau(x_1 \cdots x_n)]$
$= (\mathrm{Sgn}\sigma)(\mathrm{Sgn}\tau) D(x_1 \cdots x_n)$
(左)


⇒ sgn(στ)(1) = (sgnσ)(sgnτ)(1)
⇒ sgn(στ) = (sgnσ)(sgnτ)
定理 3.1.6 の(1)
sgn(ε) = 1
sgn(σ⁻¹) = sgn(σ)
(-1)²


証明 (1) sgn(ε) = (-1)^0
= 1
(2) sgn(σ σ⁻¹) = sgn(ε) = 1
= sgn(σ) sgn(σ⁻¹)
公式
⇒ sgn(σ⁻¹) = [sgn(σ)]⁻¹ = sgn(σ)